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Spaces of distributions with prescribed Besov wavefront set
Spazi di distribuzioni con fronte d'onda di tipo Besov fissato

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Abstract

La tesi è dedicata allo studio di una topologia di tipo Hörmander su spazi di distribuzioni con fronte d'onda di tipo Besov fissato. Dopo una revisione di alcuni concetti chiave della teoria degli spazi vettoriali topologici e della teoria delle distribuzioni, verrà introdotta la teoria dei fronti d'onda, con le sue applicazioni alla costruzione del fronte d'onda di tipo Besov e della topologia di Hörmander nel caso liscio. Successivamente, verrà introdotto il formalismo delle seminorme di tipo Hörmander-Besov su spazi di distribuzioni con fronte d'onda di tipo Besov fissato. La topologia indotta da queste seminorme sarà caratterizzata tramite un'immersione in un opportuno spazio di distribuzioni. Infine, verrà costruito un candidato per il duale di questi spazi, insieme a una dualità ipocontinua che li renda una coppia duale.

This thesis is dedicated to the study of a Hörmander-type topology on spaces of distributions with a prescribed Besov wavefront set. After a succinct review of some key concepts from the theory of topological vector spaces and distribution theory, the theory of wavefront sets will be introduced, along with its application to the construction of the Besov wavefront set and of the Hörmander's topology in the smooth setting. Then, the framework of Hörmander-Besov seminorms will be introduced on spaces of distributions with a prescribed Besov wavefront set. The topology induced by these seminorms will be characterized through an immersion in a suitable space of distributions. Lastly, a candidate for the dual of these spaces will be constructed along with a hypocontinuous duality that renders them into a dual pair.

Contents

Introduction

"Is this all real? Or is it just happening inside my head?"
"Of course it is happening inside your head, Harry. But why on Earth should that mean that it is not real?"

Harry Potter and the Deathly Hallows

When dealing with the state of a physical system, singularities often carry vital information. For instance, information about the position of a pointwise source, such as an electrically charged particle, can be encoded by the singularity of a Dirac delta distribution centered at that source. Analogously, a key feature of relativist time evolution equations, namely that information carried by a physical signal travels at finite speed, see, *e.g.*, [1], is encoded in the time evolution of the support of its initial data. More specifically, tracking the evolution of the boundary of the support, which carries a discontinuity unless the function is smooth, allows one to obtain information about the speed of propagation in the specific model. It is in this framework that the theory of wavefront sets arose.

The wavefront set of a function or distribution can be heuristically described as the collection of points in physical space and directions in momentum space at which the function or distribution fails to behave smoothly. The theory surrounding this concept was pioneered by Lars Hörmander and Johannes Jisse Duistermaat in the 1970, notably [2] an [3], as a tool for studying the propagation of singularities for solutions of Partial Differential Equations (PDEs). Their work addresses precisely the above raised question: given an evolution equation and precise information regarding the singularities of the initial data, can one determine, at any later time, where those singularities propagate? The notion of wavefront set is therefore particularly well-suited for this investigation, as it contains not only information of where the distribution at hand is singular, but also on the directions towards which the lack of regularity can spread. Conversely, one may ask whether a solution to a specific PDE can be chosen so that its singular behaviour propagates only within a prescribed region of space and along prescribed directions.

From a physical point of view, the finite propagation speed required by special and general relativity entails that information carried by a physical state must propagate within the causal cone determined by the initial event. This causal restriction is reflected by the requirement that the wavefront set of the distribution representing the physical signal needs to be contained in a corresponding conic set. One way to obtain solutions satisfying such a physical constraint is through functional-analytic methods. More specifically, in [2] and [4], Hörmander introduced a convergence underlying what is now known as "Hörmander's topology", namely a mode of convergence for sequences of distributions that controls the wavefront set of the limit. Hence, starting from the PDE that describes the physical model, one can identify a solution with a prescribed wavefront set by constructing a sequence of approximated solutions that converges with respect to Hörmander's topology. Then, the required wavefront set condition is automatically satisfied by the limit distribution.

Another key point in the study of physical models encoded by PDEs is the investigation of the global regularity of their solutions. In this regard, a common approach relies on the use of specific function spaces whose definitions encodes the required degree of regularity. Examples include Sobolev and Besov spaces, see [5] and [6] respectively. Besov spaces are particularly useful for studying solutions to non-linear PDEs, such as the Navier-Stokes equations, see [7] and [8], or Einstein's equations, see [9] and [10], due to the control they provide over products of distribution and, consequently, over polynomial non-linearities. Their widespread use also extends to the theory of Stochastic PDEs (SPDEs), *i.e.*, partial differential equations involving random coefficients, fields or source terms, see, *e.g.*, [11]. Given their large application in many differential problems of physical relevance, it is natural to ask whether the aforementioned theory of wavefront sets can be refined to encode the finer degrees of regularity captured by Besov spaces. As was shown in [12], where the authors developed the notion of wavefront set of Besov type, this is indeed the case. Analogously to its smooth counterpart, the Besov wavefront set records where and in which direction in phase space a distribution fails locally to possess a prescribed Besov regularity. It can therefore be used to track the irregular behaviour of solutions to PDEs and SPDEs with respect to finer degrees of regularity than its smooth analogue. Hence, the next natural step in this research is to construct an analogue of Hörmander's topology on the case of spaces of distributions with a prescribed Besov wavefront set. Such a tool, then, similarly to its smooth counterpart, could be used to construct solutions to differential equations while retaining control of their singularities with respect to a fixed Besov regularity.

A synopsis of the thesis is the following:

The first chapter focuses on the presentation of the mathematical tools required in the rest of the work. In the first section, we briefly recall the theory of topological vector spaces, ranging from the topology on spaces of linear operators to the construction of projective and inductive limit spaces. Then, in the second section we review the theory of distributions, with particular emphasis on Fourier analysis and the topological properties of arbitrary spaces of test functions and distributions. Last but not least, in the third section we recount the theory of Besov spaces, from their construction and characterizations to their duality properties.

The second chapter is devoted to the presentation of the theory of wavefront sets. In the first section, we review the theory of pseudodifferential operators, focusing firstly in the global case on Euclidean space and then on its local counterpart on open sets. In the second section, we recall the theory of smooth wavefront sets both through explicit Fourier estimates and through the calculus of pseudodifferential operators introduced in the former section. Furthermore, we also review the theory of wavefront sets of Besov regularity and the construction of Hörmander's topology for spaces of distributions with a prescribed smooth wavefront set.

The third chapter is centered around the development of an analogue of Hörmander's topology for spaces of distributions with a prescribed Besov wavefront set. In the first section, we construct explicitly the appropriate seminorms that induce the counterpart of Hörmander's topology on the aforementioned spaces of distributions. Then, in the second section we derive the topological properties of this constructions by means of a topological embedding in a suitably chosen space of distributions. Lastly, in the third section we construct a candidate for the dual of such spaces and its associated hypocontinuous duality. We also discuss the generalization and continuity within this framework of structural operations between distributions, such as the tensor product and the pullback by smooth functions.

Chapter 1

Mathematical Introduction

The road is wild and wicked
Winding through the wood,
Where all that's wrong is right
And all that's bad is good.

The Ballad of the Witches' Road

Before delving into the main topics of this work, we shall summarize in this first chapter some basic notions of paramount importance for the development of Chapters 2 and 3.

Aim of Section 1.1 is to review selected concepts from the theory of topological vector spaces and their applications to the characterization of the functional spaces which will be explored through the rest of this work. A more thorough discussion of these topics can be found in [13], [14] and [15]. In Section 1.2, we shall recall some key notions from the theory of distributions and their applications to the study of the distributional Fourier transform. We refer to [16, Part II] and [4] for a more complete analysis of this topics. Lastly, in Section 1.3 we shall recount the definition and properties of Besov spaces and local Besov spaces. These topics are delved into in more detail in [17] and [6].

1.1 Topological Vector Spaces

As our work is concerned with the topological properties of a space of distribution, we begin by recalling some relevant notions from functional analysis. In Subsection 1.1.1, we will introduce topological vector spaces together with generalizations of concepts familiar from the theory of normed spaces, such as disks and bounded sets. In Subsection 1.1.2, then, we shall report on the application of this theory to spaces of linear, continuous operators between topological vector spaces, with particular regards to the different topologies one can endow this space with and their interplay. Lastly, in Subsection 1.1.3 we shall discuss the construction and properties of Fréchet spaces and limit spaces, as well as the notion of duality of vector spaces, due to their applications to the topics of Section 1.2.

1.1.1 Preliminary notions

Topological vector spaces were introduced in order to generalize methods from functional analysis to a broader class of spaces than Banach ones.

Definition 1.1.1 (TVS)**.** *Let X be a vector space over an Archimedean, valued field $\mathbb{K}$ endowed with a topology τ. The pair (X,τ) is a* ***Topological Vector Space (TVS)*** *if the maps*

$$X \times X \ni (x,y) \longmapsto x + y \in X, \quad \mathbb{K} \times X \ni (\lambda, x) \longmapsto \lambda x \in X$$

are continuous with respect to the product topologies of $X \times X$ and $\mathbb{K} \times X$, where $\mathbb{K}$ is endowed with its natural metric topology, see [13, Prerequisites C]. If (X,τ) is a TVS, then we call τ a ***vector topology*** *on X.*

Remark 1.1.1. In Definition 1.1.1, as well as in the following, the field $\mathbb{K}$ is assumed to be either $\mathbb{R}$ or $\mathbb{C}$. The results presented in Section 1.1 can be slavishly extended to vector spaces over any valued Archimedean field $\mathbb{K}$, see [13, Prerequisites C]. □

Remark 1.1.2. For subsets $U, V \subset X$ of a vector space over $\mathbb{K}$ and $I \subset \mathbb{K}$, we set

$$V + U \doteq \{v + u \mid u \in U \text{ and } v \in V\},$$
$$I \cdot U \doteq \{\alpha u \mid \alpha \in I \text{ and } u \in U\}.$$

If $V = \{x\}$, then we write $x + U$ in place of $V + U$, and if $I = \{\alpha\}$, then we write αU in place of $I \cdot U$. □

Example 1.1.1. Every normed space $(X, \|\cdot\|_X)$ endowed with its norm topology is a TVS as per Definition 1.1.1. This follows from the properties of the norm, see [14, Chapter 15.1]. □

Remark 1.1.3. Let $\{X_\alpha \mid \alpha \in A\}$ be a collection of TVSs as per Definition 1.1.1 for some index set A and let $X \doteq \prod_{\alpha \in A} X_\alpha$ be their Cartesian product, *i.e.*, X is the space of functions $f : A \to \bigcup_{\alpha \in A} X_\alpha$ such that $f(\alpha) \in X_\alpha$ for every $\alpha \in A$. X carries a natural structure of vector space over $\mathbb{K}$ with addition $(f + g) : \alpha \mapsto f(\alpha) + g(\alpha)$ and scalar multiplication $(\lambda f) : \alpha \mapsto \lambda f(\alpha)$ for every $f, g \in X$ and $\lambda \in \mathbb{K}$. Furthermore, X also carries a natural topology τ, which is the unique topology for which the projection maps $\pi_\alpha : X \to X_\alpha$, $\pi_\alpha(f) = f(\alpha)$, are continuous, see [18, Theorem 3.37]. If τ_α is the vector topology of X_α and τ is the aforementioned topology on X, then (X,τ) is a TVS, see [13, Chapter I.2]. This is also an example of the construction of projective limit spaces, see Definition 1.1.17. □

In general, a TVS need be neither normable nor metrizable, as per the following proposition.

Proposition 1.1.1. *Let (X,τ) be a Hausdorff TVS as per Definition 1.1.1. The topology τ is metrizable if and only if there exists a neighbourhood U of $0 \in X$ such that for every neighbourhood V of 0 there exists $\lambda_V \in \mathbb{K}$ for which $U \subset \lambda V$ for every $\lambda \in \mathbb{K}$ whenever $|\lambda| \geq |\lambda_V|$. The topology τ is normable if and only if the aforementioned neighbourhood U can also be chosen to be convex.*

A proof of the first part of this result can be found in [13, Proposition 6.2], whereas a proof of the second part can be found in [13, Proposition 2.1].

Since a general TVS need not carry a metric structure, the role played by balls in normed spaces must be separated into several, distinguished properties. This leads to the following terminology.

Definition 1.1.2. *Let (X,τ) be a TVS over the field $\mathbb{K}$ as per Definition 1.1.1 and let $U \subset X$. The set U is said to be*

- *1. **absorbed** by $V \subset X$ (equivalently, V is said to **absorb** U), if there exists $\lambda_V \in \mathbb{K}$ such that $U \subset \lambda V$ for every $\lambda \in \mathbb{K}$ with $|\lambda| \geq |\lambda_V|$;*
- *2. **absorbing** if U absorbs every finite subset of X;*
- *3. **balanced** if $\lambda U \subset U$ for every $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$;*
- *4. a **disk** if it is convex and balanced;*
- *5. a **barrel** if it is an absorbing and closed disk.*

Remark 1.1.4. Let (X,τ) be a TVS, see Definition 1.1.1, whose vector topology is induced by a translation invariant distance $d : X \times X \to [0,+\infty)$, *e.g.*, a normed spaces $(X, \|\cdot\|_X)$. Then, for every $r > 0$, the open ball centered at the origin $B(0,r) \doteq \{x \in X \mid d(x,0) < r\}$ is an absorbing disk as per Definition 1.1.2. Similarly, every closed ball $\overline{B(0,r)} \doteq \{x \in X \mid d(x,0) \leq r\}$ is a barrel. □

To further differentiate between the fundamental properties of balls in normed spaces, one has to investigate the notion of boundedness in non-metric spaces. This additional structure is given by the notion of a bornology.

Definition 1.1.3 (Bornology)**.** *A* ***bornology*** $\mathfrak{B}$ *on a set* X *is a family of subsets of* X *with the properties:*

- $(B1)$ $\mathfrak{B}$ *covers* X*, i.e.,* $X = \bigcup_{B \in \mathfrak{B}} B$*;*
- $(B2)$ *if* $B \in \mathfrak{B}$ *and* $A \subset B$*, then* $A \in \mathfrak{B}$*;*
- $(B3)$ *if* $B_1, \dots, B_N \in \mathfrak{B}$*, then* $\bigcup_{j=1}^{N} B_j \in \mathfrak{B}$*.*

The elements of a bornology are called ***bounded sets****.*

Remark 1.1.5. Let (X, τ) be a TVS as per Definition 1.1.1. We say that $U \subset X$ is bounded if every neighbourhood of 0 absorbs it, see Definition 1.1.2. The family $\mathfrak{B}_\tau$ of all sets that are bounded in the aforementioned sense forms the natural bornology for the TVS X as per Definition 1.1.3, see [13, Chapter I.5]. In this terminology, Proposition 1.1.1 states that a Hausdorff TVS is metrizable if and only if it possesses a bounded neighbourhood of 0 and it is normable if and only if it possesses a convex, bounded neighbourhood of 0. □

We next recall the space of continuous linear maps between two TVSs and fix the notation regarding dual spaces and isomorphisms of TVSs.

Definition 1.1.4 (Continuous operators)**.** *Let* (X, τ) *and* (Y, σ) *be TVS, as per Definition 1.1.1. We denote by* $\boldsymbol{\mathfrak{L}(X, Y)}$ *the space of all continuous linear maps from* X *to* Y *and we set* $\boldsymbol{\mathfrak{L}(X)} \doteq \mathfrak{L}(X, X)$*. Whenever* $Y = \mathbb{C}$ *with its natural norm topology, we call* $\boldsymbol{X'} \doteq \mathfrak{L}(X, \mathbb{C})$ *the* ***topological dual space*** *of* X*. We say that a linear map* $F : X \to Y$ *is an* ***isomorphism*** *if it is invertible, continuous and its inverse is continuous.*

Remark 1.1.6. Let (X, τ) be a TVS and let $v \in X$ and $\lambda \in \mathbb{K}$, $\lambda \neq 0$, be fixed. By Definition 1.1.1, the map $X \ni x \mapsto \lambda x + v \in X$ is an isomorphism in the sense of Definition 1.1.4. This property is known in the literature as **translation invariance** of the vector topology τ. At the level of open sets, it entails that the topology is closed under finite addition and scalar multiplication in the sense of Remark 1.1.2. Furthermore, as is the case for normed spaces, it follows also that the topology τ can be fully characterized by identifying a base of neighbourhoods of $0 \in X$. Theorem 1.1.2 makes this statement precise. □

Definition 1.1.5 (Filter)**.** *A* ***filter*** $\mathfrak{F}$ *on a set* X *is a collection of subsets of* X *such that*

- $(F1)$ $\mathfrak{F} \neq \varnothing$ *and* $\varnothing \notin \mathfrak{F}$*;*
- $(F2)$ *if* $F \in \mathfrak{F}$ *and* $F \subset G \subset X$*, then* $G \in \mathfrak{F}$*;*

- *$(F3)$ if $F, G \in \mathfrak{F}$, then $F \cap G \in \mathfrak{F}$.*

Theorem 1.1.2. *Let X be a vector space over the field $\mathbb{K}$ and τ be a topology on X. (X, τ) is a TVS, see Definition 1.1.1,if and only if all the translation maps $X \ni x \mapsto x + v \in X$ as $v \in X$ are isomorphisms as per Definition 1.1.4 and every base of neighbourhoods $\mathscr{U}$ of $0 \in X$ satisfies:*

- *$(V1)$ for each $U \in \mathscr{U}$ there exists $V \in \mathscr{U}$ such that $V + V \subset U$;*
- *$(V2)$ every $U \in \mathscr{U}$ is absorbing and balanced as per Definition 1.1.2.*

Conversely, let $\mathscr{U}$ be a non-empty filter on X, see Definition 1.1.5, such that $0 \in U$ for every $U \in \mathscr{U}$. In addition, assume that for every $U_1, U_2 \in \mathscr{U}$ there exists $U_3 \in \mathscr{U}$ such that $U_3 \subset U_1 \cap U_2$. If $\mathscr{U}$ further satisfies properties $(V1)$ and $(V2)$, then it is a neighbourhood of 0 for a unique vector topology $\tilde{\tau}$ on X.

A proof of this theorem can be found in [13, Proposition 1.2]. Theorem 1.1.2 entails the following corollary, a proof of which can be found in [13, Chapter III.1].

Corollary 1.1.3. *Let (X, τ) and (Y, σ) be TVS as per Definition 1.1.1 and let $T : X \to Y$ be a linear map. Then T is continuous if and only if T is continuous at 0.*

In normed spaces, the notions of topology and bornology are closely related. This is further proved by the fact that all continuous operators between normed spaces are locally bounded, *i.e.*, they send bounded sets into bounded sets, and viceversa. This equivalence fails for arbitrary TVSs, but it is recovered for a smaller class of spaces. In order to define them, we require the following notion.

Definition 1.1.6 (LCTVS). *Let (X, τ) be a TVS as per Definition 1.1.1. We say that τ is **locally convex** if there exists a base of neighbourhoods of $0 \in X$ made of convex sets. We say that (X, τ) is a **Locally Convex Topological Vector Space (LCTVS)** if τ is locally convex and Hausdorff.*

Remark 1.1.7. Any normed space $(X, \|\cdot\|_X)$ is an LCTVS, as it is Hausdorff and the collection of open balls $B(0, r)$, $r > 0$, is a base of convex neighbourhoods of $0 \in X$. □

Definition 1.1.7 (Bornological spaces). *Let (X, τ) be an LCTVS as per Definition 1.1.6 and let $\mathfrak{B}$ be a bornology on X as per Definition 1.1.3. X is **$\mathfrak{B}$—bornological** if every disk which absorbs all bounded sets, see Definition 1.1.2, is a neighbourhood of 0 in the topology τ. If $\mathfrak{B} = \mathfrak{B}_\tau$, see Remark 1.1.5, then (X, τ) is called **bornological**.*

For bornological space, continuity admits the following characterization, which retraces its more familiar analogue for normed spaces.

Theorem 1.1.4. *Let (X,τ) be a bornological space as per Definition 1.1.7 and let (Y,σ) be an LCTVS as per Definition 1.1.6, both endowed with the bornologies induced by their respective topologies, see Remark 1.1.5. A linear map $T : X \to Y$ is continuous if and only if one of the following conditions holds:*

- *1. T is continuous at 0;*
- *2. T is sequentially continuous at 0;*
- *3. T is locally bounded, i.e., for every bounded $B \subset X$, $T[B] \subset Y$ is bounded.*

A proof of this result can be found in [14, Section 28.3, (4)].

Amongst the various vector topologies that can be defined, those generated by seminorms will be of primary interest within this work.

Definition 1.1.8. *Let X be a vector space over the field $\mathbb{K}$. A seminorm on X is a function $p : X \to [0,+\infty)$ with the properties that $p(x+y) \leq p(x)+p(y)$ and $p(\lambda x) = |\lambda| p(x)$ for every $x, y \in X$ and $\lambda \in \mathbb{K}$.*

Remark 1.1.8. Let X be a vector space over the field $\mathbb{K}$ and let $\{p_\alpha \mid \alpha \in A\}$, for some index set A, be a separating collection of seminorms, *i.e.*, each p_α is a seminorm as per Definition 1.1.8 and for every $x \in X$, $x \neq 0$, there exists an $\alpha \in A$ such that $p_\alpha(x) \neq 0$. For every $\varepsilon > 0$ and finite $J \subset A$, we set

$$U(\varepsilon, J) \doteq \{x \in X \mid p_\alpha(x) < \varepsilon \ \forall \ \alpha \in J\}.$$

The topology induced by the seminorms $\{p_\alpha \mid \alpha \in A\}$ is the unique vector topology τ for which the collection of $U(\varepsilon, J)$ is a base of neighbourhoods of 0, see Theorem 1.1.2. This topology is locally convex as per Definition 1.1.6 and Hausdorff, see [14, Section 18.1, (3)]. Hence, X endowed with this topology is an LCTVS. Conversely, the topology of any LCTVS is always induced by a separating collection of seminorms. obtained as the Minkowski functional of each element of a prescribed convex base of neighbourhoods of 0, see [13, Chapter II.4]. □

Remark 1.1.9. As an immediate consequence of Remark 1.1.8 and Proposition 1.1.4, one has that, for a given LCTVS X, a subset $B \subset X$ is bounded, see Remark 1.1.5, if and only if for every continuous seminorm p on X

$$\sup_{x \in B} p(x) < +\infty.$$

Analogously, a linear map $T : X \to Y$ is continuous if and only if for every continuous seminorm q on Y there exist constants $C_1, \ldots, C_N > 0$ and continuous seminorms $p_1, \ldots, p_N$ on X such that

$$q(T(x)) \leq \sum_{j=1}^{N} C_j p_j(x),$$

for all $x \in X$. □

1.1.2 Spaces of continuous operators

Theorem 1.1.2 provides a convenient method to construct vector topologies. This is especially useful when constructing one on a space of linear operators between TVSs. To this end, let (X,τ) and (Y,σ) be two TVS as per Definition 1.1.1 and let $\mathscr{M}$ be a collection of subsets of X with the following properties:

- $(M1)$ every element of $\mathscr{M}$ is bounded as per Remark 1.1.5;
- $(M2)$ if $M_1, M_2 \in \mathscr{M}$ then $M_1 \cup M_2 \in \mathscr{M}$;
- $(M3)$ $\mathscr{M}$ is total in X, *i.e.*, the closure with respect to the topology τ of the linear span of $\bigcup_{M\in\mathscr{M}} M$ coincides with X.

Then, for each $M \in \mathscr{M}$ and each V neighbourhood of $0 \in Y$, we write $U(M,V)$ for the set of maps $T \in \mathfrak{L}(X,Y)$ such that $T[M] \subset V$.

Proposition 1.1.5. *Let (X,τ) and (Y,σ) be LCTVS as per Definition 1.1.6 and let $\mathfrak{L}(X,Y)$ be as per Definition 1.1.4. Let $\mathscr{M}$ and $U(M,V)$ be as per the above construction. Then, the set of all $U(M,V)$, as M varies in $\mathscr{M}$ and V ranges over the neighbourhoods of $0 \in Y$, forms a base of neighbourhoods of $0 \in \mathfrak{L}(X,Y)$ for a unique vector topology $\tau_{\mathscr{M}}$ as per Theorem 1.1.2. The topology $\tau_{\mathscr{M}}$ is locally convex as per Definition 1.1.6.*

A proof of this proposition can be found in [15, Section 39.1].

Remark 1.1.10. Different choices of the class $\mathscr{M}$ induce different but equally important topologies. More specifically:

- 1. Let $\mathscr{M}_\sigma$ be the collection of finite subsets of X. Then $\mathscr{M}_\sigma$ induces a topology τ_σ on $\mathfrak{L}(X,Y)$ as per Proposition 1.1.5. We shall write $\mathfrak{L}(X,Y)_\sigma$ as the space $\mathfrak{L}(X,Y)$ endowed with τ_σ. This topology is that of simple or pointwise convergence, *i.e.*, a sequence $(T_n)_{n\in\mathbb{N}} \subset \mathfrak{L}(X,Y)_\sigma$ converges to $T \in \mathfrak{L}(X,Y)_\sigma$ if and only if for every $x \in X$, $T_n(x)$ converges to $T(x)$ in the topology of Y.

- 2. Let $\mathscr{M}_b$ be the collection of all bounded subsets of X as per Remark 1.1.5. Then $\mathscr{M}_b$ induces a topology τ_b on $\mathfrak{L}(X,Y)$ as per Proposition 1.1.5. We shall write $\mathfrak{L}(X,Y)_b$ as the space $\mathfrak{L}(X,Y)$ endowed with τ_b. This topology is that of bounded convergence, *i.e.*, a sequence $(T_n)_{n\in\mathbb{N}} \subset \mathfrak{L}(X,Y)_b$ converges to $T \in \mathfrak{L}(X,Y)_b$ if and only if for all bounded sets $B \subset X$ and continuous seminorm q on Y, one has that

$$\lim_{n\to+\infty} \sup_{x\in B} q(T_n(x) - T(x)) = 0.$$

- 3. Let $\mathscr{M}_p$ be the collection of all precompact subsets of X, *i.e.*, of all sets $K \subset X$ such that the closure of K is compact. Then $\mathscr{M}_p$ induces a topology τ_p on $\mathfrak{L}(X,Y)$ as per Proposition 1.1.5. We shall write $\mathfrak{L}(X,Y)_p$ as the space $\mathfrak{L}(X,Y)$ endowed with τ_p. This topology is that of uniform convergence on precompact sets, *i.e.*, a sequence $(T_n)_{n\in\mathbb{N}} \subset \mathfrak{L}(X,Y)_p$ converges to $T \in \mathfrak{L}(X,Y)_p$ if and only if for all precompact sets $K \subset X$ and continuous seminorm q on Y, one has that

$$\lim_{n\to+\infty} \sup_{x\in K} q(T_n(x) - T(x)) = 0.$$

□

Remark 1.1.11. Let X be an LCTVS as per Definition 1.1.6 and let X' be its dual space as per Definition 1.1.4. X' is usually equipped with its strong dual topology, which is that of bounded convergence in Remark 1.1.10. X' can also induce a topology on X, known as **weak topology**, which is uniquely characterized, in light of Theorem 1.1.2, by the base of neighbourhoods of 0 composed of

$$U(\varepsilon; u_1, \dots, u_N) \doteq \{x \in X \mid |u_j(x)| < \varepsilon \ \forall\, j \in \{1, \dots, N\}\},$$

for any $\varepsilon > 0$ and $u_1, \dots, u_N \in X'$. This topology is denoted by $\sigma(X, X')$. In an analogous manner, one can define an additional topology, known as **weak* topology**, on X' by declaring a weak* base of neighbourhood of $0 \in X'$ to be composed of

$$V(\varepsilon; x_1, \dots, x_N) \doteq \{u \in X' \mid |u(x_j)| < \varepsilon \ \forall\, j \in \{1, \dots, N\}\},$$

for any $\varepsilon > 0$ and $x_1, \dots, x_N \in X$. This topology is denoted by $\sigma^*(X', X)$.

□

Example 1.1.2. Let $X = Y = \mathbb{R}^d$ endowed with any norm topology. By the Heine-Borel theorem, a set $B \subset X$ is compact if and only if it is closed and bounded. Hence, the class of precompact sets coincides with that of bounded sets, which implies that the topology of bounded convergence and precompact convergence, see Remark 1.1.10, coincide. Furthermore, because X is finite dimensional, every compact subset is contained within the linear span of finitely many elements of a vector basis, which implies that the precompact convergence coincides with the pointwise convergence. Thus, for a finite dimensional normed space X, all the topologies listed in Remark 1.1.10 coincide. This property fails, however, in infinite dimensional normed spaces, as the Riesz's lemma, see [5, Theorem 6.5], implies that bounded sets need not be precompact in this scenario. □

As Example 1.1.2 shows, the topologies of Remark 1.1.10 are strictly distinct for arbitrary TVSs. However, when X and Y possess some specific, additional structure, they share a notion of boundedness. With the aim of characterizing these additional structure, we introduce the following notions.

Definition 1.1.9 (Barrelled space)**.** *Let X be an LCTVS as per Definition 1.1.6. X is a **barrelled space** if every barrel in X, see Definition 1.1.2, is a neighbourhood of* $0 \in X$.

Example 1.1.3. Every metrizable, sequentially complete LCTVS is a barrelled space, see [14, Section 27.1, (1)]. Therefore, any Banach space is a barrelled space.

Definition 1.1.10 (Montel space)**.** *Let X be an LCTVS as per Definition 1.1.6. X is a **Montel space** if it is a barrelled space as per Definition 1.1.9 and it has the Heine-Borel property, i.e., a subset $B \subset X$ is compact if and only if it is closed and bounded, see Remark 1.1.5.*

Remark 1.1.12. By Riesz's lemma, see [5, Theorem 6.5], no infinite dimensional normed space can be a Montel space as per Definition 1.1.10, whereas every finite dimensional normed space is a Montel space. Furthermore, in analogy with Remark 1.1.2, the topologies τ_b and τ_p of Remark 1.1.10 coincide on $\mathfrak{L}(X)$ whenever X is a Montel space. □

Montel spaces enjoy the following list of properties.

Definition 1.1.11 (Reflexivity and semi-reflexivity)**.** *Let X be an LCTVS as per Definition 1.1.6, let X' be its dual space as per Definition 1.1.4, and let X'' be the dual space of X'. We say that X is **semi-reflexive** if the map $J : X \to X''$, given by $J(x)(u) = u(x)$, is bijective. We say that X is **reflexive** if the map J is an isomorphism.*

Proposition 1.1.6. *Let (X, τ) be a Montel space as per Definition 1.1.10. Then*

- *1. X is reflexive as per Definition 1.1.11;*

- *2. X' is reflexive;*
- *3. the topologies τ and $\sigma(X, X')$, see Remark 1.1.11, coincide.*

A proof of these statements can be found in [14, Section 27.2, (1)-(3)].

Proposition 1.1.7. *Let X be a LCTVS as per Definition 1.1.6 and let X' be its dual space as per Definition 1.1.4. Then, if X is Montel as per Definition 1.1.10, X' is also Montel. Furthermore, if X is reflexive as per Definition 1.1.11 and X' is Montel, then X is Montel.*

A proof of this result can be found in [13, Chapter IV.5, Proposition 5.9].

The compactness properties of Montel spaces yield strong relations among the operator topologies introduced in Remark 1.1.10. The **Banach–Steinhaus** theorem, also called the **uniform boundedness principle**, is the main tool to establish these equivalences. We first recall some preliminary notions.

Definition 1.1.12 (Uniform space)**.** *A **uniformity** or **uniform structure** on a set X is a filter $\mathfrak{W}$ on X, see Definition 1.1.5, such that*

- *$(W1)$ for every $W \in \mathfrak{W}$, $\Delta \doteq \{(x,x) \mid x \in X\} \subset W$;*
- *$(W2)$ for every $W \in \mathfrak{W}$, $W^{-1} \doteq \{(x,y) \mid (y,x) \in W\} \in \mathfrak{W}$;*
- *$(W3)$ for every $W \in \mathfrak{W}$ there exists $V \in \mathfrak{W}$ such that*

$$V \circ V \doteq \{(x,z) \mid \exists y \in X \text{ such that } (x,y) \in W \text{ and } (y,z) \in V\} \subset W.$$

*Each element $W \in \mathfrak{W}$ is called a **vicinity** or **entourage** of the uniformity $\mathfrak{W}$. The set X equipped with the uniformity $\mathfrak{W}$ is called a **uniform space**.*

Definition 1.1.13 (Equicontinuity)**.** *Let (X,τ) be a topological space, let $(Y,\mathfrak{W})$ be a uniform space as per Definition 1.1.12 and let Y^X be the space of all functions $f : X \to Y$.*

- *A set $H \subset Y^X$ is **equicontinuous** at $x_0 \in X$ if for each vicinity W of $\mathfrak{W}$ there exists a neighbourhood U of x_0 such that $(f(x), f(x_0)) \in W$ for every $x \in U$ and $f \in H$.*
- *A set $H \subset Y^X$ is **equicontinuous** if it is equicontinuous at every $x_0 \in X$.*
- *If $(X,\mathfrak{U})$ is uniform, then $H \subset Y^X$ is **uniformly equicontinuous** if for every vicinity W of $\mathfrak{W}$ there exists a vicinity U of $\mathfrak{U}$ such that $(f(x_1), f(x_2)) \in W$ for every $(x_1,x_2) \in U$ and $f \in H$.*

Remark 1.1.13. Let (X,τ) be an LCTVS as per Definition 1.1.6 and let $\mathscr{U}$ be a base of neighbourhoods of 0 in X. For every $U \in \mathscr{U}$, let

$$W(U) \doteq \{(x,y) \in X \times X \mid x - y \in U\}.$$

Then, $\mathfrak{W}_\tau \doteq \{W(U) \mid U \in \mathscr{U}\}$ is a uniformity as per Definition 1.1.12, see [13, Chapter I.1]. One can adapt Definition 1.1.13 to the case of linear operators between LCTVS equipped with their vector topology induced uniformity. Employing Proposition 1.1.4, one has that, for any pair of LCTVSs (X,τ) and (Y,σ), $H \subset \mathfrak{L}(X,Y)$ is equicontinuous if and only if it is equicontinuous at $0 \in X$ with respect to the uniformities $\mathfrak{W}_\tau$ and $\mathfrak{W}_\sigma$. In particular, this is equivalent to saying that $H \subset \mathfrak{L}(X,Y)$ is equicontinuous if and only if for every continuous seminorm p on X and q on Y there exists $C > 0$ such that

$$\sup_{T\in H} q(T(x)) \le Cp(x),$$

for every $x \in X$. □

With this definitions, we may now state the uniform boundedness principle.

Theorem 1.1.8 (Banach-Steinhaus)**.** *Let X, Y be LCTVS as per Definition 1.1.6 such that X is a barrelled space. Then $H \subset \mathfrak{L}(X,Y)$ is equicontinuous, see Definition 1.1.13, if and only if it is pointwise bounded, i.e., for every $x \in X$ and continuous seminorm q on Y, $\sup_{T\in H} q(T(x)) < +\infty$.*

A proof of this result can be found at [13, Chapter III.4, 4.2]. The following proposition is an application of this theorem.

Proposition 1.1.9. *Let (X,τ) and (Y,σ) be Montel spaces as per Definition 1.1.10. Then, on the space $\mathfrak{L}(X,Y)$, the topologies τ_σ, τ_b and τ_p, see Remark 1.1.10, coincide.*

Proposition 1.1.9 is a special case of [13, Chapter III.4, 4.6] together with the observations in Remark 1.1.12. Hence, a proof can be found therein. One of its immediate consequences is the following corollary.

Corollary 1.1.10. *Let (X,τ) and (Y,σ) be Montel spaces as per Definition 1.1.10 and let $(T_n)_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{L}(X,Y)$. Then, T_n converges pointwise to $T \in \mathfrak{L}(X,Y)$, see Remark 1.1.10, if and only if it converges uniformly on bounded sets to $T \in \mathfrak{L}(X,Y)$.*

The notion of equicontinuity can also be used to distinguish more precisely the different types of continuity for bilinear maps on TVSs.

Definition 1.1.14 (Hypocontinuity). *Let (X, τ), (Y, σ) and (Z, ω) be LCTVSs as per Definition 1.1.6 and let $T : X \times Y \to Z$ be a bilinear map. We say that*

- *T is **jointly continuous** if it is continuous with respect to the product topology $\tau \times \sigma$ on $X \times Y$;*
- *T is **separately continuous** if the maps $X \ni x \mapsto T_y(x) = T(x, y) \in Z$ and $Y \ni y \mapsto T_x(y) = T(x, y) \in Z$ are continuous for all $y \in Y$ and $x \in X$, respectively;*
- *T is **hypocontinuous** if, for all bounded sets $B \subset X$ and $D \subset Y$ as per Remark 1.1.5, the collections $\{T_y \mid y \in D\}$ and $\{T_x \mid x \in B\}$ are equicontinuous as per Definition 1.1.13.*

Remark 1.1.14. Let X, Y, Z and T be as per Definition 1.1.14. As each singleton $\{x\} \subset X$ and $\{y\} \subset Y$ is bounded, then hypocontinuity of T implies separate continuity. Furthermore, joint continuity of T implies hypocontinuity, see [15, Section 40.1]. The converse to these implications is generally not true. □

Example 1.1.4. Let X be an LCTVS as per Definition 1.1.6 and let X' be its dual space as per Definition 1.1.4. Then, the bilinear map $\langle \cdot, \cdot \rangle : X' \times X \to \mathbb{C}$, $\langle u, x \rangle \doteq u(x)$ is always separately continuous. Furthermore, if X is barrelled, see Definition 1.1.9, then $\langle \cdot, \cdot \rangle$ is hypocontinuous as per Definition 1.1.14, see [15, Section 40.1, Example 2]. In general, $\langle \cdot, \cdot \rangle$ is not jointly continuous, see [15, Section 40.1, Example 3]

While the notions of continuity in Definition 1.1.14 are distinguished in general, they may coincide on particular spaces.

Proposition 1.1.11. *Let X, Y and Z be LCTVSs as per Definition 1.1.6 and let $T : X \times Y \to Z$ be a bilinear map. If X and Y are barrelled, see Definition 1.1.9, then T is hypocontinuous as per Definition 1.1.14 if and only if it is separately continuous. If X and Y are further metrizable, then T is hypocontinuous if and only if it is jointly continuous.*

A proof of this result can be found in [15, Section 40.2, (1)-(3)].

1.1.3 Fréchet and Limit Spaces

Within the broad class of TVSs, we shall mostly focus on Fréchet spaces and limit spaces, as they account for the most commonly used spaces of test functions in distribution theory amongst them, see Section 1.2. In this subsection, we shall review the particular construction of the topology of these spaces and their fundamental topological properties.

Definition 1.1.15 (Complete LCTVS)**.** *Let (X,τ) be an LCTVS X as per Definition 1.1.6, whose topology is induced by the separating family of seminorms $\{p_\alpha \mid \alpha \in A\}$, see Remark 1.1.8. A sequence $(x_n)_n \subset X$ is said to be Cauchy if for every $\alpha \in A$ and $\varepsilon > 0$, there exists $N = N(\varepsilon,\alpha) \in \mathbb{N}$ such that $p_\alpha(x_n - x_m) < \varepsilon$ for every $m, n \in \mathbb{N}$ with $m, n > N$. The space (X,τ) is said to be complete if every Cauchy sequence is convergent.*

Definition 1.1.16. *A Fréchet space is an LCTVS, see Definition 1.1.6, which is metrizable and complete as per Definition 1.1.15.*

Remark 1.1.15. By [13, Chapter I.6, 6.1], an LCTVS is metrizable if and only if it possesses a countable base of neighbourhoods of 0 composed of disks. Hence, a Fréchet space can be equivalently characterized as a complete LCTVS whose topology is induced by a countable family of seminorms. □

Example 1.1.5. Let $K \subset \mathbb{R}^d$ be a compact set and let $\mathscr{E}(K) \doteq C^\infty(K;\mathbb{C})$. For each multi-index $\alpha \in \mathbb{N}^d$, let $p_\alpha(f) \doteq \sup_{x\in K} |\partial^\alpha f(x)|$ for all $f \in \mathscr{E}(K)$. Then, the topology induced by the family $(p_\alpha)_{\alpha\in\mathbb{N}^d}$ makes $\mathscr{E}(K)$ into a Fréchet space. Analogously, for any open $\Omega \subset \mathbb{R}^d$, the space $\mathscr{E}(\Omega)$ is a Fréchet space when equipped with the seminorms $p_{\alpha,j}(f) \doteq \sup_{x\in K_j} |\partial^\alpha f(x)|$, where $\alpha \in \mathbb{N}^d$ and $(K_j)_{j\in\mathbb{N}}$ is a family of compact subsets such that $\Omega \subset \bigcup_{j\in\mathbb{N}} K_j$. A proof of the aforementioned properties can be found in [19, Chapter 10, Example I]. □

Example 1.1.6. The class of Fréchet spaces is not exhaustive of the functional spaces used in distribution theory. For instance, for $\Omega \subset \mathbb{R}^d$ and open set, let $\mathscr{D}(\Omega)$ be the set of functions $f \in \mathscr{E}(\Omega)$, see Example 1.1.5, such that the support of f is a compact subset of Ω. In analogy with Example 1.1.5, one could define the family of seminorms

$$p_{\alpha,j}(f) \doteq \sup_{x\in K_j} |\partial^\alpha f(x)|,$$

where $\alpha \in \mathbb{N}^d$ and $(K_j)_{j\in\mathbb{N}}$ is a sequence of compact sets such that $\Omega \subset \bigcup_{j\in\mathbb{N}} K_j$. While these seminorms define a locally convex topology τ on $\mathscr{D}(\Omega)$ as per Definition 1.1.6, see [16, Theorem 1.37], this topology is not complete, see [16, Section 6.2]. Therefore, $(\mathscr{D}(\Omega),\tau)$ is not a Fréchet space as per Definition 1.1.16. □

As Example 1.1.6 shows, s, the class of Fréchet spaces, although broad, excludes several spaces of central importance in functional analysis and distribution theory. These spaces can instead be described through the machinery of **projective** and **inductive limit**, hence they may be informally categorized as **limit spaces**.

Definition 1.1.17 (Projective topology)**.** *Let X be a vector space over the field $\mathbb{K}$ and let A be an index set. For every $\alpha \in A$, let (X_α, τ_α) be an LCTVS over a field $\mathbb{K}$ as per Definition 1.1.6 and let $F_\alpha : X \to X_\alpha$ be a linear map. The* ***projective topology*** *τ on X with respect to the family $\{(X_\alpha, F_\alpha, \tau_\alpha \mid \alpha \in A)\}$ is the coarsest vector topology on X for which each mapping F_α is continuous.*

The projective topology always makes the space X of Definition 1.1.17 a TVS, as is guaranteed by the following proposition. The same is not generically true for the property of being an LCTVS.

Proposition 1.1.12. *Let (X, τ) be the vector space over the field $\mathbb{K}$ equipped with the projective topology with respect to the family $\{(X_\alpha, F_\alpha, \tau_\alpha \mid \alpha \in A)\}$ as per Definition 1.1.17. Then τ is a vector topology over X as per Definition 1.1.1. (X, τ) is an LCTVS as per Definition 1.1.6 if and only if for each $x \in X$, $x \neq 0$, there exists an $\alpha \in A$ and $U_\alpha \subset X_\alpha$ a neighbourhood of $0 \in X_\alpha$ such that $f_\alpha(x) \notin U_\alpha$.*

A proof of this result can be found in [13, Chapter II.5, 5.1].

Remark 1.1.16. The notion of projective topology as per Definition 1.1.17 is the analogue in the case of TVSs of the initial topology, see, *e.g.*, [20, Chapter I.2.3]. In the same manner, given another TVS (Y, σ), a function $f : Y \to X$ is continuous if and only if $F_\alpha \circ f : Y \to X_\alpha$ is continuous for every $\alpha \in A$. □

Example 1.1.7. We now list several standard examples of projective topologies, as per Definition 1.1.17, following the exposition in [13, Chapter II.5].

- **Subspace topology**. Let (X, τ) be an LCTVS as per Definition 1.1.6 and let $M \subset X$ be a closed vector subspace. The subspace topology $\tau|_M$ on M can be characterized as the projective topology with respect to the family $\{(M, \tau|_M, \iota)\}$, where ι is the natural immersion $\iota : M \hookrightarrow X$.

- **Product topology**. Let $\{(X_\alpha, \tau_\alpha) \mid \alpha \in A\}$ be a collection of LCTVSs for some index set A and let X be their Cartesian product endowed with its vector topology τ, see Remark 1.1.3. τ is also the projective topology with respect to the family $\{(X_\alpha, \tau_\alpha, \pi_\alpha) \mid \alpha \in A\}$, where $\pi_\alpha : X \to X_\alpha$ are the natural projections defined by $\pi_\alpha(f) = f(\alpha)$ for every $f \in X$. This topology is also locally convex as a consequence of Proposition 1.1.12.

- **Weak topology**. Let (X, τ) be an LCTVS and let Y be a fixed collection of linear maps from X to $\mathbb{K}$. The weak topology induced by Y on X, denoted by $\sigma(X, Y)$ is the projective topology with respect to the family $\{(X_u, \tau_u, u) \mid u \in Y\}$, where $X_u = \mathbb{K}$, while τ_u is the norm topology on $\mathbb{K}$ for every $u \in Y$. If $\mathbb{K} = \mathbb{C}$ and $Y = X'$, this topology reduces to the weak topology $\sigma(X, X')$ as per Remark 1.1.11.

- **Weak-$*$ topology**. Let (X,τ) be an LCTVS and let X' be its strong dual space as per Definition 1.1.4. Let $M \subset X$ and let $\delta_x : X' \to \mathbb{C}$, defined by $\delta_x(u) = u(x)$ for every $x \in M$ and $u \in X'$. The weak* topology on X' induced by M, denoted by $\sigma(X',M)$, is the projective topology with respect to the family $\{(X'_x, \tau_x, \delta_x) \mid x \in M\}$, where $X'_x = \mathbb{C}$ and τ_x is the norm topology on $\mathbb{C}$ for every $x \in M$. If $M = X$, this topology reduces to the weak* topology $\sigma^*(X',X)$ as per Remark 1.1.11.

$\square$

Definition 1.1.18 (Projective limit space). *Let A be an index set equipped with a reflexive, transitive, antisymmetric order relation "$\leq$" and let $\{(X_\alpha, \tau_\alpha) \mid \alpha \in A\}$ be a collection of LCTVSs as per Definition 1.1.6. For every $\alpha, \beta \in A$ with $\alpha \leq \beta$, let $g_{\alpha,\beta} : X_\beta \to X_\alpha$ be a linear, continuous map. Let $\tilde{X} = \prod_{\alpha\in A} X_\alpha$ and let $X \subset \tilde{X}$ be the set of elements $f \in \tilde{X}$ satisfying $f(\alpha) = g_{\alpha,\beta}(f(\beta))$ for every $\alpha, \beta \in A$, $\alpha \leq \beta$. The space X endowed with the projective topology with respect to the family $\{(X_\alpha, \tau_\alpha, \pi_\alpha|_X) \mid \alpha \in A\}$, where π_α is the natural projection of $\tilde{X}$ onto X_α, is called the **projective limit** of the family $\{X_\alpha \mid \alpha \in A\}$ with respect to the **linking maps** $\{g_{\alpha,\beta} \mid \alpha, \beta \in A,\ \ \alpha \leq \beta\}$. We write*

$$X = \varprojlim g_{\alpha,\beta} X_\alpha.$$

Example 1.1.8. Projective limit spaces as per Definition 1.1.18 can be heuristically thought of as the limit of a decreasing sequence of LCTVSs. To illustrate this idea, we consider the space $C^0(\Omega)$ of continuous functions from an open set $\Omega \subset \mathbb{R}^d$ into $\mathbb{C}$. A locally convex vector topology on $C^0(\Omega)$, see Proposition 1.1.12, can be induced through the family of seminorms

$$p_K(f) \doteq \sup_{x\in K} |f(x)|,$$

for every compact subset K of Ω. One may choose to pick K only from a countable family of compact subsets that covers Ω in analogy with Remark 1.1.5 to emphasize the Fréchet nature of this locally convex vector topology, see [16, Example 1.44]. This restriction, however, is unnecessary for the present construction. Let $\mathfrak{K}_\Omega$ be the collection of compact subsets of Ω and let us fix $C_K \doteq C^0(K)$ for every $K \in \mathfrak{K}_\Omega$. The pair (C_K, τ_K), where τ_K is the norm topology induced by p_K, is a Banach space and, therefore, an LCTVS, see Remark 1.1.7. Furthermore, we may equip $\mathfrak{K}_\Omega$, which shall be our index set, with the order relation given by $K_1 \leq K_2$ if and only if $K_1 \subset K_2$, which satisfies all the requirements of Definition 1.1.18. Lastly, for every $K_1, K_2 \in \mathfrak{K}_\Omega$ with $K_1 \leq K_2$, let $g_{K_1,K_2} : C_{K_2} \to C_{K_1}$ be the natural inclusion of $C^0(K_2) \hookrightarrow C^0(K_1)$ given by the restriction to K_1, *i.e.*, $g_{K_1,K_2}(f) = f|_{K_1}$ for every $f \in C_{K_2}$. Then, the topology induced by the seminorms p_K, $K \in \mathfrak{K}_\Omega$, renders $C^0(\Omega)$ into the projective limit of the family $\{(C_K, \tau_K) \mid K \in \mathfrak{K}_\Omega\}$ with respect to the maps $\{g_{K_1,K_2} \mid K_1, K_2 \in \mathfrak{K}_\Omega\, K_1 \leq K_2\}$. □

Example 1.1.9. In analogy with Example 1.1.7, one can construct the space $\mathscr{E}(\Omega) \doteq C^\infty(\Omega;\mathbb{C})$, $\Omega \subset \mathbb{R}^d$ an arbitrary open set, with the topology of Example 1.1.5 as a projective limit of the Fréchet spaces $\mathscr{E}(K)$, $K \in \mathfrak{K}_\Omega$, with their respective topologies. □

Remark 1.1.17. In general, the projective limit space as per Definition 1.1.18 is defined with respect to an uncountable index set, see Examples 1.1.8 and 1.1.9. However, one can restrict the index set to some of its strictly smaller subsets and obtain the same result. More specifically, if $(A, \leq)$ is as per Definition 1.1.18, we say that $B \subset A$ is **cofinal** if for every $\alpha \in A$ there exists $\beta \in B$ such that $\alpha \leq \beta$. Then, the projective limit of the family $\{X_\alpha \mid \alpha \in A\}$ with respect to the linking maps $\{g_{\alpha,\beta} \mid \alpha, \beta \in A, \ \ \alpha \leq \beta\}$ is isomorphic as per Definition 1.1.4 to the same projective limit taken with the respect to any cofinal subset $B \subset A$, see [14, Section 19.8, (2)]. Thus, the two projective limits may be identified. In particular, in both Examples 1.1.8 and 1.1.9, one may substitute $\mathfrak{K}_\Omega$ with a countable collection of compact sets $(K_n)_{n\in\mathbb{N}}$ such that $\Omega \subset \bigcup_{n\in\mathbb{N}} K_n$. □

A second method for constructing functional spaces as limit of existing collection of LCTVS is given by the notion of inductive topology.

Definition 1.1.19 (Inductive topology)**.** *Let X be a vector space over the field $\mathbb{K}$ and let A be an index set. For every $\alpha \in A$, let (X_α, τ_α) be an LCTVS over a field $\mathbb{K}$ as per Definition 1.1.6 and let $G_\alpha : X_\alpha \to X$ be linear maps. The* ***inductive topology*** *on X with respect to the family $\{(X_\alpha, \tau_\alpha, G_\alpha) \mid \alpha \in A\}$ is the finest locally convex topology, see Definition 1.1.6, for which every mapping G_α is continuous.*

Example 1.1.10. We now list some standard examples of inductive topologies as per Definition 1.1.19, following the exposition of [13, Chapter II.6].

- **Quotient space**. Let (X, τ) be an LCTVS as per Definition 1.1.6 and let $M \subset X$ be a closed vector subspace. The quotient topology $\tau|_{X/M}$ on X/M, see [18, Chapter 3], is the inductive topology with respect to the family $\{(X, \tau, \iota)\}$, where $\iota : X \to X/M$ is the canonical quotient map.

- **Locally Convex Direct Sum**. Let $\{(X_\alpha, \tau_\alpha) \mid \alpha \in A\}$ be a collection of LCTVSs as per Definition 1.1.6 for some index set A. We define the **direct sum** of $\{X_\alpha \mid \alpha \in A\}$ to be the vector space of $f \in \prod_{\alpha\in A} X_\alpha$ such that $f(\alpha) = 0$ for all but a finite number of $\alpha \in A$ and we denote it by $\bigoplus_{\alpha\in A} X_\alpha$. For every $\alpha \in A$, we define $\Gamma_\alpha : X_\alpha \to \bigoplus_{\beta\in A} X_\beta$, $\Gamma_\alpha(x) = f_{\alpha,x}$ such that $f_{\alpha,x}(\beta) = x$ if $\beta = \alpha$ and $f_{\alpha,x}(\beta) = 0$ otherwise. The **locally convex**

direct sum of $\{(X_\alpha, \tau_\alpha) \mid \alpha \in A\}$ is the direct sum $\bigoplus_{\alpha\in A} X_\alpha$ endowed with the inductive topology with respect to the family $\{(X_\alpha, \tau_\alpha, \Gamma_\alpha) \mid \alpha \in A\}$. We denote this topology as $\tau = \bigoplus_{\alpha\in A} \tau_\alpha$.

