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Introduction to extriangulated categories

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Abstract

Extriangulated categories have been introduced by Nakaoka and Palu to generalize two important families of categories: exact and triangulated categories. The notion of extriangulated category allows us to prove analogue theorems in a unified framework. We define exact categories, a generalization of abelian categories. After the proof of the main theorems, the focus shifts on generalization of concepts of abelian categories in the exact category theory. An important family of additive categories are idempotent complete and weakly idempotent complete categories which possess useful properties when studied as exact categories. Another important definition in exact category theory are acyclic complexes, a generalization of exact complexes of abelian categories. We shift focus on the homotopy category of an additive category and study some properties that are central to the definition of triangulated categories. A fundamental operation in category theory is localization which has a rich theory in triangulated framework. Another important example of localization are Serre quotients, obtained by localizing abelian categories by subcategories closed under extensions. We define extriangulated categories, which generalize exact and triangulated categories, and study the links with the latter two theories. We introduce localization of extriangulated categories which unifies localization of triangulated categories and Serre quotients.

Introduction

Exact and triangulated categories are two branches of additive category theory with vast applications in both algebra and geometry.
These types of categories are equipped with a class of distinguished short sequences, or triangles, and feature many analogue aspects.
Extriangulated categories, introduced by Nakaoka and Palu in [6], generalize exact and triangulated categories by focusing on the shared ingredient that are the triangles by using a biadditive functor $\mathbb{E}$ into **Ab** and for any pair of objects A, C a correspondence $\mathbf{s}$ from $\mathbb{E}(C, A)$ to the isomorphism classes of short sequences $A \to B \to C$.

Section 1 focuses on exact categories and follows the approach of [1]. Exact categories are additive categories with a class of admissible kernel-cokernel pairs closed under isomorphisms and whose push-out and pull-back result in another kernel-cokernel pair.
One major example are abelian categories which possess a natural exact structure given by all the kernel-cokernel pairs.
An important aspect of exact categories is the variety of exact structures that can be put on an additive category: abelian categories can be studied with a different exact structure to highlight particular behaviours that we may want to focus on.
After proving some central theorems of exact categories, we concentrate on quasi-abelian categories which are proved to possess a maximal exact structure.
A significant example of quasi-abelian category is the Banach spaces category: the maximal exact structure is naturally induced by the inclusion of closed subspaces as kernels and the cokernels are given by the quotient maps by closed subspaces.
We introduce the concept of idempotent completeness: idempotent complete categories are a bridge between additive categories and abelian categories.
Moreover, any additive category admits a canonical embedding into an idempotent completion that preserves many aspects of the starting category.
Weakly idempotent complete categories further generalize idempotent categories and have useful properties when studied in the exact framework. To generalize exact sequences in abelian categories, we introduce admissible morphisms that are a central piece in the study of homological algebra in the exact setting.

Section 2 studies some properties of the homotopy category $K(\mathscr{A})$ of cochain complexes in an additive category $\mathscr{A}$ which are generalized by triangulated categories, following [7].
In Section 3 we focus on triangulated categories which are categories equipped with an autoequivalence [1], known as shift functor, and a class of distinguished triangles $X \to Y \to Z \to X[1]$ and are a major branch of homological algebra.
The triangulated structure captures and generalizes the essence of long exact sequences which are at

the foundation of many algebraic and geometric subfields of mathematics.

A fundamental operation in category theory is localization: given a category $\mathscr{C}$ and a collection of morphisms S, the localization of $\mathscr{C}$ by S consists in a category obtained by adding inverses to morphisms of S.
In section 4 we focus on localization by multiplicative systems: under this assumption, the localized category admits a calculus of fractions that eases its study.
We concentrate on the link between full triangulated subcategories and multiplicative systems: given a full triangulated subcategory $\mathscr{D}$ of a triangulated category $\mathscr{C}$ we can associate a canonical multiplicative system $S(\mathscr{D})$.
The localization of $\mathscr{C}$ by $S(\mathscr{D})$ is called Verdier quotient of $\mathscr{C}$ by $\mathscr{D}$ and has an important universal property in the triangulated framework.
Section 5 focuses on a salient example of Verdier quotient which is the derived category of an exact category $\mathscr{A}$, defined as the Verdier quotient of $K(\mathscr{A})$ by the full triangulated subcategory of acyclic complexes.
Another central example of multiplicative systems are those associated to Serre subcategories, studied as in [3], that are full extension-closed subcategories of abelian categories.