□

Definition 1.1.20 (Inductive limit space)**.** *Let A be an index set equipped with a reflexive, transitive, antisymmetric order relation "$\leq$" and let $\{(X_\alpha, \tau_\alpha) \mid \alpha \in A\}$ be a collection of LCTVSs as per Definition 1.1.6. For every $\alpha, \beta \in A$ with $\alpha \geq \beta$, let $h_{\alpha,\beta} : X_\beta \to X_\alpha$ be a linear, continuous map. Let $\tilde{X} = \bigoplus_{\alpha\in A} X_\alpha$ be the locally convex direct sum of $\{(X_\alpha, \tau_\alpha) \mid \alpha \in A\}$ and let $\Gamma_\alpha : X_\alpha \to \tilde{X}$ be as per Example 1.1.10. Let $H \subset \tilde{X}$ be the vector subspace generated by the image of the maps $\Gamma_\beta - \Gamma_\alpha \circ h_{\alpha,\beta}$ as $\alpha, \beta \in A$ with $\alpha \geq \beta$. We further assume that the mappings $h_{\alpha,\beta}$ are such that H is a closed subspace of $\tilde{X}$. Then, the **inductive limit space** of the family $\{(X_\alpha, \tau_\alpha) \mid \alpha \in A\}$ with respect to the **inductive linking maps** $\{h_{\alpha,\beta} \mid \alpha, \beta \in A, \alpha \geq \beta\}$ is the quotient space $X \doteq \tilde{X}/H$ endowed with its natural inductive topology as per Example 1.1.10. We write*

$$X = \varinjlim h_{\alpha,\beta} X_\beta.$$

*The inductive limit space X is said to be **strict** if τ_α induces the topology on τ_β on X_β whenever $\alpha \geq \beta$.*

Example 1.1.11. An important example of inductive limit spaces as per Definition 1.1.20 is given by the following example. Let $\{(X_\alpha, \tau_\alpha) \mid \alpha \in A\}$ be a collection of LCTVSs as per Definition 1.1.6 for some index set A. Assume further that the family $\{(X_\alpha, \tau_\alpha) \mid \alpha \in A\}$ is strictly increasing, *i.e.*, $X_\alpha \neq X_\beta$ for every $\alpha \neq \beta$ and that there exists a continuous immersion $h_{\alpha,\beta} : X_\beta \to X_\alpha$ for every $\alpha \geq \beta$. For every $\alpha \in A$, let $G_\alpha : X_\alpha \to X$, where $X = \bigcup_{\alpha\in A} X_\alpha$, is the canonical immersion $X_\alpha \hookrightarrow X$. Then, the space X equipped with the inductive topology with respect to the family $\{(X_\alpha, \tau_\alpha, G_\alpha) \mid \alpha \in A\}$, see Definition 1.1.19, is isomorphic in the sense of Definition 1.1.4 to the inductive limit space of the family $\{(X_\alpha, \tau_\alpha) \mid \alpha \in A\}$ with respect to the linking maps $\{h_{\alpha,\beta} \mid \alpha, \beta \in A, \alpha \geq \beta\}$, see [13, Chapter II.6]. □

Example 1.1.12. Inductive limit spaces as per Definition 1.1.20 can be heuristically thought of as the limit of an increasing sequence of LCTVSs. To illustrate this idea, we consider the space $\mathscr{D}(\Omega)$ of smooth, compactly supported functions $f : \Omega \to \mathbb{C}$. As in Example 1.1.8, let $\mathfrak{K}_\Omega$ be the collection of compact subsets of Ω endowed with the order relation "$\leq$" defined by $K_1 \leq K_2$ if and only if $K_1 \subset K_2$. For any $K \in \mathfrak{K}_\Omega$, we define $\mathscr{D}_K(\Omega)$ to be the space of smooth functions whose support is contained within K and we endow it with the topology induced by the family of seminorms:

$$p_{\alpha,K}(f) \doteq \sup_{x\in K} |\partial^\alpha f(x)|,$$

for every $\alpha \in \mathbb{N}^d$ and $K \in \mathfrak{K}_\Omega$. For each $K \in \mathfrak{K}_\Omega$, the family $\mathfrak{P}_K \doteq \{p_{\alpha,K} \mid \alpha \in \mathbb{N}^d\}$ defines a Fréchet topology τ_K on $\mathscr{D}_K(\Omega)$ in analogy to Example 1.1.5. For every $K_1, K_2 \in \mathfrak{K}_\Omega$ with $K_1 \leq K_2$, one has the continuous linear immersion $\mathscr{D}_{K_1}(\Omega) \hookrightarrow \mathscr{D}_{K_2}(\Omega)$, which we shall denote by h_{K_2,K_1}. Then, $\mathscr{D}(\Omega) = \bigcup_{K\in\mathfrak{K}_\Omega} \mathscr{D}_K(\Omega)$ is the strict inductive limit space of the family $\{(\mathscr{D}_K(\Omega), \tau_K) \mid K \in \mathfrak{K}_\Omega\}$ with respect to the linking maps $\{h_{K_2,K_1} \mid K_1, K_2 \in \mathfrak{K}_\Omega, K_1 \leq K_2\}$. □

Two particularly useful classes of functional spaces are obtained as inductive limits of Banach or Fréchet spa

Definition 1.1.21 (LB and LF spaces). *An LCTVS X, see Definition 1.1.6, is an **(LB)-space** if it is the inductive limit space as per Definition 1.1.20 of an increasing sequence of Banach spaces, see Example 1.1.11. An LCTVS X is an **(LF)-space** if it is the inductive limit space of an increasing sequence of Fréchet spaces, see Definition 1.1.16. If the inductive limit space X is also strict, then we call it, respectively, a **strict (LB)-space** or a **strict (LF)-space**.*

Remark 1.1.18. While the definition of projective and inductive limit spaces, see Definition 1.1.18 and 1.1.20, are stated for families of LCTVSs of arbitrary cardinality, we shall be mostly interested in limit spaces built out of sequences of LCTVS. The constructions in Examples 1.1.8, 1.1.9 and 1.1.12 can be slavishly reproduced by considering an exhaustion of Ω of compact sets, *i.e.*, a sequence of compact sets $(K_j)_{j\in\mathbb{N}}$ such that $K_j \subset K_{j+1}$ and $\Omega \subset \bigcup_{j\in\mathbb{N}} K_j$. Consequently, the space $C^0(\Omega)$ equipped with the topology of Example 1.1.8 is a Fréchet space, see [16, Example 1.44], whereas the space $\mathscr{D}(\Omega)$ equipped with the topology of Example 1.1.12 is an (LF)-space, see [13, Chapter II.6, Example 2] □

We shall now present a series of fundamental properties that hold for limit spaces.

Proposition 1.1.13. *Let X be a Fréchet space as per Definition 1.1.16, let Y be the (LF)-limit space as per Definition 1.1.21 of a strictly increasing sequence $(Y_n)_{n\in\mathbb{N}}$ and let $T \in \mathfrak{L}(X, Y)$. Then, there exists $\overline{n}$ such that $T[X] \subset Y_{\overline{n}}$ and $T \in \mathfrak{L}(X, Y_{\overline{n}})$.*

A proof of this result can be found in [14, Section 19.6, (4)].

Theorem 1.1.14. *Let X be the strict inductive limit of a strictly increasing sequence of LCTVSs $(X_n)_{n\in\mathbb{N}}$, see Example 1.1.11. Let us also assume that, for every $n \in \mathbb{N}$, the image of X_n under the inclusion $X_n \hookrightarrow X_{n+1}$ is closed. Then:*

- *$B \subset X$ is bounded, see Remark 1.1.5, if and only if there exists $\overline{n} \in \mathbb{N}$ such that $B \subset X_{\overline{n}}$ is bounded;*
- *if each X_n is complete as per Definition 1.1.15, X is complete;*
- *$K \subset X$ is precompact if and only if there exists $\overline{n} \in \mathbb{N}$ such that $k \subset X_{\overline{n}}$ is precompact.*

Proofs of these results can be found, respectively, in [13, Chapter II.6, 6.5–6.6] and [15, Section 43.4, (5)].

We conclude this subsection with a short introduction to the notion of duality, which will be further explored in Section 1.2 for specific choices of LCTVSs. We shall also introduce the notion of adjoint map, along with some comments regarding its existence and properties.

Definition 1.1.22 (Duality of TVSs)**.** *Let X and Y be vector spaces over the field $\mathbb{K}$ as per Definition 1.1.1. A* ***duality*** *on X and Y is a bilinear map*

$$ {}_X\langle \cdot\,,\,\cdot \rangle_Y : X \times Y \to \mathbb{K}, $$

with the properties that:

- *if ${}_X\langle x, y\rangle_Y = 0$ for all $y \in Y$, then $x = 0 \in X$;*
- *if ${}_X\langle x, y\rangle_Y = 0$ for all $x \in X$, then $y = 0 \in Y$.*

The triple $(X, Y, {}_X\langle \cdot\,,\,\cdot \rangle_Y)$ is known as a ***dual system****.*

Remark 1.1.19. It is standard convention throughout the literature, see, *e.g.*, [13, Chapter IV.1], to denote the dual system $(X, Y, {}_X\langle \cdot\,,\,\cdot \rangle_Y)$ by ${}_X\langle X\,,\,Y \rangle_Y$ and to call it a **dual pair**. Whenever the spaces X and Y are implicit, the suffixes ${}_X$ and ${}_Y$ in the duality shall be dropped. □

Example 1.1.13. Let X be a TVS over $\mathbb{C}$ as per Definition 1.1.1 and let X' be its dual space as per Definition 1.1.4. Then, one has a natural duality $\langle X, X' \rangle$ given by the map

$$ \begin{aligned} {}_X\langle \cdot\,,\,\cdot \rangle_Y : & X \times X' \longrightarrow \mathbb{C} \\ & (x, u) \longmapsto u(x). \end{aligned} $$

Analogously, for any closed $Y \subset X'$, the map ${}_X\langle \cdot\,,\,\cdot \rangle_Y$ defines the dual pair ${}_X\langle X\,,\,Y \rangle_{X'}$. □

Remark 1.1.20. Given V, W vector spaces over $\mathbb{K}$ and $\langle\cdot,\cdot\rangle$ a duality on them, one can induce a topology on V by requiring that the maps $V \ni v \mapsto \langle v, w\rangle \in \mathbb{K}$, $w \in W$, are continuous. This is the weak topology of Example 1.1.7 induced by the space of mappings $\mathscr{W} \doteq \{V \ni v \mapsto \langle v, w\rangle \in \mathbb{K} \mid w \in W\}$. As $\mathscr{W}$ is naturally isomorphic to W, the topology may be equivalently denoted by $\sigma(V, \mathscr{W})$ or $\sigma(V, W)$. If V is equipped with the weak topology $\sigma(V, W)$, then W is its dual space and the duality map $\langle\cdot,\cdot\rangle$ is continuous. Furthermore, if $\tilde{W} \subsetneq W$ and $\langle V, \tilde{W}\rangle$ is a dual pair as per Remark 1.1.19, then the topology $\sigma(V, \tilde{W})$ is strictly coarser than $\sigma(V, W)$, see [13, Chapter IV.1, 1.2]. □

We report the following results, as it shall be useful for its application in Section 1.2.

Proposition 1.1.15. *Let $\langle X, Y\rangle$ be a dual pair as per Remark 1.1.19 and let $\tilde{Y} \subsetneq Y$. Then, $\langle X, \tilde{Y}\rangle$ is a dual pair if and only if $\tilde{Y}$ is dense in Y with respect to $\sigma(X, Y)$*

A proof of this result can be found in [13, Chapter IV.1, 1.3].

Definition 1.1.23 (Adjoint map)**.** *Let ${}_{X_1}\langle X_1\,, X_2\rangle_{X_2}$ and ${}_{Y_1}\langle Y_1\,, Y_2\rangle_{Y_2}$ be dual pairs as per Remark 1.1.19 and let $T : X_1 \to Y_1$ be a linear map. The* ***adjoint map or adjoint of T*** *with respect to the dualities ${}_{X_1}\langle\cdot\,,\cdot\rangle_{X_2}$ and ${}_{Y_1}\langle\cdot\,,\cdot\rangle_{Y_2}$ is the linear operator $T^* : Y_2 \to X_2$ defined by*

$$
{}_{Y_1}\langle T(x)\,, y\rangle_{Y_2} = {}_{X_1}\langle x\,, T^*(y)\rangle_{X_2}.
$$

If $X_1 = X$ and $Y_1 = Y$ are TVSs, see Definition 1.1.1, and $X_2 = X'$, $Y_2 = Y'$ are their dual spaces, see Definition 1.1.4, with their natural continuous dualities as per Remark 1.1.22, then the adjoint of the operator $T : X \to Y$ is called the ***dual operator or dual of T*** *and is denoted by $T' : Y' \to X'$.*

Remark 1.1.21. For an arbitrary linear operator $T : X_1 \to Y_1$, its adjoint with respect to the dual pairs ${}_{X_1}\langle X_1\,, X_2\rangle_{X_2}$ and ${}_{Y_1}\langle Y_1\,, Y_2\rangle_{Y_2}$ as per Definition 1.1.23 need not exist. In general, T^* is well-defined if and only if T is continuous with respect to the weak topologies $\sigma(X_1, X_2)$ and $\sigma(Y_1, Y_2)$ as per Remark 1.1.20, see [13, Chapter IV.2, 2.1].

Remark 1.1.22. Let $\langle X_1\,, X_2\rangle_X$ and $\langle Y_1\,, Y_2\rangle_Y$ be dual pairs, see Remark 1.1.19, of TVSs as per Definition 1.1.1. Assume that the dualities $\langle\cdot\,,\cdot\rangle_X$ and $\langle\cdot\,,\cdot\rangle_Y$ are continuous and let $T \in \mathfrak{L}(X_1, Y_1)$. Then, T^* is well-defined as per Remark 1.1.21 and, by continuity of the duality, if $(y_n)_n \subset Y_2$ satisfies $y_n \to y \in Y_2$, then, for every $x \in X_1$, one has that

$$
\langle x, T^*(y_n)\rangle_X = \langle T(x), y_n\rangle_Y \to \langle T(x), y\rangle_Y = \langle x, T^*(y)\rangle.
$$

Hence, T^* is sequentially continuous with respect to the weak topology $\sigma(X_2, X_1)$, see Remark 1.1.20. If X and Y are LCTVSs, see Definition 1.1.6, then the continuity of T is suffices to imply continuity of T^*, see [15, Section 32.2, (4)]. □

1.2 Distribution theory

Amongst the TVSs explored in Section 1.1, the spaces $\mathscr{E}(\Omega)$ and $\mathscr{D}(\Omega)$, see respectively Remark 1.1.5 and Example 1.1.12, and their dual spaces will be of paramount importance in the rest of this work. This section shall be devoted to recalling the fundamental properties of these spaces within the context of distribution theory. In Subsection 1.2.1, we shall review some preliminary notions regarding test functions and distributions. We shall also present the extension to distributions of concepts that are well-defined for functions, such as differentiation and the notion of support. In Subsection 1.2.2, we shall review the spaces of Schwartz functions and tempered distributions, as well as the construction of the Fourier transform and its properties on such spaces. Lastly, we shall also recall the notion of convolution product of distributions and its functional properties.

1.2.1 Spaces of test functions and distributions

We shall begin by recalling the standard spaces of test functions and distributions which shall be frequently employed in the rest of this work.

Definition 1.2.1. *Let $\Omega \subset \mathbb{R}^d$ be an open set. We denote by:*

- *$\mathscr{E}(\Omega)$ the space of smooth functions $f : \Omega \to \mathbb{C}$ and $\mathscr{E}'(\Omega)$ its dual space as per Definition 1.1.4;*
- *$\mathscr{D}(\Omega)$ the space of smooth, compactly supported functions $f : \Omega \to \mathbb{C}$ and $\mathscr{D}'(\Omega)$ its dual space.*

For each pair of spaces, we denote by $\langle \cdot, \cdot \rangle$ the natural duality on them as per Example 1.1.13.

Remark 1.2.1. In the following, we always assume the spaces $\mathscr{E}(\Omega)$ and $\mathscr{D}(\Omega)$, see Definition 1.2.1, to be endowed respectively with the Fréchet topology of Example 1.1.5 and with the LF-topology of Example 1.1.12. Analogously, the spaces $\mathscr{E}'(\Omega)$ and $\mathscr{D}'(\Omega)$ will always be assumed to be equipped with their respective strong dual topology, see Remark 1.1.11. We wish to recount also that the continuity of elements in $\mathscr{D}'(\Omega)$ can be characterized by either of the following equivalent conditions, see [4, Theorem 2.1.4]:

$(C1)$ a linear map $u : \mathscr{D}(\Omega) \to \mathbb{C}$ is continuous if and only if for every compact $K \subset \Omega$ there exists $C > 0$ and $n \in \mathbb{N}$ such that, for every $\varphi \in \mathscr{D}(K) \subset \mathscr{D}(\Omega)$,

$$|\langle u, \varphi \rangle| \leq C \sum_{|\alpha| \leq k} \sup_{x \in K} |\partial^\alpha \varphi(x)|;$$

$(C2)$ a linear map $u : \mathscr{D}(\Omega) \to \mathbb{C}$ is continuous if and only if, for every sequence $(\varphi_j)_{j\in\mathbb{N}} \subset \mathscr{D}(\Omega)$ such that there exists $K \subset \Omega$ compact for which $supp(\varphi_j) \subset K$ for all $j \in \mathbb{N}$ and $\sup_{x\in K} |\partial^\alpha \varphi_j(x)| \to 0$ as $j \to +\infty$ for every $\alpha \in \mathbb{N}^d$, one has that

$$\lim_{j\to+\infty} |\langle u, \varphi_j\rangle| = 0.$$

The constants C and N in item $(C1)$ act as the continuity parameter of u and allow to classify distributions with the following definition. □

Definition 1.2.2 (Order of a distribution)**.** *Let $u \in \mathscr{D}'(\Omega)$ as per Definition 1.2.1. If there exists $N \in \mathbb{N}$ such that, for all compact sets $K \subset \Omega$,*

$$|\langle u, \varphi\rangle| \leq C \sum_{|\alpha|\leq N} \sup_{x\in K} |\partial^\alpha \varphi(x)|,$$

for some constant $C > 0$ that may depend on K and every $\varphi \in \mathscr{D}(K)$, then N is called the ***order of u****. We denote the order of u by $ord(u)$.*

Remark 1.2.2. Every compactly supported distribution, see Definition 1.2.4, has finite order as per Definition 1.2.2, see [16, Theorem 6.24(d)]. □

Remark 1.2.3. For any $f \in L^1_{loc}(\Omega)$, let $u_f : \mathscr{D}(\Omega) \to \mathbb{C}$ be the map

$$\varphi \mapsto u_f(\varphi) \doteq \int_\Omega f(x)\varphi(x)dx.$$

For each f, u_f is a well-defined, linear map. Furthermore, it follows from condition $(C2)$ of Remark 1.2.1 and Hölder's inequality, see [5, Theorem 4.6], that u_f is continuous on $\mathscr{D}(\Omega)$ and, hence, identifies a distribution in $\mathscr{D}'(\Omega)$. Therefore, $L^1_{loc}(\Omega)$ may be regarded as a subspace of $\mathscr{D}'(\Omega)$ by way of the continuous immersion $L^1_{loc}(\Omega) \ni f \mapsto u_f \in \mathscr{D}'(\Omega)$. As smooth functions are locally bounded by Weierstrass theorem and, hence, locally integrable, one also has that $\mathscr{E}(\Omega) \hookrightarrow \mathscr{D}'(\Omega)$ via $\mathscr{E}(\Omega) \ni \varphi \mapsto u_\varphi \in \mathscr{D}'(\Omega)$.

Similarly, if $L^1_c(\Omega)$ is the space of integrable, compactly supported functions $f : \Omega \to \mathbb{C}$, then $L^1_c(\Omega)$ can be viewed as a subspace of $\mathscr{E}'(\Omega)$ via the continuous immersion $L^1_c(\Omega) \ni f \mapsto u_f \in \mathscr{E}'(\Omega)$. As $\mathscr{D}(\Omega) \hookrightarrow L^1_c(\Omega)$ continuously, one also has the continuous immersion $\mathscr{D}(\Omega) \hookrightarrow \mathscr{E}'(\Omega)$ induced by the map $\mathscr{D}(\Omega) \ni \varphi \mapsto u_\varphi \in \mathscr{E}'(\Omega)$. □

Remark 1.2.4. With the notation of Definition 1.2.1, we note that $\mathscr{D}(\Omega) \hookrightarrow \mathscr{E}(\Omega)$ continuously, as the topology of $\mathscr{E}(\Omega)$ is coarser than that of $\mathscr{D}(\Omega)$, see Examples 1.1.5 and 1.1.12. Therefore, one also has the continuous inclusion $\mathscr{E}'(\Omega) \hookrightarrow \mathscr{D}'(\Omega)$. Accordingly, in the following, many notions will be defined solely for the space of distributions $\mathscr{D}'(\Omega)$, while their analogues for $\mathscr{E}'(\Omega)$ will be tacitly understood to be identically constructed. □

We shall now recall a collection of topological properties for the spaces of test functions and distributions. These will be of paramount importance in the constructions of Chapters 2 and 3.

Theorem 1.2.1 (Topological properties of distributions)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $\mathscr{D}(\Omega)$, $\mathscr{E}(\Omega)$, $\mathscr{D}'(\Omega)$ and $\mathscr{E}'(\Omega)$ be as per Definition 1.2.1 endowed with their strong topology, see Remark 1.2.1.*

- *The space $\mathscr{D}(\Omega)$ is a complete, bornological, Montel strict (LF)-space, see Definitions 1.1.15, 1.1.7, 1.1.10 and 1.1.21.*
- *The space $\mathscr{E}(\Omega)$ is a bornological, Montel Fréchet space, see Definition 1.1.16.*
- *The spaces $\mathscr{D}'(\Omega)$ and $\mathscr{E}'(\Omega)$ are barreled, bornological, Montel and non-metrizable LCTVSs.*
- *The space $\mathscr{D}(\Omega)$ is sequentially dense in $\mathscr{D}'(\Omega)$, i.e., for every $u \in \mathscr{D}'(\Omega)$ there exists a sequence $(\chi_n)_{n\in\mathbb{N}} \subset \mathscr{D}(\Omega)$ such that $u_{\chi_n} \to u$ in $\mathscr{D}'(\Omega)$, see Remark 1.2.3. Analogously, $\mathscr{D}(\Omega)$ is sequentially dense in $\mathscr{E}'(\Omega)$.*

Proof for the previous results can be found, respectively, in [13, Chapters II.6, II.8, III.8], [19, Chapters 10, 14.6, 34.4] [21, Chapter 4.1, pp. 314], [13, Chapters IV.5, IV.6] and [19, Theorem 28.2].

Historically, distributions were first introduced as a tool to obtain weak solutions to linear PDEs. The success of this approach is due to the extension to distributions of some fundamental operations and properties of functions, which we shall now recall.

Remark 1.2.5. Let $\varphi \in \mathscr{D}(\Omega)$. Then, for every $j \in \{1, \dots, d\}$, $\partial_j \varphi \in \mathscr{D}(\Omega)$. Therefore, for a fixed $u \in \mathscr{D}'(\Omega)$, the map

$$\mathscr{D}(\Omega) \ni \varphi \longmapsto -\langle u, \partial_j \varphi \rangle \in \mathbb{C},$$

is well-defined. It is also linear and continuous in the sense of Remark 1.2.1, see [4, Chapter 3.1], hence it identifies an element of $\mathscr{D}'(\Omega)$. In light of this, we have the following definition. □

Definition 1.2.3 (Distributional derivative)**.** *For any $u \in \mathscr{D}'(\Omega)$, the distributional derivative of order $\alpha \in \mathbb{N}^d$ of u is the distribution $\partial^\alpha u$ given by*

$$\langle \partial^\alpha u, \varphi \rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha \varphi \rangle,$$

for all $\varphi \in \mathscr{D}(\Omega)$.

Remark 1.2.6. Let $\psi \in \mathscr{E}(\Omega)$ be fixed. Then, the map

$$\mathscr{D}(\Omega) \ni \varphi \longmapsto \langle u, \psi\varphi \rangle \in \mathbb{C}$$

is well-defined, linear and continuous, see [4, Chapter 3.1]. Hence it defines an element of $\mathscr{D}'(\Omega)$. We shall denote this distribution by ψu and call it the **product between ψ and u**. Multiplication by $\psi \in \mathscr{E}(\Omega)$ is a continuous operator from $\mathscr{D}'(\Omega)$ into itself and multiplication by $\varphi \in \mathscr{D}(\Omega)$ is a continuous operator from $\mathscr{D}'(\Omega)$ into $\mathscr{E}'(\Omega)$. □

Definition 1.2.4 (Support). *Let $u \in \mathscr{D}'(\Omega)$ and $x \in \Omega$. The point x lies in the distributional support of u, written as $x \in supp(u)$, if for every open neighbourhood O of x there exists a function $\varphi \in \mathscr{D}(O)$ such that $\langle u, \varphi \rangle \neq 0$.*

Example 1.2.1. Let $x \in \Omega$ be fixed and let $\delta_x \in \mathscr{D}'(\Omega)$ be the distribution given by $\langle \delta_x, \varphi \rangle = \varphi(x)$, for every $\varphi \in \mathscr{D}(\Omega)$. For any $O \subset \Omega$ open neighbourhood of x, there exists $\varphi \in \mathscr{D}(O)$ such that $\langle \delta_x, \varphi \rangle = \varphi(x) = 1$. Hence, $x \in supp(\delta_x)$ by Definition 1.2.4. Furthermore, for every $y \in \Omega$, $y \neq x$, $O \subset \Omega$ open neighbourhood of y such that $x \notin O$ and $\varphi \in \mathscr{D}(O)$, one has that $\langle \delta_x, \varphi \rangle = 0$. This implies that $y \notin supp(\delta_x)$ for any $y \neq x$, hence $supp(\delta_x) = \{x\}$. □

Remark 1.2.7. The support of a distribution as per Definition 1.2.4 behaves in a similar manner to the support of functions. Indeed, if $u = u_f$ for some $f \in L^1_{loc}(\Omega)$, see Remark 1.2.3, then $supp(u) = supp(f)$. To show this, we note that, by Definition 1.2.4, $\Omega \setminus supp(u)$ is the set of points $x \in \Omega$ for which there exists an open neighbourhood O such that $\langle u, \varphi \rangle = 0$ for all $\varphi \in \mathscr{D}(O)$. Therefore, if $u = u_f$, by the Fundamental Lemma of the Calculus of Variations, see [4, Theorem 1.2.5], one has that $f(x) = 0$ almost everywhere in O, hence, $x \notin supp(f)$. Conversely, if $x \notin supp(f)$, then there exists an open neighbourhood O of x such that $f(x) = 0$ almost everywhere in O. Then, for any $\varphi \in \mathscr{D}(O)$, one has that $\langle u, \varphi \rangle = 0$, hence $x \notin supp(u)$. □

Remark 1.2.8. Let $u \in \mathscr{D}'(\Omega)$ and $\varphi \in \mathscr{D}(\Omega)$. Let us further assume that $supp(u) \cap supp(\varphi) = \varnothing$ as per Definition 1.2.4. Then, $\langle u, \varphi \rangle = 0$.Therefore, since $supp(u) = \varnothing$ if and only if $u = 0 \in \mathscr{D}'(\Omega)$, see [4, Theorem 2.2.1], it follows that, for all $u \in \mathscr{D}'(\Omega)$ and $\chi \in \mathscr{D}(\Omega)$,

$$supp(\chi u) \subset supp(u) \cap supp(\chi).$$

□

Remark 1.2.9. The space of test functions $\mathscr{D}(\Omega)$, see Definition 1.2.1, plays a primary role in the theory of distributions, as the compactness of the support of its elements grants them a multitude of additional properties that arbitrary smooth functions do not possess. Similarly, one would expect compactly supported distributions to be of paramount importance. Therefore, it is natural to study the properties of the space $\mathscr{K}'(\Omega)$ of distributions $u \in \mathscr{D}'(\Omega)$ with compact support. It turns out that $\mathscr{K}'(\Omega) = \mathscr{E}'(\Omega)$, see [4, Theorem 2.3.1]. □

Remarks 1.2.7 and 1.2.8 show how localization of functions, usually encoded in the restriction to a subset of their domain or multiplication by a smooth, compactly supported function, translates to distributions through their notion of support. Similarly, one can characterize the set of points at which a distribution behaves as the integral distribution of a smooth function.

Definition 1.2.5 (Singular support). *Let $u \in \mathscr{D}'(\Omega)$ and $x \in \Omega$. The point x lies outside the singular support of u, written as $x \notin singsupp(u)$, if there exist $\chi \in \mathscr{D}(\Omega)$, $\chi(x) \neq 0$, and $\psi \in \mathscr{E}(\Omega)$ such that $\chi u = u_\psi$, where u_ψ is as per Remark 1.2.3.*

Remark 1.2.10. Let $u \in \mathscr{D}'(\Omega)$ and let $O \doteq supp(u) \subset \Omega$ as per Definition 1.2.4. Let us further assume that $O \subsetneq \Omega$. For any $x \notin O$, let $V \subsetneq \Omega \setminus O$ be an open neighbourhood such that $\langle u, \varphi \rangle = 0$ for all $\varphi \in \mathscr{D}(V)$. Let $\chi \in \mathscr{D}(\Omega)$ be such that $supp(\chi) \subset V$ and $\chi(x) \neq 0$. Then, as $\chi\varphi \in \mathscr{D}(V)$ for all $\varphi \in \mathscr{D}(\Omega)$, one has that

$$\langle \chi u, \varphi \rangle = \langle u, \chi\varphi \rangle = 0,$$

for all $\varphi \in \mathscr{D}(\Omega)$, which implies that $\chi u = 0 \in \mathscr{D}'(\Omega)$. Hence, by Remark 1.2.3, one has that $\chi u = u_0$ for $0 \in \mathscr{E}(\Omega)$. Therefore, $x \notin singsupp(u)$ as per Definition 1.2.5, which implies that $singsupp(u) \subset supp(u)$. □

Example 1.2.2. Let $\Omega \subset \mathbb{R}^d$ be an open set and $x \in \Omega$ be fixed. Let $\delta_x : \mathscr{D}(\Omega) \to \mathbb{C}$ be the map defined by $\langle \delta_x, \varphi \rangle = \varphi(x)$ for every $\varphi \in \mathscr{D}(\Omega)$. By Example 1.2.1, $supp(\delta_x) = \{x\}$ as per Definition 1.2.4,while Remark 1.2.10, gives $singsupp(\delta_x) \subset supp(\delta_x) = \{x\}$. Hence, $singsupp(\delta_x)$ is either the empty set or $\{x\}$. We shall now briefly prove that $singsupp(\delta_x) = \{x\}$.

If $singsupp(\delta_x) = \emptyset$, then there exist $\chi \in \mathscr{D}(\Omega)$ with $\chi(x) \neq 0$ and $\psi \in \mathscr{E}(\Omega)$ such that $\chi\delta_x = u_\psi$, see Definition 1.2.5. We note that $\chi\delta_x \neq 0 \in \mathscr{D}'(\Omega)$, as $\langle \chi\delta_x, \chi \rangle = \chi(x)^2 \neq 0$. By Remarks 1.2.7 and 1.2.8, this implies that

$$supp(\psi) = supp(u_\psi) = supp(\chi\delta_x) \subset supp(\chi) \cap supp(\delta_x) = \{x\}.$$

Therefore, $\psi(y) = 0$ for almost every $y \in \Omega$, which implies that $\psi \equiv 0$ as ψ is a continuous function on Ω. Then, $\chi\delta_x = u_\psi = 0 \in \mathscr{D}'(\Omega)$, which is a contradiction. Hence, we have that $singsupp(\delta_x) = \{x\}$. □

Another structure typical of function spaces is that of tensor product. We shall recall it briefly for test functions in the following remark and extend this notion to spaces of distributions in Definition 1.2.6.

Remark 1.2.11. Let $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ be open sets. For any $f \in \mathscr{D}(\Omega_1)$ and $g \in \mathscr{D}(\Omega_2)$, their tensor product is the function $f \otimes g \in \mathscr{D}(\Omega_1 \times \Omega_2)$ given by

$$(f \otimes g)(x, y) = f(x)g(y).$$

The space of finite linear combinations of functions of type $f \otimes g$, $f \in \mathscr{D}(\Omega_1)$ and $g \in \mathscr{D}(\Omega_2)$, is denoted by $\mathscr{D}(\Omega_1) \otimes \mathscr{D}(\Omega_2)$. In general, this is a strict subspace of $\mathscr{D}(\Omega_1 \times \Omega_2)$. However, as is proven in [19, Theorem 39.2], the former space is sequentially dense in the latter, *i.e.*, for every $h \in \mathscr{D}(\Omega_1 \times \Omega_2)$ there exist a sequence $(h_n)_{n\in\mathbb{N}} \subset \mathscr{D}(\Omega_1) \otimes \mathscr{D}(\Omega_2)$ such that $h_n \to h$ in the topology of $\mathscr{D}(\Omega_1 \times \Omega_2)$, see Remark 1.2.1. In particular, this implies that any linear, continuous map from $\mathscr{D}(\Omega_1) \otimes \mathscr{D}(\Omega_2)$ into $\mathscr{D}(\Omega_1 \times \Omega_2)$ extends uniquely to a linear, continuous map from $\mathscr{D}(\Omega_1 \times \Omega_2)$ into itself. Furthermore, every linear, continuous map from $\mathscr{D}(\Omega_1 \times \Omega_2)$ into itself is uniquely determined by its values over $\mathscr{D}(\Omega_1) \otimes \mathscr{D}(\Omega_2)$, see [19, Theorem 5.1]. □

Definition 1.2.6 (Tensor product). *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ be open sets. For every $u \in \mathscr{D}'(\Omega_1)$ and $v \in \mathscr{D}'(\Omega_2)$, $u \otimes v$ is the distribution in $\mathscr{D}'(\Omega_1 \times \Omega_2)$ uniquely determined in light of Remark 1.2.11 by the property that*

$$\langle u \otimes v, \varphi \otimes \psi \rangle = \langle u, \varphi \rangle \langle v, \psi \rangle,$$

for all $\varphi \in \mathscr{D}(\Omega_1)$ and $\psi \in \mathscr{D}(\Omega_2)$.

Remark 1.2.12. Let $f \in L^1_{loc}(\Omega_1)$, $g \in L^1_{loc}(\Omega_2)$ and let $u_f \in \mathscr{D}'(\Omega_1)$, $u_g \in \mathscr{D}'(\Omega_2)$ be as per Remark 1.2.3. We note that, given representatives $\tilde{f}$ and $\tilde{g}$ of f in $L^1_{loc}(\Omega_1)$ and g in $L^1_{loc}(\Omega_2)$ respectively,

$$(f \otimes g)(x, y) \doteq f(x)g(y) = \tilde{f}(x)\tilde{g}(y) = (\tilde{f} \otimes \tilde{g})(x, y), \tag{1.1}$$

for almost all $x \in \Omega_1$ and $y \in \Omega_2$, hence for almost all $(x, y) \in \Omega_1 \times \Omega_2$. Therefore, the tensor product of f and g, defined by Equation (1.1), is a well-defined element of $L^1_{loc}(\Omega_1 \times \Omega_2)$ and the map $u_{f\otimes g} : \mathscr{D}(\Omega_1 \times \Omega_2) \to \mathbb{C}$ given by

$$u_{f\otimes g} : \varphi \longmapsto \int_{\Omega_1 \times \Omega_2} f(x)g(y)\varphi(x, y)dxdy,$$

for all $\varphi \in \mathscr{D}(\Omega_1 \times \Omega_2)$ defines an element of $\mathscr{D}'(\Omega_1 \times \Omega_2)$ as per Remark 1.2.3. Then, for any $\varphi \in \mathscr{D}(\Omega_1)$ and $\psi \in \mathscr{D}(\Omega_2)$, using Definition 1.2.6, one has that

$$\langle u_f \otimes u_g, \varphi \otimes \psi \rangle = \langle u, \varphi \rangle \langle v, \psi \rangle = \left(\int_{\Omega_1} f(x)\varphi(x)dx \right) \left(\int_{\Omega_2} g(y)\psi(y)dy \right) =$$

$$= \int_{\Omega_1 \times \Omega_2} f(x)g(y)\varphi(x)\psi(y)dxdy = \langle u_{f\otimes g}, \varphi \otimes \psi \rangle.$$

By density of $\mathscr{D}(\Omega_1) \otimes \mathscr{D}(\Omega_2)$ in $\mathscr{D}(\Omega_1 \times \Omega_2)$, see Remark 1.2.11, this implies that $u_f \otimes u_g = u_{f\otimes g}$. □

The notion of tensor product allows one to reproduce in terms of distributions and duality, certain operations on integrals of multi-variable functions, such as Fubini's theorem for multiple integrations, see, *e.g.*, [5, Theorem 4.5]. To clarify the nature of this procedure, we shall require the following lemma.

Lemma 1.2.2 (Partial evaluation)**.** *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ be open sets. Let $u \in \mathscr{D}'(\Omega_1)$ and $\varphi \in \mathscr{E}(\Omega_1 \times \Omega_2)$ be such that $\varphi(x, y) = 0$ for every $y \in \Omega_2$ whenever $x \notin K$, for some compact set $K \subset \Omega_1$. Then, the map*

$$\Omega_2 \ni y \mapsto \langle u, \varphi(\cdot, y)\rangle \in \mathbb{C},$$

where $\varphi(\cdot, y) : x \mapsto \varphi(x, y)$ for every $x \in \Omega_1$ and $y \in \Omega_2$, is smooth. Furthermore, for every $\alpha \in \mathbb{N}^d$ and $y \in \Omega_2$,

$$\partial_y^\alpha \langle u, \varphi(\cdot, y)\rangle = \langle u, \partial_y^\alpha \varphi(\cdot, y)\rangle.$$

A proof of this result can be found in [4, Theorem 2.1.3].

Remark 1.2.13. Lemma 1.2.2 shows that the partial evaluation of a multi-variable test function $\varphi \in \mathscr{D}(\Omega_1 \times \Omega_2)$ against a distribution $u \in \mathscr{D}'(\Omega_1)$ yields a well-defined, smooth and compactly supported function. For the sake of clarity, whenever a multi-variable test functions is evaluated successively against several single-variable distributions, we shall write the arguments for both explicitly. For instance,

$$\langle u(x), \langle v(y), \varphi(x, y)\rangle\rangle,$$

for any $\varphi \in \mathscr{D}(\Omega_1 \times \Omega_2)$, $u \in \mathscr{D}'(\Omega_1)$ and $v \in \mathscr{D}'(\Omega_2)$, is the evaluation against u of the smooth, compactly supported function $\Omega_1 \ni x \mapsto \langle v, \varphi(x, \cdot)\rangle \in \mathbb{C}$. Hence, given $u \in \mathscr{D}'(\Omega_1)$ and $v \in \mathscr{D}'(\Omega_2)$, the distribution $u \otimes v$ acts by successive evaluations, in the sense that

$$\langle u \otimes v, \varphi\rangle = \langle u(x), \langle v(y), \varphi(x, y)\rangle\rangle = \langle v(y), \langle u(x), \varphi(x, y)\rangle\rangle,$$

see [4, Theorem 5.1.1]. □

We now conclude this subsection with the introduction of the Schwartz kernel, which is a useful tensor product representation for operators between spaces of distributions. For a more thorough exploration of this subject, we refer to [19, Chapter 51] and [4, Chapter 5.2]. The fundamental result regarding this notion is the following.

Theorem 1.2.3 (Schwartz kernel representation). *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ be open sets and let $\mathscr{K} \in \mathscr{D}'(\Omega_1 \times \Omega_2)$ be fixed. Then, the linear map $K : \mathscr{D}(\Omega_2) \to \mathscr{D}'(\Omega_1)$ defined by*

$$\langle K(\varphi), \psi \rangle = \langle \mathscr{K}, \psi \otimes \varphi \rangle, \tag{1.2}$$

for all $\psi \in \mathscr{D}'(\Omega_1)$ and $\varphi \in \mathscr{D}'(\Omega_2)$, is continuous. Conversely, if $K \in \mathfrak{L}(\mathscr{D}(\Omega_2), \mathscr{D}'(\Omega_1))$, then there exists a unique $\mathscr{K} \in \mathscr{D}'(\Omega_1 \times \Omega_2)$ which satisfies Equation (1.2) for all $\varphi \in \mathscr{D}'(\Omega_1)$ and $\psi \in \mathscr{D}'(\Omega_2)$. The distribution $\mathscr{K}$ is called the ***Schwartz kernel*** *of the operator K.*

Example 1.2.3. Let $f \in L^1_{loc}(\Omega_1 \times \Omega_2)$ and, for every $\varphi \in \mathscr{D}(\Omega_2)$, let $K(\psi)$ be function

$$K(\psi)(x) = \int_{\Omega_2} f(x,y)\varphi(y)dy,$$

for all $x \in \Omega_1$. By Fubini's theorem, $K(\psi) \in L^1_{loc}(\Omega_1)$, hence, by Remark 1.2.3, $u_{K(\psi)} \in \mathscr{D}'(\Omega_1)$ is well-defined. Therefore, for every $\psi \in \mathscr{D}(\Omega_1)$,

$$\langle u_{K(\psi)}, \varphi \rangle = \int_{\Omega_1} \int_{\Omega_2} \psi(x) f(x,y)\varphi(y)dydx = \int_{\Omega_1 \times \Omega_2} f(x,y)(\psi \otimes \varphi)(x,y)dxdy.$$

Thus, in light of the previous reasoning and Theorem 1.2.3, the distribution u_f is the Schwartz kernel of the operator K. We wish to stress, however, that the kernel representation of an operator K needs not be of integral form. For instance, for some fixed $x \in \Omega_1$ and $y \in \Omega_2$, the map $T_{x,y} : \mathscr{D}(\Omega_2) \to \mathscr{D}'(\Omega_1)$ given by $T(\varphi) = \varphi(y)\delta_x$, see Remark 1.2.1, is continuous, but its Schwartz kernel is the distribution $\mathscr{T} = \delta_x \otimes \delta_y$, which doesn't admit a representation in terms of locally integrable functions, see Remark 1.2.2. As a matter of fact, one cannot in general expect the Schwartz kernel of an operator to be a smooth function. The following proposition provides a case where such a phenomenon might occur. □

Proposition 1.2.4. *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ be open sets and let $\mathfrak{K} \in \mathscr{E}(\Omega_1 \times \Omega_2)$. Let $K : \mathscr{D}(\Omega_2) \to \mathscr{D}'(\Omega_1)$ be the linear map defined by*

$$\langle K(\varphi), \psi \rangle = \langle u_{\mathfrak{K}}, \psi \otimes \varphi \rangle,$$

for every $\psi \in \mathscr{D}(\Omega_1)$, $\varphi \in \mathscr{D}(\Omega_2)$. Then, there exists a unique continuous extension of K to $\mathscr{E}'(\Omega_2)$ such that the map

$$\Omega_1 \ni x \mapsto K(v)(x) \doteq \langle v, \mathfrak{K}(x, \cdot) \rangle \in \mathbb{C},$$

where $\mathfrak{K}(x,\cdot) : y \mapsto \mathfrak{K}(x,y)$, is smooth for all $v \in \mathscr{E}'(\Omega_1)$. Conversely, for any $K \in \mathfrak{L}(\mathscr{E}'(\Omega_2), \mathscr{E}(\Omega_1))$, there exists $\mathfrak{K} \in \mathscr{E}(\Omega_1 \times \Omega_2)$ such that

$$\langle u_{K(u_\varphi)}, \psi \rangle = \langle u_{\mathfrak{K}}, \psi \otimes \varphi \rangle,$$

for all $\psi \in \mathscr{D}(\Omega_1)$, $\varphi \in \mathscr{D}(\Omega_2)$.

1.2.2 Tempered distributions and Fourier transform

We shall begin by recalling the construction of the space of Schwartz functions. This third space of test functions is particularly well-suited for the study of the Fourier transform, which shall play a role of paramount importance in the following work.

Definition 1.2.7 (Schwartz functions and tempered distributions)**.** *We call $\mathscr{S}(\mathbb{R}^d)$ the space of all smooth functions $f : \mathbb{R}^d \to \mathbb{C}$ such that for all multi-indices $\alpha, \beta \in \mathbb{N}^d$,*

$$\sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta f(x)| < +\infty. \tag{1.3}$$

We shall denote by $\mathscr{S}'(\mathbb{R}^d)$ its dual space, see Definition 1.1.4, with respect to the locally convex topology, see Definition 1.1.6, induced by the separating family of seminorms $\{\|\cdot\|_{\alpha,\beta} \mid \alpha, \beta \in \mathbb{N}^d\}$. The space $\mathscr{S}'(\mathbb{R}^d)$ is also called the space of ***tempered distributions****.*

Remark 1.2.14. In Definition 1.2.7, Equation (1.3) defines a family of seminorms indexed by $\alpha, \beta \in \mathbb{N}^d$ given by the map

$$\mathscr{S}(\mathbb{R}^d) \ni f \mapsto \|f\|'_{\alpha,\beta} \doteq \sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta f(x)| \in [0, +\infty). \tag{1.4}$$

With this collection of seminorms, $\mathscr{S}(\mathbb{R}^d)$ is a Fréchet space as per Definition 1.1.16, see [16, Theorem 7.4]. The seminorms of Equation (1.4) are sometimes replaced by a similar family of seminorms, such as

$$\|f\|^*_{n,m} \doteq \sup_{\substack{|\alpha|\le n\\ |\beta|\le m}} \sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta f(x)|,$$

or

$$\|f\|^{**}_{n,m} \doteq \sum_{\substack{|\alpha|\le n\\ |\beta|\le m}} \sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta f(x)|,$$

for any $n, m \in \mathbb{N}$ and $f \in \mathscr{S}(\mathbb{R}^d)$. Because the seminorms $\|\cdot\|^*_{n,m}$ and $\|\cdot\|^{**}_{n,m}$ are constructed by taking finite suprema or sums of the seminorms $\|\cdot\|'_{\alpha,\beta}$, they are topologically equivalent to the family in Definition 1.2.7. In this work, we shall use a fourth equivalent family of seminorms, namely

$$\mathscr{S}(\mathbb{R}^d) \ni f \mapsto \|f\|_{n,\beta} \doteq \sup_{x\in\mathbb{R}^d} |\langle x\rangle^n \partial^\beta f(x)| \in [0, +\infty), \tag{1.5}$$

for every $n \in \mathbb{N}$ and $\beta \in \mathbb{N}$, where $\langle x\rangle \doteq \sqrt{1 + |x|^2}$ for every $x \in \mathbb{R}^d$. □

Remark 1.2.15. By Definition 1.2.1, for any $\varphi \in \mathscr{D}(\mathbb{R}^d)$, the function

$$\varphi_{\alpha,\beta}(x) \doteq x^\alpha \partial^\beta \varphi(x),$$

for every $x \in \mathbb{R}^d$ and any $\alpha, \beta \in \mathbb{N}^d$, is an element of $\mathscr{D}(\mathbb{R}^d)$. Hence, $\varphi_{\alpha,\beta}$ is bounded for all $\alpha, \beta \in \mathbb{N}$, which implies that $\varphi \in \mathscr{S}(\mathbb{R}^d)$ as per Definition 1.2.7. Thus, the inclusions $\mathscr{D}(\mathbb{R}^d) \subset \mathscr{S}(\mathbb{R}^d) \subset \mathscr{E}(\mathbb{R}^d)$ are continuous, see [16, Theorem 7.10] and [13, Chapter III.8, Example 5]. Then, one also has the continuous inclusions $\mathscr{E}'(\mathbb{R}^d) \subset \mathscr{S}'(\mathbb{R}^d) \subset \mathscr{D}'(\mathbb{R}^d)$. Furthermore, by Remark 1.2.14 for any $\varphi \in \mathscr{S}(\mathbb{R}^d)$, one has that

$$\sup_{x\in\mathbb{R}^d} |\varphi(x)\langle x\rangle^{\frac{1+d}{p}}| < +\infty,$$

for any $p \in [1, +\infty)$. Hence, using the seminorms $\|\cdot\|_{n,\beta}$, $n \in \mathbb{N}$ and $\beta \in \mathbb{N}^d$ of Remark 1.2.14, one has that

$$\|\varphi\|_{L^p(\mathbb{R}^d)} \leq \sup_{x\in\mathbb{R}^d} \left|\langle x\rangle^{\frac{1+d}{p}} \varphi(x)\right| \left(\int_{\mathbb{R}^d} \langle x\rangle^{-d-1} dx\right)^{1/p} = \|\langle\cdot\rangle^{-d-1}\|^{1/p}_{L^1(\mathbb{R}^d)} \|\varphi\|_{c_d,0}. \tag{1.6}$$

As $\langle\cdot\rangle^{-d-1} \in L^1(\mathbb{R}^d)$, Equation (1.6) entails that $\mathscr{S}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ continuously. Since $\mathscr{D}(\mathbb{R}^d) \hookrightarrow \mathscr{S}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ and $\mathscr{D}(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, see [5, Corollary 4.23], one also has that $\mathscr{S}(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$. □

Another functional space that plays an important role in the theory of tempered distributions is that of smooth functions with polynomial growth.

Definition 1.2.8 (Functions with polynomial growth). *A function $f \in \mathscr{E}(\mathbb{R}^d)$ has polynomial growth if for every $\alpha \in \mathbb{N}^d$, there exists $C_\alpha > 0$ and $N_\alpha \in \mathbb{N}$ such that*

$$|\partial^\alpha f(x)| \leq C_\alpha \langle x\rangle^{N_\alpha},$$

where $\langle x\rangle \doteq \sqrt{1+|x|^2}$, for all $x \in \mathbb{R}^d$. The space of functions with polynomial growth is denoted by $\mathscr{E}_{pol}(\mathbb{R}^d)$.

Remark 1.2.16. Let $\varphi \in \mathscr{S}(\mathbb{R}^d)$ as per Definition 1.2.7 and let $f \in \mathscr{E}_{pol}(\mathbb{R}^d)$ as per Definition 1.2.8. For any $\alpha, \beta \in \mathbb{N}^d$ and $n \in \mathbb{N}$,

$$\sup_{x\in\mathbb{R}^d} |\langle x\rangle^n \partial^\alpha \varphi(x) \partial^\beta f(x)| \leq C_\beta \sup_{x\in\mathbb{R}^d} |\langle x\rangle^{n+N_\beta} \partial^\alpha \varphi(x)| < +\infty.$$

Thus, using Leibniz rule and Remark 1.2.14, this implies that $\varphi f \in \mathscr{S}(\mathbb{R}^d)$ and that the multiplication map $\mathscr{S}(\mathbb{R}^d) \ni \varphi \mapsto f\varphi \in \mathscr{S}(\mathbb{R}^d)$ is continuous for any $f \in \mathscr{E}_{pol}(\mathbb{R}^d)$. We also note that, as $\mathscr{S}(\mathbb{R}^d) \subset \mathscr{E}_{pol}(\mathbb{R}^d)$, one has that the map $\mathscr{S}(\mathbb{R}^d) \times \mathscr{S}(\mathbb{R}^d) \ni (\varphi, \psi) \mapsto \varphi\psi \in \mathscr{S}(\mathbb{R}^d)$ is continuous, see [16, Theorem 7.4]. Furthermore, one also has that $\varphi f \in L^1(\mathbb{R}^d)$, see Remark 1.2.15. Hence, the map $u_f : \varphi \mapsto \int_{\mathbb{R}^d} \varphi(x) f(x) dx$ is a well-defined element of $\mathscr{S}'(\mathbb{R}^d)$, see Remark 1.2.3. Additionally, for any $u \in \mathscr{S}'(\mathbb{R}^d)$, the map $\mathscr{S}(\mathbb{R}^d) \ni \varphi \mapsto \langle u, f\varphi\rangle \in \mathbb{C}$ is a well-defined element of $\mathscr{S}'(\mathbb{R}^d)$ for every $f \in \mathscr{E}_{pol}(\mathbb{R}^d)$. We denote this distribution by fu in analogy with Remark 1.2.6.

We also note that, as $\mathscr{S}(\mathbb{R}^d) \hookrightarrow L^p(\Omega)$, see Remark 1.2.15, and the dual space of $L^p(\Omega)$ as per Definition 1.1.4 is isomorphic to $L^q(\Omega)$, where $p, q \in (1, +\infty)$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, then one has that $L^q(\mathbb{R}^d) \hookrightarrow \mathscr{S}'(\mathbb{R}^d)$. □

Remark 1.2.17. Let $X \subset \mathscr{E}(\mathbb{R}^d)$ as per Definition 1.2.1 By [19, Theorem 25.5], the following are equivalent:

- for every $f \in X$ and $u \in \mathscr{S}'(\mathbb{R}^d)$, $fu \in \mathscr{S}'(\mathbb{R}^d)$, see Remark 1.2.6, and $(f, u) \mapsto fu$ is continuous;
- for every $f \in X$ and $\varphi \in \mathscr{S}(\mathbb{R}^d)$, $f\varphi \in \mathscr{S}(\mathbb{R}^d)$ and $(f, \varphi) \mapsto f\varphi$ is continuous;
- for every $f \in X$ and every $\alpha \in \mathbb{N}^d$, there exists a polynomial P_α such that $|\partial^\alpha f(x)| \le P_\alpha(x)$ for every $x \in \mathbb{R}^d$.

In particular, taking Remark 1.2.16 into account, the largest space $X \subset \mathscr{E}(\mathbb{R}^d)$ for which the multiplication $X \times \mathscr{S}'(\mathbb{R}^d) \ni (f, u) \mapsto fu \in \mathscr{S}'(\mathbb{R}^d)$ is well-defined and continuous is exactly $X = \mathscr{E}_{pol}(\mathbb{R}^d)$.

Definition 1.2.9 (Fourier transform)**.** *For any $f \in \mathscr{S}(\mathbb{R}^d)$, see Definition 1.2.7, its **Fourier transform**, denoted equivalently by $\mathscr{F}(f)$ or $\hat{f}$, is the function*

$$\mathscr{F}(f)(\xi) \doteq (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-i\xi\cdot x} dx,$$

*for all $\xi \in \mathbb{R}^d$, where $\xi \cdot x$ is the Euclidean scalar product on $\mathbb{R}^d$. The **inverse Fourier transform** of f, denoted by $\check{f}$ or $\mathscr{F}^{-1}(f)$, is the function*

$$\mathscr{F}^{-1}(f)(\xi) \doteq (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{i\xi\cdot x} dx,$$

for all $\xi \in \mathbb{R}^d$.

While the notion of Fourier transform as per Definition 1.2.9 can be defined for any $f \in L^1(\mathbb{R}^d)$, the choice of $\mathscr{S}(\mathbb{R}^d)$ is prompted by the following proposition.

Proposition 1.2.5 (Properties of Fourier transform)**.** *Let $\mathscr{F}$ be the Fourier transform on $\mathscr{S}(\mathbb{R}^d)$ as per Definition 1.2.9.*

(F1) For any $f \in \mathscr{S}(\mathbb{R}^d)$, $\mathscr{F}(f)$ is a smooth function and, for every $\xi \in \mathbb{R}^d$,

$$\partial^\alpha \mathscr{F}(f)(\xi) = (-i)^{|\alpha|}\mathscr{F}(f_\alpha)(\xi),$$
$$\mathscr{F}(\partial^\alpha f)(\xi) = i^{|\alpha|}\xi^\alpha \mathscr{F}(f)(\xi),$$

where $f_\alpha(x) \doteq x^\alpha f(x)$ for any $\alpha \in \mathbb{N}^d$.

(F2) $\mathscr{F} : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ is continuous.

(F3) $\mathscr{F}$ is invertible with inverse given by $\mathscr{F}^{-1}$, see Definition 1.2.9, and $\mathscr{F}^{-1} : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ is continuous. In particular, $\mathscr{F}$ is an isomorphism of $\mathscr{S}(\mathbb{R}^d)$ into itself, see Definition 1.1.4.

(F4) For any $f \in \mathscr{S}(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$, let $\tau_y(f) : x \mapsto f(x+y)$. Then,

$$\mathscr{F}(\tau_y(f))(\xi) = e^{i\xi\cdot y}\mathscr{F}(f)(\xi),$$
$$\mathscr{F}(e^{i\eta\cdot} f)(\xi) = \tau_{-\eta}(\mathscr{F}(f))(\xi).$$

A proof of results $(F1)-(F3)$ can be found in [16, Theorem 7.4], whereas a proof of $(F4)$ can be found in [16, Theorem 7.2 (a)-(b)].

Remark 1.2.18. We note that, by Definition 1.2.9, for any $f \in \mathscr{S}(\mathbb{R}^d)$

$$\mathscr{F}^{-1}(f)(\xi) = \mathscr{F}(f)(-\xi),$$

for all $\xi \in \mathbb{R}^d$. Therefore, all the properties of the Fourier transform of Proposition 1.2.5 can be slavishly adapted to the inverse Fourier transform. □

Remark 1.2.19. The space of Schwartz functions as per Definition 1.2.7 is usually defined for functions on the whole Euclidean space $\mathbb{R}^d$. For an open set $\Omega \subset \mathbb{R}^d$, $\mathscr{S}(\Omega)$ is used in the literature to indicate either the space of functions $f \in \mathscr{S}(\mathbb{R}^d)$ restricted to Ω or those $f \in \mathscr{S}(\mathbb{R}^d)$ such that $supp(f) \subset \Omega$. These definitions are inequivalent and, in general, the former space is strictly larger than the latter.

A direct construction of $\mathscr{S}(\Omega)$ is not standard because the space of Schwartz functions is built to be invariant under the action of the Fourier transform as per Definition 1.2.9, both as a set and as a LCTVS, see Proposition 1.2.5. However, the Fourier transform generally does not preserve the support of its argument, in the sense that, if $f \in L^1(\Omega)$, $supp(f) \subset \Omega$ does not imply $supp(\mathscr{F}(f)) \subset \Omega$. A classic example of this phenomenon is given by the function $1_{[-1,1]} : \mathbb{R} \to \mathbb{R}$, $1_{[-1,1]}(x) = 1$ if $x \in [-1,1]$ and $1_{[-1,1]}(x) = 0$ otherwise. Then, $supp(1_{[-1,1]}) = [-1,1]$, while $\mathscr{F}(1_{[-1,1]})(\xi) = \sqrt{\frac{2}{\pi}}\frac{sin(\xi)}{\xi}$ for any $\xi \in \mathbb{R}\setminus\{0\}$ and $\mathscr{F}(1_{[-1,1]})(0) = \sqrt{\frac{2}{\pi}}$, which has unbounded support. □

We now recall another key operation between functions, the convolution product, which will be subsequently extended to distributions. We shall begin by briefly reviewing its construction and properties for integrable functions.

Definition 1.2.10 (Convolution of functions). *Let $p, q \in [1, +\infty]$ satisfy $\frac{1}{p} + \frac{1}{q} \geq 1$, where we adopt the convention that, if $p = +\infty$, then $\frac{1}{p} = 0$ and viceversa. Let $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$. Then, the* ***convolution or convolution product of f with g*** *is the function $f \star g : \mathbb{R}^d \to \mathbb{C}$ given by*

$$(f \star g)(x) \doteq \int_{\mathbb{R}^d} f(x-y)g(y)dy, \tag{1.7}$$

for almost all $x \in \mathbb{R}^d$.

Remark 1.2.20. As proven in [5, Theorem 4.33], Equation (1.7) provides a well-defined function. Furthermore, for $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, $f \star g$ in an element of the space $L^r(\mathbb{R}^d)$, where $r \in [1, +\infty]$ is such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. □

The convolution of functions enjoys the following additional properties.

Proposition 1.2.6 (Support of convolution). *Let $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ as per Definition 1.2.10. Then, $supp(f \star g) \subset \overline{supp(f) + supp(g)}$. In particular, if both f and g are compactly supported, then so is $f \star g$.*

A proof of this result can be found in [5, Proposition 4.18].

Proposition 1.2.7 (Regularity of the convolution). *Let $f \in C^0_c(\mathbb{R}^d)$ be a continuous, compactly supported function and let $g \in L^1_{loc}(\mathbb{R}^d)$.*

- *The convolution $(f \star g)(x)$ as per Definition 1.2.10 is well-defined for every $x \in \mathbb{R}^d$ and $f \star g \in C^0(\mathbb{R}^d)$.*
- *If $f \in C^k(\mathbb{R}^d)$ for some $k \in \mathbb{N}$, then $f \star g \in C^k(\mathbb{R}^d)$. Furthermore, for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$ and all $x \in \mathbb{R}^d$,*

$$\partial^\alpha (f \star g)(x) = [(\partial^\alpha f) \star g](x).$$

Proofs of these results can be found in [5, Propositions 4.19-4.20].

The connection between the convolution product and the Fourier transform is given by the following result.

Theorem 1.2.8. *Let $f, g \in \mathscr{S}(\mathbb{R}^d)$. Then, $f \star g \in \mathscr{S}(\mathbb{R}^d)$. Furthermore,*

$$\mathscr{F}(f \star g)(\xi) = (2\pi)^{d/2} \mathscr{F}(f)(\xi)\mathscr{F}(g)(\xi),$$

for all $\xi \in \mathbb{R}^d$.

A proof of this result can be found in [4, Theorem 7.1.6].

Remark 1.2.21. Theorem 1.2.8 allows for an equivalent construction of the convolution between Schwartz functions. By [16, Theorem 7.4 (b)], the pointwise multiplication $\mathscr{S}(\mathbb{R}^d) \times \mathscr{S}(\mathbb{R}^d) \ni (f, g) \mapsto M(f, g) \doteq f g \in \mathscr{S}(\mathbb{R}^d)$ is continuous. Hence, using Theorem 1.2.8, one may equivalently define the convolution of elements in $\mathscr{S}(\mathbb{R}^d)$ as

$$(f \star g)(x) \doteq (2\pi)^{d/2} \mathscr{F}^{-1}[\mathscr{F}(f)\mathscr{F}(g)](x). \tag{1.8}$$

Then, one may see the convolution as a map $\star : \mathscr{S}(\mathbb{R}^d) \times \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ given by $\star(\cdot, \cdot) = \mathscr{F}^{-1} \circ M(\mathscr{F}(\cdot), \mathscr{F}(\cdot))$. It follows from Proposition 1.2.5 that $\star$ is continuous as it is a composition of continuous maps. □

Remark 1.2.22. As per Proposition 1.2.5, the Fourier transform $\mathscr{F}$ is an isomorphism, see Definition 1.1.4, of $\mathscr{S}(\mathbb{R}^d)$ into itself. This implies that, for any fixed $u \in \mathscr{S}'(\mathbb{R}^d)$, the map

$$\mathscr{S}(\mathbb{R}^d) \ni \varphi \mapsto \langle u, \mathscr{F}(\varphi)\rangle \in \mathbb{C},$$

is well-defined, linear and continuous as a composition of linear, continuous maps. Thus, it identifies an element of $\mathscr{S}'(\mathbb{R}^d)$, so that the following definition is well-posed. □

Definition 1.2.11 (Distributional Fourier transform)**.** *Let $u \in \mathscr{S}'(\mathbb{R}^d)$ as per Definition 1.2.7. The* ***Fourier transform of u*** *is the distribution $\mathscr{F}(u) \in \mathscr{S}'(\mathbb{R}^d)$, equivalently denoted by $\hat{u}$, defined by*

$$\langle \mathscr{F}(u), \varphi\rangle = \langle u, \mathscr{F}(\varphi)\rangle,$$

for all $\varphi \in \mathscr{S}(\mathbb{R}^d)$.

The properties of the Fourier transform with respect to differentiation, multiplication by polynomials and convolution can be extended to the distributional Fourier transform, provided these operations are interpreted appropriately. Hence, we have the following results.

Proposition 1.2.9 (Properties of the distributional Fourier transform)**.** *Let $\mathscr{F}$ be the distributional Fourier transform on $\mathscr{S}'(\mathbb{R}^d)$ as per Definition 1.2.11.*

(DF1) For any $u \in \mathscr{S}'(\mathbb{R}^d)$, $\mathscr{F}(u)$ is a tempered distribution as per Definition 1.2.7. Furthermore, if P_α is the polynomial $P_\alpha(x) = \prod_{j=1}^N x_j^{\alpha_j}$, for any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, then $P_\alpha u \in \mathscr{S}'(\mathbb{R}^d)$ and

$$\partial^\alpha \mathscr{F}(u) = (-i)^{|\alpha|} \mathscr{F}(P_\alpha u),$$
$$\mathscr{F}(\partial^\alpha u) = i^{|\alpha|} P_\alpha \mathscr{F}(u),$$

where differentiation and product by P_α are as per Definition 1.2.3 and Remark 1.2.6.

(DF2) $\mathscr{F} : \mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ *is an isomorphism as per Definition 1.1.4.*

(DF3) *For any $u \in \mathscr{S}'(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$, let $\tau_y(u)$ be the translation of u by y, i.e., the unique tempered distribution such that $\langle \tau_y(u), f \rangle = \langle u, \tau_{-y}(f)\rangle$ for all $f \in \mathscr{S}(\mathbb{R}^d)$. Furthermore, let $exp_x : \xi \mapsto e^{i\xi \cdot x}$. Then, for every $u \in \mathscr{S}'(\mathbb{R}^d)$ and $\xi \in \mathbb{R}^d$, $exp_\xi u \in \mathscr{S}'(\mathbb{R}^d)$ and*

$$\mathscr{F}(\tau_y u) = exp_y \mathscr{F}(u),$$
$$\mathscr{F}(exp_y u) = \tau_{-y}\mathscr{F}(u),$$

for every $y \in \mathbb{R}^d$, where product by exp_y is as per Remark 1.2.6.

Definition 1.2.12 (Proper maps)**.** *Let X, Y be topological spaces and $F : X \to Y$. The map F is said to be **proper** if for every compact $K \subset Y$, $F^{-1}[K] \subset X$ is compact.*

Definition 1.2.13 (Convolution of distributions)**.** *Let $u, v \in \mathscr{D}'(\mathbb{R}^d)$ be such that, for every compact set $K \subset \mathbb{R}^d$,*

$$S(u, v, K) \doteq \{(x, y) \in supp(u) \times supp(v) \mid x + y \in K\} \subset \mathbb{R}^d \times \mathbb{R}^d, \tag{1.9}$$

*is a compact set. The **convolution between u and v** is $u \star v \in \mathscr{D}'(\mathbb{R}^d)$ given by*

$$\langle u \star v, f \rangle \doteq \langle u \otimes v, f_+ \rangle,$$

where $u \otimes v$ is the tensor product between u and v, see Definition 1.2.6, and, for all $f \in \mathscr{D}(\mathbb{R}^d)$, $f_+ \in \mathscr{E}(\mathbb{R}^d)$ is given by $f_+(x, y) = f(x + y)$ for all $x, y \in \mathbb{R}^d$.

Remark 1.2.23. We note that the support condition of Equation (1.9) is symmetric with respect to the exchange $(u, v) \mapsto (v, u)$. Thus, for any $u, v \in \mathscr{D}'(\mathbb{R}^d)$ for which $u \star v$ is well-defined as per Definition 1.2.13, one has that $v \star u$ is also well-defined. Furthermore, it also holds true that the convolution product is commutative, *i.e.*, $u \star v = v \star u$, and associative, *i.e.*, $u_1 \star (u_2 \star u_3) = (u_1 \star u_2) \star u_3$, for all distributions for which these operations are well-defined, see [16, Theorem 6.37]. □

Remark 1.2.24. For any $f \in \mathscr{D}(\mathbb{R}^d)$, the function f_+, see Definition 1.2.13, is smooth but not necessarily compactly supported. Therefore, for arbitrary $u, v \in \mathscr{D}'(\mathbb{R}^d)$, as $u \otimes v \in \mathscr{D}'(\mathbb{R}^{2d})$, the expression $\langle u \otimes v, f_+ \rangle$ is ill-defined. The additional hypothesis on the supports of u and v in Definition 1.2.13 is sufficient to ensure that $u \star v$ is a well-defined distribution on $\mathscr{D}(\mathbb{R}^d)$. In particular, if either u or v has compact support as per Definition 1.2.4, then the condition on the set in Equation (1.9) is automatically satisfied, see the discussion in [4, Chapter 4.2]. □

Remark 1.2.25. Let $\chi \in \mathscr{D}(\mathbb{R}^d)$ and $u \in \mathscr{D}'(\mathbb{R}^d)$. As $supp(u_\chi) = supp(\chi)$ is compact, see Remark 1.2.7, $u \star u_\chi$ is well-defined as per Definition 1.2.13 and Remark 1.2.24. Then, with the notation of Proposition 1.2.5 ($F4$), the function

$$\mathbb{R}^d \ni x \mapsto f(x) = \langle u, \chi(x - \cdot)\rangle,$$

where $\chi(x-\cdot) : y \mapsto \chi(x-y)$, is smooth and $u \star u_\chi = u_f$, see [4, Theorem 4.2.3]. Analogously, if $u = u_\varphi$ for some $\varphi \in \mathscr{E}(\mathbb{R}^d)$, then $u_\varphi \star u_\chi = u_{\varphi\star\chi}$, where $\varphi \star \chi$ is as per Definition 1.2.10. □

By duality, all of the properties in Propositions 1.2.5-1.2.7 and Theorem 1.2.8 can be extended to the convolution of distributions. Amongst the properties of the convolution product, we wish to highlight its relationship with Schwartz functions and tempered distribution.

Proposition 1.2.10. *Let $f \in \mathscr{S}(\mathbb{R}^d)$, u_f be the associated distribution as per Remark 1.2.3 and let $u \in \mathscr{S}'(\mathbb{R}^d)$ as per Definition 1.2.7. Then, the convolution $u \star u_f$ is a well-defined tempered distribution, see Definition 1.2.13, and there exists $g \in \mathscr{E}_{pol}(\mathbb{R}^d)$, see Definition 1.2.8, such that $u \star u_f = u_g$. The function g is given by*

$$g(x) \doteq \langle u, f(x-\cdot)\rangle,$$

where $f(x-\cdot) : y \mapsto f(x-y)$, for every $x \in \mathbb{R}^d$.

A proof of this result can be found in [16, Theorem 7.19(b)].

Theorem 1.2.11 (Properties of convolution). *Let $u, v \in \mathscr{D}'(\mathbb{R}^d)$ be such that $u \star v$ is a well-defined distribution in $\mathscr{D}'(\mathbb{R}^d)$ as per Definition 1.2.13.*

(C1) For any $\alpha \in \mathbb{N}^d$,

$$\partial^\alpha(u \star v) = (\partial^\alpha u) \star v = u \star (\partial^\alpha v),$$

where the derivatives are as per Definition 1.2.3.

(C2) As per Definition 1.2.4,

$$supp(u \star v) \subset \overline{supp(u) + supp(v)}.$$

Hence, for every $u, v \in \mathscr{E}'(\mathbb{R}^d)$, $u \star v \in \mathscr{E}'(\mathbb{R}^d)$.

(C3) The convolution product, as a map $\star : \mathscr{D}'(\mathbb{R}^d) \times \mathscr{E}'(\mathbb{R}^d) \to \mathscr{D}'(\mathbb{R}^d)$, is hypocontinuous, see Definition 1.1.14. Similarly, it is hypocontinuous as a map $\star : \mathscr{S}'(\mathbb{R}^d) \times \mathscr{E}'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$.

Proofs of these results can be found, respectively, in [4, Chapter 4.2], [19, Theorem 27.6], [4, Theorem 7.1.15].

Remark 1.2.26. Amongst the properties of the Fourier transform for functions, Theorem 1.2.8 is the most delicate to extend to distributions. This difficulty stems from the fact that, in general, the Fourier transform of a distribution is a distribution and any notion of pointwise multiplication of distributions is ill-defined on arbitrary distributions, even in $\mathcal{E}'(\mathbb{R}^d)$, see [4, Chapter 8.2]. Nevertheless, appropriate versions of this property is available for distributions in the appropriate spaces. We also note that a notion of pointwise product for distributions can be defined under additional conditions on the structure of their singularities with the methods we shall explore in Chapter 2. □

We wish to present now the analogue of Theorem 1.2.8 for distributions. To this avail, we shall introduce the celebrated Paley-Wiener-Schwartz theorem. To state this result, we shall require a brief review of some additional notions from complex analysis. A more thorough exploration of these concepts can be found in [16, Part 1, Chapter 3] and [4, Chapter 7.3]

Definition 1.2.14 (Holomorphic functions)**.** *Let $\Omega \subset \mathbb{C}^d$ be an open set and let $f : \Omega \to \mathbb{C}$ be a continuous function. We say that f is **holomorphic in Ω** if, for every $j \in \mathbb{N}$ and every $(z_1, \dots, z_d) \in \Omega$, there exists $U \subset \mathbb{C}$ a neighbourhood of* 0 *such that the function*

$$\mathbb{C} \ni \lambda \mapsto g_j(\lambda) \doteq f(z_1, \dots, z_{j-1}, z_j + \lambda, z_{j+1}, \dots, z_d)$$

*is differentiable in U. If $f : \mathbb{C}^d \to \mathbb{C}$ is holomorphic in $\mathbb{C}^d$, then it is said to be **entire**.*

Holomorphic functions enjoy the following properties.

Proposition 1.2.12. *Let $\Omega \subset \mathbb{C}^d$ be an open set and let $f : \Omega \to \mathbb{C}$ be a holomorphic function as per Definition 1.2.14.*

- *$(H1)$ If f is entire and bounded, then it is constant.*
- *$(H2)$ $f|_{\mathbb{R}^d \cap \Omega}$ is real analytic, i.e., for every $x_0 \in \mathbb{R} \cap \Omega$ there exists $U \subset \mathbb{R} \cap \Omega$ a neighbourhood of x_0 and $(a_\alpha)_{\alpha \in \mathbb{N}^d} \subset \mathbb{C}$ such that*

  $$f(x) = \sum_{\alpha \in \mathbb{N}^d} a_\alpha (x - x_0)^\alpha,$$

  for every $x \in U_0$.
- *$(H3)$ If $f|_{\mathbb{R}^d \cap \Omega} \equiv 0$, then $f \equiv 0$ in Ω.*

Proofs of these results can be found, respectively, in [22, Theorem 10.23], [4, Lemma 1.3.9] and [16, Lemma 7.21]. The following property is a consequence of Proposition 1.2.12, item ($H2$).

Corollary 1.2.13. *Let $f : \mathbb{C}^d \to \mathbb{C}$ be an holomorphic function as per Definition 1.2.14 and let $\tilde{f} \doteq f|_{\mathbb{R}^d}$. If there exists $\Omega \subset \mathbb{R}^d$ with positive Lebesgue measure such that $\tilde{f}|_\Omega \equiv 0$, then $f \equiv 0$ everywhere on $\mathbb{C}^d$. In particular, if f has compact support and is holomorphic, then it is identically* 0.

A proof of this result can be found in [23].

Lemma 1.2.14. *Let $u \in \mathscr{E}'(\mathbb{R}^d)$ as per Definition 1.2.1. Then there exists a unique smooth function $g : \mathbb{R}^d \to \mathbb{C}$ such that $\mathscr{F}(u) = u_g$, see Definition 1.2.11 and Remark 1.2.15. The function g is given by*

$$g(\xi) \doteq \langle u, exp_{-\xi} \rangle, \tag{1.10}$$

for all $\xi \in \mathbb{R}^d$, where $exp_\eta : x \mapsto e^{i\eta \cdot x}$ for all $\eta \in \mathbb{R}^d$.

A proof of this result can be found in [4, Theorem 7.1.14].

Remark 1.2.27. In order to state the Paley-Wiener-Schwartz theorem, we need to introduce the notion of distributions defined on complex number. However, we note that the space $\mathscr{D}(\Omega)$ in Definition 1.2.1 is well-defined as the space of smooth, compactly supported functions from Ω to $\mathbb{C}$ even if $\Omega \subset \mathbb{C}^d$. Furthermore, the topology of Remark 1.2.1 carries over to the complex case unchanged. Hence, the space $\mathscr{D}'(\mathbb{C}^d)$ is the dual space as per Definition 1.1.4 of $\mathscr{D}(\mathbb{C}^d)$ with respect to its locally convex topology, and the space $\mathscr{E}'(\mathbb{C}^d)$ is defined analogously.

We also note that, for all $u \in \mathscr{E}'(\mathbb{R}^d)$, Equation (1.10) defines a smooth function that extends uniquely to an entire function $\tilde{g}$ on $\mathbb{C}^d$ as per Definition 1.2.14, see [4, Theorem 7.1.14] and [16, Lemma 7.21]. We call $u_{\tilde{g}}$ the **Fourier-Laplace transform** of u and we denote it by $\tilde{\mathscr{F}}(u)$.

Theorem 1.2.15 (Paley-Wiener). *Let $\varphi \in \mathscr{D}(\mathbb{R}^d)$ as per Definition 1.2.1 and let $K \subset \mathbb{R}^d$ be a convex, compact set such that $supp(\varphi) \subset K$. Let $\tilde{\mathscr{F}}(\varphi)$ denote the unique entire extension of $\mathscr{F}(\varphi)$, see Definition 1.2.9, to $\mathbb{C}^d$ as per Remark 1.2.27. Then, for every $N \in \mathbb{N}$, there exists $C_N > 0$ such that*

$$|\tilde{\mathscr{F}}(\varphi)(\xi)| \leq C_N \langle \xi \rangle^{-N} e^{H_K(Im(\xi))}, \tag{1.11}$$

for every $\xi \in \mathbb{C}^d$, where $H_K : \mathbb{R}^d \to \mathbb{R}$ is given by $H_K(\eta) = \sup_{x \in K} x \cdot \eta$ for every $\eta \in \mathbb{R}^d$. Conversely, if $\tilde{F} : \mathbb{C}^d \to \mathbb{C}$ is an entire function which satisfies Equation (1.11), then there exists $\varphi \in \mathscr{D}(\mathbb{R}^d)$ such that $\tilde{F}|_{\mathbb{R}^d} \equiv \mathscr{F}(\varphi)$.

Theorem 1.2.16 (**Paley-Wiener-Schwartz**). *Let $u \in \mathscr{E}'(\mathbb{R}^d)$ as per Definition 1.2.1 and let $K \subset \mathbb{R}^d$ be a convex, compact set such that $supp(u) \subset K$, see Definition 1.2.4. Let $N = ord(u) < +\infty$, see Definition 1.2.2 and Remark 1.2.2, and let $\tilde{g}$ be the unique entire extension to $\mathbb{C}^d$ of the smooth function g for which $\mathscr{F}(u) = u_g$, see Remark 1.2.27. Then, there exists $C > 0$ such that*

$$|\tilde{g}(\xi)| \leq C\langle\xi\rangle^N e^{H_K(Im(\xi))}, \tag{1.12}$$

for every $\xi \in \mathbb{C}^d$, where $H_K : \mathbb{R}^d \to \mathbb{R}$ is the function given by $H_K(\xi) = \sup_{x\in K} x\cdot\eta$ for every $\eta \in \mathbb{R}^d$.

Conversely, if $\tilde{g} : \mathbb{C}^d \to \mathbb{C}$ is an entire function as per Definition 1.2.14 which satisfies Equation 1.12 for some choice of a compact set $K \subset \mathbb{R}^d$ and a natural number N, then there exists a unique $u \in \mathscr{E}'(\mathbb{R}^d)$ such that $supp(u) \subset K$, $ord(u) \leq N$ and, given $g = \tilde{g}|_{\mathbb{R}^d}$, $\mathscr{F}(u) = u_g$.

A proof of Theorems 1.2.16 and 1.2.15 can be found in [4, Theorem 7.3.1].

Remark 1.2.28. Let $u \in \mathscr{E}'(\mathbb{R}^d)$ as per Definition 1.2.1 and let $N \doteq ord(u)$ as per Definition 1.2.2. Let $g \in \mathscr{E}(\mathbb{R}^d)$ be such that $\mathscr{F}(u) = u_g$ as per Lemma 1.2.14. Then, as a consequence of Theorem 1.2.16, there exists $C > 0$ such that

$$|g(\xi)| \leq C\langle\xi\rangle^N, \tag{1.13}$$

for every $\xi \in \mathbb{R}^d$. Furthermore, the map $F : \mathscr{E}'(\mathbb{R}^d) \to \mathscr{E}(\mathbb{R}^d)$ given by $F(u) = g$, where $\mathscr{F}(u) = u_g$, is continuous. In fact, let $u \in \mathscr{E}'(\mathbb{R}^d)$, let $K \subset \mathbb{R}^d$ be a compact set and let $\alpha \in \mathbb{N}^d$. Then, by Lemma 1.2.14 and Proposition 1.2.9, item $(DF1)$, we have that

$$p_{K,\alpha}(F(u)) = \sup_{x\in K} |\partial_\xi^\alpha F(u)(\xi)| = \sup_{x\in K} |\partial_\xi^\alpha \langle u(x), e^{-ix\cdot\xi}\rangle| = \\ \sup_{x\in K} |\langle u(x), (-i)^{|\alpha|} x^\alpha e^{-ix\cdot\xi}\rangle| \leq \sup_{\varphi\in B_{K,\alpha}} |\langle u, \varphi\rangle|,$$

where $B_{K,\alpha}$ is the set of functions $\varphi : x \mapsto x^\alpha e^{-ix\cdot\xi}$, $\xi \in K$. As $B_{K,\alpha} \subset \mathscr{E}(\mathbb{R}^d)$ is a bounded set, the map

$$\mathscr{E}'(\mathbb{R}^d) \ni u \mapsto \sup_{\varphi\in B_{K,\alpha}} |\langle u, \varphi\rangle| \in [0, +\infty),$$

is a continuous seminorm on $\mathscr{E}'(\mathbb{R}^d)$ by Remark 1.2.1. Hence, by arbitrariness of $K \subset\subset \mathbb{R}^d$ and $\alpha \in \mathbb{N}^d$, it follows that F is continuous. □

Remark 1.2.29. As a consequence of Remark 1.2.28, the Fourier transform of a compactly supported distribution is, up to the identification $u_g \mapsto g$, a smooth function of polynomial growth as per Definition 1.2.8. Thus, let $u \in \mathscr{E}'(\mathbb{R}^d)$

and $g \in \mathscr{E}_{pol}(\mathbb{R}^d)$ be such that $\mathscr{F}(u) = u_g$ and let $v \in \mathscr{S}'(\mathbb{R}^d)$. As per Theorem 1.2.11, the convolution $u \star v$ is a well-defined tempered distribution. Furthermore, the product $g\mathscr{F}(v) \in \mathscr{S}'(\mathbb{R}^d)$ is well-defined by Theorem 1.2.9 and Remark 1.2.16. Hence, the equality

$$\mathscr{F}(u \star v) = (2\pi)^{d/2} g\mathscr{F}(v),$$

is well-defined and avoids the problem discussed in Remark 1.2.26. Therefore, up to the identification $\mathscr{F}(u) \mapsto g$, we have that, for all $u \in \mathscr{E}'(\mathbb{R}^d)$ and all $v \in \mathscr{S}'(\mathbb{R}^d)$,

$$\mathscr{F}(u \star v) = (2\pi)^{d/2} \mathscr{F}(u)\mathscr{F}(v).$$

□

1.3 Besov Spaces

We conclude this first chapter by recalling the construction and basic properties of Besov spaces. We shall review the Littlewood-Paley decomposition for tempered distribution and a collection of embeddings between Besov and other functional spaces. Then, we shall recall the properties of the convolution product and the duality one can construct between Besov spaces. Lastly, we shall recount the construction of local Besov spaces through their local characterization. Before delving into these arguments, we wish, for the sake of clarity, to fix some notation regarding functions and distributions. More specifically, the identification of some functional spaces with subsets of distributional spaces in agreement with what has been explored throughout Section 1.2.

Remark 1.3.1 (Notation). In light of Remark 1.2.3, we henceforth identify the distributions $u_f \in \mathscr{D}'(\Omega)$, $\Omega \subset \mathbb{R}^d$ an open set, with their generating functions $f \in L^1_{loc}(\Omega)$. Analogously, each space

$$\mathscr{D}(\Omega) \hookrightarrow \mathscr{D}(\mathbb{R}^d) \subset \mathscr{S}(\mathbb{R}^d) \subset \mathscr{E}_{pol}(\mathbb{R}^d) \subset \mathscr{E}(\mathbb{R}^d) \hookrightarrow \mathscr{E}(\Omega),$$

where the immersion $\mathscr{D}(\Omega) \hookrightarrow \mathscr{D}(\mathbb{R}^d)$ is given by the extension as 0 on $\mathbb{R}^d \setminus \Omega$ while the immersion $\mathscr{E}(\mathbb{R}^d) \hookrightarrow \mathscr{E}(\Omega)$ is given by the restriction to Ω, see Definitions 1.2.1, 1.2.7 and 1.2.8, will be identified with the respective subspaces of

$$\mathscr{E}'(\Omega) \hookrightarrow \mathscr{S}'(\mathbb{R}^d) \hookrightarrow \mathscr{D}'(\Omega),$$

in accordance with Remarks 1.2.3, 1.2.15 and 1.2.16. Similarly, for any function $f \in L^1_{loc}(\Omega)$ and any multi-index $\alpha \in \mathbb{N}^d$, $\partial^\alpha f$ will denote the distributional derivative of order α of f, see Definition 1.2.3. Statements concerning pointwise or functional properties of f are to be interpreted at the level of representatives. For instance, $\partial^\alpha f \in L^1(\Omega)$, $f \in L^1_{loc}(\Omega)$ and $\alpha \in \mathbb{N}^d$, entails that there exists $g \in L^1(\Omega)$ such that $\partial^\alpha u_f = u_g$. Likewise, the Fourier transform of a distribution $u_f \in \mathscr{S}'(\mathbb{R}^d)$, for $f : \mathbb{R}^d \to \mathbb{C}$, shall be denoted as $\mathscr{F}(f)$, equivalently $\hat{f}$, without distinguishing notationally between the distribution or its functional representative. For instance, in light of Lemma 1.2.14, for any $u \in \mathscr{E}'(\mathbb{R}^d)$, we shall write $\mathscr{F}(u) \in \mathscr{E}_{pol}(\mathbb{R}^d)$ if there exists $g \in \mathscr{E}_{pol}(\mathbb{R}^d)$ such that $\mathscr{F}(u) = u_g$. □

We shall now review the notion of Littlewood-Paley partition of unity, which shall be of paramount importance for the construction of Besov spaces.

Definition 1.3.1 (Littlewood-Paley). *Let $N \in \mathbb{N}$ and let $\psi \in \mathscr{D}(\mathbb{R}^d)$ be a positive function satisfying*

$$supp(\psi) \subset \overline{B(0, 2^N)} \setminus \overline{B(0, 2^{-N})} \text{ and } \psi(-x) = \psi(x) \text{ for all } x \in \mathbb{R}^d.$$

The ***Littlewood-Paley partition of unity*** *associated with ψ is the sequence $(\psi_k)_{k\in\mathbb{N}}$ of smooth, compactly supported functions which satisfies*

(LP1) $supp(\psi_0) \subset \overline{B(0, 2^N)}$ *and* $\psi_0(-x) = \psi_0(x)$ *for all* $x \in \mathbb{R}^d$*;*

(LP2) $\psi_k(x) = \psi(2^{-k}x)$ *for all* $x \in \mathbb{R}^d$ *and* $k \geq 1$*;*

(LP3) for every $x \in \mathbb{R}^d$,

$$\sum_{k=0}^{+\infty} \psi_k(x) = 1.$$

For any $k \in \mathbb{N}$, we write Δ_k for the operators given by

$$\Delta_k(u) = \mathscr{F}^{-1}[\psi_k \mathscr{F}(u)],$$

for any $u \in \mathscr{S}'(\mathbb{R}^d)$, see Remark 1.3.1.

Remark 1.3.2. Let $\psi \in \mathscr{D}(\mathbb{R}^d)$ be as per Definition 1.3.1. Then, a Littlewood-Paley partition of unity associated with ψ exists, see [17, Proposition 2.10]. We also note that the operators Δ_k are well-defined as linear, continuous maps from $\mathscr{S}'(\mathbb{R}^d)$ into $\mathscr{E}(\mathbb{R}^d)$. In fact, for any $u \in \mathscr{S}'(\mathbb{R}^d)$, $\mathscr{F}(u) \in \mathscr{S}'(\mathbb{R}^d)$, see Theorem 1.2.9, item $(DF2)$. Then, as $supp(\psi_k)$ is compact, $\psi_k \mathscr{F}(u) \in \mathscr{E}'(\mathbb{R}^d)$, which implies $\Delta_k(u) \in \mathscr{E}_{pol}(\mathbb{R}^d) \subset \mathscr{E}(\mathbb{R}^d)$ by Remark 1.2.28. The continuity of Δ_k follows from the continuity of the above operations. We also note that, for all $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}^d$,

$$|\partial^\alpha \psi_k(x)| = |\partial_x^\alpha \psi(2^{-k}x)| = |2^{-k|\alpha|} \partial^\alpha(\psi)(2^{-k}x)| \leq 2^{-k|\alpha|} \|\psi\|_{0,\alpha},$$

where $\|\cdot\|_{n,\beta}$, $n \in \mathbb{N}$ and $\beta \in \mathbb{N}^d$ are as per Remark 1.2.14. Then, the condition $supp(\psi_k) \subset \{2^{k-N} \leq \|x\| \leq 2^{k+N} \mid x \in \mathbb{R}^d\}$, implies that

$$|\partial^\alpha \psi_k(x)| \leq 2^{-k|\alpha|}\|\psi\|_{0,\alpha} \leq C_\alpha \langle x\rangle^{-|\alpha|},$$

for all $k \in \mathbb{N}$ and some fixed $C_\alpha > 0$ independent of k.

□

Remark 1.3.3. The Littlewood-Paley partition of unity $(\psi_k)_{k\in\mathbb{N}}$ associated to an appropriately chosen $\psi \in \mathscr{D}(\mathbb{R}^d)$, see Definition 1.3.1, reproduces an identity operator. In fact, for any $u \in \mathscr{S}'(\mathbb{R}^d)$, let $S_M(u) \doteq \sum_{k=0}^{M} \Delta_k(u) \in \mathscr{S}'(\mathbb{R}^d)$, $M \in \mathbb{N}$. Then, one has that $S_M(u) \to u$ in $\mathscr{S}'(\mathbb{R}^d)$, see [17, Proposition 2.12]. Hence, the sequence $(S_M)_{M\in\mathbb{N}}$ converges to the identity of $\mathscr{S}'(\mathbb{R}^d)$ in $\mathfrak{L}(\mathscr{S}'(\mathbb{R}^d))_\sigma$, see Remark 1.1.10. Analogously, S_M converges to the identity of $\mathscr{S}(\mathbb{R}^d)$ in $\mathfrak{L}(\mathscr{S}(\mathbb{R}^d))_\sigma$. As $\mathscr{S}(\mathbb{R}^d)$ and $\mathscr{S}'(\mathbb{R}^d)$ are Montel spaces as per Definition 1.1.10, see [21, Chapter 3.9, Example 5] and Proposition 1.1.7, Proposition 1.1.9 implies that S_M converges to the identity respectively in $\mathfrak{L}(\mathscr{S}(\mathbb{R}^d))_b$ and in $\mathfrak{L}(\mathscr{S}'(\mathbb{R}^d))_b$. □

Definition 1.3.2 (Banach valued ℓ^q-spaces)**.** *Let $(X, \|\cdot\|_X)$ be a complex Banach space, let $a = (a_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ be a sequence satisfying $a_n \neq 0$ for every $n \in \mathbb{N}$ and let $q \in [1, +\infty]$. We call $\ell^q(aX)$ the space of sequences $f = (f_n)_{n\in\mathbb{N}} \subset X$ for which the norm*

$$\|f\|_{\ell^q(aX)} \doteq \begin{cases} \left(\sum_{n\in\mathbb{N}} \|a_n f_n\|_X^q\right)^{\frac{1}{q}} & q < +\infty, \\ \sup_{n\in\mathbb{N}} \|a_n f_n\|_X & q = +\infty. \end{cases}$$

is finite. We fix $\ell^q \doteq \ell^q(\mathbf{1}\mathbb{C})$, where $\mathbf{1}$ is the constant sequence where each entry is equal to 1.

Definition 1.3.3 (Besov spaces)**.** *Let $p, q \in [1, +\infty]$ and let $\alpha \in \mathbb{R}$. Let $\psi \in \mathscr{D}(\mathbb{R}^d)$, let $(\psi_k)_{k\in\mathbb{N}} \subset \mathscr{D}(\mathbb{R}^d)$ be its associated Littlewood-Paley partition of unity and let Δ_k be as per Definition 1.3.1. The Besov space $B^\alpha_{p,q}(\mathbb{R}^d)$ is the set of $u \in \mathscr{S}'(\mathbb{R}^d)$ such that*

$$\|u\|^\psi_{B^\alpha_{p,q}(\mathbb{R}^d)} \doteq \|(2^{k\alpha}\|\Delta_k(u)\|_{L^p(\mathbb{R}^d)})_{k\in\mathbb{N}}\|_{\ell^q} < +\infty, \tag{1.14}$$

where ℓ^q is as per Definition 1.3.2.

Remark 1.3.4. By Remark 1.3.2, Equation (1.14) is a well-defined norm on $B^\alpha_{p,q}(\mathbb{R}^d)$. Furthermore, it is independent of the choice of Littlewood-Paley partition of unity, in the sense that, if $(\chi_k)_{k\in\mathbb{N}}$ is a Littlewood-Paley partition of unity associated with $\chi \in \mathscr{D}(\mathbb{R}^d)$ as per Definition 1.3.1 with $\chi \neq \psi$, then there exists $C_1, C_2 > 0$ such that

$$C_1\|u\|^\chi_{B^\alpha_{p,q}(\mathbb{R}^d)} \leq \|u\|^\psi_{B^\alpha_{p,q}(\mathbb{R}^d)} \leq C_2\|u\|^\chi_{B^\alpha_{p,q}(\mathbb{R}^d)},$$

for all $u \in B^{\alpha}_{p,q}(\mathbb{R}^d)$, see [6, Theorem 2.1]. Thus, we shall henceforth omit the index ψ in the norm $\|\cdot\|^{\psi}_{B^{\alpha}_{p,q}(\mathbb{R}^d)}$. □

Remark 1.3.5. In light of Remark 1.3.3, one has the decomposition

$$u = \sum_{k=0}^{+\infty} \Delta_k(u), \tag{1.15}$$

for all $u \in \mathscr{S}'(\mathbb{R}^d)$, where the series converges in the topology of $\mathscr{S}'(\mathbb{R}^d)$ in the sense of Remark 1.3.2. Thus, $B^{\alpha}_{p,q}(\mathbb{R}^d)$ in Definition 1.3.3 consists of the tempered distributions whose decomposition as per Equation (1.15) converges in a weighted ℓ^q space. More rigorously, Definitions 1.3.3 and 1.3.2 imply that $u \in B^{\alpha}_{p,q}(\mathbb{R}^d)$ if and only if the sequence $(\Delta_k(u))_{k\in\mathbb{N}}$ is an element of $\ell^q(2^{(\cdot)\alpha}L^p(\mathbb{R}^d))$, where $2^{(\cdot)\alpha} = (2^{k\alpha})_{k\in\mathbb{N}}$. □

Besov spaces enjoy the following useful properties.

Proposition 1.3.1 (Properties of Besov spaces). *Let $\alpha \in \mathbb{R}$, $p,q \in [1,+\infty]$ and let $B^{\alpha}_{p,q}(\mathbb{R}^d)$ be as per Definition 1.3.3.*

(B1) The space $(B^{\alpha}_{p,q}(\mathbb{R}^d), \|\cdot\|_{B^{\alpha}_{p,q}(\mathbb{R}^d)})$ is a Banach space.

(B2) $\mathscr{S}(\mathbb{R}^d) \hookrightarrow B^{\alpha}_{p,q}(\mathbb{R}^d) \hookrightarrow \mathscr{S}'(\mathbb{R}^d)$, see Definition 1.2.7, and the inclusions are continuous.

(B3) If $p,q < +\infty$, then $\mathscr{D}(\mathbb{R}^d)$ is sequentially dense in $B^{\alpha}_{p,q}(\mathbb{R}^d)$.

Proofs of these results can be found in [6, Theorem 2.4].

Proposition 1.3.2 (Monotonic inclusions of Besov spaces). *For all $\alpha \in \mathbb{R}$ and $p,q \in [1,+\infty]$, let $B^{\alpha}_{p,q}(\mathbb{R}^d)$ be a Besov space as per Definition 1.3.3.*

(BM1) If $1 \le q_1 \le q_2 \le +\infty$, then $B^{\alpha}_{p,q_1}(\mathbb{R}^d) \hookrightarrow B^{\alpha}_{p,q_2}(\mathbb{R}^d)$ for any $\alpha \in \mathbb{R}$ and $p \in [1,+\infty]$.

(BM2) If $\alpha > \beta$, then $B^{\alpha}_{p,q_1}(\mathbb{R}^d) \hookrightarrow B^{\beta}_{p,q_2}(\mathbb{R}^d)$ for any $p, q_1, q_2 \in [1,+\infty]$.

(BM3) If $\alpha > \beta$ and $1 \le p_1 < p_2 \le +\infty$ are such that $\alpha - \frac{d}{p_1} = \beta - \frac{d}{p_2}$, then $B^{\alpha}_{p_1,q}(\mathbb{R}^d) \hookrightarrow B^{\beta}_{p_2,q}(\mathbb{R}^d)$ for all $q \in [1,+\infty]$.

Proofs of these results can be found in [6, Propositions 2.2, 2.3] and [6, Theorem 2.5]

In applications to PDEs, Besov spaces contain or interpolate between many classical function spaces, including Lebesgue and Sobolev spaces. We briefly recall the classes needed to state the relevant embeddings.

Definition 1.3.4 (Hölder-Zygmund spaces). *Let $\Omega \subset \mathbb{R}^d$ be an open set, let $k \in \mathbb{N}$ and let $r \in [0,1]$. A function $f : \Omega \to \mathbb{C}$ is said to be **r- Hölder continuous** if*

$$\sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|f(x)-f(y)|}{\|x-y\|^r} < +\infty.$$

*The function f is said to be of **Hölder-Zygmund class** of order (k,r), written $f \in C^{k,r}(\Omega)$, it f is differentiable up to order k, all of its derivatives are bounded while the derivatives of f of order k are r-Hölder continuous. We assume $C^{k,r}(\Omega)$ to be equipped with the Banach topology induced by the norm*

$$C^{k,r}(\Omega) \ni f \mapsto \|f\|_{C^{k,r}(\Omega)} \doteq \|f\|_{C^k(\Omega)} + \sum_{|\alpha|=k} \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|\partial^\alpha f(x)-\partial^\alpha f(y)|}{\|x-y\|^r}.$$

Definition 1.3.5 (Sobolev and Sobolev-Slobodeckij spaces). *Let $\Omega \subset \mathbb{R}^d$, let $k \in \mathbb{N}$ and let $p \in [1,+\infty]$. The **Sobolev space of order (k,p)** over Ω, denoted by $W^{k,p}(\Omega)$, is the set of functions $f \in L^p(\Omega)$ such that, for every $\alpha \in \mathbb{N}^d$ with $0 \le |\alpha| \le k$, $\partial^\alpha f \in L^p(\Omega)$ as per Remark 1.3.1. We assume $W^{k,p}(\Omega)$ to be equipped with the Banach topology induced by the norm*

$$W^{k,p}(\Omega) \ni f \mapsto \|f\|_{W^{k,p}(\Omega)} \doteq \sum_{|\alpha|\le k} \|\partial^\alpha f\|_{L^p(\Omega)}.$$

*For any non integer $s \in (0,+\infty)$ and $p \in [1,+\infty)$, the **Sobolev-Slobodeckij space of order (s,p)** over Ω, denoted by $W^{s,p}(\Omega)$, is the set of $f \in W^{\lfloor s\rfloor,p}(\Omega)$ such that*

$$\sum_{|\alpha|=\lfloor s\rfloor} \left(\int_\Omega \int_\Omega \frac{|\partial^\alpha f(x)-\partial^\alpha f(y)|^p}{\|x-y\|^{d+(s-\lfloor s\rfloor)p}} dxdy \right) < +\infty.$$

We assume $W^{s,p}(\Omega)$ to be endowed with the Banach topology induced by the norm

$$W^{s,p}(\Omega) \ni f \mapsto \|f\|_{W_S^{s,p}(\Omega)} \doteq \|f\|_{W^{\lfloor s\rfloor,p}(\Omega)} + \sum_{|\alpha|=\lfloor s\rfloor} \left(\int_\Omega \int_\Omega \frac{|\partial^\alpha f(x)-\partial^\alpha f(y)|^p}{\|x-y\|^{d+(s-\lfloor s\rfloor)p}} dxdy \right)^{1/p}.$$

Remark 1.3.6. The Sobolev-Slobodeckij spaces as per Definition 1.3.5 are generalization of Sobolev spaces for non-integer $s \in (0,+\infty)$. However, for $s \in \mathbb{N}$, the norm $\|\cdot\|_{W_S^{s,p}(\Omega)}$ is possibly ill-defined and it may diverge, see [24]. Thus, we shall henceforth assume $W^{s,p}(\Omega)$ to be the Sobolev space of order (s,p) for any $s \in \mathbb{N}$ and $p \in [1,+\infty]$ and the Sobolev-Slobodeckij space of order (s,p) for any non-integer $s \in (0,+\infty)$ and $p \in [1,+\infty]$. □

Proposition 1.3.3 (Functional inclusions of Besov spaces). *For all $\alpha \in \mathbb{R}$ and $p,q \in [1,+\infty]$, let $B^\alpha_{p,q}(\mathbb{R}^d)$ be a Besov space as per Definition 1.3.3.*

(BF1) *For every $p \in [1, +\infty]$, $B^0_{p,1}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \hookrightarrow B^0_{p,\infty}(\mathbb{R}^d)$, see Definition 1.3.3.*

(BF2) *Let $C_{b,u}(\mathbb{R}^d)$ be the space of bounded, uniformly continuous functions from $\mathbb{R}^d$ into $\mathbb{C}$. Then, $B^0_{\infty,1} \hookrightarrow C_{b,u}(\mathbb{R}^d) \hookrightarrow B^0_{\infty,\infty}$.*

(BF3) *For every $p \in (1, +\infty)$ and $\alpha \in \mathbb{N}$, $B^\alpha_{p,1}(\mathbb{R}^d) \hookrightarrow W^{\alpha,p}(\mathbb{R}^d) \hookrightarrow B^\alpha_{p,\infty}$, see Definition 1.3.5. For every $p \in (1, +\infty)$ and every positive, non-integer α, the spaces $B^\alpha_{p,p}(\mathbb{R}^d)$ and $W^{\alpha,p}(\mathbb{R}^d)$ are isomorphic as per Definition 1.1.4.*

(BF4) *For every non-integer $\alpha \in (0, +\infty)$, $B^\alpha_{\infty,\infty}(\mathbb{R}^d)$ is isomorphic to the space $C^{\lfloor\alpha\rfloor,\alpha-\lfloor\alpha\rfloor}(\mathbb{R}^d)$, see Definition 1.3.4.*

Proofs of these results can be found, respectively, in [25, Section 2.5.7, pp. 89-90], [26, Section 1.2.4] and [26, Section 2.6.5].

Remark 1.3.7. Each of the three parameters α, p, q in Definition 1.3.3 encodes a different type of information.

- The parameter $\alpha \in \mathbb{R}$ measures the regularity of the distribution, as can be deduced from the fact that

$$\partial_j : B^\alpha_{p,q}(\mathbb{R}^d) \to B^{\alpha-1}_{p,q}(\mathbb{R}^d),$$

  where ∂_j is the derivative with respect to the j^{th} coordinate as per Definition 1.2.3, is continuous, see [6, Theorem 2.2]. For non-integer $\alpha > 0$, the identification $B^\alpha_{\infty,\infty}(\mathbb{R}^d) \equiv C^{\lfloor\alpha\rfloor,\alpha-\lfloor\alpha\rfloor}(\mathbb{R}^d)$ further supports this interpretation.

- To understand the properties regulated by the parameters p and q, let $u \in B^\alpha_{p,q}(\mathbb{R}^d)$. Then, $(2^{\alpha k}\|\Delta_k(u)\|_{L^p(\mathbb{R}^d)}) \in \ell^q$, which implies that, for some $(c_k)_{k\in\mathbb{N}} \in \ell^q$, $\|\Delta_k(u)\|_{L^p(\mathbb{R}^d)}) \leq 2^{-\alpha k}|c_k|$. As Δ_k localizes u in Fourier space in the annular region $\{2^{k-N} \leq \|x\| \leq 2^{k+N} \mid x \in \mathbb{R}^d\}$, the limit $k \to +\infty$ can be read as a limit in the frequencies of u. Hence, at high frequencies, the decay of u is regulated by the L^p norm and the convergence in the space ℓ^q.

- The parameter q also provides finer information regarding regularity. In fact, using Proposition 1.3.2, items $(BM2)$ and $(BM3)$, and Proposition 1.3.3, item $BF(4)$, one has the following chain of inclusions

$$C^{\lfloor\alpha\rfloor,\alpha-\lfloor\alpha\rfloor}(\mathbb{R}^d) \hookrightarrow B^\beta_{\infty,q_1}(\mathbb{R}^d) \hookrightarrow B^\beta_{\infty,q_2}(\mathbb{R}^d) \hookrightarrow C^{\lfloor\beta\rfloor,\beta-\lfloor\beta\rfloor}(\mathbb{R}^d), \tag{1.16}$$

  where every inclusion is strict for every positive, non-integer $\alpha > \beta$ and $q_1 < q_2 < +\infty$. Hence, the parameter q may be interpreted as controlling the finer variation in regularity between $C^{\lfloor\alpha\rfloor,\alpha-\lfloor\alpha\rfloor}(\mathbb{R}^d)$ and $C^{\lfloor\beta\rfloor,\beta-\lfloor\beta\rfloor}(\mathbb{R}^d)$ for $\alpha>\beta$. Furthermore, Proposition 1.3.2, items $(BM2)$ and $(BM3)$, also imply that

$$B^{\alpha+\varepsilon}_{p,q_1}(\mathbb{R}^d)\hookrightarrow B^{\alpha+\varepsilon}_{p,\infty}(\mathbb{R}^d)\hookrightarrow B^{\alpha}_{p,1}(\mathbb{R}^d)\hookrightarrow B^{\alpha}_{p,q_2}(\mathbb{R}^d),$$

for any $\varepsilon>0$, $\alpha\in\mathbb{R}$ and $p,q_1,q_2\in[1,+\infty]$. Thus, arbitrarily small changes in the parameter α affect the regularity encoded by the space $B^{\alpha}_{p,q}(\mathbb{R}^d)$ more than any arbitrarily large changes in q. This points to the fact that the regularity encoded by q is much finer than that encoded by α.

□

As Besov spaces are subsets of tempered distributions, see Definition 1.3.3, it is natural to ponder whether the convolution product, see Definition 1.2.13 and Proposition 1.2.10, is well-defined and continuous. The following result provides sufficient conditions for these properties.

Proposition 1.3.4 (Continuity of convolution on Besov spaces)**.** *Let $\alpha\in\mathbb{R}$ and $q,p,p_1,p_2\in[1,+\infty]$ be such that $1+\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$. Then, for any $f\in B^{\alpha}_{p_1,q}(\mathbb{R}^d)$ and $g\in L^{p_2}(\mathbb{R}^d)$ their convolution $f\star g$, see Definition 1.2.13, is well defined. Furthermore, there exists $C>0$ such that*

$$\|f\star g\|_{B^{\alpha}_{p,q}(\mathbb{R}^d)}\leq C\|f\|_{B^{\alpha}_{p_1,q}(\mathbb{R}^d)}\|g\|_{L^{p_2}(\mathbb{R}^d)},$$

for all $f\in B^{\alpha}_{p_1,q}(\mathbb{R}^d)$ and $g\in L^{p_2}(\mathbb{R}^d)$.

A proof of this result can be found in [27].

Another key characteristic of Besov spaces are their duality properties. While in Chapter 3 we shall mostly be concerned with spaces of type $B^{\alpha}_{\infty,\infty}$ as per Definition 1.3.3, for which a characterization of the dual is not at present known, a dual pairing can still be constructed with their conjugate space, as per the following result.

Theorem 1.3.5 (Duality of Besov spaces)**.** *Let $\alpha\in\mathbb{R}$ and $p,q\in[1,+\infty]$ be fixed. Let $p',q'\in[1,+\infty]$ be such that $\frac{1}{\mathfrak{r}}+\frac{1}{\mathfrak{r}'}=1$ for $\mathfrak{r}=p,q$. Let $(\Delta_k)_{k\in\mathbb{N}}$ be the sequence of linear operators associated with the Littlewood-Paley partition of unity $(\chi_k)_{k\in\mathbb{N}}$ with $N=1$ as per Definition 1.3.1 and let*

$$\langle\cdot,\cdot\rangle_B:B^{\alpha}_{p,q}(\mathbb{R}^d)\times B^{-\alpha}_{p',q'}(\mathbb{R}^d)\to\mathbb{C},$$

be the bilinear map given by

$$\langle u,v\rangle_B\doteq\sum_{k,k'\in\mathbb{N}}\int_{\mathbb{R}^d}\Delta_k(u)(x)\Delta_{k'}(v)(x)dx,\tag{1.17}$$

for all $u \in B^{\alpha}_{p,q}(\mathbb{R}^d)$ *and* $v \in B^{-\alpha}_{p',q'}(\mathbb{R}^d)$. *Then,* $\langle B^{\alpha}_{p,q}(\mathbb{R}^d), B^{-\alpha}_{p',q'}(\mathbb{R}^d)\rangle_B$ *is a dual pair as per Remark 1.1.19. Furthermore, the map* $\langle \cdot, \cdot\rangle_B$ *is jointly continuous as per Definition 1.1.14 and it identifies* $B^{-\alpha}_{p',q'}(\mathbb{R}^d)$ *as the dual space of* $B^{\alpha}_{p,q}(\mathbb{R}^d)$ *whenever* $p,q < +\infty$.

A proof of this result can be found in [17, Proposition 2.76].

Remark 1.3.8. We note that the bilinear map in Equation (1.17) could be reformulated as

$$\langle u, v\rangle_B{}' \doteq \sum_{\substack{k,k'\in\mathbb{N}\\ |k-k'|\leq 1}} \int_{\mathbb{R}^d} \Delta_k(u)(x)\Delta_{k'}(v)(x)dx. \tag{1.18}$$

The equivalence between $\langle \cdot, \cdot\rangle_B$ and $\langle \cdot, \cdot\rangle_B{}'$ follows from the fact that, by Definition 1.3.1, $supp(\psi_k) \subset \{2^{k-1} < \|x\| \leq 2^{k+1} \mid x \in \mathbb{R}^d\}$. In fact, $u \in B^{\alpha}_{p,q}(\mathbb{R}^d)$ implies $\Delta_k(u) \in L^p(\mathbb{R}^d)$, while $v \in B^{-\alpha}_{p',q'}(\mathbb{R}^d)$ implies $\Delta_{k'}(v) \in L^{p'}(\mathbb{R}^d)$ for any $k, k' \in \mathbb{N}$ by Definition 1.3.3. Then, for every $k \in \mathbb{N}$, let $\chi_k \in \mathscr{D}(\mathbb{R}^d)$ be a function such that $\chi_k \equiv 1$ in a neighbourhood of $supp(\psi_k)$ and $\chi_k \equiv 0$ on $supp(\psi_{k'})$ for all $k' \in \mathbb{N}$ such that $|k-k'| > 1$. As $supp(\psi_k)$ and $supp(\psi_{k'})$ are strictly separated for all $|k-k'| > 1$, χ_k exists. Furthermore, using Proposition 1.2.10, we note that

$$\Delta_k(u) = \mathscr{F}^{-1}[\psi_k \mathscr{F}(u)] = \mathscr{F}^{-1}[\chi_k\psi_k\mathscr{F}(u)] = F_k \star \Delta_k(u),$$

where $F_k = \mathscr{F}^{-1}(\chi_k)$. Thus, using [5, Proposition 4.16], one has that

$$\begin{aligned}\int_{\mathbb{R}^d} \Delta_k(u)(x)\Delta_{k'}(v)(x)dx &= \int_{\mathbb{R}^d} (F_k \star \Delta_k(u))(x)\Delta_{k'}(v)(x)dx = \\ &= \int_{\mathbb{R}^d} \Delta_k(u)(x)(\check{F}_k \star \Delta_{k'}(v))(x)dx.\end{aligned}$$

Then, as

$$\mathscr{F}\left[\check{F}_k \star \Delta_{k'}(v)\right] = \chi_k\psi_{k'}\mathscr{F}(v) \equiv 0,$$

for all $k, k' \in \mathbb{N}$ such that $|k-k'| > 1$ by our choice of χ_k, it follows that $\check{F}_k \star \Delta_{k'}(v) \equiv 0$ whenever $|k-k'| > 1$. This implies that all terms in Equation (1.17) with $|k-k'| > 1$ are identically 0. In turn, the dualities $\langle \cdot, \cdot\rangle_B$ and $\langle \cdot, \cdot\rangle_B{}'$ coincide on $B^{\alpha}_{p,q}(\mathbb{R}^d) \times B^{-\alpha}_{p',q'}(\mathbb{R}^d)$.