Section 6 focuses on extriangulated categories, also known as externally triangulated, which generalize exact categories and triangulated categories at the same time.
The main purpose of extriangulated categories is the study of analogue results for exact and triangulated categories. Furthermore, they provide a common language to deal with problems that link these theories.
An extriangulated category $\mathscr{C}$ is an additive category equipped with a biadditive functor $\mathbb{E} : \mathscr{C}^{op} \times \mathscr{C} \to Ab$, the external component, and a realization $\mathbf{s}$ that to every $\delta \in \mathbb{E}(C, A)$ associates a short sequence $A \xrightarrow{x} B \xrightarrow{y} C$.
The definition of extriangulated category strongly resembles the definition of triangulated category hence some proofs for triangulated categories can be repeated in the extriangulated framework.
After proving that exact and triangulated categories induce an extriangulated structure, we prove also that the extriangulated structure induced by an exact structure can be used to reconstruct the original exact structure and a similar result is true for triangulated categories.
The result is surprising since extriangulated categories (like triangulated categories) in general do not have push-outs or pull-backs.
Hence extriangulated categories are a faithful generalization since the induced extriangulated structure does not lose information when switching from the exact framework to the extriangulated framework and similarly for the triangulated aspect.
Section 7 concerns localization of extriangulated categories, as defined in [5]. Before introducing the main theorems regarding localization, it is necessary to introduce weakly extriangulated categories which are the type of categories obtained by localization.
We conclude by proving that extriangulated localization generalizes both Serre and Verdier quotients.

1 Exact categories

Definition 1.1 *Let $\mathscr{A}$ be an additive category. A kernel-cokernel pair (i, p) in $\mathscr{A}$ is a pair of composable morphisms*

$$A' \xrightarrow{i} A \xrightarrow{p} A''$$

such that i is a kernel of p and p is a cokernel of i. By fixing $\mathscr{E}$, a class of kernel-cokernel pairs, an admissible monic i is a morphism for which there exists p such that $(i,p) \in \mathscr{E}$. Admissible epics are defined by duality. Admissible monics will be depicted with $\rightarrowtail$ and admissible epics by $\twoheadrightarrow$ in diagrams.

We call an exact structure on $\mathscr{A}$ a class $\mathscr{E}$ of kernel-cokernel pairs closed under isomorphisms and satisfying the following axioms:

- ***(E0)*** *For all objects $A \in \mathscr{A}$, the identity morphism 1_A is an admissible monic.*
- ***(E0)^{op}*** *For all objects $A \in \mathscr{A}$, the identity morphism 1_A is an admissible epic.*
- ***(E1)*** *The class of admissible monics is closed under composition.*
- ***(E1)^{op}*** *The class of admissible epics is closed under composition.*
- ***(E2)*** *The push-out of an admissible monic along an arbitrary morphism exists and yields an admissible monic.*
- ***(E2)^{op}*** *The pull-back of an admissible epic along an arbitrary morphism exists and yields an admissible epic.*

An exact category is a pair $(\mathscr{A}, \mathscr{E})$ where $\mathscr{A}$ is an additive category and $\mathscr{E}$ is an exact structure on $\mathscr{A}$. The elements of $\mathscr{E}$ are called short exact sequences.

Remark 1.2 ***(E2)*** *and* ***(E2)^{op}*** *can be represented by the diagrams*

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & PO & \downarrow \\ A' & \dashrightarrow & B' \end{array} \qquad \begin{array}{ccc} A & \dashrightarrow\!\!\!\!\!\rightarrow & B \\ \downarrow & PB & \downarrow \\ A' & \twoheadrightarrow & B' \end{array}$$

Remark 1.3 *Because the axioms are self dual, $\mathscr{E}$ is an exact structure on $\mathscr{A}$ if and only if $\mathscr{E}^{op}$ is an exact structure on $\mathscr{A}^{op}$*