□

We conclude this section outlining of an equivalent characterization of Besov spaces, known in the literature as "**local means characterization**".

Definition 1.3.6. *Let $s \in \mathbb{Z}$. We call $\mathscr{B}_s$ the subset of $\mathscr{D}(B(0,1))$, see Remark 1.3.1, whose elements γ satisfy, for some $\varepsilon > 0$,*

$$\check{\gamma}(x) \neq 0 \text{ for all } x \in \left\{\frac{\varepsilon}{2} < \|x\| \leq 2\varepsilon\right\} \text{ and } \partial^\alpha \check{\gamma}(0) = 0 \text{ for all } \alpha \in \mathbb{N}^d, |\alpha| \leq s,$$

where the second condition is empty for $s < 0$.

Definition 1.3.7 (Local means norm)**.** *Let $\alpha \in \mathbb{R}$ and let $p,q \in [1,+\infty]$. Let $\gamma \in \mathscr{B}_{\lfloor\alpha\rfloor}$ as per Definition 1.3.6 and let $\beta \in \mathscr{D}(B(0,1))$ such that $\mathscr{F}^{-1}(\beta)(0) \neq 0$. We call **local means norm** the map $\|\cdot\|_{lm}^{\alpha,p,q,\beta,\gamma} : B^\alpha_{p,q}(\mathbb{R}^d) \to [0,+\infty)$ given by*

$$\|u\|_{lm}^{\alpha,p,q,\beta,\gamma} \doteq \|\langle u\,,\,\beta^1_x\rangle\|_{L^p_x(\mathbb{R}^d)} + \left\| \frac{\left\|\langle u\,,\,\gamma^\lambda_x\rangle\right\|_{L^p_x(\mathbb{R}^d)}}{\lambda^\alpha} \right\|_{\Lambda^q_\lambda(0,1)},$$

where $\Lambda^q(0,1) \doteq L^q((0,1),x^{-1}dx)$ for $q < +\infty$, $\Lambda^\infty(0,1) \doteq L^\infty(0,1)$ and , for any function $f : \mathbb{R}^d \to \mathbb{C}$, $\lambda \in (0,1)$ and $y \in \mathbb{R}^d$, $f^\lambda_y : x \mapsto \lambda^{-d} f\left(\frac{x-y}{\lambda}\right)$.

Remark 1.3.9. The norm $\|\cdot\|_{lm}^{\alpha,p,q,\beta,\gamma}$ as per Definition 1.3.7 mirrors the Littlewood-Paley structure of the Besov norm as per Definition 1.3.3 by replacing the factors 2^{-k}, $k \in \mathbb{N}$, with $\lambda \in (0,1)$ and the related ℓ^q sum over $k \in \mathbb{N}$ with an integral norm over the interval $(0,1)$. □

Proposition 1.3.6 (Equivalence of local means characterization)**.** *Let $\alpha \in \mathbb{R}$ and let $p,q \in [1,+\infty]$. Then, the local means norm $\|\cdot\|_{lm}^{\alpha,p,q,\beta,\gamma}$ as per Definition 1.3.7 is equivalent on $B^\alpha_{p,q}(\mathbb{R}^d)$ to the Besov norm $\|\cdot\|_{B^\alpha_{p,q}(\mathbb{R}^d)}$ as per Remark 1.3.4.*

A proof of this result can be found in [26, Section 2.5.3]. We state now a key characterization of endpoint Besov spaces, which follows from Proposition 1.3.6.

Corollary 1.3.7 (Local means for endpoint Besov spaces)**.** *Let $u \in \mathscr{S}'(\mathbb{R}^d)$ and let $\alpha \in \mathbb{R}$. Then $u \in B^\alpha_{\infty,\infty}(\mathbb{R}^d)$ if and only if*

$$\sup_{x\in\mathbb{R}^d} |\langle \hat{u}(\xi), e^{ix\cdot\xi}\check{\beta}(\xi)\rangle| < +\infty \text{ and } \sup_{\lambda\in(0,1)} \sup_{x\in\mathbb{R}^d} \lambda^{-\alpha} |\langle \hat{u}(\xi), e^{ix\cdot\xi}\check{\gamma}(\lambda\xi)\rangle| < +\infty,$$

for all $\beta \in \mathscr{D}(B(0,1))$, see Remark 1.3.1, such that $\check{\beta}(0) \neq 0$ and for all $\gamma \in \mathscr{B}_{\lfloor\alpha\rfloor}$, see Definition 1.3.6.

A proof of this result can be found in [12, Proposition 9].

Remark 1.3.10. An important consequence of Proposition 1.3.6 is that the choice of functions β, γ in Definition 1.3.7 is arbitrary, in the sense that each choice generates a norm equivalent to the Besov one as per Definition 1.3.3. Hence, the indices β, γ in the local means norm, see Definition 1.3.7, shall be dropped henceforth. Proposition 1.3.6 also allows one to define a wider class of Besov spaces. For instance, the local means norm in Definition 1.3.7 is still well-defined for $p, q \in (0, 1]$. Hence, one may define the space $B^{\alpha}_{p,q}(\mathbb{R}^d)$, $\alpha \in \mathbb{R}$, $p, q \in (0, +\infty]$, as the collection of $u \in \mathscr{S}'(\mathbb{R}^d)$ for which $\|u\|^{\alpha,p,q,\beta,\gamma}_{lm} < +\infty$, see [28, Theorem 2.10]. Furthermore, they also allow to define the spaces of locally Besov distributions in the following manner. □

Definition 1.3.8 (Local Besov spaces). *Let $\alpha \in \mathbb{R}$ and let $p, q \in [1, +\infty]$. A distribution $u \in \mathscr{D}'(\mathbb{R}^d)$ as per Definition 1.2.1 is an element of $B^{\alpha,loc}_{p,q}(\mathbb{R}^d)$ if and only if for every compact subset $K \subset \mathbb{R}^d$,*

$$\|u\|_{B^{\alpha}_{p,q}(K)} \doteq \|\langle u\,,\, \beta^1_x\rangle\|_{L^p_x(K)} + \left\| \frac{\left\|\langle u\,,\, \gamma^\lambda_x\rangle\right\|_{L^p_x(K)}}{\lambda^{\alpha+1/q}} \right\|_{\Lambda^q(0,1)} < +\infty, \tag{1.19}$$

where $\Lambda^q(0, 1)$ is as per Definition 1.3.7.

Remark 1.3.11. The space $B^{\alpha,loc}_{p,q}(\mathbb{R}^d)$ in Definition 1.3.8 can be equivalently characterized as the space of distributions $u \in \mathscr{D}'(\mathbb{R}^d)$ such that $\varphi u \in B^{\alpha}_{p,q}(\mathbb{R}^d)$ for all $\varphi \in \mathscr{D}(\mathbb{R}^d)$, see [29]. Furthermore, as this definition relies on localizing the distribution u by multiplication with a smooth, compactly supported function φ, it can be equivalently defined for elements of $\mathscr{D}'(\Omega)$, $\Omega \subset \mathbb{R}^d$ an open set. Hence, we shall call $B^{\alpha,loc}_{p,q}(\Omega)$ the set of $u \in \mathscr{D}'(\Omega)$ such that φu, extended as 0 to $\mathbb{R}^d \setminus \Omega$, is an element of $B^{\alpha}_{p,q}(\mathbb{R}^d)$ for all $\varphi \in \mathscr{D}(\Omega)$. The space $B^{\alpha,loc}_{p,q}(\Omega)$ is a Fréchet space as per Definition 1.1.16 when endowed with the topology generated by the family of seminorms $\{\|\cdot\|_{B^{\alpha}_{p,q}(K_n)} \mid n \in \mathbb{N}\}$ as per Equation (1.19), for $(K_n)_{n\in\mathbb{N}}$ an exhaustion of Ω comprised of compact sets, see [29, Remark 2.9]. For $p = q = +\infty$, by Corollary 1.3.7 we have

$$\|u\|_{B^{\alpha}_{\infty,\infty}(K)} \doteq \sup_{x\in K} |\langle \hat{u}(\xi), e^{ix\cdot\xi}\check{\beta}(\xi)\rangle| + \sup_{\lambda\in(0,1)} \sup_{x\in K} \lambda^{-\alpha} |\langle \hat{u}(\xi), e^{ix\cdot\xi}\check{\gamma}(\lambda\xi)\rangle|. \tag{1.20}$$

We note that Equation (1.20) implies that, for any compact subsets $K_1, K_2 \subset \Omega$ such that $K_1 \subset K_2$, one has that

$$\|u\|_{B^{\alpha}_{\infty,\infty}(K_1)} \leq \|u\|_{B^{\alpha}_{\infty,\infty}(K_2)},$$

for every $u \in B^{\alpha,loc}_{\infty,\infty}(\Omega)$. Furthermore, as the inclusion $B^{\alpha}_{\infty,\infty}(\mathbb{R}^d) \hookrightarrow B^{\alpha,loc}_{\infty,\infty}(\Omega)$ is continuous, Proposition 1.3.1 implies that $\mathscr{S}(\mathbb{R}^d) \hookrightarrow B^{\alpha,loc}_{\infty,\infty}(\Omega)$ continuously. □

Chapter 2

Wavefront Set Theory

Through many miles of tricks and trials
We'll wander high and low.
Tame your fears, a door appears:
The time has come to go.

The Ballad of the Witches' Road

In this chapter, we shall recall the theory of wavefront sets. To this avail, we shall review the pseudodifferential calculus, in both the global and local setting. Then, we shall recall the extension of the notion of wavefront set to the Besov regularity and the construction of Hörmander's topology for the smooth case. These notions are the fundamental tools on which the results presented in Chapter 3 are built.

Aim of Section 2.1 is to review the theory of pseudodifferential operators, both in the global case of Euclidean space and its local analogue on open sets. A more thorough discussion of these topics can be found in [30] and [31] In Section 2.2, we shall review the theory of smooth wavefront sets and their connections with the theory of pseudodifferential operators, following the presentation of [30] and [31]. Furthermore, we shall also recount the construction of wavefront sets of Besov regularity and of Hörmander's topology in the smooth setting. We refer to [4], [12], [2] and [3] for a more complete analysis of these topics.

2.1 Pseudodifferential Operators

A key element of the constructions of Sections 2.2 and Chapter 3 lies in the notion of pseudodifferential operators, or ΨDOs. These are linear operators between spaces of functions and distributions that generalize the classical notion of differential operators by replicating their behaviour under the Fourier transform. In Subsection 2.1.1, we shall review this theory for pseudodifferential operators on Euclidean space, while in Subsection 2.1.2 we shall extend these results to the local theory on open sets.

2.1.1 ΨDO calculus on $\mathbb{R}^d$

In order to outline the notion of pseudodifferential operators, we begin by reviewing the symbol calculus on $\mathbb{R}^d$.

Definition 2.1.1 (Symbols). *Let $m \in \mathbb{R}$ and let $d, n \in \mathbb{N}$ be fixed. The space of **symbols of order m**, denoted by $S^m(\mathbb{R}^d, \mathbb{R}^n)$, is the set of functions $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^n)$ such that for all $\alpha \in \mathbb{N}^d$ and $\beta \in \mathbb{N}^n$ there exists $C_{\alpha,\beta} > 0$ for which*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C\langle\xi\rangle^{m-|\beta|},$$

for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^n$. We endow this space with the Fréchet topology as per Definition 1.1.16 induced by the seminorms

$$S^m(\mathbb{R}^d, \mathbb{R}^d) \ni a \mapsto \|a\|_{k,m} \doteq \sup_{(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^n} \max_{|\alpha|+|\beta|\leq k} \langle\xi\rangle^{|\beta|-m} |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)|,$$

*for every $k \in \mathbb{N}$. We fix $S^{-\infty}(\mathbb{R}^d, \mathbb{R}^n) = \bigcap_{m\in\mathbb{R}} S^m(\mathbb{R}^d, \mathbb{R}^n)$ the space of **residual symbols**, endowed with the Fréchet topology induced by the seminorms $\|\cdot\|_{k,m}$ as $k \in \mathbb{N}$ and $m \in \mathbb{Z}$. We denote by $S^m(\mathbb{R}^n)$ the space of symbols of order m that are independent of x, i.e., $\psi \in S^m(\mathbb{R}^n)$ if and only if $\psi \in S^m(\mathbb{R}^d, \mathbb{R}^n)$ and $\partial_x^\alpha \psi(x,\xi) = 0$ for all $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}^d$ with $|\alpha| > 0$.*

Remark 2.1.1. We note that the space $S^{-\infty]}(\mathbb{R}^d,\mathbb{R}^n)$ as per Definition 2.1.1 can be equivalently constructed as the projective limit of the spaces $S^m(\mathbb{R}^d, \mathbb{R}^n)$ as m varies in $\mathbb{R}$ as per Definition 1.1.18. However, as $\mathbb{Z}$ is cofinal in $\mathbb{R}$, this topology is equivalent to the Fréchet one we endowed $S^{-\infty}(\mathbb{R}^d, \mathbb{R}^n)$ with as per Remark 1.1.17. □

Proposition 2.1.1. *Let $m \in \mathbb{R}$ and let $S^m(\mathbb{R}^d, \mathbb{R}^n)$ be as per Definition 2.1.1.*

(S1) For any $\alpha \in \mathbb{N}^d$ and $\beta \in \mathbb{N}^n$, the maps

$$\partial_x^\alpha : S^m(\mathbb{R}^d, \mathbb{R}^n) \to S^m(\mathbb{R}^d, \mathbb{R}^n) \text{ and } \partial_\xi^\beta : S^m(\mathbb{R}^d, \mathbb{R}^n) \to S^{m-|\beta|}(\mathbb{R}^d, \mathbb{R}^n),$$

are continuous.

(S2) $\mathscr{S}(\mathbb{R}^d \times \mathbb{R}^n) \hookrightarrow S^{-\infty}(\mathbb{R}^d, \mathbb{R}^n)$*, see Remark 1.3.1.*

(S3) For any $m' \in \mathbb{R}$*, the map*

$$S^m(\mathbb{R}^d, \mathbb{R}^n) \times S^{m'}(\mathbb{R}^d, \mathbb{R}^n) \ni (a, b) \mapsto ab \in S^{m+m'}(\mathbb{R}^d, \mathbb{R}^n)$$

is continuous.

(S4) For any $m' > m$ *and any* $a \in S^m(\mathbb{R}^d, \mathbb{R}^n)$*, there exists a sequence* $(a_k)_{k\in\mathbb{N}} \subset S^{-\infty}(\mathbb{R}^d, \mathbb{R}^n)$ *such that* $a_k \to a$ *in the topology of* $S^{m'}(\mathbb{R}^d, \mathbb{R}^n)$*.*

Proofs of these results can be found in [30, Chapter 3.1, pp.29-30].

Remark 2.1.2. Let $A : \mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$, see Remark 1.3.1, be a differential operator of order $m \in \mathbb{N}$ with constant coefficients, *i.e.*, there exists $C_\alpha \in \mathbb{R}$ for every $\alpha \in \mathbb{N}^d$, $|\alpha| \le m$, such that

$$A = \sum_{|\alpha|\le m} C_\alpha \partial^\alpha,$$

where the derivatives are as per Definition 1.2.3. Then, using Theorem 1.2.9, we may write for any $u \in \mathscr{S}'(\mathbb{R}^d)$,

$$A(u) = \mathscr{F}^{-1}\{\mathscr{F}[A(u)]\} = \sum_{|\alpha|\le m} C_\alpha \mathscr{F}^{-1}[\mathscr{F}(\partial^\alpha u)] = \sum_{|\alpha|\le m} C_\alpha \mathscr{F}^{-1}\left[i^{|\alpha|} P_\alpha \mathscr{F}(u)\right].$$

Hence, for the polynomial $\sigma_A \doteq \sum_{|\alpha|\le m} i^{|\alpha|} C_\alpha P_\alpha$, one has that

$$A(u) = \mathscr{F}^{-1}[\sigma_A \mathscr{F}(u)].$$

If $u \in \mathscr{D}(\mathbb{R}^d)$, $A(u) \in \mathscr{S}(\mathbb{R}^d)$ and the Fourier transform and its inverse can be written explicitly as integrals as per Definition 1.2.9. Hence,

$$A(u)(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sigma_A(\xi) u(y) e^{i\xi\cdot(x-y)} dy\, d\xi.$$

This integral representation is particularly useful in the theory of linear PDEs. In fact, if one is interested in solving the differential problem $A(u) = f$, for a given source term $f : \mathbb{R}^d \to \mathbb{C}$, under the appropriate conditions on f and σ_D one has that $\mathscr{F}[A(u)] = \sigma_A \mathscr{F}(u) = \mathscr{F}(f)$. Hence, u is given by

$$u(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{f(y)}{\sigma_A(\xi)} e^{i\xi\cdot(x-y)} dy\, d\xi, \quad \text{for all } x \in \mathbb{R}^d. \tag{2.1}$$

We note that, for Equation (2.1) to define a function, the hypothesis on f and σ_A are relatively strong. For instance, $f \in L^1(\mathbb{R}^d)$ and $\frac{1}{\sigma_A} \in L^1(\mathbb{R}^d)$ are sufficient conditions for the integrals to be finite, but they are particularly restrictive. The theory of pseudodifferential operators, then, is concerned with the possibility of defining integral representations analogous to Equation (2.1) for a broader class of differential operators and source terms. $\square$

In order to introduce the notion of pseudodifferential operator, we now recall the quantization of symbols.

Definition 2.1.2 (Quantization). *Let $m, w \in \mathbb{R}$ and let $a \in C^\infty(\mathbb{R}^{3d})$ be such that the map*

$$\mathbb{R}^{3d} \ni (x, y, \xi) \mapsto \langle x - y\rangle^{-w} a(x, y, \xi) \in \mathbb{C},$$

is an element of $S^m(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$ as per Definition 2.1.1. We denote this class of symbols by $\langle x - y\rangle^w S^m(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$. The ***quantization of a*** *is the operator* $\mathrm{Op}(a) : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ *such that, for all $u \in \mathscr{S}(\mathbb{R}^d)$,*

$$(\mathrm{Op}(a)u)(x) \doteq (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(x, y, \xi) u(y) e^{i\xi\cdot(x-y)} dy\, d\xi. \tag{2.2}$$

If $a \in S^m(\mathbb{R}^d)$, hence independent of both x and y variables, then $\mathrm{Op}(a)$ *is called the* ***Fourier multiplier of a****.*

Remark 2.1.3. It is not immediately clear whether the integral in Equation (2.2) must be absolutely convergent for every choice of $m, w \in \mathbb{R}$. However, it defines the jointly continuous map, see Definition 1.1.14,

$$\mathrm{Op} : \langle x - y\rangle^w S^{-\infty}(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d) \times \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d),$$

for every $w \in \mathbb{R}$, see [30, Proposition 4.4]. Hence, by Proposition 2.1.1, item ($S4$), it extends to a continuous map from $\langle x - y\rangle^w S^m(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d) \times \mathscr{S}(\mathbb{R}^d)$ into $\mathscr{S}(\mathbb{R}^d)$. More specifically, for any $\chi \in \mathscr{D}(\mathbb{R}^d)$ such that $\chi \equiv 1$ in a neighbourhood of 0, $a \in \langle x - y\rangle^w S^m(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d)$ and $u \in \mathscr{S}(\mathbb{R}^d)$, one has that

$$(\mathrm{Op}(a)u)(x) = \lim_{j\to+\infty} (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \chi(\xi/j) a(x, y, \xi) u(y) e^{i\xi\cdot(x-y)} dy\, d\xi,$$

where the limit converges in $\mathscr{S}(\mathbb{R}^d)$ and is independent of the choice of χ, see [30, Remark 4.6]. In particular, the quantization operator Op is also a continuous map $\mathrm{Op} : \langle x - y\rangle^w S^m(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d \times \mathbb{R}^d)$. Furthermore, for any $a \in \langle x-y\rangle^w S^m(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d)$, as $\mathrm{Op}(a) \in \mathfrak{L}(\mathscr{S}(\mathbb{R}^d))$, one may define its adjoint $\mathrm{Op}(a)^* \in \mathfrak{L}(\mathscr{S}'(\mathbb{R}^d))$, see Definition 1.1.23. In particular, as $\mathscr{S}(\mathbb{R}^d)$ is sequentially dense in $\mathscr{S}'(\mathbb{R}^d)$, $Op(a)^*$ is fixed by its actions on $a \in S^{-\infty}(\mathbb{R}^d \times$

$\mathbb{R}^d;\mathbb{R}^d)$ and $u,v\in\mathscr{S}(\mathbb{R}^d)$, namely

$$\begin{aligned}\langle \mathrm{Op}(a)u\,,\,v\rangle_S &= (2\pi)^{-d}\int_{\mathbb{R}^{3d}} a(x,y,\xi)u(y)v(x)e^{i\xi\cdot(x-y)}dy\,d\xi\,dx =\\ &= (2\pi)^{-d}\int_{\mathbb{R}^{3d}} a(y,x,-\xi)u(x)v(y)e^{i\xi\cdot(x-y)}dy\,d\xi\,dx = \langle u\,,\,\mathrm{Op}(a^\dagger)v\rangle_S,\end{aligned}$$

where we have swapped the integrals in x and y in the second equality and we have fixed $a^\dagger\in S^{-\infty}(\mathbb{R}^d\times\mathbb{R}^d;\mathbb{R}^d)$, $a^\dagger(x,y,\xi)\doteq a(y,x,-\xi)$ for all $x,y,\xi\in\mathbb{R}^d$. Since $a\mapsto a^\dagger$ is an isomorphism of $\langle x-y\rangle^w S^m(\mathbb{R}^d\times\mathbb{R}^d;\mathbb{R}^d)$ into itself, we have that $\mathrm{Op}(a)^*\doteq\mathrm{Op}(a^\dagger)$. In particular, we may extend $\mathrm{Op}(a)$ to an operator in $\mathfrak{L}(\mathscr{S}'(\mathbb{R}^d))$ by the relation

$$\langle \mathrm{Op}(a)\psi\,,\,\varphi\rangle_S \doteq \langle\psi\,,\,\mathrm{Op}(a^\dagger)\varphi\rangle_S,$$

for all $\psi,\varphi\in\mathscr{S}(\mathbb{R}^d)$. □

Remark 2.1.4. By Remark 2.1.3, for any $a\in\langle x-y\rangle^w S^m(\mathbb{R}^d\times\mathbb{R}^d,\mathbb{R}^d)$, $w,m\in\mathbb{R}$, $\mathrm{Op}(a)\in\mathfrak{L}(\mathscr{S}(\mathbb{R}^d),\mathscr{S}'(\mathbb{R}^d))$. Thus, by Remark 1.3.1, $\mathrm{Op}(a)$ is an element of $\mathfrak{L}(\mathscr{D}(\mathbb{R}^d),\mathscr{D}'(\mathbb{R}^d))$, where we identify $\mathrm{Op}(a)$ with its restriction to $\mathscr{D}(\mathbb{R}^d)$, which entails that $\mathrm{Op}(a)$ admits a Schwartz kernel $\mathscr{K}_a\in\mathscr{D}'(\mathbb{R}^d\times\mathbb{R}^d)$ by Theorem 1.2.3. Furthermore, one also has that $\mathscr{K}_a\in\mathscr{S}'(\mathbb{R}^d\times\mathbb{R}^d)$, see [30, Theorem 2.19]. □

Remark 2.1.5. The quantization procedure we have outlined in Definition 2.1.2 owes its name to quantum mechanics, as it mirrors the process of associating an operator to a given physical observable, see, *e.g.*, [32, Chapter 13]. It should be noted that the two processes are distinct in nature and that the quantization we discuss in this work should not be interpreted as anything more than a procedure to construct operators with useful properties from smooth functions. However, we also note that there is a connection between the ΨDO calculus, and thus the quantization procedure, and the symplectic structure that regulates both classical and quantum mechanics. We refer to [30, Chapter 5.14] for a more thorough exploration of this connection. □

Remark 2.1.6. The quantization procedure for a symbol, see Definition 2.1.2, aims to generalize the association $\sigma_A\mapsto A$ we presented in Remark 2.1.2. In fact, if $a(x,y,\xi)=\xi^\alpha$ for all $x,y,\xi\in\mathbb{R}^d$ and for some $\alpha\in\mathbb{N}^d$, then $\mathrm{Op}(a)=i^{-|\alpha|}\partial^\alpha$ as a consequence of Proposition 1.2.5, item *(F1)*. Hence, by the linearity of the map $\mathrm{Op}(\cdot)$, any differential operator with constant coefficients can be seen as the quantization of a polynomial. To generalize this procedure to a broader class of operators, one may consider two natural intermediate steps before allowing the symbol a to depends on all variables. One is to assume $a(x,y,\xi)=a_L(x,\xi)$. In this case, the operator

$$(\mathrm{Op}_L(a_L)(u))(x) \doteq (2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a_L(x,\xi)u(y)e^{i\xi\cdot(x-y)}dyd\xi,$$

is called the **left quantization of a_L**. At the same time, if $a(x,y,\xi)=a_R(y,\xi)$, the operator

$$(\mathrm{Op}_R(a_R)(u))(x) \doteq (2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a_R(y,\xi)u(y)e^{i\xi\cdot(x-y)}dyd\xi,$$

is called the **right quantization of a_R**. The distinction between these procedures becomes relevant when describing a differential operator with non-constant coefficients as the quantization of some symbol, as the Fourier transforms used in Remark 2.1.2 would then also act on the coefficients. However, for any symbol $a\in\langle x-y\rangle^w S^m(\mathbb{R}^d\times\mathbb{R}^d,\mathbb{R}^d)$, there exist $a_L,a_R\in S^m(\mathbb{R}^d,\mathbb{R}^d)$ such that

$$\mathrm{Op}(a)=\mathrm{Op}_L(a_L)=\mathrm{Op}_R(a_R),$$

see [30, Theorem 4.11].

□

Definition 2.1.3 (ΨDOs)**.** *Let $m\in\mathbb{R}$, let $S^m(\mathbb{R}^d\times\mathbb{R}^d,\mathbb{R}^d)$ be as per Definition 2.1.1 and let* Op *be as per Definition 2.1.2. The space of* ***pseudodifferential operators*** *or* ***ΨDOs of order m*** *is the set*

$$\Psi^m(\mathbb{R}^d)\doteq\{\mathrm{Op}(a)\mid a\in\langle x-y\rangle^w S^m(\mathbb{R}^d\times\mathbb{R}^d,\mathbb{R}^d),\ w\in\mathbb{R}\}.$$

We fix $\Psi^{-\infty}(\mathbb{R}^d)\doteq\bigcap_{m\in\mathbb{R}}\Psi^m(\mathbb{R}^d)$ the space of ***residual pseudodifferential operators****.*

Example 2.1.1. Let $(\psi_k)_{k\in\mathbb{N}}\subset\mathscr{D}(\mathbb{R}^d)$ be a Littlewood-Paley partition of unity as per Definition 1.3.1 and let $(\Delta_k)_{k\in\mathbb{N}}$ be the associated sequence of operators. By Remark 1.3.2, we have that $(\psi_k)_{k\in\mathbb{N}}\subset S^0(\mathbb{R}^d,\mathbb{R}^d)$, where we identify ψ_k with the map $\mathbb{R}^d\times\mathbb{R}^d\ni(x,\xi)\mapsto\psi_k(\xi)\in\mathbb{C}$. Then, for every $\varphi\in\mathscr{S}(\mathbb{R}^d)$,

$$\Delta_k(\varphi)(x)=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\psi_k(\xi)\varphi(y)e^{i\xi\cdot(x-y)}dyd\xi.$$

Hence, $\Delta_k=\mathrm{Op}(\psi_k)$ as per Definition 2.1.2 and $(\Delta_k)_{k\in\mathbb{N}}\subset\Psi^0(\mathbb{R}^d)$ as per Definition 2.1.3.

Remark 2.1.7. By [30, Corollary 4.15], one may equivalently define the spaces $\Psi^m(\mathbb{R}^d)$, $m \in \mathbb{R}$, of Definition 2.1.3 using the left or right quantization of Remark 2.1.6. In particular, one may fix $w = 0$ in Definition 2.1.3 without loss of generality. Additionally, one may also define the operators

$$\sigma_L : \Psi^m(\mathbb{R}^d) \to S^m(\mathbb{R}^d, \mathbb{R}^d), \quad \sigma_R : \Psi^m(\mathbb{R}^d) \to S^m(\mathbb{R}^d, \mathbb{R}^d),$$

given by the relations $\mathrm{Op}_L(\sigma_L(A)) = A$ and $\mathrm{Op}_R(\sigma_R(A)) = A$ for all $A \in \Psi^m(\mathbb{R}^d)$. These are well-defined inverses for the operators Op_L and Op_R by [30, Theorem 4.11] and are called, respectively, the **left reduction** of A and the **right reduction** of A. Then, the topology of $\Psi^m(\mathbb{R}^d)$ may be induced by imposing that either Op_L or Op_R is an isomorphism of $S^m(\mathbb{R}^d, \mathbb{R}^d)$ into $\Psi^m(\mathbb{R}^d)$. Both choices generate the same Fréchet topology on $\Psi^m(\mathbb{R}^d)$, see [30, Chapter 4.3]. □

In principle the space of residual operators as per Definition 2.1.3 need not coincide with the set of quantizations of residual symbols as per Definitions 2.1.2 and 2.1.1. However, the following result proves that this is the case.

Proposition 2.1.2. *Let $A \in \mathfrak{L}(\mathscr{S}(\mathbb{R}^d), \mathscr{S}'(\mathbb{R}^d))$ and let $\mathscr{K}_A \in \mathscr{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ be its Schwartz kernel as per Remark 2.1.4. A is a residual ΨDO as per Definition 2.1.3 if and only if $\mathscr{K}_A \in \mathscr{E}(\mathbb{R}^d \times \mathbb{R}^d)$ and*

$$|\partial_x^\alpha \partial_y^\beta \mathscr{K}_A(x, y)| \le C_{\alpha,\beta,N} \langle x - y \rangle^{-N},$$

for every $\alpha, \beta \in \mathbb{N}^d$ and $N \in \mathbb{N}$. Furthermore, there exist $a_L, a_R \in S^{-\infty}(\mathbb{R}^d, \mathbb{R}^d)$ such that $A = \mathrm{Op}_L(a_L) = \mathrm{Op}_R(a_R)$.

A proof of this result can be found in [30, Proposition 4.13].

Furthermore, residual operators are smoothing in the following sense.

Proposition 2.1.3. *For any $A \in \Psi^{-\infty}(\mathbb{R}^d)$ as per Definition 2.1.3, one has that $A \in \mathfrak{L}(\mathscr{S}'(\mathbb{R}^d), \mathscr{S}'(\mathbb{R}^d) \cap \mathscr{E}(\mathbb{R}^d))$, see Remark 1.3.1. More precisely, for any tempered distribution u, there exists $N \in \mathbb{N}$ such that $\langle \cdot \rangle^{-N} A(u)$ is a smooth, bounded function.*

A proof of this result can be found in [30, Lemma 4.17].

As with differential operators, one may also consider compositions of ΨDOs. Furthermore, their order behave analogously to differential operators, as per the following result.

Theorem 2.1.4. *Let $m, m' \in \mathbb{R}$, $A \in \Psi^m(\mathbb{R}^d)$ and $B \in \Psi^{m'}(\mathbb{R}^d)$ as per Definition 2.1.3. Then $A \circ B \in \Psi^{m+m'}(\mathbb{R}^d)$. Furthermore, the map*

$$\circ : \Psi^m(\mathbb{R}^d) \times \Psi^{m'}(\mathbb{R}^d) \to \Psi^{m+m'}(\mathbb{R}^d),$$

given by $\circ(A,B) \doteq A \circ B$, *is continuous, see Remark 2.1.7. In addition, one has that, for all* $j \in \mathbb{N}$,

$$\sigma_L(A \circ B) - \sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha| \le j-1}} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \sigma_L(A) \cdot \partial_x^\alpha \sigma_L(B) \in S^{m+m'-j}(\mathbb{R}^d, \mathbb{R}^d),$$

where σ_L *is the left reduction of* $A \circ B$ *as per Remark 2.1.3.*

A proof of this result can be found in [30, Theorem 4.22].

When analyzing a differential operator, the higher order derivatives often play a prominent role. Similarly, we seek an analogous notion for ΨDOs, which leads us to the following definition.

Definition 2.1.4 (Principal symbol). *Let* $A \in \Psi^m(\mathbb{R}^d)$ *and let* $\sigma_L(A)$ *be its left reduction as per Remark 2.1.7. The **principal symbol of A** is the equivalence class*

$$\sigma^m(A) \doteq [\sigma_L(A)] \in S^m(\mathbb{R}^d, \mathbb{R}^d)/S^{m-1}(\mathbb{R}^d, \mathbb{R}^d).$$

Remark 2.1.8. Let $A = \sum_{|\alpha| \le m} C_\alpha \partial^\alpha$ for some $m \in \mathbb{N}$ and $C_\alpha \in \mathbb{R}$ for all $\alpha \in \mathbb{N}^d$. Then, by Remarks 2.1.2 and 2.1.7, we have that

$$\sigma_L(A)(x, y, \xi) = \sigma_R(A)(x, y, \xi) = \sum_{|\alpha| \le m} i^{|\alpha|} C_\alpha \xi^\alpha = i^m \sum_{|\alpha| = m} C_\alpha \xi^\alpha + \sum_{|\alpha| < m} i^{|\alpha|} C_\alpha \xi^\alpha.$$

As the left reduction $\sigma_L(A)$ is the sum of the polynomial $P_m(\xi) \doteq i^m \sum_{|\alpha|=m} C_\alpha \xi^\alpha$ of degree m with the polynomial $P_{m-1}(\xi) \doteq \sum_{|\alpha|<m} i^{|\alpha|} C_\alpha \xi^\alpha$ of degree $m-1$, the principal symbol of A as per Definition 2.1.4 has P_m as a natural class representative. In particular, the quantization of P_m is $\sum_{|\alpha|=m} C_\alpha \partial^\alpha$, which is exactly the leading order component of A. For the sake of clarity, in the following we shall identify $\sigma^m(A)$ with one of its representatives $P_m \in S^m(\mathbb{R}^d, \mathbb{R}^d)$ whenever this identification is clear. For instance, if $a \in S^m(\mathbb{R}^d, \mathbb{R}^d)$ is such that $a = a_m + a_{m-1}$, for $a_m \in S^m(\mathbb{R}^d, \mathbb{R}^d)$ and $a_{m-1} \in S^{m-1}(\mathbb{R}^d, \mathbb{R}^d)$, then we may identify $\sigma^m(\mathrm{Op}_L(a)) = [a]$ with a_m. We note that this identification might not be straightforward in all cases and, hence, will be avoided whenever confusion might arise from it. □

The calculus of principal symbols is compatible with both left and right quantizations, as well as with the composition of ΨDOs, as per the following result.

Proposition 2.1.5. *Let* $\sigma^m : \Psi^m(\mathbb{R}^d) \to S^m(\mathbb{R}^d, \mathbb{R}^d)/S^{m-1}(\mathbb{R}^d, \mathbb{R}^d)$ *be the principal symbol map as per Definition 2.1.4 for any* $m \in \mathbb{R}$.

(PS1) $\sigma^m(\mathrm{Op}_L(a)) = \sigma^m(\mathrm{Op}_R(a)) = [a]$ *for any* $a \in S^m(\mathbb{R}^d, \mathbb{R}^d)$ *as per Definitions 2.1.1 and 2.1.4.*

(PS2) *For any* $A \in \Psi^m(\mathbb{R}^d)$ *and* $B \in \Psi^{m'}(\mathbb{R}^d)$*,* $\sigma^{m+m'}(A \circ B) = \sigma^m(A)\sigma^{m'}(B)$*, in the sense that for any* $a \in \sigma^m(A)$ *and* $b \in \sigma^{m'}(B)$*,* $ab \in \sigma^{m+m'}(A \circ B)$*.*

A proof of this result can be found in [30, Proposition 4.24].

Amongst all symbols, those of elliptic kind play a key role in the formulation of the topics discussed in Section 2.2.

Definition 2.1.5 (Elliptic symbols)**.** *Let* $m \in \mathbb{R}$ *and* $d, n \in \mathbb{N}$ *be fixed. A symbol* $a \in S^m(\mathbb{R}^d, \mathbb{R}^n)$ *as per Definition 2.1.1 is said to be* ***elliptic*** *if there exists* $b \in S^{-m}(\mathbb{R}^d, \mathbb{R}^n)$ *such that* $ab - 1 \in S^{-1}(\mathbb{R}^d, \mathbb{R}^n)$*.*

Proposition 2.1.6 (Equivalent characterization of ellipticity)**.** *Let* $m \in \mathbb{R}$ *and* $a \in S^m(\mathbb{R}^d, \mathbb{R}^n)$ *as per Definition 2.1.1. The following are equivalent:*

(E1) *a is elliptic as per Definition 2.1.5;*

(E2) *there exist* $R, c > 0$ *such that, if* $|\xi| \geq R$*, then* $|a(x,\xi)| \geq c|\xi|^m$ *for all* $x \in \mathbb{R}^d$*;*

(E3) *there exists* $C, c > 0$ *such that* $|a(x,\xi)| \geq c|\xi|^m - C|\xi|^{m-1}$ *for all* $\xi \in \mathbb{R}^n$*,* $|\xi| \geq 1$ *and* $x \in \mathbb{R}^d$*.*

A proof of this result can be found in [30, Proposition 3.12].

Elliptic symbols are used to generalize the notion of elliptic differential operators used in classical PDE theory, see, *e.g.*, [33], through the following notion.

Definition 2.1.6 (Elliptic ΨDO)**.** *A ΨDO* $A \in \Psi^m(\mathbb{R}^d)$ *is called* ***elliptic*** *if there exists* $a \in S^m(\mathbb{R}^d, \mathbb{R}^d)$ *such that a is elliptic as per Definition 2.1.5 and* $\sigma^m(A) = [a]$ *as per Definition 2.1.4.*

The power of elliptic ΨDOs comes from the fact that they are invertible up to a residual remainder, in the sense of the following result.

Theorem 2.1.7 (Elliptic parametrix)**.** *Let* $A \in \Psi^m(\mathbb{R}^d)$ *be an elliptic ΨDO as per Definition 2.1.6. Then, there exists* $B \in \Psi^{-m}(\mathbb{R}^d)$*, called the* ***parametrix of A****, such that*

$$A \circ B - I, \;\; B \circ A - I \in \Psi^{-\infty}(\mathbb{R}^d).$$

Furthermore, if $B' \in \Psi^{-m}(\mathbb{R}^d)$ *is another parametrix of A, then* $B - B' \in \Psi^{-\infty}(\mathbb{R}^d)$*.*

Elliptic ΨDOs, analogously to their differential counterparts, also enjoy regularization properties. Firstly, they maintain exactly the singularities of a distribution, in the sense of the following proposition.

Proposition 2.1.8 (Pseudolocality and elliptic regularity)**.** *Let $m \in \mathbb{R}$ be fixed and let $A \in \Psi^m(\mathbb{R}^d)$ as per Definition 2.1.3. Then,*

$$singsupp(Au) \subset singsupp(u),$$

see Definition 1.2.5, for all $u \in \mathscr{S}'(\mathbb{R}^d)$. Furthermore, if A is elliptic as per Definiton 2.1.6, then

$$singsupp(Au) = singsupp(u).$$

Proofs of these results can be found in [30, Proposition 4.16] and [30, Proposition 4.32] respectively.

In order to state the appropriate regularizing properties of ΨDOs and elliptic ΨDOs, we first review the notion of Sobolev-Bessel spaces.

Definition 2.1.7 (Sobolev-Bessel spaces)**.** *Let $s \in \mathbb{R}$. The* ***Sobolev-Bessel space of order s*** *is the collection of $f \in \mathscr{S}'(\mathbb{R}^d)$ such that $\langle\cdot\rangle^s \mathscr{F}(f) \in L^2(\mathbb{R}^d)$ as per Remark 1.3.1, where $\langle x\rangle^s \doteq \left(\sqrt{1+\|x\|^2}\right)^s$ for all $x \in \mathbb{R}^d$. We denote this set by $H^s(\mathbb{R}^d)$ and we endow it with the Banach topology induced by the norm*

$$H^s(\mathbb{R}^d) \ni f \mapsto \|\langle\cdot\rangle^s \mathscr{F}(f)\|_{L^2(\mathbb{R}^d)} \in [0, +\infty).$$

For any $r, s \in \mathbb{R}$, we call ***weighted Sobolev-Bessel space of order s, r*** *the collection of $f \in \mathscr{S}'(\mathbb{R}^d)$ such that $\langle\cdot\rangle^{-r} f \in H^s(\mathbb{R}^d)$. We denote this space by $\langle x\rangle^r H^s(\mathbb{R}^d)$ and we endow it with the Banach topology induced by the norm*

$$\langle x\rangle^r H^s(\mathbb{R}^d) \ni f \mapsto \|\langle\cdot\rangle^s \mathscr{F}(\langle\cdot\rangle^{-r} f)\|_{L^2(\mathbb{R}^d)} \in [0, +\infty)$$

Remark 2.1.9. Sobolev-Bessel spaces $H^s(\mathbb{R}^d)$ as per Definition 2.1.7 coincide with Sobolev-Slobodeckij spaces $W^{s,2}(\mathbb{R}^d)$, see Definition 1.3.5, for every positive s, see [26, Sections 1.2.3, 1.2.4]. Therefore, while we defined $H^s(\mathbb{R}^d)$ as a subspace of $\mathscr{S}'(\mathbb{R}^d)$, it is in fact a space of functions whenever $s \geq 0$. For negative s, however, Sobolev-Bessel spaces are spaces of distributions. For instance, let $\delta \in \mathscr{S}'(\mathbb{R}^d)$ be such that $\langle \delta, \varphi\rangle = \varphi(0)$ for every $\varphi \in \mathscr{S}(\mathbb{R}^d)$. Then, $\mathscr{F}(\delta) = 1$, in the sense that, for every $\varphi \in \mathscr{S}(\mathbb{R}^d)$,

$$\langle \mathscr{F}(\delta), \varphi\rangle = \int_{\mathbb{R}^d} \varphi(x)dx.$$

Hence, for all $s < -d/2$, one has that $\langle\cdot\rangle^s \mathscr{F}(\delta) = \langle\cdot\rangle^s \in L^2(\mathbb{R}^d)$, which implies that $\delta \in H^s(\mathbb{R}^d)$. □

Remark 2.1.10. Sobolev-Bessel spaces are closely related to the theory of ΨDOs through one of their equivalent characterizations. By the Fourier-Plancherel theorem, see [16, Theorem 7.9], the Fourier transform extends to an isomorphism $\mathscr{F} \in \mathfrak{L}(L^2(\mathbb{R}^d))$ as per Definition 1.1.4. Hence, $\langle\cdot\rangle^s \mathscr{F}(u) \in L^2(\mathbb{R}^d)$ if and only if $\mathscr{F}^{-1}[\langle\cdot\rangle^s \mathscr{F}(u)] \in L^2(\mathbb{R}^d)$. By Definition 2.1.2, let $\langle D\rangle^s \doteq \mathrm{Op}(\langle\cdot\rangle^s)$ be the Fourier multiplier of $\langle\cdot\rangle^s$. Then, for all $u \in \mathscr{S}'(\mathbb{R}^d)$, $u \in H^s(\mathbb{R}^d)$ if and only if $\langle D\rangle^s u \in L^2(\mathbb{R}^d)$. Analogously, for any $u \in \mathscr{S}'(\mathbb{R}^d)$, one has that $u \in \langle x\rangle^r H^s(\mathbb{R}^d)$ if and only if $\langle D\rangle^s(\langle\cdot\rangle^{-r}u) \in L^2(\mathbb{R}^d)$. □

We now state a series of results regarding the properties of ΨDOs between weighted Sobolev-Bessel spaces.

Theorem 2.1.9 (Continuity between Sobolev spaces). *Let $s, r, m \in \mathbb{R}$ be fixed and let $A \in \Psi^m(\mathbb{R}^d)$ as per Definition 2.1.3. Then, one has that A is a continuous operator $\langle x\rangle^r H^s(\mathbb{R}^d) \to \langle x\rangle^r H^{s-m}(\mathbb{R}^d)$, see Definition 2.1.7, and the immersion $\Psi^m \hookrightarrow \mathfrak{L}(\langle x\rangle^r H^s(\mathbb{R}^d), \langle x\rangle^r H^{s-m}(\mathbb{R}^d))$ is continuous, see Remark 2.1.7.*

A proof of this result can be found in [30, Theorem 4.40].

Corollary 2.1.10 (Elliptic regularity). *Let $s, r, m \in \mathbb{R}$ be fixed. Let $A \in \Psi^m(\mathbb{R}^d)$ be elliptic as per Definition 2.1.6 and let $u \in \langle x\rangle^r H^s(\mathbb{R}^d)$ as per Definition 2.1.7. If $A(u) \in \langle x\rangle^r H^t(\mathbb{R}^d)$ for some $t \in \mathbb{R}$, then $u \in \langle x\rangle^r H^{t+m}(\mathbb{R}^d)$.*

A proof of this result can be found in [30, Corollary 4.41].

As ΨDOs are naturally defined on tempered distributions, the previous results can be adapted to the case of Besov spaces.

Theorem 2.1.11 (Continuity between Besov spaces). *Let $m \in \mathbb{R}$ be fixed and let $A \in \Psi^m(\mathbb{R}^d)$ as per Definition 2.1.3. Then, for all $\alpha \in \mathbb{R}$ and $p, q \in [1, +\infty]$, $A \in \mathfrak{L}(B^\alpha_{p,q}(\mathbb{R}^d)), B^{\alpha-m}_{p,q}(\mathbb{R}^d)$, see Definition 1.3.3.*

A proof of this result can be found in [34, Theorem 6.19].

Corollary 2.1.12 (Elliptic regularity on Besov spaces). *Let $\alpha, m \in \mathbb{R}$ and $p, q \in [1, +\infty]$ be fixed. Let $A \in \Psi^m(\mathbb{R}^d)$ be elliptic as per Definition 2.1.6 and let $u \in B^\alpha_{p,\infty}(\mathbb{R}^d)$. If $A(u) \in B^\beta_{p,q}(\mathbb{R}^d)$ for some $\beta \in \mathbb{R}$, then $u \in B^{\beta+m}_{p,q}(\mathbb{R}^d)$.*

A proof of this result can be found in [34, Corollary 6.23].

We introduce now the notion of properly supported ΨDOs, which shall be used extensively in the characterization of pseudodifferential operators on open sets.

Definition 2.1.8. *Let $A \in \Psi^m(\mathbb{R}^d)$ as per Definition 2.1.3 and let $\mathscr{K}_A \in \mathscr{D}'(\mathbb{R}^d \times \mathbb{R}^d)$ be its Schwartz kernel as per Remark 2.1.4. Let $\mathrm{pr}_1, \mathrm{pr}_2 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be the projections onto the first and second factor respectively. Then, A is said to be properly supported if $\mathrm{pr}_1|_{supp(\mathscr{K}_A)}$ and $\mathrm{pr}_2|_{supp(\mathscr{K}_A)}$ are proper maps as per Definition 1.2.12.*

A more intuitive and equivalent definition of properly supported ΨDOs is given by the following proposition.

Proposition 2.1.13. *Let $A \in \Psi^m(\mathbb{R}^d)$ as per Definition 2.1.3 and let A^* be its adjoint as per Remark 2.1.3. Then, A is properly supported as per Definition 2.1.8, if and only if for any compact set $K \subset \mathbb{R}^d$, there exist compact sets $K', K'' \subset \mathbb{R}^d$ such that $supp(u) \subset K$ implies $supp(Au) \subset K'$ and $supp(A^*u) \subset K''$ for all $u \in \mathscr{D}(\mathbb{R}^d)$.*

A proof of this result can be found in [35, Proposition 3.4].

Remark 2.1.11. We note that the characterization given by Proposition 2.1.13 depends only on the action of a ΨDO on smooth compactly supported functions. Thus, when we generalize this class of operators to ΨDOs on open sets, which cannot, in general, be defined through Schwartz functions and tempered distributions in the way that their counterparts on Euclidean space can, this notion shall remain unchanged. Therefore, we shall use Proposition 2.1.13 as the definition of properly supported ΨDOs henceforth. □

Example 2.1.2. Let $\varphi, \Phi \in \mathscr{D}(\mathbb{R}^d)$ and let $\psi \in \mathscr{S}(\mathbb{R}^d)$ as per Remark 1.3.1. Let $a : (x, y, \xi) \mapsto \Phi(x)\psi(\xi)\varphi(y)$ for all $x, y, \xi \in \mathbb{R}^d$. Then, by Remark 1.2.14 and Definition 2.1.1, one has that $a \in S^{-\infty}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$, so that $A = \mathrm{Op}(a)$ lies in $\Psi^{-\infty}(\mathbb{R}^d)$ as per Definition 2.1.3. It also holds true that, for all $u \in \mathscr{D}(\mathbb{R}^d)$, $supp(A(u)) \subset supp(\Phi) \subset\subset \mathbb{R}^d$. Furthermore, by Remark 2.1.3, $A^* = \mathrm{Op}(a^\dagger)$, where $a^\dagger : (x, y, \xi) \mapsto \varphi(x)\psi(-\xi)\Phi(y)$. Hence, $supp(A^*u) \subset supp(\varphi) \subset\subset \mathbb{R}^d$ for all $u \in \mathscr{D}(\mathbb{R}^d)$. Thus, A is properly supported by Proposition 2.1.13. □

While the construction in Example 2.1.2 might seem sufficiently special to suggest that properly supported ΨDOs are not common, the following result proves the converse.

Proposition 2.1.14. *Let $A \in \Psi^m(\mathbb{R}^d)$ as per Definition 2.1.3. Then, there exists $A_0 \in \Psi^m(\mathbb{R}^d)$ properly supported as per Definition 2.1.8 and $A_1 \in \Psi^{-\infty}(\mathbb{R}^d)$ such that $A = A_0 + A_1$.*

A proof of this result can be found in [35, Proposition 3.3].

Remark 2.1.12. The theory of ΨDOs is particularly powerful because it allows one to have a strong control over the singularities of a distributional solution to a PDE, in particular through the theory of propagation of singularities, see [30, Chapter 7]. Thus, in many applications, smooth remainders are negligible. By Proposition 2.1.14, every ΨDO A differs from a properly supported ΨDO A_0 by a residual remainder and, by Proposition 2.1.3, this remainder is smoothing, in the sense that it sends tempered distributions as per Definition 1.2.7 into smooth functions. Hence, whenever one is interested only in the singular structure of a distribution and how it is changed by a ΨDO, one may assume that the operator is properly supported, as the difference is a smooth, thus non-singular, remainder. Furthermore, we note that this decomposition is not unique. In fact, let $A, A_0 \in \Psi^m(\mathbb{R}^d)$and $A_1 \in \Psi^{-\infty}(\mathbb{R}^d)$ be as per Proposition 2.1.14. Then, for any $B \in \Psi^{-\infty}$, let $B_0 \doteq A_0 + B$ and $B_1 \doteq A_1 - B$. By Definition 2.1.3, one has that $B_0 \in \Psi^m(\mathbb{R}^d)$, $B_1 \in \Psi^{-\infty}(\mathbb{R}^d)$ and $A = B_0 + B_1$. □

2.1.2 Local ΨDO calculus on open sets

In Section 2.2, we shall review the theory of wavefront sets and its connection to the theory of ΨDOs. The notion of wavefront set, as presented in Definition 2.2.4, is concerned only with the local behaviour of a distribution. Hence it can be developed for distributions defined solely on an open subset of Euclidean space. This, however, requires the use of pseudodifferential operators defined on an open set. Thus, we shall now recount the relevant parts of this theory, adapting the discussion of [30, Chapter 5] from the smooth manifold setting.

Definition 2.1.9. *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $m \in \mathbb{R}$ be fixed. For any $A \in \Psi^m(\mathbb{R}^d)$, let $\mathscr{K}_A \in \mathscr{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ be its Schwartz kernel as per Remark 2.1.4. We call $\Psi^m_c(\Omega)$ the set of $A \in \Psi^m(\mathbb{R}^d)$ such that $\mathrm{supp}(\mathscr{K}_A)$ is a compact subset of $\Omega \times \Omega$.*

Definition 2.1.10. *Let $m \in \mathbb{R}$ be fixed and let $\Omega \subset \mathbb{R}^d$ be an open set. Then, $\Psi^m(\Omega)$ is the space of linear maps $A : \mathscr{D}(\Omega) \to \mathscr{E}(\Omega)$, see Remark 1.3.1, such that:*

- *for every $\varphi \in \mathscr{D}(\Omega)$, there exists $A_\varphi \in \Psi^m_c(\Omega)$ as per Definition 2.1.9 such that, for all $u \in \mathscr{D}(\Omega)$,*

$$\varphi A(\varphi u) = A_\varphi(\tilde{u})|_\Omega \in \mathscr{E}(\Omega),$$

  where $\tilde{u} \in \mathscr{D}(\mathbb{R}^d)$ is the extension by 0 of u on $\mathbb{R}^d \setminus \Omega$;

- *for every $\varphi, \psi \in \mathscr{D}(\Omega)$ such that $\mathrm{supp}(\varphi) \cap \mathrm{supp}(\psi) = \emptyset$, there exists $K \in \mathscr{E}(\Omega \times \Omega)$ such that*

$$\varphi(x) A(\psi u)(x) = \int_\Omega K(x, y) u(y) dy,$$

  for every $u \in \mathscr{D}(\Omega)$.

*We call the **space of residual ΨDOs** on Ω the set $\Psi^{-\infty}(\Omega) \doteq \bigcap_{m \in \mathbb{R}} \Psi^m(\Omega)$.*

Remark 2.1.13. The space of residual ΨDOs $\Psi^{-\infty}(\Omega)$ as per Definition 2.1.10 coincides with that of linear operators with a smooth Schwartz kernel as per Remark 2.1.4, see [30, Remark 5.23]. Thus, by Proposition 1.2.4, it coincides also with $\mathfrak{L}(\mathscr{E}'(\Omega), \mathscr{E}(\Omega))$. This reproduces the smoothing result of Proposition 2.1.3. Furthermore, if $A \in \Psi^{-\infty}(\Omega)$ is properly supported as per Remark 2.1.11, then it is a continuous operator $\mathscr{D}'(\Omega) \to \mathscr{E}(\Omega)$ and $\mathscr{E}'(\Omega) \to \mathscr{D}(\Omega)$. □

Remark 2.1.14. The class $\Psi^m(\Omega)$ as per Definition 2.1.10 is strictly larger than $\Psi^m(\mathbb{R}^d)$ for $\Omega = \mathbb{R}^d$. The difference between the two spaces is that in the construction of the latter, see Remark 2.1.3, one also requires the Schwartz kernel of the operator to decay sufficiently fast away from the diagonal $\{(x,x) \mid x \in \mathbb{R}^d\}$, see [30, Example 5.25]. To distinguish these spaces, we shall henceforth denote the latter by $\Psi^m_\infty(\mathbb{R}^d)$. Some of the results that we have previously stated for $\Psi^m_\infty(\mathbb{R}^d)$, *e.g.*, Proposition 2.1.4 and Remark 2.1.6, require additional care when extended to $\Psi^m(\Omega)$. Thus, we shall now review the constructions and statements that cannot be carried over directly to the case of ΨDOs on open sets. □

Definition 2.1.11 (Local symbol)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $m \in \mathbb{R}$ be fixed. A function $a \in C^\infty(\Omega \times \mathbb{R}^d)$ is a* ***local symbol of order m*** *if, for every $\varphi \in \mathscr{D}(\Omega)$, $\varphi a \in S^m(\mathbb{R}^d, \mathbb{R}^d)$ as per Definition 2.1.1, where we identify $\varphi a \in \mathscr{E}(\Omega \times \mathbb{R}^d)$ with its extension by 0 to $\mathbb{R}^d \times \mathbb{R}^d$. We denote the set of all local symbols on Ω of order m by $S^m(\Omega, \mathbb{R}^d)$. We call the space of* ***residual local symbols*** *the set $S^{-\infty}(\Omega, \mathbb{R}^d) \doteq \bigcap_{m\in\mathbb{R}} S^m(\Omega, \mathbb{R}^d)$.*

Remark 2.1.15. We note that the class of symbols on $\mathbb{R}^d$ as per Definition 2.1.1 is a strict subset of $S^m(\Omega, \mathbb{R}^d)$ even for $\Omega = \mathbb{R}^d$. For instance, the smooth function $a : (x, \xi) \mapsto e^{\|x\|^2} e^{-\|\xi\|^2}$ is a residual local symbol as per Definition 2.1.11, but it is not an element of $S^m(\mathbb{R}^d, \mathbb{R}^d)$ as per Definition 2.1.1 for any $m \in \mathbb{R}$ due to the unboundedness in the x variable. Thus, we shall denote the class of symbols as per Definition 2.1.1 by $S^m_\infty(\mathbb{R}^d, \mathbb{R}^d)$ henceforth. □

Remark 2.1.16. The space $S^m(\Omega, \mathbb{R}^d)$, see Definition 2.1.11, is constructed as a local counterpart of its analogue on $\mathbb{R}^d$. Thus, let $(K_n)_{n\in\mathbb{N}}$ be an exhaustion of Ω by compact sets and let $(\chi_n)_{n\in\mathbb{N}} \subset \mathscr{D}(\mathbb{R}^d)$ be a partition of unity on Ω subordinate to $(K_n)_{n\in\mathbb{N}}$, see [30, Chapter 5.2, p. 77]. The topology of $S^m(\Omega, \mathbb{R}^d)$ is the Fréchet topology induced by the family of seminorms

$$\|a\|_{k,m,n} \doteq \|\chi_n a\|_{k,m},$$

for all $k, n \in \mathbb{N}$, where $\|\cdot\|_{k,m}$ are as per Definition 2.1.1. The topology on the space of residual local symbols is the Fréchet topology induced by the collection of seminorms $\{\|\cdot\|_{k,m,n} \mid k, n \in \mathbb{N},\ m \in \mathbb{Z}\}$. Analogously, one can characterize the Fréchet topology of $\Psi^m(\Omega)$ by suitable localizations using a partition of unity and the seminorms generating the topology of $\Psi^m_\infty(\mathbb{R}^d)$, see Remark 2.1.7.

Remark 2.1.17. On an arbitrary open set, the quantization process is not globally defined. Instead, one has that every ΨDO is locally the quantization of a local symbol. More specifically, let $(\chi_k)_{k\in\mathbb{N}}$ be a smooth partition of unity on Ω and let $\tilde{\chi}_k \in \mathscr{D}(\mathbb{R}^d)$ be such that $\tilde{\chi}_k \equiv 1$ in a neighbourhood of $supp(\chi_k)$ for every $k \in \mathbb{N}$. Then, for any local symbol $a \in S^m(\Omega, \mathbb{R}^d)$, we may define its quantization by

$$\mathrm{Op}(a) : u \mapsto \sum_{k\in\mathbb{N}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \chi_k(x) a(x,\xi) \tilde{\chi}_k(y) u(y) e^{i\xi\cdot(x-y)} dy d\xi,$$

for all $u \in \mathscr{D}(\Omega)$. We note that this construction amounts to reconstructing the quantization of a from that of the localized symbols $(x, y, \xi) \mapsto \chi_k(x)a(x,\xi)\tilde{\chi}_k(y)$ in the sense of Definition 2.1.2. Thus, one expects the properties of quantization on an open set to be analogous to their counterparts on Euclidean space. The following proposition clarifies the analogy between the two. □

Proposition 2.1.15. *Let $\Psi^m(\Omega)$ be as per Definition 2.1.10 and let $S^m(\Omega, \mathbb{R}^d)$ be as per Definition 2.1.11. The map $Op : S^m(\Omega, \mathbb{R}^d) \to \Psi^m(\Omega)$ as per Remark 2.1.17 is a well-defined, linear and continuous map, see Remark 2.1.16, with the property that $Op(\mathbf{1}) = Id$, where $\mathbf{1}$ is the constant function equal to 1 and Id is the identity operator on $\mathscr{D}(\Omega)$. Furthermore, $Op(a)$ is properly supported for every $a \in S^m(\Omega, \mathbb{R}^d)$ and for all $B \in \Psi^m(\Omega)$ there exist $b \in S^m(\Omega, \mathbb{R}^d)$ and $B_0 \in \Psi^{-\infty}(\Omega)$ such that $B = Op(b) + B_0$.*

A proof of this result can be found in [30, Proposition 5.41].

Remark 2.1.18. The notion of principal symbol as per Definition 2.1.4 carries over to the case of $\Psi^m(\Omega)$ unchanged due to Proposition 2.1.15. In fact, for any $A \in \Psi^m(\Omega)$, let $a \in S^m(\Omega, \mathbb{R}^d)$ and $A_0 \in \Psi^{-\infty}(\Omega)$ such that $A = Op(a) + A_0$. Then, the principal symbol of A, denoted by $\sigma^m(A)$, is

$$\sigma^m(A) = [a] \in S^m(\Omega, \mathbb{R}^d)/S^{m-1}(\Omega, \mathbb{R}^d).$$

If $b \in S^m(\Omega, \mathbb{R}^d)$ and $A_1 \in \Psi^{-\infty}(\Omega)$ provide another reduction for A, *i.e.*, they are such that $A = Op(b) + A_1$ with $b \neq a$ and $A_1 \neq A_0$, then $Op(b-a) \in \Psi^{-\infty}(\Omega)$. This implies that $b - a \in S^{-\infty}(\Omega, \mathbb{R}^d)$, see [31, Proof of Proposition 18.1.19], which entails that $[a] = [b]$ in $S^m(\Omega, \mathbb{R}^d)/S^{m-1}(\Omega, \mathbb{R}^d)$. Hence, the principal symbol of A is well-defined. □

The composition of ΨDOs on an open set is not always well-defined in contrast to Theorem 2.1.4. Thus, we have the following result.

Theorem 2.1.16. *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $m, m' \in \mathbb{R}$ be fixed. Let $A \in \Psi^m(\Omega)$ and let $B \in \Psi^{m'}(\Omega)$ as per Definition 2.1.10. If at least one between A or B is properly supported, see Remark 2.1.11, then $A \circ B \in \Psi^{m+m'}(\Omega)$. If both are properly supported, then so is $A \circ B$.*

A proof of this result can be found in [30, Theorem 5.32].

Proposition 2.1.17. *Let $A \in \Psi^m(\Omega)$ and $B \in \Psi^{m'}(\Omega)$ be as per Theorem 2.1.16. Let $\sigma^m(A)$ and $\sigma^{m'}(B)$ be their principal symbols as per Remark 2.1.18. Then, one has that $\sigma^{m+m'}(A \circ B) = \sigma^m(A)\sigma^{m'}(B)$, in the sense that for any $a \in \sigma^m(A)$ and $b \in \sigma^{m'}(B)$, $ab \in \sigma^{m+m'}(A \circ B)$.*

A proof of this result can be found in [30, Proposition 5.40].

Remark 2.1.19. The notion of ellipticity as per Definitions 2.1.5 and 2.1.6 carries over to the case of ΨDOs on an open set Ω, see Definition 2.1.10, by simply replacing the space of symbols $S^m(\mathbb{R}^d, \mathbb{R}^d)$ with its local analogue $S^m(\Omega, \mathbb{R}^d)$. Then, Proposition 2.1.6 is to be understood at the level of local symbols, *i.e.*, either by replacing in its statement the symbol a with χa, for any $\chi \in \mathscr{D}(\Omega)$, or by requiring that the estimates hold on every compact subset $K \subset \Omega$. There is, however, an important change in the properties of the elliptic parametrix as per Theorem 2.1.7, as highlighted by the following result. □

Theorem 2.1.18 (Local elliptic parametrix)**.** *Let $A \in \Psi^m(\Omega)$ be elliptic as per Remark 2.1.19. Then, there exists $B \in \Psi^{-m}(\Omega)$ properly supported as per Remark 2.1.11 such that*

$$A \circ B - I, \ \ B \circ A - I \in \Psi^{-\infty}(\Omega),$$

see Definition 2.1.10. Furthermore, if $B' \in \Psi^{-m}(\Omega)$ is another such properly supported ΨDO, then $B' - B \in \Psi^{-\infty}(\Omega)$.

We conclude this section with a brief review of the continuity of ΨDOs between Sobolev and Besov spaces. To this avail, we introduce the following notions.

Definition 2.1.12 (Local Sobolev spaces)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $s \in \mathbb{R}$ be fixed. The space of* ***compactly supported Sobolev functions of order s on Ω****, denoted by $H^s_c(\Omega)$, is the set of all $u \in H^s(\mathbb{R}^d)$, see Definition 2.1.7, such that $\mathrm{supp}(u) \subset\subset \Omega$. The space of* ***locally Sobolev functions of order s on Ω****, denoted by $H^s_{loc}(\Omega)$, is the set of all $u \in \mathscr{D}'(\Omega)$ such that $\varphi u \in H^s(\mathbb{R}^d)$ for all $\varphi \in \mathscr{D}(\Omega)$, where we identify $\varphi u \in \mathscr{E}'(\Omega)$ with its extension by 0 to $\mathbb{R}^d$.*

Remark 2.1.20. $H^s_{loc}(\Omega)$ is endowed with the Fréchet topology induced by the family of seminorms

$$H^s_{loc}(\Omega) \ni u \mapsto \|\chi_k u\|_{H^s(\mathbb{R}^d)} \in [0, +\infty),$$

for all $k \in \mathbb{N}$, where $(\chi_k)_{k\in\mathbb{N}} \subset \mathscr{D}(\mathbb{R}^d)$ is a partition of unity on Ω. In analogy with the topology of $\mathscr{D}(\Omega)$, see Example 1.1.12, that of $H^s_c(\Omega)$ as per Definition 2.1.12 is the inductive limit, see Definition 1.1.20, of the spaces $H^s(K) \doteq \{u \in H^s_{loc}(\Omega) \mid supp(u) \subset K\}$, for $K \subset\subset \Omega$, endowed with the subspace topology, that it inherits from $H^s_{loc}(\Omega)$. □

In analogy to Theorem 2.1.9 and Corollary 2.1.10, we have the following results.

Theorem 2.1.19 (Continuity on local Sobolev spaces)**.** *Let $A \in \Psi^m(\Omega)$ as per Definition 2.1.10. Then, $A \in \mathfrak{L}(H^s_c(\Omega), H^{s-m}_{loc}(\Omega))$ as per Definitions 1.1.4 and 2.1.12. If A is properly supported as per Remark 2.1.11, then it is also continuous $H^s_c(\Omega) \to H^{s-m}_c(\Omega)$ and $H^s_{loc}(\Omega) \to H^{s-m}_{loc}(\Omega)$.*

A proof of this result can be found in [30, Corollary 5.60].

Theorem 2.1.20 (Elliptic regularity on local Sobolev spaces)**.** *Let $A \in \Psi^m(\Omega)$ as per Definition 2.1.10 be elliptic as per Remark 2.1.19 and let $u \in \mathscr{E}'(\Omega)$ as per Definition 1.2.1. If $Au \in H^s_{loc}(\Omega)$ for some $s \in \mathbb{R}$, then $u \in H^{s+m}_{loc}(\Omega)$. If A is properly supported as per Remark 2.1.11, then this property holds even if $u \in \mathscr{D}'(\Omega)$.*

A proof of this result can be found in [30, Theorem 5.70].

Theorem 2.1.21 (Continuity on local Besov spaces)**.** *Let $A \in \Psi^m(\Omega)$ as per Definition 2.1.10 be properly supported as per Remark 2.1.11 and let $B^{\alpha,loc}_{p,q}(\Omega)$ be as per Remark 1.3.11 for any $\alpha \in \mathbb{R}$ and $p, q \in [1, +\infty]$. Then, A is continuous $B^{\alpha,loc}_{p,q}(\Omega) \to B^{\alpha-m,loc}_{p,q}(\Omega)$.*

A proof of this result can be found in [12, Remark 22].

Example 2.1.3. Let $\Omega \subset \mathbb{R}^d$ be a bounded, open set of class C^2, see [5, Section 9.6] and let Δ be the Laplacian on $\mathbb{R}^d$, *i.e.*,

$$\Delta\varphi(x) \doteq \sum_{j=1}^{d} \frac{\partial^2 \varphi}{\partial x_j^2}(x),$$

for all $\varphi \in \mathscr{D}(\Omega)$. For a fixed $\mu \in \mathbb{R}$, one may wish to investigate the existence and regularity of weak eigenfunctions of the Dirichlet Laplacian with eigenvalue μ, *i.e.*, functions $\varphi \in C^2(\overline{\Omega})$, where $\overline{\Omega}$ is the closure of Ω in $\mathbb{R}^d$, such that

$$\begin{cases} -\Delta\varphi(x) = \mu\varphi(x) & x \in \Omega, \\ \varphi = 0. & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

We first notice the following. Let $\varphi \in C^2(\overline{\Omega})$ be a solution to Equation (2.3). Then, using Equation (2.3) and Green's formula, see [5, Section 9, p. 296],

$$\mu \int_\Omega |\varphi(x)|^2 dx = -\int_\Omega \Delta\varphi(x)\overline{\varphi(x)}dx = \int_\Omega \|\nabla\varphi(x)\|^2 dx \geq 0.$$

If $\mu = 0$, then $\nabla\varphi \equiv 0$, which implies that φ is constant on every connected component of Ω. As $\varphi|_{\partial\Omega} \equiv 0$, the only possible constant is 0. Thus, if $\mu \leq 0$, then $\varphi \equiv 0$ is the only solution to Equation (2.3), and we can limit our search to $\mu > 0$. In particular, there exists only at most countably many values of $\mu > 0$ for which Equation 2.3 admits a non trivial solution. In fact, by [5, Theorem 9.31], there exist $(\mu_n)_{n\in\mathbb{N}} \subset (0, +\infty)$ and $(\varphi_n)_{n\in\mathbb{N}}$ a basis for $H^1_0(\Omega)$ such that $\mu_n \to +\infty$ as $n \to +\infty$ and $-\Delta\varphi_n = \mu_n\varphi_n$ for all $n \in \mathbb{N}$. Here, $H^1_0(\Omega)$ is the closure of $\mathscr{D}(\Omega)$, see Definition 1.2.1, with respect to the Sobolev norm $\|\cdot\|_{H^1(\mathbb{R}^d)}$, see Definition 2.1.7, endowed with the Banach topology induced by this norm. We note that $C^2(\overline{\Omega}) \subset H^1_0(\Omega) \subsetneq H^1(\Omega)$, where the last inclusion is strict as the constant function $\varphi \equiv 1$ is an element of the latter space but not of the former. Here, the Laplacian is acting as a distributional differential operator, see Definition 1.2.3, on φ_n and the equivalence is to be read as $-\Delta\varphi_n - \mu_n\varphi_n = 0 \in \mathscr{D}'(\Omega)$. We now note that $-\Delta$ is the quantization of $\sigma_{-\Delta}(\xi) = |\xi|^2$ as per Remark 2.1.2 and Definition 2.1.2. Thus, $-\Delta$ is an elliptic operator by Proposition 2.1.6 and Definition 2.1.6. This implies that $-\Delta$ is also an elliptic, second order ΨDO on Ω in the sense of Definition 2.1.10 and Remark 2.1.19. Then, as $\varphi_n \in H^1_0(\Omega) \subset H^1(\Omega)$, Theorem 2.1.20 implies that $\varphi_n \in H^3(\Omega)$ for all $n \in \mathbb{N}$. By induction, it follows that $\varphi_n \in \bigcap_{n\in\mathbb{N}} H^n(\mathbb{R}^d)$. Therefore, using Sobolev's embeddings theorem, see [30, Exercise 2.8], it follows that $\varphi_n \in \mathscr{E}(\mathbb{R}^d)$ for all $n \in \mathbb{N}$. Then, as Ω is of class C^2, by [5, Theorem 9.25], each φ_n admits a unique extension to a function $\tilde{\varphi}_n \in C^2(\overline{\Omega}) \cap C^\infty(\Omega)$, which is the sought eigenfunction. □

2.2 Wavefront Set theory

A key application of the theory we presented in Section 2.1 is to the regularity theory of PDEs and to the propagation of singularities. Within this context, the notion of wavefront set is of paramount importance, as it describes the microlocal regularity of a distribution. In Subsection 2.2.1, we shall recall the theory of smooth wavefront sets and the applications of the ΨDO calculus reviewed in Section 2.1 to this framework. In Subsection 2.2.2, we shall review the known results regarding wavefront sets of Besov regularity and Hörmander's topology for the smooth wavefront sets. These results shall guide the developments presented in Chapter 3.

2.2.1 Pointwise estimates and ΨDO calculus

We begin by recalling some notions that will be used frequently in the following.

Definition 2.2.1 (Conic set). *A subset $V \subset \mathbb{R}^d$ is said to be a* ***cone*** *if for every $\lambda > 0$ and $v \in V$, $\lambda v \in V$. For a fixed $\Omega \subset \mathbb{R}^d$ open set, we shall say that a subset $L \subset \Omega \times \mathbb{R}^d$ is* ***conic*** *if for every $\lambda > 0$ and $(x,\xi) \in L$, $(x,\lambda\xi) \in L$.*

Definition 2.2.2 (Local diffeomorphism). *Let $O, U \subset \mathbb{R}^d$ be open sets. A function $f : O \to U$ is said to be a* ***local diffeomorphism*** *if for every $x \in O$ there exists O_x a neighbourhood of x such that $f|_{O_x} : O_x \to f[O_x]$ is smooth, invertible and its inverse is smooth. If one may choose $O_x = O$ for every $x \in O$ and $f[O] = U$, then f is said to be a* ***diffeomorphism****.*

Definition 2.2.3 (Pullback). *Let $O, U \subset \mathbb{R}^d$ be open sets and let $f \in C^\infty(O,U)$. The* ***pullback by f*** *is the map $f^* : \mathscr{E}(U) \to \mathscr{E}(O)$, see Remark 1.3.1, defined by $f^*(\varphi) = \varphi \circ f$ for every $\varphi \in \mathscr{E}(U)$. We call $f^*\varphi$ the* ***pullback of φ by f****.*

Remark 2.2.1. For any $f \in C^\infty(O,U)$ as per Definition 2.2.3, the composition $\varphi \circ f$ is smooth for any $\varphi \in \mathscr{E}(U)$. Thus, the pullback of smooth functions is well-defined. In general, this operation cannot be extended to distributions without requiring f to satisfy additional hypotheses. In particular, if f is a local diffeomorphism as per Definition 2.2.2, then the pullback by f extends to a unique continuous map $f^* : \mathscr{D}'(U) \to \mathscr{D}'(O)$, see [4, Theorem 6.1.2]. □

We now introduce the notion of smooth wavefront set.

Definition 2.2.4 (Smooth wavefront set). *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $u \in \mathscr{D}'(\Omega)$ as per Definition 1.2.1. A point $(y,\eta) \in \Omega \times \mathbb{R}^d \setminus \{0\}$ is a* ***regular point for u*** *if there exists $\varphi_y \in \mathscr{D}(\Omega)$ with $\varphi_y(y) \neq 0$ and $V_\eta \subset \mathbb{R}^d \setminus \{0\}$ an open conic neighbourhood of η as per Definition 2.2.1 such that*

$$\sup_{\xi \in V_\eta} |\langle \xi \rangle^N \mathscr{F}(\varphi_y u)(\xi)| < +\infty,$$

for all $N \in \mathbb{N}$. The ***smooth wavefront set of u****, denoted by* ***$WF(u)$****, is the collection of points $(y,\eta) \in \Omega \times \mathbb{R}^d \setminus \{0\}$ such that (y,η) is not a regular point for u.*

Remark 2.2.2. Let $u \in \mathscr{D}'(\Omega)$ as per Definition 1.2.1 and let $(y,\eta) \notin WF(u)$ as per Definition 2.2.4. Let φ_y and V_η be as per Definition 2.2.4. Then, for all $N \in \mathbb{N}$, there exists $C_N > 0$ such that

$$|1_{V_\eta}(\xi)\mathscr{F}(\varphi_y u)(\xi)| \leq \langle \xi \rangle^{-N}, \tag{2.4}$$

for all $\xi \in \mathbb{R}^d$. By the Paley-Wiener Theorem 1.2.15, this decay is exactly that of the Fourier-Laplace transform of smooth, compactly supported functions. Hence, the regular points (y,ξ) of u are the points y in physical space and the directions ξ in momentum space at which an appropriate localization of u, here given by $\mathscr{F}^{-1}[1_{V_\eta}\mathscr{F}(\varphi_y u)]$, behaves as a smooth function. We note that, since the function 1_{V_η} is discontinuous at the boundary of V_η, the Paley-Wiener theorem is inapplicable, as the function $1_{V_\eta}\mathscr{F}(\varphi_y u)$ cannot have an entire extension to $\mathbb{C}^d$, unless $V_\eta = \mathbb{R}^d$. However, the estimate in Equation (2.4) implies that, for any $\alpha \in \mathbb{N}^d$, $\langle\cdot\rangle^{|\alpha|}1_{V_\eta}\mathscr{F}(\varphi_y u) \in L^1(\mathbb{R}^d)$. Hence, by Proposition 1.2.9, item *(DF1)*, and [36, Theorem 8.22(d)] one has that $\mathscr{F}^{-1}[1_{V_\eta}\mathscr{F}(\varphi_y u)]$ is a smooth function. □

Remark 2.2.3. In light of Remark 2.2.2, the notion of singular support, see Definition 1.2.5, can be heuristically reinterpreted a posteriori as a version of the wavefront set that does not distinguish between different directions in momentum space. More precisely, let $u \in \mathscr{D}'(\Omega)$ as per Definition 1.2.1 and let $WF(u)$ be as per Definition 2.2.4. Let $\mathrm{pr}_1 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be the projection onto the first factor. Then, using the Paley-Wiener-Schwartz theorem, see Theorem 1.2.16, one has that

$$\mathrm{pr}_1[WF(u)] = singsupp(u),$$

see [4, Chapter 8.1, pp. 253-254]. In particular, $WF(u) = \varnothing$ if and only if $u \in \mathscr{E}(\Omega)$. However, the notion of wavefront set is not trivial in general, in the sense that there exists $u \in \mathscr{D}'(\Omega)$ such that $\mathrm{pr}_2[WF(u)] \subsetneq \mathbb{R}^d \setminus \{0\}$, see, *e.g.*, [30, Example 6.26]. Furthermore, one also has that, for any $L \subset \Omega \times \mathbb{R}^d \setminus \{0\}$ closed, conic set, see Definition 2.2.1, there exists $u \in \mathscr{D}'(\Omega)$ such that $WF(u) = L$, see [4, Theorem 8.14]. □

Remark 2.2.4. The theory of the wavefront set is constructed by working on an open set $\Omega \subset \mathbb{R}^d$, which contains physical points, and on the open cone $\mathbb{R}^d \setminus \{0\}$, which parametrizes directions in momentum space. This formulation is intrinsically linked to the construction of wavefront sets on manifolds. Specifically, for a smooth manifold M and a distribution $u \in \mathscr{D}'(M)$, its wavefront set is a subset of the punctured cotangent bundle $T^*M \setminus \{0\}$. We refer the interested reader [30, Chapter 5] for a more thorough discussion of this topic. □

The notion of smooth wavefront set can be more conveniently studied through the use of ΨDOs. Thus, we shall now review the theory of the operator wavefront set, which is a key component of this application.

Definition 2.2.5 (Essential support). *Let $a \in S^m(\Omega, \mathbb{R}^d)$ as per Definition 2.1.11. A point $(y,\eta) \in \Omega \times \mathbb{R}^d \setminus \{0\}$ does not lie in the* ***essential support of a****, written as $(y,\eta) \notin ess\ supp(a)$, if there exists $L_{(y,\eta)} \subset \Omega \times \mathbb{R}^d \setminus \{0\}$ a conic neighbourhood* *of (y,η) as per Definition 2.2.1 such that, for all $\nu \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{N}^d$,*

$$\sup_{(x,\xi)\in L_{(y,\eta)}} |\langle \xi \rangle^{\nu} \partial_x^{\alpha} \partial_{\xi}^{\beta} a(x,\xi)| < +\infty. \tag{2.5}$$

Remark 2.2.5. If $a \in S^m(\Omega, \mathbb{R}^d)$ as per Definition 2.1.11 is elliptic as per Remark 2.1.19, then its essential support is $\Omega \times \mathbb{R}^d \setminus \{0\}$. To prove this, we note that, by Proposition 2.1.6, for every $\chi \in \mathscr{D}(\Omega)$ there exists $R_\chi, c_\chi > 0$ such that

$$|\chi(x)a(x,\xi)| \geq c_\chi |\xi|^m,$$

for every $\xi \in \mathbb{R}^d$ with $|\xi| \geq R_\chi$. Let $(y,\eta) \in \Omega \times \mathbb{R}^d \setminus \{0\}$ and let $L_{(y,\eta)}$ be a conic neighbourhood of (y,η) as per Definition 2.2.1. Then, for any $\chi \in \mathscr{D}(\Omega)$ and for any $\nu > -m$,

$$\begin{aligned}\sup_{(x,\xi)\in L_{(y,\eta)}} |\langle \xi \rangle^{\nu} \chi(x) a(x,\xi)| \geq & \sup_{\substack{(x,\xi)\in L_{(y,\eta)} \\ |\xi|\geq R_\chi}} |\langle \xi \rangle^{\nu} \chi(x) a(x,\xi)| \geq \\ & \geq c_\chi \sup_{\substack{(x,\xi)\in L_{(y,\eta)} \\ |\xi|\geq R_\chi}} \langle \xi \rangle^{\nu} |\xi|^m = +\infty.\end{aligned}$$

Hence, the estimate in Equation (2.5) cannot be satisfied for every $\nu \in \mathbb{R}$ even for $\alpha = \beta = 0$. This implies that $(y,\eta) \in ess\ supp(a)$, which entails the sought result. □

Definition 2.2.6 (Global operator wavefront set). *Let $A \in \Psi^m_\infty(\mathbb{R}^d)$ as per Remark 2.1.14 and let $a \in S^m(\mathbb{R}^d, \mathbb{R}^d)$ as per Definition 2.1.1 be such that $A = Op_L(a)$ as per Remark 2.1.6. Then, the **global operator wavefront set of A**, denoted by $WF'_\infty(A)$, is the essential support of a as per Definition 2.2.5.*

Remark 2.2.6. While Definition 2.2.6 is equivalent in content to Definition 2.2.5, it is useful when discussing properties of operators without having to rely on their symbols or reductions as per Definition 2.1.3 and Remark 2.1.6. In particular, for ΨDOs on open sets Ω as per Definition 2.1.10, since the quantization is not an exact bijection between $S^m(\Omega, \mathbb{R}^d)$ and $\Psi^m(\Omega)$ as per Proposition 2.1.15, one wishes to work with a tool at the level of operators rather than symbols. □

The operator wavefront set enjoys the following properties.

Proposition 2.2.1. *Let $A \in \Psi^m_\infty(\mathbb{R}^d)$ and $B \in \Psi^{m'}_\infty(\mathbb{R}^d)$, $m, m' \in \mathbb{R}$, as per Remark 2.1.14 and let $WF'_\infty(A)$ and $WF'_\infty(B)$ be as per Definition 2.2.6.*

(OWF1) If the Schwartz kernel of A, see Remark 2.1.4, has compact support, then $WF'_\infty(A) = \varnothing$ if and only if $A \in \Psi^{-\infty}_\infty(\mathbb{R}^d)$.

(OWF2) *Let $a_L, a_R \in S^m(\mathbb{R}^d, \mathbb{R}^d)$ be such that $A = \mathrm{Op}_L(a_L) = \mathrm{Op}_R(a_R)$. Then, $WF'_\infty(A) = ess\ supp(a_L) = ess\ supp(a_R)$.*

(OWF3) $WF'_\infty(A+B) \subset WF'_\infty(A) \cup WF'_\infty(B)$.

(OWF4) $WF'_\infty(A \circ B) \subset WF'_\infty(A) \cap WF'_\infty(B)$.

(OWF5) $WF'_\infty(A^*) = WF'_\infty(A)$, *where A^* is the adjoint of A as per Remark 2.1.3.*

(OWF6) *Let $O, U \subset \mathbb{R}^d$ be open sets and let $f : O \to U$ be a diffeomorphism as per Definition 2.2.2. Let f^* and $(f^{-1})^*$ be the pullback by f and f^{-1} respectively as per Remark 2.2.1. Then, for any $A \in \Psi^m_c(U)$ as per Definition 2.1.9,*

$$WF'_\infty(A) = f_*[WF'_\infty(f^* \circ A \circ (f^{-1})^*)],$$

where $f_*[WF'_\infty(A)] \doteq \{(f(x), (f'(x)^T)^{-1}\xi) \mid (x,\xi) \in WF'_\infty(A)\}$, $f'(x)$ is the Jacobian matrix of f evaluated at x and $f'(x)^T$ is its transpose matrix.

A proof of this results can be found in [30, Proposition 6.4].

Definition 2.2.7 (Operator Wavefront set)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $A \in \Psi^m(\Omega)$ as per Definition 2.1.10 For any $\psi \in \mathscr{D}(\Omega)$, let $A_\psi \in \Psi^m_\infty(\mathbb{R}^d)$ be the operator defined by $A_\psi : u \mapsto \psi A(\psi u)$ for every $u \in \mathscr{S}(\mathbb{R}^d)$ and let $WF'_\infty(A_\psi)$ be as per Definition 2.2.6. Then, the **operator wavefront set of A** is the set $WF'(A)$ given by*

$$WF'(A) = \bigcup_{\psi \in \mathscr{D}(\Omega)} WF'_\infty(A_\psi).$$

Remark 2.2.7. Items *(OWF3)-(OWF6)* in Proposition 2.2.1 also hold for the operator wavefront set of $A \in \Psi^m(\Omega)$ and $B \in \Psi^{m'}(\Omega)$ as per Definition 2.2.7, with the additional hypothesis of Theorem 2.1.16 for item *(OWF4)* to be well-defined. □

Remark 2.2.8. Let $A \in \Psi^m(\Omega)$ as per Definition 2.1.10. If $A \in \Psi^{-\infty}(\Omega)$, then A_ψ as per Definition 2.2.7 is a residual ΨDO on $\mathbb{R}^d$ for every $\psi \in \mathscr{D}(\Omega)$. Thus, as the Schwartz kernel of A_ψ is compactly supported, Proposition 2.2.1, item *(OWF1)*, implies that $WF'_\infty(A_\psi) = \varnothing$. Therefore, $WF'(A) = \varnothing$. Conversely, if $WF'(A) = \varnothing$, then $WF'_\infty(A_\psi) = \varnothing$ for every $\psi \in \mathscr{D}(\Omega)$. Thus, $A_\psi \in \Psi^{-\infty}_\infty(\mathbb{R}^d)$ for every $\psi \in \mathscr{D}(\Omega)$ by Proposition 2.2.1, item *(OWF1)*. In particular, the Schwartz kernel of A_ψ is the smooth function $K_{A_\psi} : (x,y) \mapsto \psi(x)K_A(x,y)\psi(y)$ as per Remark 2.1.4 and Propositions 1.2.4 and 2.1.3. Since $\psi \in \mathscr{D}(\Omega)$ is arbitrary, a partition of unity argument yields that K_A is also smooth, which implies that $A \in \Psi^{-\infty}(\Omega)$. This proves that $WF'(A) = \emptyset$ if and only if A is a residual ΨDO. In particular, we note that item *(OWF1)* of Proposition 2.2.1 holds for elements of $\Psi^m(\Omega)$ without the hypothesis of compact support for their kernels. This follows because their operator wavefront set is defined from localizations on compact sets as per Definition 2.2.7. □

Example 2.2.1. Let $\varphi, \chi \in \mathscr{D}(\Omega)$ as per Definition 1.2.1 and let $\psi \in S^0(\mathbb{R}^d)$ as per Definition 2.1.1. Let $A \in \Psi^0(\Omega)$ be the quantization of the local symbol $(x, y, \xi) \mapsto \chi(x)\psi(\xi)\varphi(y)$ as per Remark 2.1.17. Then,

$$WF'(A) = (supp(\chi) \cap supp(\varphi)) \times ess\ supp(\psi),$$

see [30, Examples 6.8-6.10]. □

Remark 2.2.9. Remark 2.2.5 also implies that, for any $A \in \Psi^m(\Omega)$ as per Definition 2.1.10, if A is elliptic as per Remark 2.1.19, then $WF'(A) = \Omega \times \mathbb{R}^d \setminus \{0\}$. □

Remark 2.2.10. Heuristically speaking, the essential support of a symbol as per Definition 2.2.5 is the collection of points in physical space and directions in momentum space at which the symbol does not behave like a residual one, see Definition 2.1.11. Hence, if $a \in S^m(\Omega, \mathbb{R}^d)$ and $u \in \mathscr{D}'(\Omega)$ are such that $supp(u) \cap \mathrm{pr}_1[ess\ supp(a)] = \emptyset$, where $\mathrm{pr}_1 : \Omega \times \mathbb{R}^d \to \Omega$ is the projection on the first factor, then one would expect $\mathrm{Op}(a)u$ to be a smooth function by Proposition 2.1.15 and Remark 2.1.13. Item *(OWF1)* of Proposition 2.2.1 and Remark 2.2.7 entail that this property is true when working with local ΨDOs on an open set. Actually, a stronger statement holds true, which is given by the following proposition. □

Theorem 2.2.2 (Microlocality of ΨDOs)**.** *Let $u \in \mathscr{D}'(\Omega)$ as per Definition 1.2.1 and let $A \in \Psi^m(\Omega)$ as per Definition 2.1.10. Then,*

$$WF(Au) \subset WF'(A) \cap WF(u),$$

see Definitions 2.2.4 and 2.2.7.

A proof of this result can be found in [30, Proposition 6.32].

While Theorem 2.2.2 is a powerful instrument, one can render the analysis of local singularities even more precise through the notion of elliptic and characteristic sets of a ΨDO.

Definition 2.2.8 (Elliptic and characteristic sets for symbols)**.** *Let $a \in S^m(\Omega, \mathbb{R}^d)$ as per Definition 2.1.11. A point $(y, \eta) \in \Omega \times \mathbb{R}^d \setminus 0$ lies in the **elliptic set of a**, written as $(y, \eta) \in Ell(a)$, if there exists $U_y \subset \Omega$ a precompact neighbourhood of y, $V_\eta \subset \mathbb{R}^d \setminus \{0\}$ a conic neighbourhood of η as per Definition 2.2.1 and two constants $R, c > 0$ such that*

$$|a(x, \xi)| \geq c|\xi|^m,$$

for every $(x,\xi) \in U_y \times V_\eta$ *such that* $|\xi| \geq R$*. The complement of* $Ell(a)$ *is* $\Omega \times \mathbb{R}^d \setminus \{0\}$ *is called the* ***characteristic set of a*** *and it is denoted by* $Char(a)$*.*

Definition 2.2.9 (Elliptic and characteristic sets for ΨDOs)**.** *Let* $A \in \Psi^m(\Omega)$ *be as per Definition 2.1.10. For every* $\psi \in \mathscr{D}(\Omega)$*, let* $A_\psi \in \Psi^m_\infty(\mathbb{R}^d)$*, see Remark 2.1.14, be the operator given by* $A_\psi : u \mapsto \psi A(\psi u)$ *and let* $a_\psi \in S^m(\mathbb{R}^d,\mathbb{R}^d)$ *be a representative of the principal symbol of* A_ψ *as per Definition 2.1.4. The* ***elliptic set of A*** *is*

$$Ell(A) = \bigcup_{\psi \in \mathscr{D}(\Omega)} Ell(a_\psi),$$

where $Ell(a)$ *is as per Definition 2.2.8 for every* $a \in S^m(\mathbb{R}^d,\mathbb{R}^d)$*. The* ***characteristic set of A*** *is the complement of* $Ell(A)$ *in* $\Omega \times \mathbb{R}^d \setminus \{0\}$*.*

Remark 2.2.11. In analogy with Remark 2.2.10, Definition 2.2.8 defines $Ell(a)$ for $a \in S^m(\Omega,\mathbb{R}^d)$ as the set of points $y \in \Omega$ and directions $\xi \in \mathbb{R}^d \setminus \{0\}$ at which a behaves as an elliptic symbol in the sense of Proposition 2.1.6. Thus, one may equivalently characterize $Ell(a)$ as the set of points $(y,\eta) \in \Omega \times \mathbb{R}^d \setminus \{0\}$ for which there exist $\chi \in S^0(\mathbb{R}^d,\mathbb{R}^d)$ supported on a conic neighbourhood of (y,η), see Definition 2.2.1, and $b \in S^{-m}(\Omega,\mathbb{R}^d)$ such that

$$(ab-1)\chi,\ (ba-1)\chi \in S^{-1}(\mathbb{R}^d,\mathbb{R}^d).$$

In particular, this implies that Definition 2.2.9, which seems to depend explicitly on a representative of $\sigma^m(A_\psi)$, is well-defined. In fact, let $a,\tilde{a} \in S^m(\mathbb{R}^d,\mathbb{R}^d)$ be representatives of $\sigma^m(A)$ for a given $A \in \Psi^m(\mathbb{R}^d)$ and let $(y,\eta) \in Ell(a)$. Let $b \in S^{-m}(\mathbb{R}^d,\mathbb{R}^d)$ and let $\chi \in S^0(\mathbb{R}^d,\mathbb{R}^d)$ with $\chi(y,\eta) \neq 0$ be such that $(ab-1)\chi$ and $(ba-1)\chi$ are symbols of order -1 on $\mathbb{R}^d$ modulo extension by 0. We note that, as $a,\tilde{a} \in \sigma^m(A) \in S^m(\mathbb{R}^d,\mathbb{R}^d)/S^{m-1}(\mathbb{R}^d,\mathbb{R}^d)$ implies that $a-\tilde{a} \in S^{m-1}(\mathbb{R}^d,\mathbb{R}^d)$, one has that $(\tilde{a}-a)b\chi,\ b(\tilde{a}-a)\chi \in S^{-1}(\mathbb{R}^d,\mathbb{R}^d)$ by Proposition 2.1.1, item *(S3)*. Thus,

$$(\tilde{a}b-1)\chi = (ab-1)\chi + (\tilde{a}-a)b\chi \in S^{-1}(\mathbb{R}^d,\mathbb{R}^d),$$

and

$$(b\tilde{a}-1)\chi = (ba-1)\chi + b(\tilde{a}-a)\chi \in S^{-1}(\mathbb{R}^d,\mathbb{R}^d).$$

This implies that $(y,\eta) \in Ell(\tilde{a})$. Hence, $Ell(a) = Ell(\tilde{a})$ and Definition 2.2.9 is well-posed. □

Remark 2.2.12. We note that the set $Ell(A)$ as per Definition 2.2.9 depends on the order of A, as it is defined from its principal symbol. For this reason, whenever confusion may arise, it is customary in the literature to write $Ell^m(A)$ for $A \in \Psi^m(\Omega)$. To clarify why it might be necessary to specify the order in the notation of the elliptic set, let us consider an elliptic $A \in \Psi^{m-1}(\Omega) \subset \Psi^m(\Omega)$. Then, $A_\psi \in \Psi^{m-1}(\mathbb{R}^d)$, see Definition 2.2.9, for every $\psi \in \mathscr{D}(\Omega)$, which implies that $\sigma^m(A_\psi) = 0$ for every $\psi \in \mathscr{D}(\Omega)$. Thus, $Ell^m(A) = \emptyset$. By contrast, Remarks 2.2.5 and 2.2.11 imply that $Ell^{m-1}(A) = \Omega \times \mathbb{R}^d \setminus \{0\}$. In the following, we shall write the order of $Ell^m(\cdot)$ explicitly only when confusion might arise. □

Remark 2.2.13. In analogy with Remark 2.2.11, for a given $A \in \Psi^m(\Omega)$ as per Definition 2.1.10, $Ell(A)$ is the set of points and directions at which A behaves as an elliptic ΨDO as per Remark 2.1.19. Thanks to this notion, Theorem 2.1.18 can be generalized to any ΨDO with $Ell(A) \neq \emptyset$ through an appropriate localization of A inside $Ell(A)$. Thus, we have the following result. □

Theorem 2.2.3 (Microlocal elliptic parametrix). *Let $A \in \Psi^m(\Omega)$ as per Definition 2.1.10 be properly supported as per Remark 2.1.11 and let $Ell(A) \neq \emptyset$ be its elliptic set as per Definition 2.2.9. Then, for every closed subset $K \subset Ell(A)$, there exists $B_K \in \Psi^{-m}(\Omega)$ such that*

$$K \cap WF'(A \circ B_K - Id) = K \cap WF'(B_K \circ A - Id) = \emptyset,$$

*where $WF'(\cdot)$ is as per Definition 2.2.7 and Id is the identity operator on $\mathscr{D}(\Omega)$. We call B_K a **microlocal elliptic parametrix** for A on K.*

A proof of this result can be found in [30, Proposition 6.17]

Remark 2.2.14. In light of Theorem 2.2.3, we say that two given ΨDOs A and B are **microlocally equivalent** on a set $L \subset \Omega \times \mathbb{R}^d \setminus \{0\}$ if $L \cap WF'(A - B) = \emptyset$. Thus, a microlocal elliptic parametrix B_K for A on K is a ΨDO of order $-m$ such that $A \circ B$ and $B \circ A$ are microlocally equivalent to the identity operator on K. □

We provide now the characterization of the wavefront set of a distribution by means of ΨDOs. The following results are a consequence of the constructions of the elliptic and characteristic sets as per Definition 2.2.9.

Theorem 2.2.4 (Pseudodifferential characterizations of the smooth wavefront set). *Let $u \in \mathscr{D}'(\Omega)$ as per Definition 1.2.1. A point $(y, \eta) \in \Omega \times \mathbb{R}^d \setminus \{0\}$ does not lie in $WF(u)$ as per Definition 2.2.4 if and only if one of the following conditions holds:*

(SWF1) there exists $L \subset \Omega \times \mathbb{R}^d \setminus \{0\}$ a conic neighbourhood of (y, η), see Definition 2.2.1, such that $Au \in \mathscr{E}(\Omega)$ for every properly supported $A \in \Psi^0(\Omega)$ as per Remark 2.1.11 with $WF'(A) \subset L$ as per Definition 2.2.7;

(SWF2) there exists $A \in \Psi^0(\Omega)$ properly supported such that $(y, \eta) \in Ell(A)$ and $Au \in \mathscr{E}(\Omega)$.

Proofs of these results can be found in [30, Lemma 6.19] and [30, Proposition 6.21].

Corollary 2.2.5. *Let $u \in \mathscr{D}'(\Omega)$ as per Definition 1.2.1 and let $WF(u)$ be as per Definition 2.2.4. Let $\mathfrak{PS}(\Omega)$ be the set of $A \in \Psi^0(\Omega)$ that are properly supported, see Definition 2.1.10 and Remark 2.1.11. Then,*

$$WF(u) = \bigcap_{\substack{A \in \mathfrak{PS}(\Omega), \\ Au \in \mathscr{E}(\Omega)}} Char(A),$$

where $Char(A)$ is the set of characteristic points of A as per Definition 2.2.9.

A proof of this result can be found in [30, Corollary 6.20]. With these results in hand, we can now state the microlocal analogue of elliptic regularity as per Corollary 2.1.10 and Theorem 2.1.20.

Theorem 2.2.6. *Let $u \in \mathscr{D}'(\Omega)$ as per Definition 1.2.1. Then, for every $A \in \Psi^m(\Omega)$ properly supported as per Remark 2.1.11,*

$$WF(u) \subset WF(Au) \cup Char(A),$$

where $Char(A)$ is as per Definition 2.2.9. In particular, for every $A \in \Psi^m(\Omega)$ that is also elliptic, one has that

$$WF(Au) = WF(u).$$

A proof of this result can be found in [30, Proposition 6.33].

We conclude this subsection with the construction of the pointwise product for distributions, following the exposition in [4, Chapter 8.2].

Remark 2.2.15. In Remark 1.2.26, we noted that the pointwise product of arbitrary distributions is, in general, ill-defined. However, under some additional hypotheses, one can define it properly in a novel way. First and foremost, let us fix $\Delta : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ the function given by $\Delta(x) = (x, x)$. We call this function the **diagonal map** and we note that $\Delta \in C^\infty(\mathbb{R}^d, \mathbb{R}^{2d})$. Then, for any $f, g \in \mathscr{E}(\mathbb{R}^d)$, one has that

$$(fg)(x) = f(x)g(x) = f(x)g(y)|_{y=x} = (f \otimes g)(x, x) = \Delta^*(f \otimes g)(x),$$

for any $x \in \mathbb{R}^d$ by Remarks 1.2.11 and Definition 2.2.3. By Remark 2.2.1, we know that the pullback can be extended to a well-defined operation between spaces of distributions under additional hypotheses and by Definition 1.2.6, we know that the tensor product of distributions is well-defined. Thus, a natural way to define the pointwise product of two distributions u and v would be

$$uv \doteq \Delta^*(u \otimes v).$$

However, this is not always a well-posed distribution. More specifically, the map Δ is not a local diffeomorphism as per Definition 2.2.2, hence its pullback might not be well-defined for any arbitrary distribution. The main obstacles, heuristically speaking, are the regions where the distributions u and v are singular, *i.e.*, where they don't behave as smooth functions, and the possibility of these regions having a non empty intersection. In fact, if either u or v was a smooth function, their product would be perfectly well-posed as per Remark 1.2.6. Thus, for the pointwise product uv to make sense, one needs first to have some form of control over the regions where both u and v do not behave as smooth functions. The theory of wavefront sets, therefore, is uniquely apt to overcome this problem, as per the following results. □

Theorem 2.2.7 (Wavefront set of tensor product). *Let $u \in \mathscr{D}'(\Omega_1)$ and $v \in \mathscr{D}'(\Omega_2)$ as per Definition 1.2.1. Let $u \otimes v \in \mathscr{D}'(\Omega_1 \times \Omega_2)$ as per Definition 1.2.6, $WF_u(v) = supp(u) \times \{0\} \times WF(v)$ and $WF_v(u) \doteq WF(u) \times supp(v) \times \{0\}$ as per Definition 2.2.4. Then,*

$$WF(u \otimes v) \subset [WF(u) \times WF(v)] \cup WF_u(v) \cup WF_v(u).$$

A proof of this result can be found in [4, Theorem 8.2.9].

Theorem 2.2.8 (Pullback of distribution). *Let $O \subset \mathbb{R}^d$ and $U \subset \mathbb{R}^n$ be open sets and let $f \in C^\infty(O, U)$. Furthermore, let*

$$\mathscr{N}_f \doteq \{(f(x), \eta) \in U \times \mathbb{R}^n \mid f'(x)^T \eta = 0\},$$

where $f'(x)$ is the Jacobian matrix of f evaluated at $x \in O$ and $f'(x)^T$ is its transpose matrix. Then, for any $u \in \mathscr{D}'(U)$, if $\mathscr{N}_f \cap WF(u) = \emptyset$, see Definition 2.2.4, then there exists a unique distribution $f^*u \in \mathscr{D}'(O)$ such that, whenever $u \in \mathscr{E}(U)$, $f^*u = u \circ f$. Moreover,

$$WF(f^*u) \subset \{(x, f'(x)^T \eta) \mid (f(x), \eta) \in WF(u)\}.$$

A proof of this result can be found in [4, Theorem 8.2.4].

Theorem 2.2.9 (Pointwise product of distributions). *Let $u, v \in \mathscr{D}'(\Omega)$ as per Definition 1.2.1 and let $WF(u)$ and $WF(v)$ be as per Definition 2.2.4. If u and v are such that $WF(u) \cap -WF(v) = \emptyset$, where $-WF(v)$ is the set of (x, η) such that $(x, -\eta) \in WF(v)\}$, then $uv \doteq \Delta^*(u \otimes v)$, see Remark 2.2.15, is a well-defined distribution over Ω. Furthermore,*

$$WF(uv) \subset \{(x, \xi + \eta) \mid (x, \xi) \in WF(u) \text{ and } (x, \eta) \in WF(v)\} \cup S_u \cup S_v,$$

where $S_u \doteq \{(x, \eta) \in WF(u) \mid x \in supp(v)\}$ and $S_v \doteq \{(x, \eta) \in WF(v) \mid x \in supp(u)\}$.

A proof of this result can be found in [4, Theorem 8.2.10].

2.2.2 Besov wavefront sets and smooth Hörmander's Topology

In the first part of this subsection, we shall review the notion and properties of the wavefront set of Besov type, following the analysis in [12]. This construction will be the starting point of the developments in Chapter 3.

Remark 2.2.16. In Remark 2.2.2, we noted that a point $(y,\eta)\in\Omega\times\mathbb{R}^d\setminus\{0\}$ does not belong to the wavefront set of a distribution u if $\mathscr{F}^{-1}[1_V\mathscr{F}(\varphi u)]$ is smooth, where $\varphi\in\mathscr{D}(\Omega)$ is supported in a neighbourhood of y and $V\subset\mathbb{R}^d\setminus\{0\}$ is an open, conic neighbourhood of η, see Definition 2.2.1. This definition relies on [36, Theorem 8.22], which provides an equivalent characterization of smoothness in terms of the decay of the Fourier transform and it depends only on the properties of the function space $\mathscr{E}(\Omega)$. Hence, one can imagine that it can be generalized to other spaces of functions and distributions, as long as they possess a characterization in terms of Fourier transform. This can lead to the notion of **wavefront set of Banach or Fréchet type**, which are defined in terms of abstract functional spaces. We refer the interested reader to [37] and [38] for an investigation of these topics. Alternatively, one may develop a more precise notion of wavefront set that is constructed from fixed functional spaces, such as Sobolev-Bessel spaces, see Definition 2.1.7. We shall report on this latter approach, in the specific context of Besov spaces, see Definition 1.3.3. □

Definition 2.2.10. *Let $u\in\mathscr{D}'(\Omega)$ as per Definition 1.2.1 and let $\alpha\in\mathbb{R}$ be fixed. We say that $(y,\eta)\in\Omega\times\mathbb{R}^d\setminus\{0\}$ does not lie in the **α-Besov wavefront set of u**, written as $(y,\eta)\notin WF^\alpha(u)$, if there exists $\varphi_y\in\mathscr{D}(\Omega)$ with $\varphi_y(y)\neq 0$ and an open conic neighbourhood $V_\eta\subset\mathbb{R}^d\setminus\{0\}$ of η as per Definition 2.2.1 such that, for any compact $K\subset\Omega$,*

$$\sup_{x\in K}\left|\int_{V_\eta}\widehat{\varphi_y u}(\xi)\check{\beta}(\xi)e^{ix\cdot\xi}d\xi\right|<+\infty,\tag{2.6}$$

and

$$\sup_{\lambda\in(0,1)}\sup_{x\in K}\left|\int_{V_\eta}\widehat{\varphi_y u}(\xi)\lambda^{-\alpha}\check{\gamma}(\lambda\xi)e^{ix\cdot\xi}d\xi\right|<+\infty,\tag{2.7}$$

for every $\beta\in\mathscr{D}(B(0,1))$ with $\check{\beta}(0)\neq 0$ and $\gamma\in\mathscr{B}_{\lfloor\alpha\rfloor}$ as per Definition 1.3.6.