An exact category stands between additive categories and abelian categories but we can put many different exact structures on the same category $\mathscr{A}$.
For example, we can take an additive category and put the exact structure given by the class of (i, p) such that $(i, p) \cong (i', p')$ where (i', p') is such that

$$A \xrightarrow{i'} A \oplus B \xrightarrow{p'} B.$$

Remark 1.4 *Isomorphisms are admissible monics and admissible epics.*
This is proven by the following diagram

$$\begin{array}{ccccc} A & \xrightarrow[\cong]{f} & B & \longrightarrow & 0 \\ {\scriptstyle 1_A}\downarrow{\scriptstyle \cong} & & {\scriptstyle f^{-1}}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow \\ A & \rightarrowtail^{1_A} & A & \twoheadrightarrow & 0 \end{array}$$

and by the closedness of the exact structures under isomorphism and by self duality of the axioms.

Remark 1.5 *The axioms* ***(E0)*** *and* $\textbf{(E0)}^{op}$ *can be weakened: instead we could require that* 1_0 *is an admissible epic.*
We first must observe that 1_0 *is the kernel and the cokernel of itself.*
By the diagram above obtained through $\textbf{(E2)}^{op}$, 1_A *is an admissible epic. By the observation above and* ***(E2)****,* ***(E0)*** *holds as well.*

Lemma 1.6

$$A \overset{\binom{1}{0}}{\rightarrowtail} A \oplus B \overset{\begin{pmatrix}0 & 1\end{pmatrix}}{\twoheadrightarrow} B$$

is short exact.

Proof:

$$\begin{array}{ccc} 0 & \rightarrowtail & B \\ \downarrow & PO & \downarrow \\ A & \rightarrowtail & A \oplus B \end{array}$$

Since the diagram is a push-out square and the top arrow and the left hand-arrow are admissible monics, by $\textbf{(E0)}^{op}$, the bottom and right-hand arrows are admissible monics by ***(E2)***. Now, since they are in a kernel-cokernel pair and the pairs are closed under isomorphisms, we can take as cokernel the projection.

Proposition 1.7 *The direct sum of two short exact sequences is short exact.*

Proof Let $A' \rightarrowtail A \twoheadrightarrow A''$ and $B' \rightarrowtail B \twoheadrightarrow B''$ be two short exact sequences.
The first step is to observe that for every object C the sequence

$$A' \oplus C \rightarrowtail A \oplus C \twoheadrightarrow A''$$

is exact. The second morphism is the composition of $(1\ 0) : A \oplus C \twoheadrightarrow A$ and $A \twoheadrightarrow A''$ and then it is and admissible epic by 1.6 and $\textbf{(E1)}^{op}$. Since the first morphism of the sequence is a kernel of the second one, it is an admissible monic.
By taking $C = B'$ and by taking the composition of $A' \oplus B' \rightarrowtail A \oplus B'$ with $A \oplus B' \rightarrowtail A \oplus B$ we have an admissible monic.
It follows that the sequence

$$A' \oplus B' \rightarrowtail A \oplus B \twoheadrightarrow A'' \oplus B''$$

is an exact sequence since it is a kernel-cokernet pair.

Corollary 1.8 *The exact structure* $\mathscr{E}$ *is an additive subcategory of the additive category* $\textbf{Fun}(\cdot \to \cdot \to \cdot,\ \mathscr{A})$

Proposition 1.9 *Consider a commutative square*

$$\begin{array}{ccc} A & \overset{i}{\rightarrowtail} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ A' & \overset{i'}{\rightarrowtail} & B' \end{array}$$

in which the horizontal arrows are admissible monics. The following assertions are equivalent:

1. *The square is a push-out.*

2. *The sequence*

$$A \overset{\begin{pmatrix} i \\ -f \end{pmatrix}}{\rightarrowtail} B \oplus A' \xrightarrow{(f'\ i')} B'$$

is exact.