Remark 2.2.17. In light of Remarks 1.3.11 and 2.2.16, Definition 2.2.10 states that a point $(y,\eta)\in\Omega\times\mathbb{R}^d\setminus\{0\}$ does not lie in $WF^\alpha(u)$ for a given $u\in\mathscr{D}'(\Omega)$ if $\mathscr{F}^{-1}[1_{V_\eta}\mathscr{F}(\varphi_y u)]\in B^{\alpha,loc}_{\infty,\infty}(\Omega)$ for the appropriate choices of $\chi_y\in$

$\mathscr{D}(\Omega)$ and $V_\eta \subset \mathbb{R}^d \setminus \{0\}$. Thus, one has that $u \in B^{\alpha,loc}_{\infty,\infty}(\Omega)$ if and only if $WF^\alpha(u) = \emptyset$, see [12, Proposition 26]. We note, however, that the notion of Besov wavefront set allows for finer control on the regularity of a distribution. For instance, let $\alpha \in \mathbb{R}\setminus\mathbb{Z}$ be fixed and let $u \in C^{\lfloor\alpha\rfloor,\alpha-\lfloor\alpha\rfloor}(\mathbb{R}^d)$. Then, Proposition 1.3.3 entails that $u \in B^\alpha_{\infty,\infty}(\mathbb{R}^d) \hookrightarrow B^{\alpha,loc}_{\infty,\infty}(\Omega)$. Thus, $WF^\alpha(u) = \emptyset$. However, if $u \notin B^{\beta,loc}_{\infty,\infty}(\Omega)$ for a given $\beta > \alpha$, then $WF^\beta(u) \neq \emptyset$. Hence, when analyzing the singular behaviour of a distribution through the notion of Besov wavefront sets, one obtains a much more variable result than with its smooth analogue as per Definition 2.2.4. □

Example 2.2.2. Let $y \in \Omega$ and let $\delta_y \in \mathscr{D}'(\Omega)$, see Definition 1.2.1, be as per Example 1.2.2. Inserting it into Equation (2.6), we obtain for any choice of $\varphi \in \mathscr{D}(\Omega)$ and $V \subset \mathbb{R}^d \setminus \{0\}$,

$$\left|\int_V \widehat{\varphi\delta_y}(\xi)\check{\beta}(\xi)e^{ix\cdot\xi}d\xi\right| = \left|\int_V \varphi(y)\check{\beta}(\xi)e^{i(x-y)\cdot\xi}d\xi\right| \leq C\|\check{\beta}\|_{L^1(\mathbb{R}^d)}, \tag{2.8}$$

for some $C > 0$, where we have used that $\varphi\delta_y = \varphi(y)\delta_y$ and $\hat{\delta}_y(\xi) = e^{-iy\cdot\xi}$, see Lemma 1.2.14, in the first equality. As $\beta \in \mathscr{D}(B(0,1))$, we have that $\check{\beta} \in \mathscr{S}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$ by Proposition 1.2.5 and Remark 1.2.15, item *(F2)*. Thus, Equation (2.8) is a finite estimate that is also uniform in $x \in \Omega$, which implies that Equation (2.6) is satisfied with $u = \delta_y$. Regarding Equation (2.7), we obtain similarly that

$$\left|\int_V \widehat{\varphi\delta_y}(\xi)\lambda^{-\alpha}\check{\gamma}(\lambda\xi)e^{ix\cdot\xi}d\xi\right| \leq C'\lambda^{-\alpha-d}, \tag{2.9}$$

for some $C' > 0$ and for any choice of $\varphi \in \mathscr{D}(\Omega)$, $V \subset \mathbb{R}^d \setminus \{0\}$ and $x \in \Omega$. Thus, for any $\alpha \leq -d$, Equation (2.9) is a finite estimate uniform in $x \in \Omega$ and $\lambda \in (0,1)$. By Definition 2.2.10, this implies that $WF^\alpha(\delta_y) = \emptyset$ for every $\alpha \leq -d$. Let $\varphi \in \mathscr{D}(\Omega)$ be such that $\varphi(y) \neq 0$ and let $V \subset \mathbb{R}^d \setminus \{0\}$ be a conic set as per Definition 2.2.1. Then, we have that

$$\left|\int_V \widehat{\varphi\delta_y}(\xi)\lambda^{-\alpha}\check{\gamma}(\lambda\xi)e^{ix\cdot\xi}d\xi\right| = \lambda^{-\alpha-d}\left|\int_V \varphi(y)\check{\gamma}(\xi)e^{i\frac{x-y}{\lambda}\cdot\xi}d\xi\right| \doteq \lambda^{-\alpha-d}C(\lambda,x). \tag{2.10}$$

We note that, in the limit $\lambda \to 0^+$, $C(\lambda,x)$ either converges to 0 if $x \neq y$ by the Riemann-Lebesgue lemma, see [36, Theorem 8.22(f)], or it is positive and constant in λ if $x = y$. In particular, since the choice of the compact set K in Equation (2.7) is arbitrary, choosing one such that $y \in K$ yields that $\lambda^{-\alpha-d}C(\lambda,x) \to +\infty$ as $\lambda \to 0^+$ whenever $\alpha > -d$. As these properties are independent of the choices of φ and V, provided that $\varphi(y) \neq 0$, this entails

$$\sup_{\lambda\in(0,1)} \sup_{x\in K} \left| \int_V \widehat{\varphi\delta_y}(\xi)\lambda^{-\alpha}\check{\gamma}(\lambda\xi)e^{ix\cdot\xi}d\xi \right| < +\infty$$

for every compact subset $K \subset \Omega$ if and only if $\alpha \leq -d$. Thus,

$$WF^\alpha(\delta_y) = \begin{cases} \varnothing & \text{if } \alpha \leq -d, \\ \{y\} \times \mathbb{R}^d \setminus \{0\} & \text{if } \alpha > -d. \end{cases}$$

By contrast, in the case of the smooth wavefront as per Definition 2.2.4, one has that

$$WF(\delta_y) = \{y\} \times \mathbb{R}^d \setminus \{0\},$$

see [30, Example 6.25]. □

As in the smooth case, see Corollary 2.2.5, and because of Theorem 2.1.21, one can also characterize the Besov wavefront set through the action of ΨDOs. Thus, we have the following result.

Theorem 2.2.10 (Characterization by ΨDOs). *Let $u \in \mathscr{D}'(\Omega)$ as per Definition 1.2.1 and let $\mathfrak{PS}(\Omega)$ be the collection of properly supported ΨDOs of order 0, see Remark 2.1.11. Let $WF^\alpha(u)$ be as per Definition 2.2.10 for any $\alpha \in \mathbb{R}$. Then,*

$$WF^\alpha(u) = \bigcap_{\substack{A\in\mathfrak{PS}(\Omega)\\ Au\in B^{\alpha,loc}_{\infty,\infty}(\Omega)}} Char(A).$$

A proof of this result can be found in [12, Proposition 34]. As a consequence of this theorem, we also have the following corollary.

Corollary 2.2.11. *Let $u \in \mathscr{D}(\Omega)$ and let $WF^\alpha(u)$ be as per Definition 2.2.10 for every $\alpha \in \mathbb{R}$. Then*

$$WF^\alpha(u) = \bigcap_{v\in B^{\alpha,loc}_{\infty,\infty}(\Omega)} WF(u-v),$$

where $WF(\cdot)$ is the smooth wavefront set as per Definition 2.2.4.

A proof of this result can be found in [12, Proposition 36].

Remark 2.2.18. Corollary 2.2.11 shows that the Besov wavefront is closely related to its smooth counterpart. As a matter of fact, we note that, for every $\alpha \in \mathbb{R}$, the inclusion $\mathscr{D}(\Omega) \hookrightarrow \mathscr{S}(\mathbb{R}^d) \hookrightarrow B^\alpha_{\infty,\infty}(\mathbb{R}^d)$, see Proposition 1.3.1, item *(B2)*, implies that $\mathscr{E}(\Omega) \hookrightarrow B^{\alpha,loc}_{\infty,\infty}(\Omega)$ by Remark 1.3.11. Since the smooth wavefront set is the collection of points at which a distribution $u \in \mathscr{D}'(\Omega)$ does not behave as an element of $\mathscr{E}(\Omega)$, see Remark 2.2.2, and since the Besov wavefront set is analogously defined with respect to $B^{\alpha,loc}_{\infty,\infty}(\Omega)$, one expects $WF^\alpha(u)$ to be a subset of $WF(u)$ for every $\alpha \in \mathbb{R}$. In fact, the following stronger result holds.

Proposition 2.2.12. *Let $u \in \mathscr{D}'(\Omega)$ as per Definition 1.2.1 and let $WF(u)$ and $WF^\alpha(u)$ be, respectively, as per Definitions 2.2.4 and 2.2.10 for any $\alpha \in \mathbb{R}$. Then,*

$$WF(u) = \overline{\bigcup_{\alpha \in \mathbb{R}} WF^\alpha(u)}.$$

A proof of this result can be found in [12, Corollary 37].

We now discuss some structural properties of the Besov wavefront set.

Theorem 2.2.13 (Behaviour under pullback)**.** *Let $O \subset \mathbb{R}^d$ and $U \subset \mathbb{R}^n$ be open sets. Let $f \in C^\infty(O, U)$, let $f^* : \mathscr{E}(U) \to \mathscr{E}(O)$ be as per Definition 2.2.3 and let*

$$\mathscr{N}_f \doteq \{(f(x), \xi) \in U \times \mathbb{R}^n \setminus \{0\} \mid x \in O \text{ and } f'(x)^T \xi = 0\},$$

where $f'(x)$ is the Jacobian matrix of f evaluated at $x \in O$, while $f'(x)^T$ is its transpose. For any $u \in \mathscr{D}'(U)$, if $\mathscr{N}_f \cap WF^\alpha(u) = \emptyset$ for some $\alpha > 0$, then there exists $f^*u \in \mathscr{D}'(O)$ as per Remark 2.2.1 and

$$WF^\beta(f^*u) \subset f^*[WF^\beta(u)],$$

for every $\beta \in \mathbb{R}$ and $u \in \mathscr{D}'(U)$, where

$$f^*[A] \doteq \{(x, f'(x)^T \xi) \mid (f(x), \xi) \in A\},$$

for every subset $A \subset U \times \mathbb{R}^n \setminus \{0\}$.

A proof of this result can be found in [12, Theorem 39]. A direct consequence of this theorem is the following corollary.

Corollary 2.2.14 (Pullback by diffeomorphism)**.** *Let $O, U \subset \mathbb{R}^d$ be open sets and let $f \in C^\infty(O, U)$ be as in the hypotheses of Theorem 2.2.13. If f is a diffeomorphism as per Definition 2.2.2, then $f^*u \in \mathscr{D}'(O)$ is well-defined for every $u \in \mathscr{D}'(U)$. Furthermore, for every $\alpha \in \mathbb{R}$,*

$$WF^\alpha(f^*u) = f^*[WF^\alpha(u)].$$

A proof of this result can be found in [12, Theorem 40].

Remark 2.2.19. We wish to note that the result of Corollary 2.2.14 also holds whenever $f \in C^\infty(O, U)$ is a local diffeomorphisms as per Definition 2.2.2. In fact, the proof of [12, Theorem 40] is carried out pointwise, *i.e.*, it is shown that, for every $(y, \eta) \in WF^\alpha(f^*u)$, one has that $(y, \eta) \in f^*[WF^\alpha(u)]$ and viceversa. The proof of the first implication requires choosing a properly supported $A \in \Psi^0(U)$ elliptic at $(f(x), (f'(x)^T)^{-1}\xi) \notin WF^\alpha(u)$ such that $Au \in B^{\alpha, loc}_{\infty,\infty}(U)$. Subsequently, one constructs the elliptic ΨDO of order 0

$$A_f : u \mapsto f^*\{A[(f^{-1})^* u]\},$$

for every $u \in \mathscr{D}'(U)$. The hypothesis of f being a diffeomorphism is used thrice, to have a well-defined distribution f^*u as $\mathscr{N}_f = \varnothing$, to rewrite $f^*[WF^\alpha(u)]$ as the set of points $(f^{-1}(y), \xi) \in O$ such that $(y, (f'(x)^T)^{-1}\xi) \in WF^\alpha(u)$ and to define the ΨDO $A_f = f^* \circ A \circ (f^{-1})^*$. However, all of these properties remain unchanged if f is only a local diffeomorphism after an appropriate localization. Firstly, if f is a local diffeomorphism, then $f'(x)$ is an invertible matrix at every $x \in O$, see [39, Theorem 9.24], which implies that $\mathscr{N}_f = \varnothing$. Thus, by Theorem 2.2.13, f^*u is a well-defined distribution over O for every $u \in \mathscr{D}'(U)$. Then, as f is invertible over a neighbourhood of x for every $x \in O$, see Definition 2.2.2, the second property also follows. Lastly, we note that both ΨDOs A and A_f need only be defined and elliptic on an arbitrary neighbourhood of the point $(f(x), (f'(x)^T)^{-1}\xi)$. Thus, one may take it to be small enough for f to be bijective onto its image. For the reverse implication, the same reasoning yields that the proof remains valid for $f \in C^\infty(O, U)$ a local diffeomorphism. □

Theorem 2.2.15 (Microlocality and regularity). *Let $A \in \Psi^m(\Omega)$ be properly supported as per Remark 2.1.11, let $u \in \mathscr{D}'(\Omega)$ and let $\alpha \in \mathbb{R}$ be fixed. Let $WF'(A)$ be as per Definition 2.2.7 and let $WF^\alpha(u)$ be as per Definition 2.2.10. Then one has that*

(MLB) $WF^{\alpha-m}(Au) \subset WF'(A) \cap WF^\alpha(u)$.;

(MLCB) $WF^\alpha(u) \subset Char(A) \cup WF^{\alpha-m}(Au)$;

(MLEB) if A is elliptic as per Remark 2.1.19, then $WF^\alpha(u) = WF^{\alpha-m}(Au)$.

Proofs of these results can be found in [12, Propositions 42, 42] and [12, Corollary 44].

In analogy with Theorems 2.2.7 and 2.2.9, analogous results can be formulated in terms of the Besov wavefront set.

Theorem 2.2.16 (Besov wavefront set of tensor product). *Let $u, v \in \mathscr{D}'(\Omega)$ as per Definition 1.2.1 and let $WF^\alpha(u)$ and $WF^\beta(v)$ be as per Definition 2.2.10 for every $\alpha, \beta \in \mathbb{R}$. Then,*

$$WF^{\alpha+\beta}(u \otimes v) \subset WF_0^\alpha(u) \times WF(v) \cup WF(u) \times WF_0^\beta(v),$$

where $WF(\cdot)$ is as per Definition 2.2.4 and $WF_0^\alpha(\cdot) \doteq WF^\alpha(\cdot) \cup supp(\cdot) \times \{0\}$. Furthermore, for $\gamma \doteq \min\{\alpha, \beta, \alpha + \beta\}$,

$$WF^\gamma(u \otimes v) \subset WF^\alpha(u) \times WF_0(v) \cup WF_0(u) \times WF^\beta(v).$$

A proof of this result can be found in [12, Proposition 45].

Theorem 2.2.17 (Product of distributions)**.** *Let $u, v \in \mathscr{D}'(\Omega)$ as per Definition 1.2.1 and let $WF^\alpha(u)$ and $WF^\beta(v)$ be as per Definition 2.2.10 for every $\alpha, \beta \in \mathbb{R}$. If for every $(x,\xi) \in \Omega \times \mathbb{R}^d \setminus \{0\}$ there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha+\beta > 0$ such that*

$$(x,\xi) \notin WF^\alpha(u) \cup -WF^\beta(v),$$

where $-WF^\beta(v) \doteq \{(y,\eta) \mid (y,-\eta) \in WF^\beta(v)\}$, then the pointwise product $uv \doteq \Delta^*(u \otimes v)$, see Remark 2.2.15, is a distribution over Ω. Furthermore, for $\gamma = \min\{\alpha, \beta, \alpha+\beta\}$,

$$\begin{aligned} WF^\gamma(uv) \subset \{(x, \xi+\eta) \mid (x,\xi) \in WF^\alpha(u), (x,\eta) \in WF_0(v) \\ \text{or } (x,\xi) \in WF_0(u), (x,\eta) \in WF^\beta(v)\}, \end{aligned}$$

where $WF_0(\cdot) = WF(\cdot) \cup supp(\cdot) \times \{0\}$.

A proof of this result can be found in [12, Theorem 46].

Remark 2.2.20. The additional condition on the indexes α and β in Theorem 2.2.17 are due to the fact that, in general, elements of $B^\alpha_{\infty,\infty}(\mathbb{R}^d)$ as per Definition 1.3.3 are themselves distributions. Thus, they cannot be multiplied unless additional conditions are imposed similarly to the case of Remark 1.2.26. The condition $\alpha+\beta > 0$ is a consequence of the renowned Young's theorem on the product of Besov distributions, see [17, Theorem 2.85]. □

We shall now review the concept of Hörmander's topology on spaces of distributions with a prescribed smooth wavefront set. The results we recall in this second part of this subsection are what we aim to generalize in Chapter 3.

Definition 2.2.11. *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $\Gamma \subset \Omega \times \mathbb{R}^d \setminus \{0\}$ be a closed, conic set as per Definition 2.2.1. We call $\mathscr{D}'_\Gamma(\Omega)$ to be the collection of all $u \in \mathscr{D}'(\Omega)$ such that $WF(u) \subset \Gamma$ as per Definition 2.2.4.*

The space $\mathscr{D}'_\Gamma(\Omega)$ can be equipped with a natural topology induced by a a collection of appropriately constructed seminorms through the following result.

Lemma 2.2.18. *Let $\mathscr{D}'_\Gamma(\Omega)$ be as per Definition 2.2.11. For every $\varphi \in \mathscr{D}(\Omega)$, $V \subset \mathbb{R}^d \setminus \{0\}$ closed cone as per Definition 2.2.1 and $\nu > 0$, let*

$$q^\nu_{\varphi,V}(u) \doteq \sup_{\xi \in V} |\langle \xi \rangle^\nu \widehat{\varphi u}(\xi)| \in [0, +\infty],$$

for every $u \in \mathscr{D}'(\Omega)$. Then, $u \in \mathscr{D}'_\Gamma(\Omega)$ if and only if $q^\nu_{\varphi,V}(u) < +\infty$ for $\nu > 0$ and for every $\varphi \in \mathscr{D}(\Omega)$ and $V \subset \mathbb{R}^d \setminus \{0\}$ such that $supp(\varphi) \times V \cap \Gamma = \varnothing$.

Definition 2.2.12. *Let $\mathscr{D}'_\Gamma(\Omega)$ be as per Definition 2.2.11. For every $\nu > 0$ and for every $\varphi \in \mathscr{D}(\Omega)$ and $V \subset \mathbb{R}^d \setminus \{0\}$ such that $\operatorname{supp}(\varphi) \times V \cap \Gamma = \varnothing$, let $q^\nu_{\varphi,V}$ be as per Lemma 2.2.18. Then, the* ***Hörmander's topology on $\mathscr{D}'_\Gamma(\Omega)$*** *is the locally convex vector topology, see Definition 1.1.6, induced by the seminorms $q^\nu_{\varphi,V}$ along with any continuous seminorm $\mathfrak{p}$ on $\mathscr{D}'(\Omega)$.*

Remark 2.2.21. By Remark 1.1.10, $\mathfrak{p}$ is a continuous seminorm on $\mathscr{D}'(\Omega)$ if and only if there exists a bounded set $B \subset \mathscr{D}(\Omega)$ such that

$$\mathfrak{p}(u) = \sup_{\chi \in B} |\langle u, \chi \rangle|.$$

Thus, in the following, we shall denote every continuous seminorm on $\mathscr{D}'(\Omega)$ by $\mathfrak{p}_B$, where B is the associated bounded subset of $\mathscr{D}(\Omega)$. Furthermore, in the following we shall denote the family of all continuous seminorms on $\mathscr{D}'(\Omega)$ by $\mathfrak{sn}(\Omega)$. Analogously, we shall denote the family of seminorms $q^\nu_{\varphi,V}$ as per Definition 2.2.12 on $\mathscr{D}'_\Gamma(\Omega)$ by $\mathfrak{sn}_\Gamma(\Omega)$. □

Remark 2.2.22. In [40], the authors distinguish between Hörmander's topology and normal topology. In their work, the former is induced by seminorms of the form $q^\nu_{\varphi,V}$ as per Definition 2.2.12 together with seminorms generated by individual elements of $\mathscr{D}(\Omega)$ rather than by bounded sets, namely

$$\mathscr{D}'(\Omega) \ni u \mapsto \mathfrak{p}_\varphi(u) \doteq |\langle u, \varphi \rangle| \in [0, +\infty),$$

for every $\varphi \in \mathscr{D}(\Omega)$, whereas the latter coincides with our notion of Hörmander's topology. However, as we shall only be interested in the latter and we will not need to distinguish between the two, we shall retain this nomenclature to emphasize the connection to Hörmander's work in [4].

Many topological properties of $\mathscr{D}'_\Gamma(\Omega)$ equipped with its Hörmander's topology are consequences of the following result.

Proposition 2.2.19. *Let $\mathscr{D}'_\Gamma(\Omega)$ be as per Definition 2.2.11, endowed with its Hörmander's topology as per Definition 2.2.12. Then, $\mathscr{D}'_\Gamma(\Omega)$ is isomorphic, see Definition 1.1.4, to a closed subspace of $\mathscr{D}'(\Omega) \times \left(\mathscr{S}(\mathbb{R}^d)\right)^{\mathbb{N}}$.*

A proof of this result can be found in [41, Corollary 13].

Remark 2.2.23. Proposition 2.2.19 allows us to deduce many topological properties for $\mathscr{D}'_\Gamma(\Omega)$. Firstly, as $\mathscr{D}'(\Omega)$ is complete, see Definition 1.1.15, and $\left(\mathscr{S}(\mathbb{R}^d)\right)^{\mathbb{N}}$ is Fréchet, see Definition 1.1.16, one also has that $\mathscr{D}'_\Gamma(\Omega)$ is complete. Then, by Theorem 1.2.1, the space $\mathscr{D}'(\Omega)$ is Montel, see Definition 1.1.10. Furthermore, the space $\mathscr{S}(\mathbb{R}^d)$, see Definition 1.2.7, is also Montel, see [21, Chapter 3.15, Example 4], and countable Cartesian product of Montel spaces is Montel, see [14, Chapter 27.2, (4)]. Thus, by Proposition 2.2.19, $\mathscr{D}'_\Gamma(\Omega)$ is isomorphic to a closed subspace of a Montel space, which entails that it is semi-Montel, *i.e.*, it possesses the Heine-Borel property, and semi-reflexive as per Definition 1.1.11. However, while $\mathscr{D}'(\Omega) \times \left(\mathscr{S}(\mathbb{R}^d)\right)^{\mathbb{N}}$ is both bornological and barrelled on account of being the countable Cartesian product of bornological, barrelled spaces, see [14, Chapter 28.4, (4)] and [14, Chapter 27.1, (5)], $\mathscr{D}'_\Gamma(\Omega)$ is neither bornological nor barrelled unless $\Gamma = \Omega \times \mathbb{R}^d \setminus \{0\}$ or $\Gamma = \varnothing$, see [41, Corollary 36].

Proposition 2.2.20 (Density of $\mathscr{D}(\Omega)$). *Let $\mathscr{D}'_\Gamma(\Omega)$ be as per Definition 2.2.11. Then, for every $u \in \mathscr{D}'_\Gamma(\Omega)$, there exists $(u_j)_{j\in\mathbb{N}} \subset \mathscr{D}(\Omega)$ such that $u_j \to u$ with respect to Hörmander's topology as per Definition 2.2.12.*

A proof of this result can be found in [4, Theorem 8.2.3].

Remark 2.2.24. As $\mathscr{D}'_\Gamma(\Omega)$ is not reflexive, see Remark 2.2.23, the identification of its dual space is a delicate process. If $\Gamma = \varnothing$, $\mathscr{D}'_\Gamma(\Omega)$ as per Definition 2.2.11 reduces to $\mathscr{E}(\Omega)$, whereas if $\Gamma = \Omega \times \mathbb{R}^d \setminus \{0\}$, it reduces to $\mathscr{D}'(\Omega)$. In both of these extremal cases, the dual space of $\mathscr{D}'_\Gamma(\Omega)$ as per Definition 1.1.4 is well-known, see Section 1.1.2. Furthermore, for any $\Gamma_1, \Gamma_2 \subset \Omega \times \mathbb{R}^d \setminus \{0\}$ closed conic sets as per Definition 2.2.1, if $\Gamma_1 \subset \Gamma_2$, then $\mathscr{D}'_{\Gamma_1}(\Omega) \subset \mathscr{D}'_{\Gamma_2}(\Omega)$. If $\mathfrak{sn}_\Gamma(\Omega)$ is the collection of all seminorms of type $q^{\nu}_{\varphi,V}$ as per Definition 2.2.12, then one also has that $\mathfrak{sn}_{\Gamma_2}(\Omega) \subset \mathfrak{sn}_{\Gamma_1}(\Omega)$, which implies that the immersion $\mathscr{D}'_{\Gamma_1}(\Omega) \hookrightarrow \mathscr{D}'_{\Gamma_2}(\Omega)$ is continuous. In particular, one has the continuous immersions

$$\mathscr{E}(\Omega) \hookrightarrow \mathscr{D}'_{\Gamma_1}(\Omega) \hookrightarrow \mathscr{D}'_{\Gamma_2}(\Omega) \hookrightarrow \mathscr{D}'(\Omega).$$

Thus, one also has that

$$\mathscr{D}(\Omega) \hookrightarrow \left(\mathscr{D}'_{\Gamma_2}(\Omega)\right)' \hookrightarrow \left(\mathscr{D}'_{\Gamma_1}(\Omega)\right)' \hookrightarrow \mathscr{E}'(\Omega).$$

This entails in particular that, for any non-trivial Γ, the dual of $\mathscr{D}'_\Gamma(\Omega)$ is a space of distributions with compact support, see Remark 1.2.9. Furthermore, the continuous inclusion $\left(\mathscr{D}'_{\Gamma_2}(\Omega)\right)' \hookrightarrow \left(\mathscr{D}'_{\Gamma_1}(\Omega)\right)'$ for $\Gamma_1 \subset \Gamma_2$ hints that the construction of the dual of $\mathscr{D}'_\Gamma(\Omega)$ can be achieved as a strictly increasing, inductive limit of functional spaces, see Definition 1.1.20 and Examples 1.1.11 and 1.1.12. Thus, we have the following construction, which we shall adapt in Chapter 3 for the Besov case. □

Let $\Omega \subset \mathbb{R}^d$ be an open set and let $\mathfrak{K}_\Omega$ be the collection of its compact subsets. Let $\mathfrak{V}(\Omega)$ be the collection of closed conic subsets of $\Omega \times \mathbb{R}^d \setminus \{0\}$ as per Definition 2.2.1 and let W be an open conic subset of $\Omega \times \mathbb{R}^d \setminus \{0\}$.

Let $\mathfrak{Wk}(W)$ be the collection of pairs $(K,L) \in \mathfrak{K}_\Omega \times \mathfrak{V}_\Omega$ such that $L \subset W$ and $\mathrm{pr}_1(L) \subset K$, where $\mathrm{pr}_1 : \Omega \times \mathbb{R}^d \to \Omega$ is the projection onto the first factor. Then, for every $(K,L) \in \mathfrak{Wk}(W)$, let

$$\mathscr{E}'_{K,L}(\Omega) \doteq \{u \in \mathscr{E}'(\Omega) \mid supp(u) \subset K \text{ and } WF(u) \subset L\}.$$

We equip the space $\mathfrak{Wk}(W)$ with the ordering $\leqq$ given by $(K_1, L_1) \leqq (K_2, L_2)$ if and only if $K_1 \subseteq K_2$ and $L_1 \subseteq L_2$. Furthermore, we equip each space $\mathscr{E}'_{K,L}(\Omega)$ with the topology induced by $\mathscr{D}'_L(\Omega)$ as its closed subspace, that is, with the topology induced by the families of seminorms $\mathfrak{sn}(\Omega)$ and $\mathfrak{sn}_\Gamma(\Omega)$ as per Remarks 2.2.21 and 2.2.23. For every $\alpha = (K_\alpha, L_\alpha), \beta = (K_\beta, L_\beta) \in \mathfrak{Wk}(W)$ with $\beta \leqq \alpha$, let $\pi_{\alpha,\beta} : \mathscr{E}'_{K_\beta,L_\beta}(\Omega) \to \mathscr{E}'_{K_\alpha,L_\alpha}(\Omega)$ be the natural immersion as per Remark 2.2.24. Then, we may define the following.

Definition 2.2.13. *Let $\mathfrak{Wk}(W)$ and $\mathscr{E}'_{K,L}(\Omega)$, $(K,L) \in \mathfrak{Wk}(W)$, be as per the above construction for some fixed open conic set $W \subset \Omega \times \mathbb{R}^d \setminus \{0\}$, see Definition 2.2.1. We denote by $\mathscr{E}'_W(\Omega)$ the inductive limit space as per Definition 1.1.20 of the family $\{\mathscr{E}'_{K,L}(\Omega) \mid (K,L) \in \mathfrak{Wk}(W)\}$ with respect to the inductive linking maps $\{\pi_{\alpha,\beta} \mid \alpha, \beta \in \mathfrak{Wk}(W),\ \beta \leqq \alpha\}$.*

Then, the following result holds.

Theorem 2.2.21. *Let $\Gamma \subset \Omega \times \mathbb{R}^d \setminus \{0\}$ be a closed conic set as per Definition 2.2.1 and let Λ be the complement in $\Omega \times \mathbb{R}^d \setminus \{0\}$ of $-\Gamma \doteq \{(x, -\xi) \mid (x,\xi) \in \Gamma\}$. Let $\mathscr{D}'_\Gamma(\Omega)$ and $\mathscr{E}'_\Lambda(\Omega)$ be as per Definitions 2.2.11 and 2.2.13 respectively. Then, $\mathscr{E}'_\Lambda(\Omega)$ is the dual space as per Definition 1.1.4 of $\mathscr{D}'_\Gamma(\Omega)$.*

A proof of this result can be found in [41, Proposition 7].

We conclude this subsection with the structural properties of the space $\mathscr{D}'_\Gamma(\Omega)$, as for the Besov wavefront set. In particular, we recall the continuity properties of the tensor product, pullback and pointwise multiplication between the appropriately chosen spaces of distributions with a prescribed smooth wavefront set.

Theorem 2.2.22 (Continuity of the tensor product)**.** *Let $O \subset \mathbb{R}^d$ and $U \subset \mathbb{R}^n$ be open sets, let $L \subset O \times \mathbb{R}^d \setminus \{0\}$ and $M \subset U \times \mathbb{R}^n \setminus \{0\}$ be closed conic sets as per Definition 2.2.1 and let $L \otimes M$ be the set*

$$L \otimes M \doteq (L \times M) \cup (O \times \{0\} \times M) \cup (L \times U \times \{0\}).$$

Then, the tensor product map

$$\otimes : \mathscr{D}'(O) \times \mathscr{D}'(U) \to \mathscr{D}'(O \times U),$$

see Definition 1.2.6, restricts to a hypocontinuous map

$$\otimes : \mathscr{D}'_L(O) \times \mathscr{D}'_M(U) \to \mathscr{D}'_{L\otimes M}(O \times U),$$

see Definition 1.1.14.

A proof of this result can be found in [40, Theorem 5.5]

Theorem 2.2.23 (Continuity of the pullback)**.** *Let $O \subset \mathbb{R}^d$ and $U \subset \mathbb{R}^n$ be open sets and let $f \in C^\infty(O,U)$. Let $\Gamma \subset U \times \mathbb{R}^n \setminus \{0\}$ be a closed conic set as per Definition 2.2.1, let $\mathscr{N}_f$ be as per Theorem 2.2.8 and let*

$$f^*\Gamma \doteq \{(x, f'(x)^T\eta) \mid (f(x),\eta) \in \Gamma\},$$

where $f'(x)$ is the Jacobian matrix of f evaluated at $x \in O$ and $f'(x)^T$ is its transpose matrix. Then, the pullback by f as per Definition 2.2.3 extends uniquely to a linear, continuous map $f^* \in \mathfrak{L}(\mathscr{D}'_\Gamma(U), \mathscr{D}'_{f^*\Gamma}(O))$ with respect to their Hörmander's topology as per Definition 2.2.12.

A proof of this result can be found in [40, Proposition 6.1].

Theorem 2.2.24 (Continuity of pointwise product)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $\Gamma_1, \Gamma_2 \subset \Omega \times \mathbb{R}^d \setminus \{0\}$ be closed conic sets as per Definition 2.2.1 such that $\Gamma_1 \cap -\Gamma_2 = \emptyset$, where $-\Gamma_2 \doteq \{(x,-\eta) \mid (x,\eta) \in \Gamma_2\}$. Let $\Gamma \subset \Omega \times \mathbb{R}^d \setminus \{0\}$ be the set*

$$\Gamma \doteq \{(x, \xi+\eta) \mid (x,\xi) \in \Gamma_1,\ (x,\eta) \in \Gamma_2\} \cup \Gamma_1 \cup \Gamma_2.$$

Then, the pointwise product

$$\mathscr{D}'_{\Gamma_1}(\Omega) \times \mathscr{D}'_{\Gamma_2}(\Omega) \ni (u,v) \mapsto uv \doteq \Delta^*(u \otimes v) \in \mathscr{D}'_\Gamma(\Omega),$$

where $\Delta^*(u \otimes v)$ is as per Remark 2.2.15, is hypocontinuous as per Definition 1.1.14.

A proof of this result can be found in [40, Theorem 7.1].

Chapter 3

Spaces of distributions with a prescribed Besov wavefront set

> No one stands on their own two feet. If they look like they do, it's because they're standing on top of people you cannot see.
>
> Bolaire Lathalia - Critical Role Campaign 4

In this chapter, we shall construct an analogue of Hörmander's topology reviewed in Subsection 2.2.2 for spaces of distributions with a prescribed Besov wavefront set as per Definition 2.2.10. We shall then prove a collection of results regarding the topological properties of these spaces equipped with their own Hörmander-Besov topology.

In Section 3.1, we shall characterize a Hörmander-type topology for distributions with a prescribed Besov wavefront set by means of appropriately chosen seminorms in analogy with Definition 2.2.12. We refer to this as the **Hörmander-Besov topology**. In Section 3.2, then, we shall present some key topological properties for the spaces of distributions endowed with their Hörmander-Besov topology by means of a suitable identification of these sets as subsets of better-known functional spaces. Lastly, in Section 3.3 we shall construct a microlocal variant of the Besov duality, see Theorem 1.3.5, and we shall prove it to be a hypocontinuous duality between suitably chosen spaces of distributions with a prescribed Besov wavefront set.

3.1 Hörmander-Besov seminorms

In this section, we shall construct the seminorm topology that we shall endow our spaces of distributions with. This structure, along with the characterization of its properties, will be an adaptation to the Besov setting of the approach discussed in [41] and [42]. Thus, we fix the following.

Definition 3.1.1. *Let $\Omega \subset \mathbb{R}^d$ be an open set, let $L \subset \Omega \times \mathbb{R}^d \setminus \{0\}$ be a closed conic set as per Definition 2.2.1 and let $\alpha \in \mathbb{R}$ be fixed. We denote by $\mathscr{D}_L'^{\alpha}(\Omega)$ the collection of distributions $u \in \mathscr{D}'(\Omega)$ as per Definition 1.2.1 such that $WF^\alpha(u) \subset L$, where $WF^\alpha(\cdot)$ is as per Definition 2.2.10.*

Remark 3.1.1. While we aim to replicate the properties of spaces of distributions with a prescribed wavefront set as in [40] and [42], we shall follow a slightly different approach motivated by the following. In Remark 2.2.16, we noted that the wavefront set construction for some Banach functional space $(X, \|\cdot\|_X)$, *e.g.*, the Sobolev-Bessel space $H^s(\mathbb{R}^d)$ as per Definition 2.1.7 or the Besov space $B^\alpha_{\infty,\infty}(\mathbb{R}^d)$ as per Definition 1.3.3, amounts to the statement that (y,η) is not an element of the X-wavefront set for a given $u \in \mathscr{D}'(\Omega)$ if and only if there exists $\varphi \in \mathscr{D}(\Omega)$ with $\varphi(y) \neq 0$ and $V \subset \mathbb{R}^d \setminus \{0\}$ a conic neighbourhood of η such that $\mathscr{F}^{-1}[1_V \mathscr{F}(\varphi u)] \in X$. As the aforementioned spaces are defined as collections of distributions $v \in \mathscr{D}'(\Omega)$ for which $\|v\|_X < +\infty$, this is equivalent to requiring that $\|\mathscr{F}^{-1}[1_V \mathscr{F}(\varphi u)]\|_X < +\infty$. Thus, the natural candidates for the seminorms on spaces of distributions whose wavefront set of type X is prescribed within a fixed region L are

$$u \mapsto \mathfrak{q}^X_{\varphi,V}(u) \doteq \|\mathscr{F}^{-1}[1_V \mathscr{F}(\varphi u)]\|_X \in [0,+\infty),$$

indexed by suitably chosen $\varphi \in \mathscr{D}(\Omega)$ and $V \subset \mathbb{R}^d \setminus \{0\}$. However, the functions 1_V are discontinuous and this can be problematic when dealing with functional spaces whose characterization depends on some regularity features. An equivalent approach can be formulated by replacing 1_V with $\psi \in S^0(\mathbb{R}^d)$, see Definition 2.1.1, chosen with an appropriate support condition. Thus, our collection of seminorms shall be indexed by the following collection. □

Definition 3.1.2. *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $L \subset \Omega \times (\mathbb{R}^d \setminus \{0\})$ be a closed conic set. A pair $(\varphi, \psi) \in \mathscr{D}(\Omega) \times S^0(\mathbb{R}^d)$, see Definitions 1.2.1 and 2.1.1, is said to be* ***admissible*** *if there exists a closed cone $V \subset \mathbb{R}^d \setminus \{0\}$ such that $supp(\psi) \subset V$ and $(supp(\varphi) \times V) \cap L = \emptyset$. We shall write V_ψ for the smallest closed cone V such that $supp(\psi) \subset V$. We shall denote by $\mathfrak{A}(\Omega, L)$ the collection of all admissible pairs $(\varphi, \psi) \in \mathscr{D}(\Omega) \times S^0(\mathbb{R}^d)$.*

In order to prove the characterization of $\mathscr{D}'^{\alpha}_{L}(\Omega)$ as per Definition 3.1.1 in terms of the appropriately constructed seminorms as per Remark 3.1.1, we shall require the following lemma.

Lemma 3.1.1. *Let $V \subsetneq \mathbb{R}^d \setminus \{0\}$ be an open conic set as per Definition 2.2.1 and let $W \subsetneq V$ be a closed conic subset of V. Then there exists $c \in (0,1)$ such that*

$$|\xi - \eta| > c(|\xi| + |\eta|),$$

for every $\eta \notin V$ and $\xi \in W$.

Proof of lemma. Let $\tilde{V} = V \cap \mathbb{S}^{d-1}$ and $\tilde{W} = W \cap \mathbb{S}^{d-1}$. For every $\xi \in \mathbb{R}^d \setminus \{0\}$, let $\omega_\xi = \frac{\xi}{|\xi|}$. As both $\mathbb{S}^{d-1} \setminus \tilde{V} = \mathbb{S}^{d-1} \cap \mathscr{C} V$ and $\tilde{W}$ are disjoint, compact sets, there exists $\tilde{c} \in (0,2)$ such that $|\omega_\xi - \omega_\eta| > \tilde{c}$, see [39, Exercise 21], or, equivalently, $\left| \frac{\xi}{|\xi|} - \frac{\eta}{|\eta|} \right| > \tilde{c}$, for any $\xi \in W$ and $\eta \notin V$, where $\tilde{c} < 2$ is a consequence of the triangle inequality and the fact that $|\omega_\xi| = 1$ for any $\xi \in \mathbb{R}^d$. For any $\xi \in W$ and $\eta \notin V$, we have that

$$\begin{aligned} |\xi - \eta| &= (|\xi| + |\eta|) \left| \omega_\xi \frac{|\xi|}{|\xi| + |\eta|} - \omega_\eta \frac{|\eta|}{|\xi| + |\eta|} \right| = \\ &= (|\xi| + |\eta|) \left(\frac{|\xi|^2}{(|\xi| + |\eta|)^2} + \frac{|\eta|^2}{(|\xi| + |\eta|)^2} - 2 \frac{|\xi||\eta|}{(|\xi| + |\eta|)^2} \omega_\xi \cdot \omega_\eta \right)^{1/2} = (\star), \end{aligned} \tag{3.1}$$

where we used that $\xi = \omega_\xi |\xi|$ for every $\xi \in \mathbb{R}^d$ in the first equality and the identity $|a + b| = \sqrt{(a+b)\cdot(a+b)}$, $a, b \in \mathbb{R}^d$, in the second equality. Furthermore, as $|\omega_\xi - \omega_\eta| > \tilde{c}$, we deduce that

$$\omega_\xi \cdot \omega_\eta = \frac{1}{2}(|\omega_\xi|^2 + |\omega_\eta|^2 - |\omega_\xi - \omega_\eta|^2) = 1 - \frac{1}{2}|\omega_\xi - \omega_\eta|^2 < 1 - \frac{\tilde{c}^2}{2}.$$

Substituting this into Equation (3.1) and through some algebraic manipulation, we obtain that

$$\begin{aligned} (\star) &\geq (|\xi| + |\eta|) \left(\frac{|\xi|^2}{(|\xi| + |\eta|)^2} + \frac{|\eta|^2}{(|\xi| + |\eta|)^2} - 2 \frac{|\xi||\eta|}{(|\xi| + |\eta|)^2} \left(1 - \frac{\tilde{c}^2}{2} \right) \right)^{1/2} = \\ &= (|\xi| + |\eta|) \left(\left(\frac{|\xi|}{|\xi| + |\eta|} - \frac{|\eta|}{|\xi| + |\eta|} \right)^2 + \frac{|\xi||\eta|}{(|\xi| + |\eta|)^2} \tilde{c}^2 \right)^{1/2} = \\ &= (|\xi| + |\eta|) \left(\left(\frac{|\xi|}{|\xi| + |\eta|} + \frac{|\eta|}{|\xi| + |\eta|} \right)^2 - (4 - \tilde{c}^2) \frac{|\xi||\eta|}{(|\xi| + |\eta|)^2} \right)^{1/2} = \\ &= (|\xi| + |\eta|) \left(1 - (4 - \tilde{c}^2) \frac{|\xi||\eta|}{(|\xi| + |\eta|)^2} \right)^{1/2} = (\boxdot). \end{aligned}$$

We observe that

$$0 \le \left(\frac{|\xi|}{|\xi|+|\eta|} - \frac{|\eta|}{|\xi|+|\eta|} \right)^2 = 1 - 4\frac{|\xi||\eta|}{(|\xi|+|\eta|)^2}, \tag{3.2}$$

which entails that

$$\frac{|\xi||\eta|}{(|\xi|+|\eta|)^2} \le \frac{1}{4}. \tag{3.3}$$

Thus, using that $\tilde{c}^2 < 4$, we obtain

$$(\boxdot) \ge (|\xi|+|\eta|)\left(1-(4-\tilde{c}^2)\frac{1}{4}\right)^{1/2} = (|\xi|+|\eta|)\frac{\tilde{c}}{2}. \tag{3.4}$$

Therefore, the sought result follows for any $c \in (0,1)$ with $c < \frac{\tilde{c}}{2}$. □

Proposition 3.1.2 (Seminorm characterization). *Let $\Omega \subset \mathbb{R}^d$ be an open set, let $L \subset \Omega \times \mathbb{R}^d \setminus \{0\}$ be a closed conic set and let $u \in \mathscr{D}'(\Omega)$. Then $u \in \mathscr{D}'^{\alpha}_L(\Omega)$ if and only if for every $(\varphi,\psi) \in \mathfrak{A}(\Omega, L)$, see Definition 3.1.2, and for every compact subset $U \subset \Omega$:*

$$\sup_{x\in U}\left|\int_{\mathbb{R}^d} \psi(\xi)\widehat{\varphi u}(\xi)\check{\beta}(\xi)e^{i\xi x}d\xi\right| < +\infty, \tag{3.5}$$

$$\sup_{\lambda\in(0,1]}\sup_{x\in U}\left|\int_{\mathbb{R}^d} \psi(\xi)\widehat{\varphi u}(\xi)\lambda^{-\alpha}\check{\gamma}(\lambda\xi)e^{i\xi x}d\xi\right| < +\infty, \tag{3.6}$$

for every $\beta \in \mathscr{D}(B(0,1))$ such that $\check{\beta}(0) \neq 0$ and $\gamma \in \mathscr{B}_{\lfloor\alpha\rfloor}$, see Definition 1.3.6.

Proof. Let $(\varphi,\psi) \in \mathfrak{A}(\Omega, L)$ as per Definition 3.1.2 and let $U \subset \Omega$ be a compact set. Let us first observe that Equations (3.5) and (3.6) are equivalent to requiring that

$$\sup_{x\in U}\left|\int_{\mathbb{R}^d} \psi(\xi)\widehat{\varphi u}(\xi)\check{\beta}(\xi)e^{i\xi x}d\xi\right| + \\ + \sup_{\lambda\in(0,1]}\sup_{x\in U}\left|\int_{\mathbb{R}^d} \psi(\xi)\widehat{\varphi u}(\xi)\lambda^{-\alpha}\check{\gamma}(\lambda\xi)e^{i\xi x}d\xi\right| < +\infty \tag{3.7}$$

This, in turn, is equivalent to

$$\left\|\mathscr{F}^{-1}[\psi\mathscr{F}(\varphi u)]\right\|_{B^{\alpha}_{\infty,\infty}(U)} < +\infty, \tag{3.8}$$

see Remark 1.3.11. Let us now assume that, given a fixed $u \in \mathscr{D}'(\Omega)$, for every admissible pair $\varphi \in \mathscr{D}(\Omega)$ and $\psi \in S^0(\mathbb{R}^d)$ and for every compact subset $U \subset \Omega$, Equation (3.8) holds true. We claim that this yields $WF^{\alpha}(u) \subset L$. Indeed, let

$(x_0, \xi_0) \notin L$ and let (φ, ψ) be an admissible pair such that $\varphi(x_0) = 1$ and $\psi(\lambda\xi_0) = 1$ for every $\lambda > R$, for some fixed $R > 0$. Let $\Phi \in \mathcal{D}(\Omega)$ be such that $\Phi|_U \equiv 1$ for some compact neighbourhood $U \subset \Omega$ of x_0 and let A be the ΨDO on Ω, see Definition 2.1.10, given by

$$Au \doteq \Phi\mathcal{F}^{-1}[\psi\mathcal{F}(\varphi u)]. \tag{3.9}$$

Then, A lies in $\Psi^0(\Omega)$, it is properly supported and it is elliptic at (x_0, ξ_0). Thus, as $Au \in B^{\alpha,loc}_{\infty,\infty}(\Omega)$, Proposition 2.2.10 implies that $(x_0, \xi_0) \notin WF^\alpha(u)$. Thus, the complement of L is a subset of the complement of WF^α, which is equivalent to saying that $WF^\alpha(u) \subset L$.

Let us now consider $u \in \mathcal{D}'^{\alpha}_L(\Omega)$. We shall prove that, for every choice of admissible (φ, ψ) and for every compact subset $U \subset \Omega$, we can decompose $\mathcal{F}^{-1}[\psi\mathcal{F}(\varphi u)]$ as a finite sum of elements of $B^{\alpha,loc}_{\infty,\infty}(\Omega)$ and smooth, rapidly decaying remainder terms, which, by Proposition 1.3.1, are also elements of $B^{\alpha,loc}_{\infty,\infty}(\Omega)$. This will imply that

$$\left\|\mathcal{F}^{-1}[\psi\mathcal{F}(\varphi u)]\right\|_{B^\alpha_{\infty,\infty}(U)} < +\infty,$$

which will entail the claim. Let $(\varphi, \psi) \in \mathfrak{A}(\Omega, L)$ be an admissible pair and fix $V \doteq V_\psi$ and $K \doteq supp(\varphi)$. We shall now construct a decomposition of φ and V in terms of the factors of Definition 2.2.10. Thus, as $K \times V \cap WF^\alpha(u) = \emptyset$, for every point $(y, \eta) \in (K \times V)$ there exist a non-negative function $\varphi_{(y,\eta)} \in \mathcal{D}(\Omega)$ and $V_{(y,\eta)} \subset \mathbb{R}^d \setminus \{0\}$ an open cone such that $\varphi_{(y,\eta)}(y) = 1$, $\eta \in V_{(y,\eta)}$ and

$$\|\mathcal{F}^{-1}[1_{V_{y,\eta}}\mathcal{F}(\varphi_{y,\eta}u)]\|_{B^\alpha_{\infty,\infty}(U)} < +\infty \tag{3.10}$$

for every compact subset $U \subset \Omega$. As $V \cap \mathbb{S}^{d-1}$ is compact, for every $y \in K$ there exist $\eta^{(l)} \in \mathbb{R}^d \setminus \{0\}$, $l = 1, \dots, k_y$, such that $V \cap \mathbb{S}^{d-1} \subset \bigcup_{l=1}^{k_y} V_{(y,\eta^{(l)})} \cap \mathbb{S}^{d-1}$, which implies that $V \subset \bigcup_{l=1}^{k_y} V_{(y,\eta^{(l)})}$. For each $y \in K$, let $\varphi_y \doteq \prod_{l=1}^{k_y} \varphi_{(y,\eta^{(l)})}$. Then, we have that $\varphi_y \in \mathcal{D}(\Omega)$ and $\varphi_y(y) = 1$ for every $y \in K$. As K is compact, there exist $y^{(j)} \in \Omega$, $j = 1, \dots, m$, such that $K \subset \bigcup_{j=1}^m \{x \in \Omega \mid \varphi_{y^{(j)}} > \frac{1}{2}\}$. Fix $\varphi_{j,l} \doteq \varphi_{(y^{(j)},\eta^{(l)})}$, $\varphi_j \doteq \varphi_{y^{(j)}}$ and $f_j \doteq \frac{\varphi}{\sum_{k=1}^m \varphi_k}\varphi_j$ for each $j \in \{1, \dots, m\}$ and $l \in \{1, \dots, k_j\}$, where $k_j = k_{y^{(j)}}$. Then, we have that $\varphi = \sum_{j=1}^m f_j$. Furthermore, for any $l \in \{1, \dots, k_j\}$, one has $f_j = \chi_{j,l}\varphi_{j,l}$, where $\chi_{j,l} = \frac{\varphi}{\sum_{k=1}^m \varphi_k}\prod_{l' \neq l} \varphi_{j,l'} \in \mathcal{D}(\Omega)$.

Let $\{\varrho_{j,l} \mid j \in \{1, \dots, m\},\ l \in \{1, \dots, k_j\}\}$ be a finite collection of symbols of order 0 on $\mathbb{R}^d$ as per Definition 2.1.1 such that $0 \leq \varrho_{j,l} \leq 1$, $supp(\varrho_{j,l}) \subsetneq V_{j,l}$ and let $\varrho_0 \in \mathcal{D}(B(0,1))$ be such that

$$\left.\left(\varrho_0 + \sum_{l=1}^{k_j} \varrho_{j,l}\right)\right|_{W_j} \equiv 1 \text{ for every } j \in \{1, \dots, m\}, \tag{3.11}$$

where $W_j = \bigcup_{l=1}^{k_j} V_{j,l}$. We fix $\psi_{j,l} \doteq \psi\varrho_{j,l}$ and $\psi_0 = \psi\varrho_0 \in \mathscr{D}(B(0,1))$ and we observe that

$$\psi = \sum_{l=1}^{k_j} \psi_{j,l} + \psi_0,$$

as $supp(\psi) \subset V \subset W_j$ for every $j \in \{1, \dots, m\}$. Therefore, we have that

$$\begin{aligned}\mathscr{F}^{-1}[\psi\mathscr{F}(\varphi u)] &= \mathscr{F}^{-1}[\psi_0\mathscr{F}(\varphi u)] + \sum_{j=1}^{m}\sum_{l=1}^{k_j} \mathscr{F}^{-1}[\psi_{j,l}\mathscr{F}(f_j u)] = \\ &= \mathscr{F}^{-1}[\psi_0\mathscr{F}(\varphi u)] + \sum_{j=1}^{m}\sum_{l=1}^{k_j} \mathscr{F}^{-1}[\psi_{j,l}\mathscr{F}(\chi_{j,l}\phi_{j,l}u)] = \\ &= \mathscr{F}^{-1}[\psi_0\mathscr{F}(\varphi u)] + (2\pi)^{d/2}\sum_{j=1}^{m}\sum_{l=1}^{k_j} \mathscr{F}^{-1}[\psi_{j,l}\hat{\chi}_{j,l} * \widehat{\phi_{j,l}u}],\end{aligned}$$

where we have used the above decompositions of φ and ψ in the first equality and Remark 1.2.29 in the third. Thus, we may write

$$\mathscr{F}^{-1}[\psi\mathscr{F}(\varphi u)] = \mathscr{F}^{-1}[\psi_0\mathscr{F}(\varphi u)] + (2\pi)^{d/2}\sum_{j=1}^{m}\sum_{l=1}^{k_j} \mathscr{F}^{-1}[A_{j,l} + B_{j,l}], \quad (3.12)$$

for $A_{j,l} = \psi_{j,l}\hat{\chi}_{j,l} * (1_{V_{j,l}}\widehat{\phi_{j,l}u})$ and $B_{j,l} = \psi_{j,l}\hat{\chi}_{j,l} * (1_{\mathscr{C}V_{j,l}}\widehat{\phi_{j,l}u})$, where $\mathscr{C}V_{j,l}$ is the complement of $V_{j,l}$ in $\mathbb{R}^d \setminus \{0\}$.

For the first term in Equation (3.12), we note that, as $\varphi u \in \mathscr{E}'(\Omega)$, $\mathscr{F}(\varphi u)$ is a smooth function with polynomial growth by Remark 1.2.28. Thus, as $\psi_0 \in \mathscr{D}(B(0,1))$, we have that $\psi_0\mathscr{F}(\varphi u) \in \mathscr{D}(\mathbb{R}^d)$. Hence, by Proposition 1.3.1, $\mathscr{F}^{-1}[\psi_0\mathscr{F}(\varphi u)] \in \mathscr{S}(\mathbb{R}^d) \hookrightarrow B^{\alpha}_{\infty,\infty}(\mathbb{R}^d)$. In particular, this implies that, for every compact subset $U \subset \Omega$,

$$C_0 \doteq \|\mathscr{F}^{-1}[\psi_0\mathscr{F}(\varphi u)]\|_{B^{\alpha}_{\infty,\infty}(U)} < +\infty. \quad (3.13)$$

For the term $A_{j,l}$ in Equation (3.12), one has that, for any $\Phi \in \mathscr{D}(\Omega)$,

$$\Phi\mathscr{F}^{-1}(A_{j,l}) = \Phi\mathscr{F}^{-1}\{\psi_{j,l}\mathscr{F}[\chi_{j,l}\mathscr{F}^{-1}(1_{V_{j,l}}\widehat{\varphi_{j,l}u})]\} = \mathscr{A}_{j,l}[\mathscr{F}^{-1}(1_{V_{j,l}}\widehat{\varphi_{j,l}u})], \quad (3.14)$$

where $\mathscr{A}_{j,l} = \mathrm{Op}(a)$ for $a(x,\xi,y) = \Phi(x)\psi_{j,l}(\xi)\chi_{j,l}(y)$, see Definition 2.1.2, is a properly supported pseudodifferential operator of class $\Psi^0(\Omega)$. We fix Φ such that its support is contained in a compact set $U' \supset U$ and such that $\Phi|_U \equiv 1$. Then, $\mathscr{A}_{j,l}$ is a continuous operator from $B^{\alpha,loc}_{\infty,\infty}(\Omega)$ to itself by Theorem 2.1.21

and $\mathscr{F}^{-1}(A_{j,l}) = \mathscr{A}_{j,l}[\mathscr{F}^{-1}(1_{V_{j,l}}\widehat{\varphi_{j,l}u})]|_U$. Exploiting Remark 1.3.11, we then have that there exist $U_1, ..., U_N$ compact subsets of Ω and $D_{j,l} > 0$ such that

$$C_{j,l} \doteq \|\mathscr{F}^{-1}(A_{j,l})\|_{B^{\alpha}_{\infty,\infty}(U)} \leq D_{j,l}\sum_{k=1}^{N}\|\mathscr{F}^{-1}(1_{V_{j,l}}\widehat{\varphi_{j,l}u})\|_{B^{\alpha}_{\infty,\infty}(U_k)} < +\infty, \quad (3.15)$$

as a consequence of Equation (3.10).

We shall now prove that $B_{j,l} \in \mathscr{S}(\mathbb{R}^d)$, where $B_{j,l}$ is as per Equation (3.12). To this avail, we first note that we only have to estimate derivatives of $B_{j,l}(\xi)$ for $\xi \in supp(\psi_{j,l}) \subsetneq V_{j,l}$. Furthermore, as $\varphi_{j,l}u \in \mathscr{E}'(\Omega)$, Remark 1.2.28 implies that there exist constants $E_{j,l}, a_{j,l} > 0$ such that $|\widehat{\varphi_{j,l}u}(x)| \leq E_{j,l}(1+|x|)^{a_{j,l}}$ for every $x \in \mathbb{R}^d$. Let us now consider $\xi \in supp(\psi_{j,l})$. By Lemma 3.1.1, there exists $c_{j,l} \in (0,1)$ such that $|\xi - \eta| > c_{j,l}(|\xi| + |\eta|)$ for every $\xi \in supp(\psi_{j,l})$ and $\eta \notin V_{j,l}$. As $\hat{\chi}_{j,l} \in \mathscr{S}(\mathbb{R}^d)$, this implies that for every $N > 0$ there exists $F_{j,l,N} > 0$ such that, for every $\eta \notin V_{j,l}$:

$$|\hat{\chi}_{j,l}(\xi-\eta)| \leq F_{N,j,l}(1+|\xi-\eta|)^{-a_{j,l}-d-1-N} \leq F_{N,j,l}(1+c_{j,l}|\xi|+c_{j,l}|\eta|)^{-a_{j,l}-d-1-N}. \quad (3.16)$$

Thus, we have that

$$\begin{aligned} |B_{j,l}(\xi)| = &\left|\psi_{j,l}(\xi)\int_{\mathbb{R}^d}\hat{\chi}_{j,l}(\xi-\eta)1_{\mathscr{C}V_{j,l}}(\eta)\widehat{\phi_{j,l}u}(\eta)d\eta\right| \leq \\ &\leq |\psi_{j,l}(\xi)|\int_{\mathscr{C}V_{j,l}}|\hat{\chi}_{j,l}(\xi-\eta)\widehat{\phi_{j,l}u}(\eta)|d\eta \leq \\ &\leq F_{N,j,l}E_{j,l}|\psi_{j,l}(\xi)|\int_{\mathscr{C}V_{j,l}}\frac{(1+|\eta|)^{a_{j,l}}}{(1+c_{j,l}|\xi|+c_{j,l}|\eta|)^{a_{j,l}+d+1+N}}d\eta = (\boxdot). \quad (3.17)\end{aligned}$$

As $|\xi|, |\eta| \geq 0$, one has that:

$$(1+c_{j,l}|\xi|+c_{j,l}|\eta|)^{-a_{j,l}-d-1-N} \leq (1+c_{j,l}|\eta|)^{-a_{j,l}-d}(1+c_{j,l}|\xi|)^{-N}. \quad (3.18)$$

Substituting this inequality into Equation (3.17), we obtain:

$$\begin{aligned}(\boxdot) &\leq F_{N,j,l}E_{j,l}|\psi_{j,l}(\xi)|\int_{\mathscr{C}V_{j,l}}\frac{(1+|\eta|)^{a_{j,l}}}{(1+c_{j,l}|\eta|)^{a_{j,l}+d+1}}d\eta(1+c_{j,l}|\xi|)^{-N} \leq \\ &\leq F_{N,j,l}E_{j,l}|\psi_{j,l}(\xi)|\left\|\frac{(1+|\cdot|)^{a_{j,l}}}{(1+c_{j,l}|\cdot|)^{a_{j,l}+d+1}}\right\|_{L^1(\mathbb{R}^d)}(1+c_{j,l}|\xi|)^{-N}. \quad (3.19)\end{aligned}$$

As $(1+|\cdot|)^{-d-1} \in L^1(\mathbb{R}^d)$ and $\frac{(1+|\cdot|)^{a_{j,l}}}{(1+c_{j,l}|\cdot|)^{-a_{j,l}-d-1}} \asymp (1+|\cdot|)^{-d-1}$ and $\psi_{j,l}$ is bounded by Definition 2.1.1, this implies that $\hat{\chi}_{j,l} * (1_{\mathscr{C}V_{j,l}} \widehat{\varphi_{j,l} u})$ is rapidly decreasing on $supp(\psi_{j,l})$, in the sense that for any $N \in \mathbb{N}$,

$$\sup_{\xi \in supp(\psi_{j,l})} |B_{j,l}(\xi)(1+c_{j,l}|\xi|)^N| < +\infty.$$

Then, for any multi-index $b \in \mathbb{N}^d$, by Proposition 1.2.11, item *(C2)*,

$$\partial^b \hat{\chi}_{j,l} * (1_{\mathscr{C}V_{j,l}} \widehat{\varphi_{j,l} u}) = (\partial^b \hat{\chi}_{j,l}) * (1_{\mathscr{C}V_{j,l}} \widehat{\varphi_{j,l} u}). \tag{3.20}$$

Since $\hat{\chi}_{j,l} \in \mathscr{S}(\mathbb{R}^d)$ implies that $\partial^b \hat{\chi}_{j,l} \in \mathscr{S}(\mathbb{R}^d)$, Equation (3.17) also shows that $\hat{\chi}_{j,l} * (1_{\mathscr{C}V_{j,l}} \widehat{\varphi_{j,l} u}) \in \mathscr{S}(\mathbb{R}^d)$. As $\psi_{j,l} \in S^0(\mathbb{R}^d)$ is bounded and its derivatives have negative polynomial growth, see Definition 2.1.1, Remark 1.2.16 implies that $B_{j,l} = \psi_{j,l} \hat{\chi}_{j,l} * (1_{\mathscr{C}V_{j,l}} \widehat{\varphi_{j,l} u}) \in \mathscr{S}(\mathbb{R}^d)$. The Fourier transform is an isomorphism from $\mathscr{S}(\mathbb{R}^d)$ into itself and $\mathscr{S}(\mathbb{R}^d)$ is continuously immersed into $B^{\alpha,loc}_{\infty,\infty}(\Omega)$ as per Proposition 1.3.1 and Remark 1.3.11. Therefore, for any compact subset $U \subset \Omega$, we have that

$$C'_{j,l} \doteq \|\mathscr{F}^{-1}(B_{j,l})\|_{B^{\alpha}_{\infty,\infty}(U)} < +\infty. \tag{3.21}$$

Plugging Equations (3.13), (3.15) and (3.21) into Equation (3.12), one obtains that:

$$\left\|\mathscr{F}^{-1}[\psi \mathscr{F}(\varphi u)]\right\|_{B^{\alpha}_{\infty,\infty}(U)} \leq C_0 + (2\pi)^{d/2} \sum_{j=1}^{m} \sum_{l=1}^{k_j} C_{j,l} + C'_{j,l} < +\infty, \tag{3.22}$$

which entails the sought result. □

As a consequence of Proposition 3.1.2, we may define the following.

Corollary 3.1.3. *For any $U \subset\subset \Omega$ and any $(\varphi, \psi) \in \mathfrak{A}(\Omega, L)$, see Definition 3.1.2, let*

$$p^{\alpha}_{\varphi,\psi,U}(u) \doteq \|\mathscr{F}^{-1}[\psi \mathscr{F}(\varphi u)]\|_{B^{\alpha}_{\infty,\infty}(U)}, \tag{3.23}$$

see Remark 1.3.11. Then, $p^{\alpha}_{\varphi,\psi,U}$ is a well-defined seminorm on $\mathscr{D}'^{\alpha}_{L}(\Omega)$.

Proof. By Proposition 3.1.2, $p^{\alpha}_{\varphi,\psi,U}(u) < +\infty$ for any $u \in \mathscr{D}'^{\alpha}_{L}(\Omega)$. Then, the map

$$\mathscr{D}'^{\alpha}_{L}(\Omega) \ni u \mapsto \mathscr{F}^{-1}[\psi \mathscr{F}(\varphi u)] \in B^{\alpha,loc}_{\infty,\infty}(\Omega)$$

is linear, while $\|\cdot\|_{B^{\alpha}_{\infty,\infty}(U)}$ is a seminorm for every compact subset $U \subset \Omega$, see Remark 1.3.11. Thus, $p^{\alpha}_{\varphi,\psi,U}$ is a well-defined seminorm as per Definition 1.1.8. □

Definition 3.1.3 (Hörmander-Besov topology). *Let $\mathscr{D}'^{\alpha}_L(\Omega)$ be as per Definition 3.1.1. We denote by $\mathfrak{sn}^{\alpha}_L(\Omega)$ the collection of seminorms $p^{\alpha}_{\varphi,\psi,U}$ as per Corollary 3.1.3 as (φ,ψ) varies in $\mathfrak{A}(\Omega, L)$ as per Definition 3.1.2 and U varies in $\mathfrak{K}_\Omega$, see Remark 1.1.8. The **Hörmander-Besov topology on $\mathscr{D}'^{\alpha}_L(\Omega)$** is the locally convex topology, see Definition 1.1.6, induced by the collections of seminorms $\mathfrak{sn}(\Omega)$ as per Remark 2.2.21 and $\mathfrak{sn}^{\alpha}_L(\Omega)$.*

Remark 3.1.2. While the definition of the Hörmander-Besov topology, see Definition 3.1.3, is the natural sequel to Corollary 3.1.3, proving the topological properties of $\mathscr{D}'^{\alpha}_L(\Omega)$ directly is not straightforward. Hence, in Section 3.2 we shall prove that this space can be identified with a closed subspace of another functional space whose properties are well-known. □

3.2 Topological properties

In this section, we shall derive some key topological properties of the space $\mathscr{D}'^{\alpha}_L(\Omega)$. Thus, we will prove that this space can be identified with a closed subspace of a suitable functional space. To this avail, we shall require the following construction. Let $\Omega \subset \mathbb{R}^d$ be an open set and let $L \subset \Omega \times \mathbb{R}^d \setminus \{0\}$ be a closed conic set as per Definition 2.2.1. Let $\{(x^{(j)}, \eta^{(j)}) \mid j \in \mathbb{N}\}$ be a countable and dense subset of the complement of L in $\Omega \times \mathbb{R}^d \setminus \{0\}$ and let $\omega^{(j)} \doteq \eta^{(j)}/|\eta^{(j)}|$ for every $j \in \mathbb{N}$. Let $L_S \doteq \{(x, \eta/|\eta|) \mid (x,\eta) \in L\}$ be the projection of L onto $\Omega \times \mathbb{S}^{d-1}$ and let

$$s_j \doteq \min\{1, \mathrm{dist}(x^{(j)}, \partial\Omega), \mathrm{dist}((x^{(j)}, \omega^{(j)}), L_S)\} > 0,$$

where we fix the convention that $\mathrm{dist}(x, \varnothing) = +\infty$ for every $x \in \mathbb{R}^d$. For any $k \in \mathbb{N}$, we set

$$\begin{gathered} O_{j,k} \doteq B(x^{(j)}, s_j/(5k)),\ O'_{j,k} \doteq B(x^{(j)}, s_j/(4k)),\ O''_{j,k} \doteq B(x^{(j)}, s_j/(3k)), \\ V_{j,k} \doteq \mathbb{R}_+ B(\omega^{(j)}, s_j/(5k)),\ V'_{j,k} \doteq \mathbb{R}_+ B(\omega^{(j)}, s_j/(4k)),\ V''_{j,k} \doteq \mathbb{R}_+ B(\omega^{(j)}, s_j/(3k)). \end{gathered}$$

We note that

$$\begin{aligned} &\overline{O}_{j,k} \times \overline{V}_{j,k} \subset O'_{j,k} \times V'_{j,k} \cup \{0\}, \\ &\overline{O'}_{j,k} \times \overline{V'}_{j,k} \subset O''_{j,k} \times V''_{j,k} \cup \{0\}, \\ &\overline{O''}_{j,k} \times \overline{V''}_{j,k} \cap L = \varnothing. \end{aligned}$$

For each pair $(j,k) \in \mathbb{N} \times \mathbb{N}$, we choose $\varphi_{j,k} \in \mathscr{D}(\mathbb{R}^d)$ and $\psi_{j,k} \in S^0(\mathbb{R}^d)$ such that

- 1. $0 \leq \varphi_{j,k} \leq 1$ and $0 \leq \psi_{j,k} \leq 1$
- 2. $\varphi_{j,k}|_{\overline{O}_{j,k}} \equiv 1$ and $supp(\varphi_{j,k}) \subset O'_{j,k}$
- 3. $\psi_{j,k}|_{\overline{V}_{j,k}\setminus B(0,2)} \equiv 1$ and $supp(\psi_{j,k}) \subset V'_{j,k} \setminus B(0,1)$

We note that, for every $(j,k) \in \mathbb{N} \times \mathbb{N}$, $(\varphi_{j,k}, \psi_{j,k}) \in \mathfrak{A}(\Omega, L)$ as per Definition 3.1.2.

Proposition 3.2.1 (Topological identification). *Let $\mathscr{D}'^{\alpha}_{L}(\Omega)$ be as per Definition 3.1.1. Let $O_{j,k}$, $O'_{j,k}$, $O''_{j,k}$, $V_{j,k}$, $V'_{j,k}$, $V''_{j,k}$, $\varphi_{j,k}$ and $\psi_{j,k}$ be as above. Let*

$$\mathscr{I} : \mathscr{D}'^{\alpha}_{L}(\Omega) \to \mathscr{D}'(\Omega) \times (B^{\alpha,loc}_{\infty,\infty}(\Omega))^{\mathbb{N}\times\mathbb{N}}, \tag{3.24}$$

be the map given by $\mathscr{I}(u) = (u, F_u)$, where $F_u(j,k) = \mathscr{F}^{-1}[\psi_{j,k}\mathscr{F}(\varphi_{j,k}u)]$ for every $j,k \in \mathbb{N}$. Then, $\mathscr{I}$ is a topological embedding, i.e., it is an isomorphism as per Definition 1.1.4 between $\mathscr{D}'^{\alpha}_{L}(\Omega)$ and $\mathscr{I}[\mathscr{D}'^{\alpha}_{L}(\Omega)] \subset \mathscr{D}'(\Omega) \times (B^{\alpha,loc}_{\infty,\infty}(\Omega))^{\mathbb{N}\times\mathbb{N}}$ endowed with its subspace topology. Furthermore, the image of $\mathscr{I}$ is closed.

Proof. To prove that $\mathscr{I}$ is a topological embedding, we shall show that it is continuous, injective and that its inverse, defined from $\mathscr{I}[\mathscr{D}'^{\alpha}_{L}(\Omega)]$ to $\mathscr{D}'^{\alpha}_{L}(\Omega)$, is continuous as well. The topology of $B^{\alpha,loc}_{\infty,\infty}(\Omega)$ is the Fréchet topology given by the seminorms $q_n(f) \doteq \|f\|_{B^{\alpha}_{\infty,\infty}(\Omega_n)}$, see Remark 1.3.11, where $\Omega_n \subset \Omega$ is compact for every $n \in \mathbb{N}$ and $\Omega = \bigcup_{n\in\mathbb{N}} \Omega_n$. We note that, for every $(j,k) \in \mathbb{N} \times \mathbb{N}$,

$$q_n(f_u(j,k)) = p^{\alpha}_{\varphi_{j,k},\psi_{j,k},\Omega_n}(u), \tag{3.25}$$

which implies that $\mathscr{I}$ is continuous. Furthermore, $\mathscr{I}$ is injective as it is linear and $\mathscr{I}(u) = 0$ if and only if $(u, F_u) = 0$, which implies $u = 0$. Therefore, to prove that $\mathscr{I}$ is a topological embedding, it is sufficient to prove that for every $(\phi, \psi) \in \mathfrak{A}(\Omega, L)$ and every compact subset $U \subset \Omega$ there exist $N \in \mathbb{N}$, $J \in \mathbb{N} \times \mathbb{N}$ of finite cardinality, $C > 0$ and $\mathfrak{p} \in \mathfrak{sn}(\Omega)$ such that

$$p^{\alpha}_{\phi,\psi,U}(u) \leq C \sum_{n=1}^{N} \sum_{(j,k)\in J} q_n(F_u(j,k)) + C\mathfrak{p}(u). \tag{3.26}$$

Let (ϕ, ψ) be an admissible pair with $K \doteq supp(\phi)$ and $V \doteq V_{\psi}$, see Definition 3.1.2, and let $U \subset\subset \Omega$ be fixed. As in the proof of Proposition 3.1.2, we shall decompose $\mathscr{F}^{-1}[\psi\mathscr{F}(\varphi u)]$ as a finite sum of terms. Then, when evaluating $\mathscr{F}^{-1}[\psi\mathscr{F}(\varphi u)]$ with respect to the local Besov seminorm $\|\cdot\|_{B^{\alpha}_{\infty,\infty}(U)}$, we shall either recover a seminorm of type q_n that evaluates $F_u(j,k)$ for finitely many n, j and k or a continuous seminorm on $\mathscr{D}'(\Omega)$. Thus, let us fix $\varepsilon > 0$ such that $supp(\phi) + \overline{B(0,\varepsilon)} \subset \Omega$ and $K'' \times V'' \cap L = \emptyset$, where we define

$$K' \doteq \left(supp(\phi) + \overline{B(0,\varepsilon/15)}\right),\ K'' \doteq \left(supp(\phi) + \overline{B(0,\varepsilon)}\right), \tag{3.27}$$

$$V' = \mathbb{R}_+\left(V \cap \mathbb{S}^{d-1} + \overline{B(0,\varepsilon/15)}\right),\ V'' \doteq \mathbb{R}_+\left(V \cap \mathbb{S}^{d-1} + \overline{B(0,\varepsilon)}\right). \tag{3.28}$$

We also set

$$\tilde{J} \doteq \{(j,k) \in \mathbb{N} \times \mathbb{N} \mid k \geq 15\varepsilon,\ \left(O_{j,k} \times B(\omega^{(j)}, s_j/5k)\right) \cap (K \times V) \neq \emptyset\}.^{1} \tag{3.29}$$

Then, as proven in [42, Proposition 3.7][2], $\tilde{J} \neq \emptyset$ and there exists a finite index set $J \subset \mathbb{N} \times \mathbb{N}$ such that

$$K \times (V \setminus \{0\}) \subset \bigcup_{(j,k)\in J} O_{j,k} \times V_{j,k} \subset \bigcup_{(j,k)\in \tilde{J}} \overline{O'_{j,k}} \times \overline{V'_{j,k}} \subset K' \times V'. \tag{3.30}$$

For any $(j,k) \in J$, we note that, per construction,

$$\sum_{(a,b)\in J} \varphi_{a,b}\Big|_{\bigcup_{(j,k)\in J} \overline{O}_{j,k}} \geq 1 \text{ and } \sum_{(a,b)\in J} \psi_{a,b}\Big|_{\bigcup_{(j,k)\in J} \overline{V}_{j,k}} \geq 1.$$

Thus, we may fix $\phi_{j,k} \doteq \frac{\phi}{\sum_{(a,b)\in J} \varphi_{a,b}} \varphi_{j,k}$ and $\tilde{\psi}_{j,k} \doteq \frac{\psi}{\sum_{(a,b)\in J} \psi_{a,b}} \psi_{j,k}$, which entails that $\phi = \sum_{(j,k)\in J} \phi_{j,k}$ and $\psi = \sum_{(j,k)\in J} \tilde{\psi}_{j,k} + \psi_0$, where ψ_0 is a smooth function supported inside $B(0,2)$. Furthermore, as proven in [42, Proposition 3.7], for every $(j,k) \in J$ there exists $m = m(j,k) \in \mathbb{N}$ such that $O'_{j,k} \subset O_{m,1}$ and $V'_{j,k} \subset V_{m,2}$. This, in turn, implies that $\phi_{j,k} = \phi_{j,k}\varphi_{m,1}$ and $\tilde{\psi}_{j,k} = \tilde{\psi}_{j,k}\psi_{m,2}$ for every $(j,k) \in J$. Using these decompositions as in the proof of Proposition 3.1.2, we obtain

$$\begin{aligned}
\mathscr{F}^{-1}[\psi\mathscr{F}(\phi u)] &= \sum_{(j,k)\in J} \left\{ \mathscr{F}^{-1}[\psi_0 \mathscr{F}(\phi_{j,k} u)] + \sum_{(a,b)\in J} \mathscr{F}^{-1}[\tilde{\psi}_{a,b} \mathscr{F}(\phi_{j,k} u)] \right\} = \\
&= \sum_{(j,k)\in J} \Bigg\{ \mathscr{F}^{-1}[\psi_0 \mathscr{F}(\phi_{j,k} u)] + \sum_{(a,b)\in J} \mathscr{F}^{-1}[\tilde{\psi}_{a,b}\psi_{m,2}\hat{\phi}_{j,k} * (\psi_{m,1}\widehat{\varphi_{m,1} u})] + \\
&\qquad + \sum_{(a,b)\in J} \mathscr{F}^{-1}[\tilde{\psi}_{a,b}\psi_{m,2}\hat{\phi}_{j,k} * ((1-\psi_{m,1})\widehat{\varphi_{m,1} u})] \Bigg\}
\end{aligned}$$

[1] Why 15? Well, because it works.
[2] And here is why 15 works.

Thus, we fix

$$\mathscr{F}^{-1}[\psi\mathscr{F}(\phi u)] = \sum_{(j,k)\in J}\left\{(1_{j,k}) + \sum_{(a,b)\in J}(2_{j,k,a,b}) + (3_{j,k,a,b})\right\}, \tag{3.31}$$

where

$$\begin{cases}(1_{j,k}) \doteq \mathscr{F}^{-1}[\psi_0\mathscr{F}(\phi_{j,k}u)],\\ (2_{j,k,a,b}) \doteq \mathscr{F}^{-1}[\tilde{\psi}_{a,b}\psi_{m,2}\hat{\phi}_{j,k} * (\psi_{m,1}\widehat{\varphi_{m,1}u})]\\ (3_{j,k,a,b}) \doteq \mathscr{F}^{-1}[\tilde{\psi}_{a,b}\psi_{m,2}\hat{\phi}_{j,k} * ((1-\psi_{m,1})\widehat{\varphi_{m,1}u})].\end{cases}$$

In order to control the term $(1_{j,k})$ in Equation (3.31), we note that, being ψ_0 smooth and compactly supported, $\mathscr{F}^{-1}[\psi_0\mathscr{F}(\phi_{j,k}u)] \in \mathscr{S}(\mathbb{R}^d)$, see the proof of Proposition 3.1.2. Furthermore, the following maps

$$\mathscr{D}'(\Omega) \ni u \mapsto \mathscr{F}(\phi_{j,k}u) \in \mathscr{E}(\mathbb{R}^d)$$
$$\mathscr{E}(\mathbb{R}^d) \ni v \mapsto \mathscr{F}^{-1}(\psi_0 v) \in \mathscr{S}(\mathbb{R}^d),$$

are continuous, see Remarks 1.2.16 and 1.2.28 and Proposition 1.2.5, item *(F3)*. Consequently, using the continuity of the embedding $\mathscr{S}(\mathbb{R}^d) \hookrightarrow B^{\alpha,loc}_{\infty,\infty}(\Omega)$, see Proposition 1.3.1 and Remark 1.3.11, we deduce that the map

$$\mathscr{D}'(\Omega) \ni u \mapsto \mathfrak{p}_0^{j,k}(u) \doteq \|\mathscr{F}^{-1}[\psi_0\mathscr{F}(\phi_{j,k}u)]\|_{B^{\alpha}_{\infty,\infty}(U)} \in [0,+\infty) \tag{3.32}$$

is a continuous seminorm.

In order to control the term $(2_{j,k,a,b})$ in Equation (3.31), we fix $\Phi \in \mathscr{D}(\Omega)$ such that $\Phi|_U \equiv 1$ and we fix $\mathscr{A}_{j,k,a,b} = \mathrm{Op}(a) \in \Psi^0(\Omega)$ for $a(x,\xi,y) = \Phi(x)\tilde{\psi}_{a,b}(\xi)\psi_{m,2}(\xi)\phi_{j,k}(y)$, see Definition 2.1.2. Then,

$$(2_{j,k,a,b}) = \mathscr{A}_{j,k,a,b}\{\mathscr{F}^{-1}[(\psi_{m,1}\widehat{\varphi_{m,1}u})]\}|_U = \mathscr{A}_{j,k,a,b}[F_u(m,1)]|_U.$$

As U is compact, there exists $N \in \mathbb{N}$ such that $U \subset \bigcup_{n=1}^N \Omega_n$. Thus, by continuity of $\mathscr{A}_{j,k,a,b}$ as an element of $\mathfrak{L}(B^{\alpha,loc}_{\infty,\infty}(\Omega))$, see Theorem 2.1.21, and by Remark 1.3.11, we deduce that there exists a constant $C_{j,k,a,b} > 0$ such that

$$\|(2_{j,k,a,b})\|_{B^{\alpha}_{\infty,\infty}(U)} \leq C_{j,k,a,b}\|\mathscr{F}^{-1}[(\psi_{m,1}\widehat{\varphi_{m,1}u})]\|_{B^{\alpha}_{\infty,\infty}(\bigcup_{n=1}^N\Omega_n)} \leq$$
$$\leq C_{j,k,a,b}\sum_{n=1}^N\|\mathscr{F}^{-1}[(\psi_{m,1}\widehat{\varphi_{m,1}u})]\|_{B^{\alpha}_{\infty,\infty}(\Omega_n)} \leq C_{j,k,a,b}\sum_{n=1}^N q_n(F_u(m,1)),$$

where in the second inequality we used the fact that

$$\sup_{x\in\bigcup_{n=1}^N\Omega_n}|f(x)| \leq \sum_{n=1}^N\sup_{x\in\Omega_n}|f(x)|.$$

Finally, to control the term $(3_{j,k,a,b})$, we note that, by construction, $supp(1-\psi_{m,1}) \subset \mathscr{C}(V_{m,1} \setminus B(0,2))$, whereas $supp(\psi_{m,2}) \subset V'_{m,2} \setminus B(0,1) \subsetneq V_{m,1} \setminus B(0,1)$. Decompose $\psi_{m,2}$ as $\psi_{m,2} = \psi^{(1)}_{m,2} + \psi^{(2)}_{m,2}$, where $\psi^{(1)}_{m,2}, \psi^{(2)}_{m,2}$ are smooth functions such that $supp(\psi^{(1)}_{m,2}) \subset V'_{m,2} \setminus B(0,2) \subsetneq V_{m,1} \setminus B(0,2)$ and $supp(\psi^{(2)}_{m,2}) \subset B(0,3)$. Let us first consider the term containing $\psi^{(2)}_{m,2}$. As $\tilde{\psi}_{a,b}\psi^{(2)}_{m,2} \in \mathscr{D}(\mathbb{R}^d)$, we note that the maps

$$\mathscr{D}'(\Omega) \ni u \mapsto (1-\psi_{m,1})\mathscr{F}(\varphi_{m,1}u) \in \mathscr{S}'(\mathbb{R}^d),$$

$$\mathscr{S}'(\mathbb{R}^d) \ni v \mapsto \hat{\phi}_{j,k} * v = \mathscr{F}[\phi_{j,k}\mathscr{F}^{-1}(v)] \in C^\infty(\mathbb{R}^d),$$

$$C^\infty(\mathbb{R}^d) \ni w \mapsto \mathscr{F}^{-1}(\tilde{\psi}_{a,b}\psi^{(2)}_{m,2}w) \in \mathscr{S}(\mathbb{R}^d) \hookrightarrow B^{\alpha,loc}_{\infty,\infty}(\Omega),$$

are continuous, see [21, Chapter 7.4, Proposition 4], [21, Chapter 3.12, p. 256], Proposition 1.2.9 and Remarks 1.2.6 and 1.3.11. Therefore, the map

$$\mathscr{D}'(\Omega) \ni u \mapsto \mathfrak{p}^{a,b,j,k}_1(u) \doteq \|\mathscr{F}^{-1}[\tilde{\psi}_{a,b}\psi^{(2)}_{m,2}\hat{\phi}_{j,k} * ((1-\psi_{m,1})\widehat{\varphi_{m,1}u})]\|_{B^\alpha_{\infty,\infty}(U)} \tag{3.33}$$

is a continuous seminorm.

Lastly, we take now into consideration the term containing $\psi^{(1)}_{m,2}$. We note that $supp(\psi^{(1)}_{m,2}) \subsetneq supp(1-\psi_{m,1})$. We claim that this implies that the map

$$\mathscr{D}'(\Omega) \ni u \mapsto \mathfrak{p}^{a,b,j,k}_2(u) \doteq \|\mathscr{F}^{-1}[\tilde{\psi}_{a,b}\psi^{(1)}_{m,2}\hat{\phi}_{j,k} * ((1-\psi_{m,1})\widehat{\varphi_{m,1}u})]\|_{B^\alpha_{\infty,\infty}(U)}. \tag{3.34}$$

is a continuous seminorm. To prove this, let $B \subset \mathscr{D}'(\Omega)$ be a bounded set. Then, by the Banach-Steinhaus Theorem 1.1.8 and Remark 1.2.28, there exist positive constants $C > 0, a > 0$ such that $|\widehat{\varphi_{m,1}u}(x)| \leq C(1+|x|)^a$ for every $x \in \mathbb{R}^d$ and every $u \in B$. Repeating the argument used in the proof of Proposition 3.1.2 regarding $B_{j,k}$, one has that, for every $N > 0$,

$$|\tilde{\psi}_{a,b}\psi^{(1)}_{m,2}\hat{\phi}_{j,k} * ((1-\psi_{m,1})\widehat{\varphi_{m,1}u})|(\xi) \leq C'_N \left\| \frac{(1+|\cdot|)^a}{(1+c|\cdot|)^{a+d+1}} \right\|_{L^1(\mathbb{R}^d)} (1+c|\xi|)^{-N} \tag{3.35}$$

for a given constant $c \in (0,1)$ that depends only on $\psi^{(1)}_{m,2}$ and $1-\psi_{m,1}$ and a given constant C'_N that depends on the Schwartz seminorms, see Remark 1.2.14, of $\hat{\phi}_{j,k}$ and the $L^\infty(\mathbb{R}^d)$-norm of $\tilde{\psi}_{a,b}\psi^{(1)}_{m,2}$. Since $\tilde{\psi}_{a,b}\psi^{(1)}_{m,2}\hat{\phi}_{j,k}$ is a Schwartz function as per Definition 1.2.7, an analogous estimate holds for derivatives of $\tilde{\psi}_{a,b}\psi^{(1)}_{m,2}\hat{\phi}_{j,k} * ((1-\psi_{m,1})\widehat{\varphi_{m,1}u})$ as in the proof of Proposition 3.1.2. Hence, for any $a, b \in \mathbb{N}^d$

$$\sup_{u\in B} \|\tilde{\psi}_{a,b}\psi^{(1)}_{m,2}\hat{\phi}_{j,k} * ((1-\psi_{m,1})\widehat{\varphi_{m,1}u})\|_{a,b} < +\infty, \tag{3.36}$$

see Remark 1.2.14. This implies that $\{\tilde{\psi}_{a,b}\psi^{(1)}_{m,2}\hat{\phi}_{j,k}*((1-\psi_{m,1})\widehat{\varphi_{m,1}u}) \mid u\in B\}$ is a bounded subset of $\mathscr{S}(\mathbb{R}^d)$. Thus, as $\mathscr{D}'(\Omega)$ is bornological, see Theorem 1.2.1, Theorem 1.1.4 implies that the map

$$\mathscr{D}'(\Omega)\ni u\mapsto \tilde{\psi}_{a,b}\psi^{(1)}_{m,2}\hat{\phi}_{j,k}*((1-\psi_{m,1})\widehat{\varphi_{m,1}u})\in\mathscr{S}(\mathbb{R}^d) \tag{3.37}$$

is continuous. Together with the continuous inclusion $\mathscr{S}(\mathbb{R}^d)\hookrightarrow B^{\alpha,loc}_{\infty,\infty}(\Omega)$, see Remark 1.3.11, this proves that $\mathfrak{p}_2^{a,b,j,k}$ as per Equation (3.34) is continuous.

To conclude, taking the $B^{\alpha}_{\infty,\infty}(U)$ norm of Equation (3.31) and noting that $\mathfrak{p}\doteq\sum_{(j,k)\in J}\left(\mathfrak{p}_0^{j,k}+\sum_{(a,b)\in J}\mathfrak{p}_1^{a,b,j,k}+\mathfrak{p}_2^{a,b,j,k}\right)\in\mathfrak{sn}(\Omega)$, we obtain that

$$p^{\alpha}_{\phi,\psi,U}(u)\le C\sum_{n=1}^{N}\sum_{(j,k)\in J}q_n(F_u(m(j,k),1))+C\mathfrak{p}(u), \tag{3.38}$$

where

$$C\doteq\max\left\{1,\sum_{(j,k)\in J}\sum_{(a,b)\in J}C_{j,k,a,n}\right\}. \tag{3.39}$$

This concludes the proof that $\mathscr{I}$ is a topological embedding.

We now claim that the image of $\mathscr{I}$ is closed in the strong topology. Let $(u_\mu,F_{u_\mu})_{\mu\in\Lambda}$ be a net in $Im[\mathscr{I}]\doteq\mathscr{I}[\mathscr{D}'^{\alpha}_L(\Omega)]$ which converges to (u,g) in the strong topology of $\mathscr{D}'(\Omega)\times\left(B^{\alpha,loc}_{\infty,\infty}(\Omega)\right)^{\mathbb{N}\times\mathbb{N}}$. Then $u_\mu\to u$ in $\mathscr{D}'(\Omega)$ and, for every $(j,k)\in\mathbb{N}\times\mathbb{N}$, $F_{u_\mu}(j,k)\to g(j,k)$ in $B^{\alpha,loc}_{\infty,\infty}(\Omega)$. We note that, as the map

$$\mathscr{D}'(\Omega)\ni v\mapsto F_v(j,k)=\mathscr{F}^{-1}[\psi_{j,k}\mathscr{F}(\varphi_{j,k}v)]\in\mathscr{S}'(\mathbb{R}^d)\hookrightarrow\mathscr{D}'(\Omega) \tag{3.40}$$

is continuous, see Proposition 1.2.9 and Remarks 1.2.16 and 1.3.1, $u_\mu\to u$ implies that $F_{u_\mu}(j,k)\to F_u(j,k)$ in $\mathscr{D}'(\mathbb{R}^d)$ for every $j,k\in\mathbb{N}$. Furthermore, as $B^{\alpha,loc}_{\infty,\infty}(\Omega)\hookrightarrow\mathscr{D}'(\Omega)$, $F_{u_\mu}(j,k)\to g(j,k)$ in $\mathscr{D}'(\Omega)$. As $\mathscr{D}'(\Omega)$ is Hausdorff, see Theorem 1.2.1, limits are unique. Thus, $g(j,k)=F_u(j,k)$ for every $(j,k)\in\mathbb{N}\times\mathbb{N}$, which proves the sought result. □

As a consequence of Proposition 3.2.1, we have the following topological properties.

Corollary 3.2.2. *For $j=1,2$, let $\Omega_j\subset\mathbb{R}^d$ be an open set, let $L_j\subset\Omega_j\times\mathbb{R}^d\setminus\{0\}$ be a closed conic set as per Definition 2.2.1 and let $\alpha_j\in\mathbb{R}$ be fixed. Then,*

(BWFI1) if $\Omega_1\subset\Omega_2$, $L_1=L_2=L$ and $\alpha_1=\alpha_2=\alpha$, then $\mathscr{D}'^{\alpha}_L(\Omega_2)\subset\mathscr{D}'^{\alpha}_L(\Omega_1)$ continuously;

(BWFI2) if $\Omega_1=\Omega_2=\Omega$, $L_1\subset L_2$ and $\alpha_1=\alpha_2=\alpha$, then $\mathscr{D}'^{\alpha}_{L_1}(\Omega)\hookrightarrow\mathscr{D}'^{\alpha}_{L_2}(\Omega)$;

(BWFI3) if $\Omega_1 = \Omega_2 = \Omega$, $L_1 = L_2 = L$ and $\alpha_1 \leq \alpha_2$, then $\mathscr{D}'^{\alpha_2}_L(\Omega) \hookrightarrow \mathscr{D}'^{\alpha_1}_L(\Omega)$.

Proof. We first note that, by Definition 3.1.3, if $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^d$ are open sets, one has that

$$\mathfrak{sn}^\alpha_L(\Omega_1) \subset \mathfrak{sn}^\alpha_L(\Omega_2),$$

for every $\alpha \in \mathbb{R}$ and $L \subset \Omega_1 \times \mathbb{R}^d \setminus \{0\}$ a closed conic set. Analogously, if $L_1 \subset L_2 \subset \Omega \times \mathbb{R}^d \setminus \{0\}$ are closed conic sets, then

$$\mathfrak{sn}^\alpha_{L_2}(\Omega) \subset \mathfrak{sn}^\alpha_{L_1}(\Omega),$$

for every $\alpha \in \mathbb{R}$. As $\mathscr{D}'^\alpha_L(\Omega)$ is precisely the collection of $u \in \mathscr{D}'(\Omega)$ for which $\mathfrak{p}(u) < +\infty$ for every $\mathfrak{p} \in \mathfrak{sn}^\alpha_L(\Omega)$, see Corollary 3.1.3, the above inclusions imply items *(BWFI1)* and *(BWFI2)* respectively. Then, item *(BWFI3)* is a consequence of Proposition 1.3.2 applied to $B^{\alpha,loc}_{\infty,\infty}(\Omega)$ as per Remark 1.3.11 and of Proposition 3.2.1. $\square$

Definition 3.2.1 (Webbed and strictly webbed)**.** *Let (X, τ) be an LCTVS as per Definition 1.1.6. Let*

$$\mathscr{W} \doteq \{C_{n_1,\dots,n_k} \mid k \in \mathbb{N},\ n_j \in \mathbb{N} \text{ for every } j \in \mathbb{N}\},$$

*be a collection of disks as per Definition 1.1.2. We call $\mathscr{W}$ a **web** if*

$$X = \bigcup_{n_1 \in \mathbb{N}} C_{n_1} \text{ and } C_{n_1,\dots,n_{k-1}} = \bigcup_{n_k \in \mathbb{N}} C_{n_1,\dots,n_k},$$

*for every $n_1, \dots, n_k \in \mathbb{N}$ and $k \geq 2$. We call $\mathscr{W}$ a **strict web** if it is a web and for any sequence $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ there exists a sequence $(\varrho_k)_{k\in\mathbb{N}} \subset (0, +\infty)$ such that*

- *for any choice of $x_k \in C_{n_1,\dots,n_k}$, the series $\sum_{k\in\mathbb{N}} \varrho_k x_k$ converges in X;*
- *the series $\sum_{k \geq k_0} \lambda_k x_k$ converges in X to an element of $C_{n_1,\dots,n_{k_0}}$ for every $k_0 \in \mathbb{N}$ and every $(\lambda_k)_{k\in\mathbb{N}} \subset [0, +\infty)$ such that $\lambda_k \leq \varrho_k$ for every $k \in \mathbb{N}$.*

*If there exists a web $\mathscr{W}$ on X, we call X a **webbed space**. If $\mathscr{W}$ is a strict web, we call X a **strictly webbed space**.*[3]

Corollary 3.2.3. *Let $\Omega \subset \mathbb{R}^d$ be an open set, let $L \subset \Omega \times (\mathbb{R}^d \setminus \{0\})$ be a closed conic set and let $\alpha \in \mathbb{R}$ be fixed. Then, the space $D'^\alpha_L(\Omega)$ is complete as per Definition 1.1.15 and strictly webbed as per Definition 3.2.1.*

[3] "I have looked upon all the universe has to hold of horror, and even the skies of spring and flowers of summer must ever afterward be poison to me" - H. P. Lovecraft

Proof. By Proposition 3.2.1, $\mathscr{D}'^{\alpha}_{L}(\Omega)$ is isomorphic as per Definition 1.1.4 to a closed subspace of $\mathscr{D}'(\Omega) \times \left(B^{\alpha,loc}_{\infty,\infty}(\Omega)\right)^{\mathbb{N}\times\mathbb{N}}$. Both spaces $\mathscr{D}'(\Omega)$ and $B^{\alpha,loc}_{\infty,\infty}(\Omega)$ are complete, see Theorem 1.2.1 and Remark 1.3.11. Thus, by [13, Chapter I.2, p. 19], $\mathscr{D}'(\Omega) \times \left(B^{\alpha,loc}_{\infty,\infty}(\Omega)\right)^{\mathbb{N}\times\mathbb{N}}$ is a complete LCTVS. As any closed subset of a complete LCTVS is complete, see [21, Chapter 2.9, Proposition 3], one has that $\mathscr{D}'^{\alpha}_{L}(\Omega)$ is complete.

We note that, as $\mathscr{D}(\Omega)$ is an (LF)-space, see Example 1.1.12 and Definition 1.1.21, [15, Chapter 35.4, (13)] implies that $\mathscr{D}'(\Omega)$ is strictly webbed as per Definition 3.2.1. Analogously, as $B^{\alpha,loc}_{\infty,\infty}(\Omega)$ is a Fréchet space, see Remark 1.3.11, [15, Chapter 35.1, (4)] entails that $B^{\alpha,loc}_{\infty,\infty}(\Omega)$ is strictly webbed. Thus, by [15, Chapter 35.4, (6)], $\mathscr{D}'(\Omega) \times \left(B^{\alpha,loc}_{\infty,\infty}(\Omega)\right)^{\mathbb{N}\times\mathbb{N}}$ is strictly webbed. As it is isomorphic to a closed subspace of a strictly webbed space, $\mathscr{D}'^{\alpha}_{L}(\Omega)$ is strictly webbed by [15, Chapter 35.4, (1)]. □

3.3 Duality and structure properties

In this section, we shall prove some structural properties of the space $\mathscr{D}'^{\alpha}_{L}(\Omega)$ as per Definition 3.1.1 in analogy with the smooth case of Subsection 2.2.2. In particular, we shall construct a hypocontinuous duality that generalizes the one between Besov spaces as per Theorem 1.17. We shall conclude this section with some remarks regarding analogous continuity results for other structural operations, specifically the tensor product and the pullback by a smooth map. We also discuss their generalizations along the direction of Remark 3.1.1.

Remark 3.3.1. In light of Corollary 3.2.2, one may heuristically interpret the spaces $\mathscr{D}'^{\alpha}_{L}(\Omega)$, as L varies in the collection of closed conic subsets of $\Omega\times\mathbb{R}^d\setminus\{0\}$, as intermediate functional spaces between $B^{\alpha,loc}_{\infty,\infty}(\Omega)$ and $\mathscr{D}'(\Omega)$. The same also holds for their smooth counterpart, namely $\mathscr{D}'_{L}(\Omega)$ as per Definition 2.2.11 between $\mathscr{E}(\Omega)$ and $\mathscr{D}'(\Omega)$. Thus, one expects spaces of distributions with a prescribed wavefront set to enjoy the properties that also hold for the endpoint cases, *i.e.*, those with $L = \Omega \times \mathbb{R}^d \setminus \{0\}$ and $L = \emptyset$. For instance, in the smooth case, the dual spaces of both $\mathscr{E}(\Omega)$ and $\mathscr{D}'(\Omega)$ consist of functions or distributions with compact support and the same holds for $\mathscr{D}'_{L}(\Omega)$ for any choice of L, see Theorem 2.2.13. The converse, however, is also true, in the sense that one expects properties that do not hold true for the extremal cases to not be enjoyed by the intermediate ones. Thus, for instance, in the Besov case, the dual space of $B^{\alpha}_{\infty,\infty}(\mathbb{R}^d)$ is strictly larger than $B^{-\alpha}_{1,1}(\mathbb{R}^d)$, see the discussion in [25, Section 2.11], and it does not belong to any Besov class. Therefore, one cannot expect to characterize the full dual space of $\mathscr{D}'^{\alpha}_{L}(\Omega)$ in analogy with its smooth counterpart. What can be achieved is the characterization of a space of distributions that forms a dual pair with $\mathscr{D}'^{\alpha}_{L}(\Omega)$ as per Remark 1.1.19. Hence, we shall now recall the construction of this space, which is analogous to that of $\mathscr{E}'_{W}(\Omega)$ as per Definition 2.2.13. □

Let $\Omega \subset \mathbb{R}^d$ be an open set and let $\mathfrak{K}_\Omega$ be the collection of all compact subsets of Ω. Let $\mathfrak{V}(\Omega)$ be the collection of closed conic subsets of $\Omega \times \mathbb{R}^d \setminus \{0\}$ as per Definition 2.2.1 and let $W \subset \Omega \times \mathbb{R}^d \setminus \{0\}$ be an open conic set. Let $\mathfrak{W}\mathfrak{f}(W)$ be the collection of pairs $(K, L) \in \mathfrak{K}_\Omega \times \mathfrak{V}_\Omega$ such that $L \subset W$ and $\mathrm{pr}_1[L] \subset K$, where $\mathrm{pr}_1 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is the projection onto the first factor. We equip the set $\mathfrak{W}\mathfrak{f}(\Omega)$ with the ordering $\leqq$ given by $(K_1, L_1) \leqq (K_2, L_2)$ if and only if $K_1 \subseteq K_2$ and $L_1 \subseteq L_2$. Then, for any $(K, L) \in \mathfrak{W}\mathfrak{f}(W)$, we define the space

$$\mathscr{E}'^{\alpha}_{K,L}(\Omega) \doteq \{v \in B^{\alpha}_{1,1}(\mathbb{R}^d) \mid supp(v) \subset K \text{ and } WF(v) \subset L\}.$$

We endow $\mathscr{E}'^{\alpha}_{K,L}(\Omega)$ with the topology induced by the norm $\|\cdot\|_{B^{\alpha}_{1,1}(K)}$, see Remark 1.3.11, together with the collection of seminorms $\mathfrak{sn}_L(\Omega)$ as per Remark 2.2.21. Furthermore, for every $a = (K_a, L_a)$, $b = (K_b, L_b) \in \mathfrak{W}\mathfrak{f}(W)$ with $b \leqq a$, let $\pi_{a,b} : \mathscr{E}'^{\alpha}_{K_b,L_b}(\Omega) \to \mathscr{E}'^{\alpha}_{K_a,L_a}(\Omega)$ be the natural immersion as per Remark 1.3.11 together with Corollary 3.2.2, item *(BWFI2)*. Thus, we may define the following.

Definition 3.3.1. *Let $\mathfrak{W}\mathfrak{f}(W)$ and $\mathscr{E}'^{\alpha}_{K,L}(\Omega)$, $(K, L) \in \mathfrak{W}\mathfrak{f}(W)$, be as per the above construction. We denote by $\mathscr{E}'^{\alpha}_{W}(\Omega)$ the inductive limit space as per Definition 1.1.20 of the family $\{\mathscr{E}'^{\alpha}_{K,L}(\Omega) \mid (K, L) \in \mathfrak{W}\mathfrak{f}(W)\}$ with respect to the inductive linking maps $\{\pi_{a,b} \mid a, b \in \mathfrak{W}\mathfrak{f}(W),\ b \leqq a\}$.*

In order to work with the space $\mathscr{E}'^{\alpha}_{W}(\Omega)$, we require some further information on its topological properties.

Remark 3.3.2. We note that, by Definitions 2.2.13 and 3.3.1, for every $\alpha \in \mathbb{R}$ and $(K, L) \in \mathfrak{W}\mathfrak{f}(W)$, we have that

$$\mathscr{E}'^{\alpha}_{K,L}(\Omega) = \mathscr{E}'_{K,L}(\Omega) \cap B^{\alpha}_{1,1}(\mathbb{R}^d).$$

Furthermore, as $B^{\alpha}_{1,1}(\mathbb{R}^d) \hookrightarrow \mathscr{D}'(\Omega)$ by Remark 1.2.15 and Proposition 1.3.1, item *(B3)*, the equivalence is also topological, in the sense that the topology of $\mathscr{E}'^{\alpha}_{K,L}(\Omega)$ agrees with the intersection topology of $\mathscr{E}'_{K,L}(\Omega)$ and $B^{\alpha}_{1,1}(\mathbb{R}^d)$, see [21, Chapter 2.1, p. 73]. This identification entails a series of topological properties for the spaces $\mathscr{E}'^{\alpha}_{K,L}(\Omega)$ and $\mathscr{E}'^{\alpha}_{W}(\Omega)$, which we shall now present. □

Proposition 3.3.1. *Let $\mathscr{E}'^{\alpha}_{K,L}(\Omega)$, $(K,L) \in \mathfrak{W}\mathfrak{f}(W)$, and let $\mathscr{E}'^{\alpha}_{W}(\Omega)$ be as per Definition 3.3.1.*

(BWFD1) There exists a continuous linear map with closed image

$$\mathscr{J} : \mathscr{E}'^{\alpha}_{K,L}(\Omega) \to B^{\alpha}_{1,1}(\mathbb{R}^d) \times \left(\mathscr{S}(\mathbb{R}^d)\right)^{\mathbb{N}\times\mathbb{N}},$$

such that $\mathscr{J}$ is an isomorphism, see Definition 1.1.4, onto its image.

(BWFD2) $\mathscr{E}'^{\alpha}_{K,L}(\Omega)$ is a Fréchet space, see Definition 1.1.16. In particular, it is also bornological and barrelled as per Definitions 1.1.7 and *1.1.9.*

(BWFD3) $\mathscr{D}(\Omega)$ is sequentially dense in $\mathscr{E}'^{\alpha}_{W}(\Omega)$, i.e., for every $u \in \mathscr{E}'^{\alpha}_{W}(\Omega)$ there exists a sequence $(u_j)_{j\in\mathbb{N}} \subset \mathscr{D}(\Omega)$ such that $u_j \to u$ in $\mathscr{E}'^{\alpha}_{W}(\Omega)$.

Proof. In order to prove item *(BWFD1)*, we shall construct the map

$$\mathscr{J} : \mathscr{E}'^{\alpha}_{K,L}(\Omega) \to B^{\alpha}_{1,1}(\mathbb{R}^d) \times \left(\mathscr{S}(\mathbb{R}^d)\right)^{\mathbb{N}\times\mathbb{N}},$$

analogously to our previously stated result regarding $\mathscr{D}'^{\alpha}_{L}(\Omega)$. Thus, let $O_{j,k}$, $O'_{j,k}$, $O''_{j,k}$, $V_{j,k}$, $V'_{j,k}$, $V''_{j,k}$, $\varphi_{j,k}$ and $\psi_{j,k}$ be as per Proposition 3.2.1. For any $u \in \mathscr{E}'^{\alpha}_{K,L}(\Omega)$, let $\mathscr{J}(u) \doteq (u, G_u) \in B^{\alpha}_{1,1}(\mathbb{R}^d) \times \left(\mathscr{S}(\mathbb{R}^d)\right)^{\mathbb{N}\times\mathbb{N}}$, where $G_u(j,k) = \psi_{j,k}\mathscr{F}(\varphi_{j,k}u)$ for every $j,k \in \mathbb{N}$. Let $\|\cdot\|_{a,b}$ be the Schwartz seminorm as per Definition 1.2.7 as a and b vary in $\mathbb{N}^d$. For any $a \in \mathbb{N}^d$, we have that

$$\sup_{\xi\in\mathbb{R}^d} |\xi^a G_u(j,k)(\xi)| = \sup_{\xi\in\mathbb{R}^d} |\xi^a \psi_{j,k}(\xi)\mathscr{F}(\varphi_{j,k}u)(\xi)| \leq q^{|a|}_{\varphi_{j,k},\psi_{j,k}}(u),$$

where $q^{\nu}_{\varphi,V}$ is as per Definition 2.2.12. Furthermore, for any $l \in \{1,\dots,d\}$, we also have that

$$\sup_{\xi\in\mathbb{R}^d} |\partial_l G_u(j,k)(\xi)| = \sup_{\xi\in\mathbb{R}^d} \left|\partial_l \left(\psi_{j,k}\mathscr{F}(\varphi_{j,k}u)\right)(\xi)\right| \leq q^{0}_{\varphi_{j,k},\partial_l\psi_{j,k}}(u) + q^{0}_{\varphi_{j,k,l},\psi_{j,k}}(u),$$

where $\varphi_{j,k,l} : x \mapsto x_l\varphi_{j,k}(x)$. Thus, employing Leibniz's rule, we have that for every $j,k \in \mathbb{N}$ and for every $a,b \in \mathbb{N}^d$ there exist $C > 0$ and $q_1,\dots,q_n \in \mathfrak{sn}_L(\Omega)$ such that

$$\|G_u(j,k)\|_{a,b} \leq C\sum_{j=1}^{n} q_j(u).$$

This proves that $\mathscr{J}$ is continuous. To prove that $\mathscr{J}$ is injective, we note that $\mathscr{J}$ is linear and if $\mathscr{J}(u) = 0$, then $u = 0$ in $B^{\alpha}_{1,1}(\mathbb{R}^d)$. This implies that $u = 0$ in $\mathscr{E}'^{\alpha}_{K,L}(\Omega)$. To conclude, we note that, as $B^{\alpha}_{1,1}(\mathbb{R}^d) \hookrightarrow \mathscr{D}'(\Omega)$ by Remark 1.2.15 and Proposition 1.3.1, item *(B2)*, for every $\mathfrak{p}_B \in \mathfrak{sn}(\Omega)$ there exists $C_B > 0$

such that $\mathfrak{p}_B(u) \leq C_B \|u\|_{B^\alpha_{1,1}(\mathbb{R}^d)}$ for every $u \in B^\alpha_{1,1}(\mathbb{R}^d)$. Thus, using the same decomposition we employed in the proof of Proposition 3.2.1, for any $q = q^\nu_{\varphi,V} \in \mathfrak{sn}_L(\Omega)$ there exist $D > 0$, a bounded subset $B \subset \mathscr{D}(\Omega)$ and finite $J \subset \mathbb{N} \times \mathbb{N}$ and $N \in \mathbb{N}$ such that

$$q^\nu_{\varphi,V}(u) \leq D\mathfrak{p}_B(u) + D \sum_{(j,k)\in J} \sum_{|a|+|b|\leq N} \|\psi_{j,k}\mathscr{F}(\varphi_{j,k}u)\|_{a,b} \leq$$
$$\leq C_B \|u\|_{B^\alpha_{1,1}(\mathbb{R}^d)} + D \sum_{(j,k)\in J} \sum_{|a|+|b|\leq N} \|\psi_{j,k}\mathscr{F}(\varphi_{j,k}u)\|_{a,b}.$$

Thus, $\mathscr{J}$ is an isomorphism onto its image. Let us now consider a generalized sequence $(f_\lambda, G_\lambda)_{\lambda\in\Lambda} \subset Im[\mathscr{J}]$ for some directed set Λ, where $G_\lambda(j,k) = \psi_{j,k}\mathscr{F}(\varphi_{j,k}f_\lambda)$, which converges to (f, g) in the space $B^\alpha_{1,1}(\mathbb{R}^d) \times \left(\mathscr{S}(\mathbb{R}^d)\right)^{\mathbb{N}\times\mathbb{N}}$. In particular, $f_\lambda \to f$ in $B^\alpha_{1,1}(\mathbb{R}^d)$ and $G_\lambda(j,k) \to G(j,k)$ for every $j,k \in \mathbb{N}$. As $B^\alpha_{1,1}(\mathbb{R}^d) \hookrightarrow \mathscr{D}'(\Omega)$, this implies that $f_\lambda \to f$ in $\mathscr{D}'(\Omega)$ and, by Remark 1.2.28, this also entails that $\psi_{j,k}\mathscr{F}(\varphi_{j,k}f_\lambda) \to \psi_{j,k}\mathscr{F}(\varphi_{j,k}f)$ in $\mathscr{E}(\mathbb{R}^d)$. Thus, as $\mathscr{S}(\mathbb{R}^d) \hookrightarrow \mathscr{E}(\mathbb{R}^d)$ and $\psi_{j,k}\mathscr{F}(\varphi_{j,k}f_\lambda) \to G(j,k)$ in $\mathscr{S}(\mathbb{R}^d)$, it follows that $G(j,k) = \psi_{j,k}\mathscr{F}(\varphi_{j,k}f)$ for every $j,k \in \mathbb{N}$. In particular, the image of $\mathscr{J}$ is closed.