3. *The square is bicartesian, i.e. both a push-out and a pull-back.*

4. *The square is part of the commutative diagrams with exact rows*

$$\begin{array}{ccccc} A & \overset{i}{\rightarrowtail} & B & \overset{p}{\twoheadrightarrow} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} & & \| \\ A' & \overset{i'}{\rightarrowtail} & B' & \overset{p'}{\twoheadrightarrow} & C \end{array}$$

Proof: (*1* $\Rightarrow$ *2*): The push-out property is equivalent to the assertion that $(f'\ i')$ is a cokernel of $\begin{pmatrix} i \\ -f \end{pmatrix}$, so it is sufficient to show that the latter is an admissible monic. This follows since it is the composition of

$$A \overset{\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}}{\rightarrowtail} A \oplus A' \xrightarrow[\cong]{\begin{pmatrix} 1 & 0 \\ -f & 1 \end{pmatrix}} A \oplus A' \xrightarrow{\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}} B \oplus A'$$

The first morphism is an admissible monic because of 1.6, the second because of 1.4 and the third because of 1.7.
(*3* $\Rightarrow$ *1*): clear.
(*2* $\Rightarrow$ *3*): since it is a kernel-cokernel pair, the square is bicartesian.
(*1* $\Rightarrow$ *4*): Let $p : B \twoheadrightarrow C$ be the cokernel of i. By using the push-out property there is a unique morphism $p' : B' \twoheadrightarrow C$ such that $p'f'=p$ and $p'i'=0$. In particular p' is epic since p is epic.
We have to show that p' is a cokernel of i'. To prove this, let $g: B' \to X$ such that $gi'=0$.
By composing with f, we have $gf'i = gi'f = 0$. Since p is the cokernel of i, there exists a unique morphism $h : C \to X$ such that $gf' = hp$.
By using the push-out property of the square and $hp'f' = hp = gf'$ and $hp'i'=0=gi'$, we obtain $hp' = g$.
Because p' is epic, the factorization h of g is unique, so p' is a cokernel of i'.
(*4* $\Rightarrow$ *2*): By forming the pull-back over p and p', we obtain

$$\begin{array}{ccccc} & & A & = & A \\ & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle i} \\ A' & \overset{j'}{\rightarrowtail} & D & \overset{q'}{\twoheadrightarrow} & B \\ \| & & {\scriptstyle q}\downarrow & PB & \downarrow{\scriptstyle p} \\ A' & \overset{i'}{\rightarrowtail} & B' & \overset{p'}{\twoheadrightarrow} & C \end{array}$$

with exact rows and columns by using the dual of the (*1* $\Rightarrow$ *4*).
Since the square

$$\begin{array}{ccc} B & = & B \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle p} \\ B' & \overset{p'}{\twoheadrightarrow} & C \end{array}$$

is commutative, we have a unique morphism $k : B \to D$ such that $q'k = 1_B$ and $qk = f'$ by using the pull-back property. Since $q'(1_D - kq')=0$, we can factor through a unique $l : D \to A'$ such that $j'l = 1_D - kq'$ because j' is a kernel of q'.
By computing $j'lk = (1_D - kq')k = 0$ and since j' is monic, $lk = 0$. The calculation $j'lj' = (1_D - kq')j' = j'$ implies $lj' = 1_{A'}$ because j' is monic.

$$i'lj = (qj')lj = q(1_D - kq')j = -(qk)(q'j) = -f'i = -i'f$$

implies lj=-f since i' is monic.
We can define the morphisms $\begin{pmatrix} k & j' \end{pmatrix}$: $B \oplus A' \to D$ and $\begin{pmatrix} q' \\ l \end{pmatrix}$: $D \to B \oplus A'$.
They are mutually inverse since
$\begin{pmatrix} k & j' \end{pmatrix} \begin{pmatrix} q' \\ l \end{pmatrix} = kq' + j'l = 1_D$ and $\begin{pmatrix} q' \\ l \end{pmatrix} \begin{pmatrix} k & j' \end{pmatrix} = \begin{pmatrix} q'k & q'j' \\ lk & lj' \end{pmatrix} = \begin{pmatrix} 1_B & 0 \\ 0 & 1_{A'} \end{pmatrix}$.
By observating that
$\begin{pmatrix} f' & i' \end{pmatrix} = q \begin{pmatrix} k & j' \end{pmatrix}$ and $\begin{pmatrix} i \\ -f \end{pmatrix} = \begin{pmatrix} q' \\ l \end{pmatrix} j$,
we have an isomorphism between $A \overset{\begin{pmatrix} i \\ -f \end{pmatrix}}{\rightarrowtail} B \oplus A' \xrightarrow{(f' \ i')} B'$ and $A \xrightarrow{j} D \xrightarrow{q} B'$.