Item *(BWFD2)* is a consequence of item *(BWFD1)*. Indeed, by [14, Chapter 18.3, (6)], the space $B^\alpha_{1,1}(\mathbb{R}^d) \times \left(\mathscr{S}(\mathbb{R}^d)\right)^{\mathbb{N}\times\mathbb{N}}$ is a Fréchet space as per Definition 1.1.16. Furthermore, by [13, Chapter II.4, p. 49], any closed subspace of a Fréchet space is Fréchet. As we can identify $\mathscr{E}'^\alpha_{K,L}(\Omega)$ with $Im[\mathscr{J}]$ by item *(BWFD1)*, this entails that $\mathscr{E}'^\alpha_{K,L}(\Omega)$ is itself a Fréchet space. Then, by [13, Chapter II.7, 7.1] and [13, Chapter II.8, 8.1], it is both bornological and barrelled.

Lastly, item *(BWFD3)* follows from the sequential density of smooth, compactly supported functions in both $B^\alpha_{1,1}(\mathbb{R}^d)$ and $\mathscr{E}'_W(\Omega)$, see Proposition 1.3.1, item *(B3)*, and [41, Lemma 22]. In fact, let $u \in \mathscr{E}'^\alpha_W(\Omega)$ and let $(\chi_j)_{j\in\mathbb{N}} \in \mathscr{D}(\Omega)$ be such that for every compact subset $K \subset \Omega$, there exists $\tilde{j} \in \mathbb{N}$ such that $\chi_j|_K \equiv 1$ for every $j \geq \tilde{j}$. Furthermore, let $(\varphi_j)_{j\in\mathbb{N}} \in \mathscr{D}(\mathbb{R}^d)$ be a sequence of positive functions such that $\|\varphi_j\|_{L^1(\mathbb{R}^d)} = 1$ and $supp(\varphi_j) + supp(\chi_j) \subset \Omega$ for every $j \in \mathbb{N}$. Let us further assume that $supp(\varphi_j) \subset B(0, 1/j)$ for every $j \in \mathbb{N}$. Then, by [4, Theorem 4.1.4], one has that $u_j \doteq (\chi_j u) \star \varphi_j$ converges to u in the topology induced by the seminorms $\mathfrak{sn}_L(\Omega)$. Analogously, by [43, Proposition 17.12], u_j converges to u in $B^\alpha_{1,1}(\mathbb{R}^d)$. Thus, it follows that $u_j \to u$ in $\mathscr{E}'^\alpha_W(\Omega)$. □

Proposition 3.3.2. *Let $\mathscr{E}'^\alpha_W(\Omega)$ be as per Definition 3.3.1. Then, it is a strict (LF)-space as per Definition 1.1.21. In particular, it is both bornological and barrelled as per Definitions 1.1.7 and 1.1.9.*

Proof. By construction, $\mathscr{E}'^{\alpha}_W(\Omega)$ is the inductive limit of the spaces $\mathscr{E}'^{\alpha}_{K,L}(\Omega)$ as $(K,L) \in \mathfrak{W}\mathfrak{f}(\Omega)$. Each $\mathscr{E}'^{\alpha}_{K,L}(\Omega)$ is a Fréchet space by Proposition 3.3.1. Furthermore, if $(K_j)_{j\in\mathbb{N}} \subset \mathfrak{K}_\Omega$ is an exhaustion of Ω by compact sets such that $K_j \subset K_{j+1}$ for every $j \in \mathbb{N}$ and $(L_j)_{j\in\mathbb{N}}$ is an exhaustion of W by closed conic sets such that $L_j \subset L_{j+1}$ for every $j \in \mathbb{N}$, then

$$\mathscr{E}'^{\alpha}_W(\Omega) = \varinjlim \pi_j \mathscr{E}'^{\alpha}_{K_j,L_j}(\Omega),$$

where $\pi_j = \pi_{(K_{j+1},L_{j+1}),(K_j,L_j)}$. Thus, $\mathscr{E}'^{\alpha}_W(\Omega)$ is an (LF)-space as per Definition 1.1.21. As the topology on $\mathscr{E}'^{\alpha}_{K,L}(\Omega)$ is induced by that of $\mathscr{E}'^{\alpha}_{\tilde{K},\tilde{L}}(\Omega)$ for every $(K,L),(\tilde{K},\tilde{L}) \in \mathfrak{W}\mathfrak{f}(W)$ with $(K,L) \leqq (\tilde{K},\tilde{L})$, the limit is also strict. By [13, Chapter II.8, Corollary 2] and [13, Chapter II.7, Corollary 2], this also entails that it is both a bornological and a barrelled space. □

Having defined the candidate space $\mathscr{E}'^{\alpha}_W(\Omega)$ and proved some of its topological properties, we can now construct our duality. To this avail, we must first introduce some notation. Let $v \in \mathscr{E}'^{\alpha}_W(\Omega)$. By construction, there exist a compact set $K \subset \Omega$ and a closed cone $L \subset W$ such that $v \in \mathscr{E}'^{\alpha}_{K,L}(\Omega)$. Then, by [42, Lemma 3.18], there exist open sets $U_1,\dots,U_N \subset \Omega$ and $\varphi_1,\dots,\varphi_N \in \mathscr{D}(\Omega)$ such that $K \subset \bigcup_{j=1}^N U_j$ and $\sum_{j=1}^N \varphi_j^2 \equiv 1$ on a neighbourhood of K.

Theorem 3.3.3 (Microlocal Besov duality). *Let $\mathscr{D}'^{\alpha}_L(\Omega)$ be as per Definition 3.1.1 and let M be the complement of $-L \doteq \{(x,-\xi) \mid (x,\xi) \in L\}$ in $\Omega \times \mathbb{R}^d \setminus \{0\}$. Let $(\chi_k)_{k\in\mathbb{N}} \subset \mathscr{D}(\mathbb{R}^d)$ be a Littlewood-Paley partition of unity as per Definition 1.3.1 and let $(\Delta_k)_{k\in\mathbb{N}}$ be the associated sequence of operators. For any $v \in \mathscr{E}'^{-\alpha}_M(\Omega)$, let $\varphi_1,\dots,\varphi_N$ be as in the above construction and let*

$$T_v : \mathscr{D}'^{\alpha}_L(\Omega) \to \mathbb{C} \tag{3.41}$$

$$u \mapsto \sum_{j=1}^N \sum_{\substack{k,k'\in\mathbb{N} \\ |k-k'|\leq 1}} \int_{\mathbb{R}^d} \Delta_k(\varphi_j u)(x)\Delta_{k'}(\varphi_j v)(x)dx. \tag{3.42}$$

Then, T_v is continuous with respect to the Hörmander-Besov topology as per Definition 3.1.3. Furthermore, the map

$$\langle\cdot\,,\,\cdot\rangle_{MB} : \mathscr{D}'^{\alpha}_L(\Omega) \times \mathscr{E}'^{-\alpha}_M(\Omega) \to \mathbb{C} \tag{3.43}$$

$$(u,v) \longmapsto \langle u\,,\,v\rangle_{MB} \doteq T_v(u), \tag{3.44}$$

is a hypocontinuous duality as per Definitions 1.1.14 and 1.1.22.

Remark 3.3.3. Before proving Theorem 3.3.3, we wish to illustrate the idea behind its construction. As per Remark 2.2.17, if $u \in \mathscr{D}'^{\alpha}_L(\Omega)$, then it behaves locally as an element of $B^{\alpha}_{\infty,\infty}(\mathbb{R}^d)$ outside of L and as an arbitrary distribution in

$\mathscr{D}'(\Omega)$ within L. Analogously, by Remark 2.2.2, if $v \in \mathscr{E}'^{-\alpha}_M(\Omega)$, then it behaves locally as a smooth, compactly supported function outside of $M = \mathscr{C}(-L)$ and as a compactly supported element of $B^{-\alpha}_{1,1}(\mathbb{R}^d)$ within M. Apart from an inversion of directions in momentum space, which is caused by the structure associated with Theorem 2.2.9, our choices of spaces imply that u behaves as a distribution where v behaves as a smooth, compactly supported function and that u behaves as a local element of $B^{\alpha}_{\infty,\infty}(\mathbb{R}^d)$ where v behaves as a compactly supported element of $B^{-\alpha}_{1,1}(\mathbb{R}^d)$. In both cases, there is a natural duality pairing between u and v, given respectively by the dual pair $\langle \mathscr{D}(\Omega), \mathscr{D}'(\Omega)\rangle_D$ as per Definition 1.2.1 and $\langle B^{\alpha}_{\infty,\infty}(\mathbb{R}^d), B^{-\alpha}_{1,1}(\mathbb{R}^d)\rangle_B$ as per Theorem 1.17. Thus, a possible duality between $\mathscr{D}'^{\alpha}_L(\Omega)$ and $\mathscr{E}'^{-\alpha}_M(\Omega)$ could be constructed as the sum of two terms, obtained as the evaluation of a microlocalization of each term against the other. We shall see that, in fact, in the proof of Theorem 3.3.3, our construction will be required to be decomposed as the sum of these two terms. $\square$

Proof of Theorem 3.3.3. The proof of this theorem is divided into 4 steps for the sake of clarity. We shall begin with some preliminary considerations in **Step 1** regarding the well-posedness of the problem. Then, in **Step 2**, we shall show that the map T_v defined through Equation (3.42) is finite and we shall characterize it by a decomposition into two terms localized in the appropriate regions of $\Omega \times \mathbb{R}^d \setminus \{0\}$. This localization follows the heuristic reasoning we presented in Remark 3.3.3. Subsequently, in **Step 3**, we shall prove the necessary continuity estimates for the terms in the aforementioned decomposition. Lastly, in **Step 4**, we shall show that these estimates are enough to imply the hypocontinuity of the map $\langle \cdot, \cdot \rangle_{MB}$ as per Equation (3.43).

Step 1: Preliminaries. Let us first prove that the expression in Equation (3.42) is not ill-defined. We recall that

$$\Delta_k(\varphi_j u)(x) = \mathscr{F}^{-1}[\chi_k \mathscr{F}(\varphi_j u)](x).$$

For every $u \in \mathscr{D}'(\Omega)$, $\varphi_j u \in \mathscr{E}'(\Omega)$. Thus, by Remark 1.2.28, $\mathscr{F}(\varphi_j u) \in \mathscr{E}_{pol}(\mathbb{R}^d)$. Then, as $\chi_k \in \mathscr{D}(\mathbb{R}^d)$, we have that $\chi_k \mathscr{F}(\varphi_j u) \in \mathscr{D}(\Omega)$. Hence, by Lemma 1.2.14, its inverse Fourier transform is a Schwartz function. Similarly, $\Delta_{k'}(\varphi_j v)$ is a Schwartz function. Therefore, the integral

$$\int_{\mathbb{R}^d} \Delta_k(\varphi_j u)(x) \Delta_{k'}(\varphi_j v)(x) dx,$$

is a well-defined and absolutely convergent integral for any $u, v \in \mathscr{D}'(\Omega)$ and for every $k, k' \in \mathbb{N}$. Let us now note that, using the property that $\chi_k(-\xi) =$

$\chi_k(\xi)$ for every $k \in \mathbb{N}$ and $\xi \in \mathbb{R}^d$ as per Definition 1.3.1 together with Remark 2.1.3, we have that

$$\int_{\mathbb{R}^d} \Delta_k(\varphi_j u)(x)\Delta_{k'}(\varphi_j v)(x)dx = \langle \varphi_j u\,,\, \Delta_k \circ \Delta_{k'}\varphi_j v\rangle_E,$$

for every $u, v \in \mathscr{D}'(\Omega)$, where $\langle \cdot\,,\, \cdot\rangle_E$ is the duality of $\mathscr{E}'(\Omega)$ and $\mathscr{E}(\Omega)$.

Step 2: Decomposition. We now fix $v \in \mathscr{E}'^{-\alpha}_M(\Omega)$. For every $M \in \mathbb{N}$, let

$$S_M \doteq \sum_{\substack{k,k'\leq M\\ |k-k'|\leq 1}} \Delta_k \circ \Delta_{k'},$$

and let, for every $u \in \mathscr{D}'^{\alpha}_L(\Omega)$,

$$T^M_v(u) \doteq \sum_{j=1}^N \langle \varphi_j u\,,\, S_M(\varphi_j v)\rangle_E = \sum_{j=1}^N \sum_{\substack{k,k'\leq M\\ |k-k'|\leq 1}} \langle \varphi_j u\,,\, \Delta_k \circ \Delta_{k'}(\varphi_j v)\rangle_E.$$

We shall construct a decomposition for T^M_v which will yield, when taking the limit for $M \to +\infty$, the aforementioned decomposition for T_v. For every fixed $j \in \mathbb{N}$, let $K_j \doteq supp(\varphi_j v)$ and let $M_j \doteq WF(\varphi_j v)$. By construction of the space $\mathscr{E}'^{-\alpha}_M(\Omega)$, there exist a compact set $K \subset \Omega$ and a closed cone $V \subset M$ such that $v \in \mathscr{E}'^{-\alpha}_{K,V}(\Omega)$. Thus, $K_j \subset K$ and $M_j \subset V \subset M$. Using a compactness argument as in the proof of Proposition 3.1.2, we consider a finite collection of functions $\theta_{j,1}, \dots, \theta_{j,m_j} \in \mathscr{D}(\Omega)$ and $g_{j,1}, \dots, g_{j,m_j} \in S^0(\mathbb{R}^d)$ as per Definitions 1.2.1 and 2.1.1 with the following properties:

- there exists $\tilde{K}_j$ a compact subset of Ω such that $K_j \Subset \tilde{K}_j$ and
$$\sum_{l=1}^{m_j} \theta_{j,l}\Big|_{\tilde{K}_j} \equiv 1;$$
- for every $j \in \{1, \dots, N\}$ and $l \in \{1, \dots, m_j\}$, $(\theta_{j,l}, g_{j,l}) \in \mathfrak{A}(\Omega, L)$ as per Definition 3.1.2;
- for every $\eta \in M_j$, there exist $R > 0$ and a conic neighbourhood V of η in $\mathbb{R}^d$ such that $g_{j,l}(-\xi) = 1$ for every $\xi \in V$ with $|\xi| \geq R$.

Subsequently, we fix the ΨDO

$$A_j(u) \doteq \sum_{l=1}^{m_j} \theta_{j,l}\mathscr{F}^{-1}[g_{j,l}\mathscr{F}(\theta_{j,l}u)],$$

for every $u \in \mathscr{D}'(\Omega)$ and we denote by B_j its adjoint as per Remark 2.1.3. We note that $g_{j,l}$ is identically 1 on $-M_j$ and that

$$WF'(B_j) \subseteq \bigcup_{l=1}^{m_j} \big(supp(\theta_{j,l}) \times ess\ supp(g_{j,l}(-\cdot))\big),$$

see Remark 2.2.1, where $g_{j,l}(-\cdot) : \xi \mapsto g_{j,l}(-\xi)$. In particular, by the proof of [31, Proposition 18.1.26], B_j is microlocally equivalent to the identity on M_j, see Remark 2.2.14. Thus, $r_j \doteq (Id - B_j)(\varphi_j v)$ is smooth, where Id is the identity on $\mathscr{D}'(\Omega)$. As both Id and B_j are properly supported, see Definition 2.1.8, we also have that r_j is compactly supported. We may now decompose T_v^M in the following manner

$$\begin{aligned} T_v^M(u) &= \sum_{j=1}^{N} \langle A_j(\varphi_j u), S_M(\varphi_j v)\rangle_E + \langle (Id - A_j)(\varphi_j u), S_M(\varphi_j v)\rangle_E = \\ &= \sum_{j=1}^{N} \langle A_j(\varphi_j u), S_M(\varphi_j v)\rangle_E + \langle \varphi_j u, (Id - B_j)[S_M(\varphi_j v)]\rangle_E. \end{aligned} \tag{3.45}$$

We claim that, for every $u \in \mathscr{D}_L'^{\alpha}(\Omega)$, the limit of $T_v^M(u)$ as $M \to +\infty$ exists and is finite in $\mathbb{C}$. Thus, as formally

$$T_v(u) = \lim_{M\to+\infty} T_v^M(u),$$

this would entail that T_v is well defined. To prove this claim, we shall work on each term in Equation (3.45) individually. We begin by noting that, by construction, whenever $u \in \mathscr{D}_L'^{\alpha}(\Omega)$,

$$\|A_j(\varphi_j u)\|_{B^{\alpha}_{\infty,\infty}(\mathbb{R}^d)} \leq \sum_{l=1}^{m_j} p^{\alpha}_{\theta_{j,l}\varphi_j, g_{j,l}, supp(\theta_{j,l})}(u). \tag{3.46}$$

Thus, $A_j(\varphi_j u) \in B^{\alpha}_{\infty,\infty}(\mathbb{R}^d)$ for every $u \in \mathscr{D}_L'^{\alpha}(\Omega)$ and, for $M \to +\infty$, we have that

$$\begin{aligned} &\langle A_j(\varphi_j u), S_M(\varphi_j v)\rangle_E = \langle A_j(\varphi_j u), S_M(\varphi_j v)\rangle_E = \\ &= \sum_{\substack{k,k' \leq M \\ |k-k'|\leq 1}} \int_{\mathbb{R}^d} \Delta_k\big[A_j(\varphi_j u)\big](x)\Delta_{k'}(\varphi_j v)(x)dx \to \langle A_j(\varphi_j u), \varphi_j v\rangle_B, \end{aligned}$$

see Theorem 1.3.5. In particular, the limit exists and it is finite. For the second term in Equation (3.45), we shall require the following lemma.

Lemma 3.3.4. *Let $j \in \{1,\dots,N\}$ be fixed and let S_M, $b \doteq \varphi_j v$ and $T \doteq Id - B_j$ be as per the above construction for every $M \in \mathbb{N}$. Let $\Phi \in \mathscr{D}(\Omega)$ be such that $\Phi \equiv 1$ in a neighbourhood of K_j. Then,*

$$\Phi(T \circ S_M)(b) \xrightarrow[M\to+\infty]{} \Phi T(b) \text{ in } \mathscr{D}(\Omega).$$

Proof of Lemma 3.3.4. As $b \in \mathscr{E}'(\Omega) \hookrightarrow \mathscr{S}'(\mathbb{R}^d)$, Remark 1.3.3 entails that $S_M(b) \to b$ in $\mathscr{S}'(\mathbb{R}^d)$. Thus, by the inclusion $\mathscr{S}'(\mathbb{R}^d) \hookrightarrow \mathscr{D}'(\Omega)$, see Remark 1.3.1, this implies that $S_M(b) \to b$ in $\mathscr{D}'(\Omega)$ as well. Let us now fix $\Gamma \doteq WF(b)$. We claim that $S_M(b) \to b$ in $\mathscr{D}'_\Gamma(\Omega)$, see Definition 2.2.12. To prove this claim, let us consider $\psi \in \mathscr{D}(\Omega)$ and $V \subset \mathbb{R}^d \setminus \{0\}$ a closed cone such that

$$supp(\psi) \times V \cap \Gamma = \varnothing.$$

Let us also consider $\tilde{\psi} \in \mathscr{D}(\Omega)$ and $W \subset \mathbb{R}^d \setminus \{0\}$ be such that $\tilde{\psi}$ is identically 1 in a neighbourhood of $supp(\psi)$, $V \cap \mathbb{S}^{d-1} \Subset int(W) \cap \mathbb{S}^{d-1}$, where $int(W)$ is the interior of W, and such that

$$supp(\tilde{\psi}) \times W \cap \Gamma = \varnothing.$$

Then, fixing $c_M \doteq S_M(b) - b$, we may write

$$\psi(c_M) = \psi(S_M - Id)(\tilde{\psi} b) + \psi S_M((1-\tilde{\psi})b) \doteq (1_M) + (2_M). \tag{3.47}$$

By Definition 2.2.12, proving that $S_M(b)$ converges to b in $\mathscr{D}'_\Gamma(\Omega)$ is equivalent to proving that the convergence holds in $\mathscr{D}'(\Omega)$, which we have already shown, and that $1_V \mathscr{F}[\psi(c_M)]$ converges to 0 faster than any polynomial and uniformly on $\mathbb{R}^d$. We shall first prove that (2_M) converges to 0 in $\mathscr{S}(\mathbb{R}^d)$, which, in light of Proposition 1.2.5, item *(F2)*, is a stronger condition than the one required. Then, we shall prove that (1_M) converges to 0 in the desired sense. For every $M \in \mathbb{N}$, let

$$Q_M(x) \doteq \sum_{\substack{k,k' \le M\\ |k-k'|\le 1}} \chi_k(x)\chi_{k'}(x).$$

Then, $(Q_M(x))_{M\in\mathbb{N}}$ is an increasing sequence and it converges to

$$\lim_{M\to+\infty} Q_M(x) = \sum_{\substack{k,k' \in \mathbb{N}\\ |k-k'|\le 1}} \chi_k(x)\chi_{k'}(x) = \sum_{k,k'\in\mathbb{N}} \chi_k(x)\chi_{k'}(x) = \left(\sum_{k\in\mathbb{N}} \chi_k(x)\right)^2 = 1,$$

as a consequence of Definition 1.3.1 and Remark 1.3.3. We also have that

$$(2_M) = \psi \mathscr{F}^{-1}[Q_M \mathscr{F}[(1-\tilde{\psi})b]].$$

By Remark 1.3.3, we have that

$$\psi S_M[(1-\tilde{\psi})b] \xrightarrow[M\to+\infty]{} \psi(1-\tilde{\psi})b = 0,$$

in the topology of $\mathscr{S}'(\mathbb{R}^d)$. Furthermore, the family of symbols $(Q_M)_{M\in\mathbb{N}}$ is bounded in $S^0(\mathbb{R}^d)$, see Definition 2.1.1, as a consequence of Remark 1.3.2. Thus, the family $(S_M)_{M\in\mathbb{N}}$ is bounded in $\Psi^0(\mathbb{R}^d)$. In particular, this implies that the family $(\psi S_M[(1-\tilde{\psi})(\cdot)])_{M\in\mathbb{N}}$ is bounded in $\Psi^{-\infty}(\mathbb{R}^d)$, as ΨDOs are smoothing outside of the diagonal, see Definition 2.1.10. Thus, the family $(\psi S_M[(1-\tilde{\psi})b])_{M\in\mathbb{N}}$ is bounded in $\mathscr{S}(\mathbb{R}^d)$. Let $(a_{M_k})_{k\in\mathbb{N}}$ be a subsequence of $(2_M)_{M\in\mathbb{N}}$. As $\mathscr{S}(\mathbb{R}^d)$ is a Montel space, see Remark 1.3.3, and $(a_{M_k})_{k\in\mathbb{N}}$ is bounded, there exists a subsequence $(a_{M_{k_j}})_{j\in\mathbb{N}} \subset (a_{M_k})_{k\in\mathbb{N}}$ and $a \in \mathscr{S}(\mathbb{R}^d)$ such that

$$a_{M_{k_j}} \xrightarrow[j\to+\infty]{} a \text{ in } \mathscr{S}(\mathbb{R}^d).$$

By the continuous inclusion $\mathscr{S}(\mathbb{R}^d) \hookrightarrow \mathscr{S}'(\mathbb{R}^d)$, see Remark 1.2.15, and the uniqueness of limits in $\mathscr{S}'(\mathbb{R}^d)$, this implies that $a = 0$. In particular, by Urysohn's lemma, see [44, Remark 2.1.17, p. 225], this entails that $(2_M) \to 0$ as $M \to +\infty$ in $\mathscr{S}(\mathbb{R}^d)$.

Regarding the term (1_M), we note that, by construction, for every $M \in \mathbb{N}$ there exists $C > 0$ such that

$$Q_M(\eta) - 1 = 0 \text{ for every } \eta \in \mathbb{R}^d \text{ with } |\eta| \leq C2^M. \tag{3.48}$$

Thus, we have that

$$\begin{aligned}
\left|\mathscr{F}[\psi(S_M - Id)(\tilde{\psi}b)](\xi)\right| &= \left|\mathscr{F}\left\{\psi\mathscr{F}^{-1}[(Q_M-1)\mathscr{F}(\tilde{\psi}b)]\right\}(\xi)\right| = \\
&= \left|(2\pi)^{-d/2}\left\{\mathscr{F}(\psi) \star [(Q_M-1)\mathscr{F}(\tilde{\psi}b)]\right\}(\xi)\right| = \\
&= (2\pi)^{-d/2}\left|\int_{\mathbb{R}^d} \hat{\psi}(\xi-\eta)(Q_M(\eta)-1)\mathscr{F}\left(\tilde{\psi}b\right)(\eta)d\eta\right| \leq \\
&\leq (2\pi)^{-d/2}\left|\int_{\{|\eta|>C2^M\}\cap W} \hat{\psi}(\xi-\eta)(Q_M(\eta)-1)\mathscr{F}\left(\tilde{\psi}b\right)(\eta)d\eta\right| + \\
&+(2\pi)^{-d/2}\left|\int_{\{|\eta|>C2^M\}\cap \mathscr{C}W} \hat{\psi}(\xi-\eta)(Q_M(\eta)-1)\mathscr{F}\left(\tilde{\psi}b\right)(\eta)d\eta\right| \doteq (1'_M) + (1''_M),
\end{aligned}$$

where we have used Remark 1.2.29 in the second equality and Equation (3.48) in the fourth step. Let us now focus on $(1'_M)$. As $supp(\tilde{\psi}) \times W$ is disjoint from $WF(b)$, $1_W \mathscr{F}\left(\tilde{\psi}b\right)$ decays faster than any polynomial. Let ν, D_ν be such that $\left|\hat{\psi}(\eta)\right| \leq D_\nu \langle\eta\rangle^{-\nu}$, let $N > 1 + d + \nu$ and let $C_N > 0$ be such that $\left|\mathscr{F}\left(\tilde{\psi}b\right)(\eta)\right| \leq$

$C_N\langle\eta\rangle^{-N}$ for every $\eta \in W$. Then, using the fact that $|Q_M(\eta)| \leq 1$ for every $\eta \in \mathbb{R}^d$, we have that

$$\begin{aligned}(1'_M) &\leq (2\pi)^{-d/2} C_N D_\nu \int_{\{|\eta|>C2^M\}\cap W} |(Q_M(\eta)-1)|\,\langle\xi-\eta\rangle^{-\nu}\langle\eta\rangle^{-N} d\eta \leq \\ &\leq 2^{1+\nu}(2\pi)^{-d/2} C_N D_\nu \langle\xi\rangle^{-\nu} \int_{\{|\eta|>C2^M\}} \langle\eta\rangle^{l-N} d\eta = \\ &= 2^{1+\nu}(2\pi)^{-d/2} C_N D_\nu \|\langle\cdot\rangle^{\nu-N} 1_{\{|\eta|>C2^M\}}\|_{L^1(\mathbb{R}^d)}\langle\xi\rangle^{-\nu}.\end{aligned}$$

As $\langle\cdot\rangle^{\nu-N} \in L^1(\mathbb{R}^d)$ for every $N > 1+d+\nu$, this entails that

$$\langle\xi\rangle^{\nu}(1'_M) \leq C'_\nu \|\langle\cdot\rangle^{\nu-N} 1_{\{|\eta|>C2^M\}}\|_{L^1(\mathbb{R}^d)} \xrightarrow[M\to+\infty]{} 0,$$

for a suitable constant $C'_\nu > 0$ and for every ν. Let us now focus on the term $(1''_M)$. By our choices of V and W, we may invoke Lemma 3.1.1 to state that there exists $c > 0$ such that

$$|\xi-\eta| \geq c(|\xi|+|\eta|),$$

for every $\xi \in V$ and $\eta \notin W$. Thus, let ν, E_ν be such that $|\hat{\psi}(\eta)| \leq E_\nu(1+|\eta|)^{-\nu}$ and let $N > 1+d+\nu$ and C_N be as above. In analogy with the proof of Proposition 3.1.2, we have that

$$\begin{aligned}(1''_M) &\leq (2\pi)^{-d/2} C_N E_\nu \int_{\{|\eta|>C2^M\}\cap \mathscr{C}W} |Q_M(\eta)-1|\,(1+c|\xi|+c|\eta|)^{-\nu}\langle\eta\rangle^{-N} d\eta \leq \\ &\leq 2(2\pi)^{-d/2} C_N E_\nu (1+c|\xi|)^{-\nu} \int_{\{|\eta|>C2^M\}} \langle\eta\rangle^{-N} d\eta \leq \\ &\leq C''_\nu \|\langle\cdot\rangle^{-N} 1_{\{|\eta|>C2^M\}}\|_{L^1(\mathbb{R}^d)}\langle\xi\rangle^{-\nu},\end{aligned}$$

for some fixed constant $C''_\nu > 0$. Hence, as in the previous case, we have that

$$\langle\xi\rangle^{\nu}(1''_M) \leq C''_\nu \|\langle\cdot\rangle^{-N} 1_{\{|\eta|>C2^M\}}\|_{L^1(\mathbb{R}^d)} \xrightarrow[M\to+\infty]{} 0,$$

for every ν. In particular, this shows that

$$\sup_{\xi\in V}\left|\langle\xi\rangle^{\nu}\mathscr{F}[\psi(S_M - Id)(\tilde{\psi}b)](\xi)\right| \xrightarrow[M\to+\infty]{} 0,$$

for every ν. Thus, together with the estimates on (2_M), we have shown that

$$q^{\nu}_{\psi,V}(c_M) = \sup_{\xi\in V}|\langle\xi\rangle^{\nu}\mathscr{F}[\psi(c_M)](\xi)| \xrightarrow[M\to+\infty]{} 0,$$

for every ν. By the arbitrariness of ψ and V, this entails that $S_M(b)$ converges to b in $\mathscr{D}'_\Gamma(\Omega)$ as we had claimed. For every $j \in \{1, ..., N\}$, $l \in \{1, ..., m_j\}$ and for every $\xi \in \mathbb{R}^d$, let us now fix

$$h_{j,l}(\xi) \doteq 1 - g_{j,l}(-\xi),$$

so that

$$\Phi T(c_M) = \sum_{l=1}^{m_j} \Phi\theta_{j,l} h_{j,l}(D)(\theta_{j,l} c_M). \tag{3.49}$$

By means of a compactness and a partition of unity argument analogous to those used in the proof of Proposition 3.1.2, let $V_{j,l,r} \subset \mathbb{R}^d \setminus \{0\}$ be closed cones for $r \in \{1, ..., n_{j,l}\}$ such that

$$supp(\theta_{j,l}) \times V_{j,l,r} \cap \Gamma = \emptyset,$$

for every $j \in \{1, ..., N\}, l \in \{1, ..., m_j\}, r \in \{1, ..., n_{j,l}\}$ and let

$$h_{j,l} = h_{j,l,0} + \sum_{r=1}^{n_{j,l}} h_{j,l,r}, \tag{3.50}$$

where $h_{j,l,0} \in \mathscr{D}(B(0,1))$ and $h_{j,l,r} \in S^0(\mathbb{R}^d)$ with $supp(h_{j,l,r}) \subset V_{j,l,r}$ for every $r \in \{1, ..., n_{j,l}\}$. Firstly, let us note that, as $c_M \to 0$ in $\mathscr{D}'_\Gamma(\Omega)$, we have that $\mathscr{F}[\theta_{j,l} c_M] \to 0$ uniformly on every compact subset of $\mathbb{R}^d$ as a consequence of Remark 1.2.28. Thus, we have that

$$\begin{aligned} &|h_{j,l,0}(D)(\theta_{j,l} c_M)(x)| \leq \\ &\leq (2\pi)^{-d} \int_{\mathbb{R}^d} |h_{j,l,0}(\xi)| \left|\mathscr{F}[\theta_{j,l} c_M](\xi)\right| d\xi \leq \\ &\leq (2\pi)^{-d} \sup_{\xi \in supp(h_{j,l,0})} \left|\mathscr{F}[\theta_{j,l} c_M](\xi)\right| \|h_{j,l,0}\|_{L^1(\mathbb{R}^d)}, \end{aligned}$$

where we have used Hölder's inequality, see [5, Theorem 4.6], in the second step. As $h_{j,l,0} \in \mathscr{D}(\Omega)$, the last term goes to 0 as $M \to +\infty$. In particular, this entails that

$$\Phi\theta_{j,l} h_{j,l,0}(D)(\theta_{j,l} c_M) \xrightarrow[M \to +\infty]{} 0 \text{ in } \mathscr{D}(\Omega).$$

Then, for any $b \in \mathbb{N}^d$, we note that

$$\begin{aligned}\left|\partial^b h_{j,l,r}(D)(\theta_{j,l}c_M)(x)\right| = \left|\mathscr{F}\left\{(\cdot)^b h_{j,l,r}\mathscr{F}[\theta_{j,l}c_M]\right\}(x)\right| \leq \\ \leq \int_{V_{j,l,r}} \langle\xi\rangle^{|b|}|h_{j,l,r}(\xi)|\left|\mathscr{F}(\theta_{j,l}c_M)(\xi)\right| d\xi \leq \\ \leq \|h_{j,l,r}\|_{L^\infty(\mathbb{R}^d)}\int_{V_{j,l,r}} \langle\xi\rangle^{|b|+1+d}\left|\mathscr{F}(\theta_{j,l}c_M)(\xi)\right|\langle\xi\rangle^{-1-d} d\xi \leq \\ \leq \|h_{j,l,r}\|_{L^\infty(\mathbb{R}^d)}\|\langle\cdot\rangle^{-1-d}\|_{L^1(\mathbb{R}^d)} q^{|b|+1+d}_{\theta_{j,l},V_{j,l,r}}(c_M).\end{aligned}$$

As $c_M \to 0$ in $\mathscr{D}'_\Gamma(\Omega)$, this implies that $h_{j,l,r}(D)(\theta_{j,l}c_M)$ goes to 0 as $M \to +\infty$ uniformly on $\mathbb{R}^d$ together with all of its derivatives. Thus, in particular,

$$\Phi\theta_{j,l}h_{j,l,r}(D)(\theta_{j,l}c_M) \xrightarrow[M\to+\infty]{} 0 \text{ in } \mathscr{D}(\Omega).$$

Substituting back into Equation (3.50), this yields that

$$\Phi\theta_{j,l}h_{j,l}(D)(\theta_{j,l}c_M) \xrightarrow[M\to+\infty]{} 0 \text{ in } \mathscr{D}(\Omega),$$

for every choice of j and l. Therefore, together with Equation (3.49), this implies that

$$\Phi T(S_M(b)-b) \xrightarrow[M\to+\infty]{} 0 \text{ in } \mathscr{D}(\Omega),$$

which is equivalent to the sought result. □

Lemma 3.3.4 entails that, for a fixed $\Phi_j \in \mathscr{D}(\Omega)$[4] which is identically equal to 1 on a neighbourhood of K_j,

$$\begin{aligned}\lim_{M\to+\infty}\langle\varphi_j u\,,\,(Id-B_j)[S_M(\varphi_j v)]\rangle_E = \lim_{M\to+\infty}\langle\varphi_j u\,,\,\Phi_j(Id-B_j)[S_M(\varphi_j v)]\rangle_D = \\ = \langle\varphi_j u\,,\,\Phi_j(Id-B_j)(\varphi_j v)\rangle_D = \langle\varphi_j u\,,\,(Id-B_j)(\varphi_j v)\rangle_E.\end{aligned}$$

Substituting this back into Equation (3.45), we obtain that

$$\exists \lim_{M\to+\infty} T^M_v(u) = \sum_{j=1}^N \langle A_j(\varphi_j u)\,,\,\varphi_j v\rangle_B + \langle\varphi_j u\,,\,(Id-B_j)(\varphi_j v)\rangle_E.$$

In particular, since the limit exists and is finite for every $u \in \mathscr{D}'^{\alpha}_L(\Omega)$, this entails that $|T_v(u)| < +\infty$ and that

$$T_v(u) = \sum_{j=1}^N \langle A_j(\varphi_j u)\,,\,\varphi_j v\rangle_B + \langle\varphi_j u\,,\,(Id-B_j)(\varphi_j v)\rangle_E,$$

[4]Is the function Φ_j even necessary? Upon further thoughts, I don't think so, as $Id-B_j$ is properly supported, but it doesn't hurt to leave it, does it?

for every $u \in \mathscr{D}'^{\alpha}_{L}(\Omega)$. Thus, we have our desired decomposition for T_v.

Step 3: Continuity estimates. By Theorem 1.3.5, we know that the map

$$\langle \cdot\,,\,\cdot\rangle_B : B^{\alpha}_{p,q}(\mathbb{R}^d)\times B^{-\alpha}_{p',q'}(\mathbb{R}^d)\to\mathbb{C},$$

is jointly continuous for every $\alpha\in\mathbb{R}$ and $p,q\in[1,+\infty]$. Thus, for every $f\in B^{\alpha}_{\infty,\infty}(\mathbb{R}^d)$ and $g\in B^{-\alpha}_{1,1}(\mathbb{R}^d)$,

$$|\langle f\,,\,g\rangle_B|\le \|f\|_{B^{\alpha}_{\infty,\infty}(\mathbb{R}^d)}\|g\|_{B^{-\alpha}_{1,1}(\mathbb{R}^d)}.$$

Thus, by Equation (3.46) there exist $p_1,\dots,p_n\in\mathfrak{sn}^{\alpha}_{L}(\Omega)$ such that

$$|\langle A_j(\varphi_j u)\,,\,\varphi_j v\rangle_B|\le \|\varphi_j v\|_{B^{-\alpha}_{1,1}(\mathbb{R}^d)}\sum_{k=1}^{n}p_k(u).$$

In particular, by continuity of the multiplication by φ_j, see [6, Theorem 4.37], there exists $C_j>0$ such that

$$|\langle A_j(\varphi_j u)\,,\,\varphi_j v\rangle_B|\le C_j\|v\|_{B^{-\alpha}_{1,1}(\mathbb{R}^d)}\sum_{k=1}^{n}p_k(u).$$

Let us consider φ_j and $r_j\doteq(Id-B_j)(\varphi_j v)\in\mathscr{D}(\Omega)$ fixed and let

$$\mathscr{D}'(\Omega)\ni u\mapsto \mathfrak{p}^j_v(u)\doteq\left|\langle\varphi_j u\,,\,r_j\rangle_E\right|\in[0,+\infty).$$

Then, $\mathfrak{p}^j_v\in\mathfrak{sn}(\Omega)$ and

$$\langle\varphi_j u\,,\,(Id-B_j)(\varphi_j v)\rangle_E\le\mathfrak{p}^j_v(u),$$

for every $u\in\mathscr{D}'^{\alpha}_{L}(\Omega)$. Therefore, we have that

$$|T_v(u)|\le\sum_{j=1}^{N}\mathfrak{p}^j_v(u)+C_j\|v\|_{B^{-\alpha}_{1,1}(\mathbb{R}^d)}\sum_{k=1}^{n}p_k(u).$$

Hence, T_v is continuous.

Step 4: Hypocontinuity. We begin by noting that, as $\langle\cdot\,,\,\cdot\rangle_B$ is a duality as per Definition 1.1.22, then so is $\langle\cdot\,,\,\cdot\rangle_{MB}$. Thus, the only remaining claim to be proven is its hypocontinuity, see Definition 1.1.14. To this avail, let us first prove that the family $\{T_v\mid v\in B\}$ is equicontinuous whenever $B\subset\mathscr{E}'^{-\alpha}_{M}(\Omega)$ is bounded. Let B be such a bounded set. By Theorem 1.1.14, there exists $(K,\Gamma)\subset\mathfrak{Wf}(M)$ such that $B\subset\mathscr{E}'^{-\alpha}_{K,\Gamma}(\Omega)$. In particular, for every $v\in B$, $supp(v)\subset K$ and $WF(v)\subset\Gamma$. We note that the constructions of the functions

$\varphi_1, ..., \varphi_N \in \mathscr{D}(\Omega)$ and of the operators $A_1, ..., A_N \in \Psi^0(\Omega)$ for T_v only depended on the support and wavefront set of v. Thus, they can be chosen uniformly in v as it varies in B. Furthermore, let

$$\tilde{B}_j \doteq \{(Id - B_j)(\varphi_j v) \mid v \in B\}.$$

As $Id - B_j$ is properly supported and of order $-\infty$ on M, it is also continuous $\mathscr{E}'_M(\Omega) \to \mathscr{D}(\Omega)$, see [42, Corollary 5.4]. Thus, by Theorem 1.1.4, $\tilde{B}_j$ is a bounded subset of $\mathscr{D}(\Omega)$. Therefore, by Remark 2.2.21, the map

$$\mathscr{D}'(\Omega) \ni u \mapsto \mathfrak{p}^j_{\tilde{B}}(u) \doteq \sup_{\chi \in \tilde{B}} |\langle u, \chi \rangle| \in [0, +\infty),$$

is a continuous seminorm on $\mathscr{D}'(\Omega)$. Furthermore, for every $j \in \{1, ..., M\}$ and for any $v \in B$,

$$\mathfrak{p}^j_v(u) \leq \mathfrak{p}^j_{\tilde{B}}(u),$$

for all $u \in \mathscr{D}'(\Omega)$. Lastly, if B is bounded in $\mathscr{E}'^{-\alpha}_{K,\Gamma}(\Omega)$, then it is also a bounded subset of $B^{-\alpha}_{1,1}(\mathbb{R}^d)$, which implies that $C \doteq \sup_{v \in B} \|v\|_{B^{-\alpha}_{1,1}(\mathbb{R}^d)} < +\infty$. Thus,

$$|T_v(u)| \leq \sum_{j=1}^{N} \mathfrak{p}^j_{\tilde{B}}(u) + C_j C \sum_{k=1}^{n} p_k(u).$$

This implies that the family $\{T_v \mid v \in B\}$ is equicontinuous as per Definition 1.1.13 and Remark 1.1.13.

Let us now prove that the family $\{T_{(\cdot)}(u) \mid u \in D\}$, where $T_{(\cdot)}(u) : v \mapsto T_v(u)$ for every $u \in \mathscr{D}'^{\alpha}_L(\Omega)$ and $v \in \mathscr{E}'^{-\alpha}_M(\Omega)$, is equicontinuous whenever $D \subset \mathscr{D}'^{\alpha}_L(\Omega)$ is bounded. Let D be such a bounded set and let

$$\tilde{D} \doteq \{\varphi_j u \mid u \in D\} \subset \mathscr{E}'(\Omega).$$

As $\mathscr{D}'^{\alpha}_L(\Omega) \hookrightarrow \mathscr{D}'(\Omega)$ and multiplication by a smooth, compactly supported function is continuous, $\tilde{D}$ is a bounded subset of $\mathscr{E}'(\Omega) \hookrightarrow \mathscr{D}'(\Omega)$. Let

$$\mathscr{E}'_M(\Omega) \ni v \mapsto \mathfrak{p}^j_{\tilde{D}}(v) \doteq \sup_{u \in \tilde{D}} |\langle u\,,\,(Id - B_j)(\varphi_j v)\rangle_E| \in [0, +\infty],$$

see Definition 2.2.13. We claim that $\mathfrak{p}^j_{\tilde{D}}$ is a continuous seminorm on $\mathscr{E}'_M(\Omega)$. Indeed, by [42, Corollary 5.4], as the operator $\varphi_j(Id - B_j)(\varphi_j \cdot)$ is of order $-\infty$ on M by construction, we have that it is continuous as a map $\mathscr{D}'_M(\Omega) \to \mathscr{D}(\Omega)$. Thus, by the construction of $\mathscr{E}'_M(\Omega)$, we have that

$$\varphi_j(Id - B_j)(\varphi_j \cdot) : \mathscr{E}'_M(\Omega) \to \mathscr{D}(\Omega),$$

is continuous. Furthermore, as $\mathscr{D}(\Omega)$ is barrelled and bornological, see Theorem 1.2.1, the Banach-Steinhaus Theorem 1.1.8 entails that the set D is equicontinuous. Thus, the map

$$\mathscr{D}(\Omega) \ni \chi \mapsto \mathfrak{q}(\chi) \doteq \sup_{u \in D} |\langle u\,,\,\chi\rangle_D|\,,$$

is a continuous seminorm. Then, as

$$\mathfrak{p}^j_{\tilde{D}}(v) = \mathfrak{q}(\varphi_j(Id - B_j)(\varphi_j v)),$$

we have that $\mathfrak{p}^j_{\tilde{D}}$ is a continuous seminorm on $\mathscr{E}'_M(\Omega)$. Furthermore, we have that $\mathfrak{p}^j_v(u) \leq \mathfrak{p}^j_{\tilde{D}}(v)$ for every $u \in D$ and that $E_k \doteq \sup_{u \in D} p_k(u)$ for every $k \in \{1, ..., n\}$. Hence,

$$|T_v(u)| \leq \sum_{j=1}^{N} \mathfrak{p}^j_{\tilde{D}}(v) + C_j \|v\|_{B^{-\alpha}_{1,1}(\mathbb{R}^d)} (\sum_{k=1}^{n} E_k),$$

for every $u \in D$ and $v \in \mathscr{E}'^{-\alpha}_M(\Omega)$. This implies that the family $\{T_{(\cdot)}(u) \mid u \in D\}$ is equicontinuous, which concludes the proof. □

Remark 3.3.4. In Remark 3.3.3, we anticipated that the duality of Theorem 3.3.3 is a generalization of the Besov duality of Theorem 1.3.5. Indeed, if $L = \emptyset$, then $\mathscr{D}'^{\alpha}_L(\Omega) = B^{\alpha,loc}_{\infty,\infty}(\Omega)$ and $\mathscr{E}'^{-\alpha}_M(\Omega) = B^{-\alpha,c}_{1,1}(\Omega)$ topologically, where $B^{\beta,c}_{1,1}(\Omega)$ is the space of compactly supported elements of $B^{\beta}_{1,1}(\mathbb{R}^d)$ whose support is contained in Ω, with the inductive limit topology induced by the collection of spaces $B^{\beta}_{1,1;K}(\mathbb{R}^d)$ for $K \in \mathfrak{K}_\Omega$, see [6, Definition 5.22]. Then, $\varphi_j u \in B^{\alpha}_{\infty,\infty}(\mathbb{R}^d)$, $\varphi_j v \in B^{-\alpha}_{1,1}(\mathbb{R}^d)$ and

$$\langle \varphi_j u\,,\,\varphi_j v\rangle_{MB} = \langle \varphi_j u\,,\,\varphi_j v\rangle_B,$$

follows from Equation (3.42). □

Remark 3.3.5. *A priori*, one may wonder if the continuity of the microlocal Besov duality as per Theorem 3.3.3 can be strengthened. More precisely, if it can be rendered into a jointly continuous map, see Definition 1.1.14. However, as was already noted in [40, Page 25], this is in general not possible for spaces of distributions. In fact, a stronger statement holds, namely that a hypocontinuous duality can be jointly continuous only between normed spaces, see [21, Page 359]. Irregardless of this obstruction, the hypocontinuous duality of Theorem 3.3.3 is a much more powerful tool than it might appear. Indeed, as it was shown in [40], it provides a path towards proving a result of paramount importance, namely the continuity of the pullback. □

Remark 3.3.6. We wish to take into consideration two other fundamental operations on distributions, the tensor product and the pullback by a smooth map. Let $\Omega \subset \mathbb{R}^d$ be an open set, let $L, M \subset \Omega \times \mathbb{R}^d \setminus \{0\}$ be closed conic sets and let $\alpha, \beta \in \mathbb{R}$ be fixed. By Theorem 2.2.16, if $u \in \mathscr{D}'^{\alpha}_L(\Omega)$ and $v \in \mathscr{D}'^{\beta}_M(\Omega)$, then $u \otimes v \in \mathscr{D}'^{\gamma}_{L\otimes M}(\Omega)$, where $\gamma = \min\{\alpha, \beta, \alpha+\beta\}$ and

$$L \otimes M \doteq (L \times M) \cup (\Omega \times \{0\} \times M) \cup (L \times \Omega \times \{0\}).$$

Thus, one has the inclusion

$$\mathscr{D}'^{\alpha}_L(\Omega) \otimes \mathscr{D}'^{\beta}_M(\Omega) \subset \mathscr{D}'^{\gamma}_{L\otimes M}(\Omega \times \Omega).$$

However, proving the continuity of this inclusion with respect to the Hörmander-Besov topologies of these spaces as per Definition 3.1.3 is a different matter entirely. A possible path would be to prove an analogue of [40, Lemma 5.2] in the Besov setting, but the non-monotonic nature of the seminorms in $\mathfrak{sn}^{\alpha}_L(\Omega)$, due to the presence of an external Fourier anti-transform, poses some technical challenges.

Let $O \subset \mathbb{R}^d$ and $U \subset \mathbb{R}^n$ be open sets and let $f \in C^\infty(O, U)$. Let $L \subset U \times \mathbb{R}^n \setminus \{0\}$ be a closed conic set as per Definition 2.2.1 and let $f^*[L]$ be as per Theorem 2.2.13. Then, if $\mathscr{N}_f \cap L = \varnothing$, we have that

$$f^* : \mathscr{D}'^{\alpha}_L(U) \to \mathscr{D}'^{\alpha}_{f^*[L]}(O),$$

is a well-defined extension of the pullback between smooth functions, see Definition 2.2.3, for every $\alpha > 0$. Analogously to the tensor product, however, proving the continuity of f^* is challenging. The main obstacle[5] in adapting the methodology employed in the proof of [40, Theorem 7.1], is in fact the continuity of the tensor product map, namely:

$$\mathscr{D}'^{\alpha}_L(O) \times \mathscr{D}'^{\beta}_M(U) \ni (u, v) \mapsto u \otimes v \in \mathscr{D}'^{\gamma}_{L\otimes M}(O \times U),$$

for any $\alpha, \beta \in \mathbb{R}$, $L \subset O \times \mathbb{R}^d \setminus \{0\}$ and $M \subset U \times \mathbb{R}^d \setminus \{0\}$ closed conic sets and where $\gamma = \min\{\alpha, \beta, \alpha+\beta\}$ and

$$L \otimes M \doteq (L \times M) \cup (O \times \{0\} \times M) \cup (L \times U \times \{0\}).$$

□

[5]There's an obstacle in particular that one would have liked to have devoted some time to, if it weren't for these pesky deadlines. But that someone will have a good part of next month free to look into it, so they won't complain... too much.

Remark 3.3.7. As we noted in Remark 3.3.1, the space $\mathscr{E}_M'^{-\alpha}(\Omega)$ we used in Theorem 3.3.3 cannot be the dual space of $\mathscr{D}_L'^{\alpha}(\Omega)$ for every choice of $\alpha \in \mathbb{R}$ and $L \subset \Omega \times \mathbb{R}^d \setminus \{0\}$. We wish to note, however, that a result on the duality could be achieved with the use of a wider class of Besov spaces. More specifically, for any $\alpha \in \mathbb{R}$ and $p, q \in [1, +\infty]$, let $b^{\alpha}_{p,q}(\mathbb{R}^d)$ be the topological closure of $\mathscr{S}(\mathbb{R}^d)$ with respect to the Besov norm $\|\cdot\|_{B^{\alpha}_{p,q}(\mathbb{R}^d)}$, see Definition 1.3.3. By Proposition 1.3.1, item *(B3)*, if $p, q < +\infty$, then $b^{\alpha}_{p,q}(\mathbb{R}^d) \equiv B^{\alpha}_{p,q}(\mathbb{R}^d)$, while $b^{\alpha}_{\infty,\infty}(\mathbb{R}^d) \subsetneq B^{\alpha}_{\infty,\infty}(\mathbb{R}^d)$. Furthermore, for every choice of α, p and q,

$$\left(b^{\alpha}_{p,q}(\mathbb{R}^d)\right)' = B^{-\alpha}_{p',q'}(\mathbb{R}^d),$$

where p' and q' are as per Theorem 1.3.5, see [25, Section 2.11.2, Remark 2]. Thus, if one was to construct a novel Besov wavefront set of type $b^{\alpha}_{\infty,\infty}$ and define its analogue space of distributions $u \in \mathscr{D}'(\Omega)$ whose $b^{\alpha}_{\infty,\infty}$-Besov wavefront set is prescribed within a conic region L, one could expect its dual space to be identified with $\mathscr{E}_M'^{-\alpha}(\Omega)$ by combining the construction in Theorem 3.3.3 with the duality arguments of [42, Theorem 3.20]. Establishing such a result would require a general theory of microlocal $b^{\alpha}_{\infty,\infty}$-regularity[6], which is at present absent in the literature. □

Remark 3.3.8. As a concluding remark, we wish to emphasize another possible direction for future research. More specifically, as we noted in Remark 3.1.1, it is already known how to generalize some of the constructions regarding the theory of wavefront sets to Banach or Fréchet functional spaces, see [37] and [38]. This procedure has the advantage of being fairly general and, thus, widely applicable. On the other hand, deriving results for abstract functional spaces is in general much more challenging than for specific ones. However, almost all of the results presented in this work are obtained in a largely abstract manner, in the sense that we have not extensively used many properties specific to Besov spaces. Thus, we believe this formalism can be successfully adapted to prove analogous results in the broader framework of spaces of distributions with a prescribed Banach or Fréchet wavefront set. □

[6]How many things can one (a non-specific one, but also the one of the previous footnote) realistically do in a month or so?

Conclusions

> "I've always hated the metaphor of the light at the end of the tunnel. Most change I've experienced in my life didn't happen in a day, and when it did it was usually less life altering then the change that took months."
>
> Taliesin Jaffe

In this thesis, we presented a novel topological construction on spaces of distributions with a prescribed Besov wavefront set, modelled on Hörmander's topology in the smooth setting. In the first chapter, we reviewed the theories of topological vector spaces, of distributions and of Besov spaces, which were of paramount importance for the rest of the work. In the second chapter, we focused on recalling some key concepts from microlocal analysis. More specifically, we recounted the theory of pseudodifferential operators, both in their global form on Euclidean space and on their local form on open sets. We also reviewed the theory of smooth wavefront sets, through their pointwise and pseudodifferential characterizations, of Besov wavefront sets and the properties of Hörmander's topology in the smooth case. Lastly, in the third chapter, we presented our adaptation of Hörmander's topology to spaces of distributions with a prescribed Besov wavefront set. Following the strategies developed in [42] and [41], we then proved some topological properties of these spaces equipped with their Hörmander-Besov topology by means of a topological embedding in a suitably chosen space of distributions. Furthermore, we identified a candidate for the dual of the aforementioned spaces and constructed a hypocontinuous duality that renders them into a dual pair.

It is worth noting that several questions regarding these spaces remain open and provide natural directions for further work. As discussed in the closing remarks of Section 3.3, we believe that both the tensor product and the pullback by a smooth function can be proven to be continuous operations with respect to the Hörmander-Besov topology. Further investigation into other structural operations, such as the push-forward by smooth maps and the product between distributions in suitably chosen spaces, warrants further research. Another topic that could prove to be fertile ground for research on this topic is the generalization to spaces of distributions with a prescribed wavefront set of Besov or Fréchet type, as we stated in Remark 3.3.8. Lastly, in [42], the authors also developed a theory of compactness in spaces of distributions with a prescribed Sobolev wavefront set, which we do not believe can be adapted into the endpoint Besov framework we based this work on. However, the concept of $b^{\alpha}_{\infty,\infty}$ wavefront set, see Remark 3.3.7, appears to be promising for developing an analogue of these compactness results in light of [45, Theorem 2.5].

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