Remark 1.10

$$\begin{array}{ccc} A' & \xrightarrow{i'} & B' \\ {\scriptstyle a}\downarrow & PO & \downarrow{\scriptstyle b} \\ A & \xrightarrow{i} & B \end{array}$$

Let $j' : B' \twoheadrightarrow C'$ be a cokernel of i' then the unique morphism $j : B \twoheadrightarrow C'$ such that $ji = 0$ and $jb = j'$ is a cokernel of i. We also have that if $j : B \twoheadrightarrow C$ is a cokernel of i then $j' = jb$ is a cokernel of i'.

Proof: The first statement is obtained by the proof of $(1 \Rightarrow 4)$ of 1.9 .
Let $g : B \to X$ be a morphism such that $gi = 0$. It follows that $gbi' = 0$ so there is a unique $l : C' \to X$ such that $gb = lj'$. By using the push-out property, we have $g = lj$ since we have $gb = ljb$ and $g - lj = 0$ since $(g - lj)i = 0$.

Corollary 1.11

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \overset{g}{\rightarrowtail} & C \\ \downarrow{\scriptstyle a} & PB & {\scriptstyle b}\downarrow & PO & \downarrow{\scriptstyle c} \\ A' & \xrightarrow{f'} & B' & \overset{g'}{\rightarrowtail} & C' \end{array}$$

is bicartesian and $A \overset{\begin{pmatrix} -a \\ gf \end{pmatrix}}{\rightarrowtail} A' \oplus C \xrightarrow{(g'f' \ c)} C'$ is an exact sequence.

Proof: By using 1.9 and its dual we have that both squares are bicartesian. Clearly gluing two bicartesian squares yields another bicartesian square and the sequence is a kernel-cokernel pair.
By observing $\begin{pmatrix} g'f' & c \end{pmatrix} = \begin{pmatrix} g' & c \end{pmatrix} \begin{pmatrix} f' & 0 \\ 0 & 1_C \end{pmatrix}$
and by using 1.9 and 1.7, it follows that $\begin{pmatrix} g'f' & c \end{pmatrix}$ is an admissible epic.

Proposition 1.12 *The pull-back of an admissible monic along an admissible epic yields an admissible monic.*

Proof:

$$\begin{array}{ccccc} A' & \xrightarrow{i'} & B' & \xrightarrow{pe} & C \\ {\scriptstyle e'}\downarrow & PB & \downarrow{\scriptstyle e} & & \| \\ A & \overset{i}{\rightarrowtail} & B & \xrightarrow{p} & C \end{array}$$

The pull-back square exists because of ***(E2)***^{op}. Let p be the cokernel of i. pe is an admissible epic because p and e are.
In general, if it exists, the pull-back of a monic is a monic.
To show the admissibility of i', we want to prove that i' is kernel of pe. Let $g' : X \to B'$ be a morphism such that $peg' = 0$. There exists a unique $f : X \to A$ such that $eg' = if$ since i is a kernel of p.

By using the pull-back property, there exists a unique $f'\colon X \to A'$ such that $e'f' = f$ and $i'f' = g'$. f' is unique morphism such that $i'f' = g'$ since i' is monic.

Proposition 1.13 *(Obscure axiom). Suppose that $i : A \to B$ is a morphism in $\mathscr{A}$ which admits a cokernel. Let $j : B \to C$ in $\mathscr{A}$ be a morphism such that $ji : A \rightarrowtail C$ is an admissible monic. Then i is an admissible monic.*

Proof: Let $k : B \to D$ be a cokernel of i. By 1.9 and the diagram

$$\begin{array}{ccc} A & \overset{ji}{\rightarrowtail} & C \\ {\scriptstyle i}\downarrow & PO & \downarrow \\ B & \rightarrowtail & E \end{array}$$

it follows that $\begin{pmatrix} i \\ ji \end{pmatrix} : A \rightarrowtail B \oplus C$ is an admissible monic. In particular $\begin{pmatrix} i \\ 0 \end{pmatrix} = \begin{pmatrix} 1_B & 0 \\ -j & 1_C \end{pmatrix} \begin{pmatrix} i \\ ji \end{pmatrix}$ is an admissible monic since $\begin{pmatrix} 1_B & 0 \\ -j & 1_C \end{pmatrix}$ is an isomorphism.
Because $\begin{pmatrix} k & 0 \\ 0 & 1_C \end{pmatrix}$ is a cokernel of $\begin{pmatrix} i \\ 0 \end{pmatrix}$, it is an admissible epic.

$$\begin{array}{ccccc} A & \overset{i}{\longrightarrow} & B & \overset{k}{\longrightarrow} & D \\ \| & & {\scriptstyle \begin{pmatrix} 1 \\ 0 \end{pmatrix}}\downarrow & PB & \downarrow{\scriptstyle \begin{pmatrix} 1 \\ 0 \end{pmatrix}} \\ A & \underset{\begin{pmatrix} i \\ 0 \end{pmatrix}}{\rightarrowtail} & B \oplus C & \underset{\begin{pmatrix} k & 0 \\ 0 & 1_C \end{pmatrix}}{\twoheadrightarrow} & D \oplus C \end{array}$$

Since the right-hand square is a pull-back, k is an admissible epic. Since i is a kernel of k, i is an admissible monic.

Corollary 1.14 *Let (i,p) and (i',p') be two pairs of composable morphisms. If ($i \oplus i'$,$p \oplus p'$) is short exact then (i,p) and (i',p') are short exact.*

Proof: since i has p as a cokernel and by using that $\begin{pmatrix} 1 \\ 0 \end{pmatrix} i = \begin{pmatrix} i & 0 \\ 0 & i' \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an admissible monic and by 1.13, i is an admissible monic.

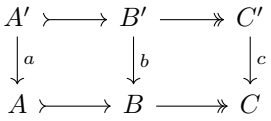
Proposition 1.15 *Let $(\mathscr{A}, \mathscr{E})$ be an exact category. A morphism from a short exact sequence $A' \rightarrowtail B' \twoheadrightarrow C'$ to another exact sequence $A \rightarrowtail B \twoheadrightarrow C$ factors through a short exact sequence $A \rightarrowtail D \twoheadrightarrow C'$*

$$\begin{array}{ccccc} A' & \overset{f'}{\rightarrowtail} & B' & \overset{g'}{\twoheadrightarrow} & C' \\ {\scriptstyle a}\downarrow & BC & \downarrow{\scriptstyle b'} & & \| \\ A & \overset{m}{\rightarrowtail} & D & \overset{e}{\twoheadrightarrow} & C' \\ \| & & \downarrow{\scriptstyle b''} & BC & \downarrow{\scriptstyle c} \\ A & \overset{f}{\rightarrowtail} & B & \overset{g}{\twoheadrightarrow} & C \end{array}$$

where D is isomorphic to the pushout $A \cup_{A'} B'$ and to the pull-back $B \times_C C'$ and the two BC-marked squares are bicartesian.

Proof: The first step is to use ***(E1)***^{op} on f' and a to obtain D and the morphisms m and b'.
We can find e: D → C' the unique morphism such that eb'=g and em=0 and let b":D → B such that b"b'=b: B' → B and b"m=f. By using 1.10, we have that e is a cokernel of m. We want to prove that the right-hand bottom square is commutative to use 1.9.
Since $gb"b' = gb = cg' = ceb'$ and $gb"m = gf = 0 = cem$, $gb" = ce$ because of the push-out property of the left-hand top square and hence the commutativity.

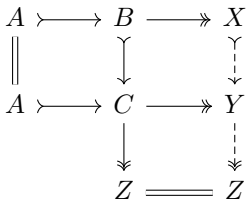
Corollary 1.16 *(Five lemma). Consider a morphism of short exact sequences.*



If a and c are isomorphisms (or admissible monics, or admissible epics), so is b.

Proof: Since pull-backs and push-outs preserve isomorphisms, applying 1.15 we have that b' and b'' are isomorphisms and since $b''b' = b$, b is an isomorphism.
Assume now that a and c are admissible monics. By using again 1.15, b' is an admissible monic because of ***(E2)***. b'' is an admissible monic because of 1.12 and so b is an admissible monic. The case of admissible epics is obtained by duality.

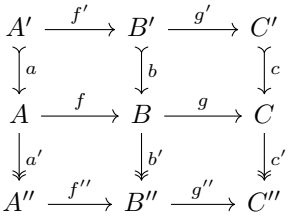
Lemma 1.17 *Consider the diagram*



in which the first two horizontal rows and the middle column are short exact. Then the diagram can be uniquely completed to obtain a commutative diagram and the third column is exact. In addition, the upper right-hand square is bicartesian.

Proof: The morphism $X \to Y$ exists unique because $A \to C \to Y$ is zero and the first row is exact. The unique morphism $Y \to Z$ is given by the exactness of the second row and because $B \to C \to Z$ is zero. Because of 1.9 we have that the upper right-hand square is bicartesian. It follows that $X \to Y$ is an admissible monic and that $Y \to Z$ is its kernel.

Corollary 1.18 *Consider a commutative diagram*



in which the columns are exact, then if any of the following conditions holds:

1. *the middle row and either one of outer rows is short exact;*

2. *the outer rows are short exact and gf=0*

then the remaining row is short exact as well.

Proof: Assume we have (1). By duality we may consider the case in which the first two rows are exact. By applying 1.15, we obtain the commutative diagram

$$\begin{array}{ccccc} A' & \overset{f'}{\rightarrowtail} & B' & \overset{g'}{\twoheadrightarrow} & C' \\ {\scriptstyle a}\downarrow & BC & \downarrow{\scriptstyle i} & & \| \\ A & \overset{\bar{f}}{\rightarrowtail} & D & \overset{\bar{g}}{\twoheadrightarrow} & C' \\ \| & & {\scriptstyle j}\downarrow & BC & \downarrow{\scriptstyle c} \\ A & \overset{f}{\rightarrowtail} & B & \overset{g}{\twoheadrightarrow} & C \end{array}$$

We have that i and j are admissible monic because of ***(E2)*** and 1.12 and $ji = b$. Thanks to 1.10, the morphism $i' : D \twoheadrightarrow A''$ determined by $i'i = 0$ and $i'\bar{f} = a'$ is a cokernel of i. Because of 1.10 we also find $j' : B \twoheadrightarrow C''$ such that $j' = c'g = g''b'$ and j' is a cokernel of j.
Given the diagram

$$\begin{array}{ccccc} B' & \overset{i}{\rightarrowtail} & D & \overset{i'}{\twoheadrightarrow} & A'' \\ \| & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle f''} \\ B' & \overset{b}{\rightarrowtail} & B & \overset{b'}{\twoheadrightarrow} & B'' \\ & & {\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle g''} \\ & & C'' & = & C'' \end{array}$$

by proving its commutativity, thanks to 1.17, we would obtain that (f'', g'') is a short exact sequence. By construction we have commutativity of the upper left-hand and of the bottom squares so we only need to show $f''i' = b'j$.
We are going to use the push-out property of the upper left-hand square of the diagram obtained through 1.15.
By calculations we have

$$(f''i')i = 0 = b'b = (b'j)i \quad \textit{and} \quad (b'j)\bar{f} = b'f = f''a' = (f''i')\bar{f}.$$

Since both $f''i'$ and $b'j$ are solutions to the same push-out problem, they must coincide.

Assume we are now in the second case.
By applying (**E2**) to g' and b, we obtain the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccc} A' & \overset{f'}{\rightarrowtail} & B' & \overset{g'}{\twoheadrightarrow} & C' \\ \| & & {\scriptstyle b}\downarrow & PO & \downarrow{\scriptstyle k} \\ A' & \overset{i}{\rightarrowtail} & B & \overset{j}{\twoheadrightarrow} & D \\ & & {\scriptstyle b'}\downarrow & & \downarrow{\scriptstyle k'} \\ & & B'' & = & B'' \end{array}$$

in which the cokernel k' of k is determined by $k'j = b'$ and $k'k = 0$ thanks to 1.10 and in which $i = bf'$ is a kernel of the admissible epic j, using 1.12. We can now apply the push-out property of the square $B'C'BD$ to the square $B'C'BC$. We have $d' : D \to C$ such that $d'k = c$ and $d'j = g$.
The diagram

$$\begin{array}{ccccc} C' & = & C' \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle c} \\ D & \overset{d'}{\longrightarrow} & C \\ {\scriptstyle k'}\downarrow & & \downarrow{\scriptstyle c'} \\ B'' & \overset{g''}{\twoheadrightarrow} & C'' \end{array}$$

has exact rows and is commutative. The commutativity of the upper square is because of construction since $c = d'k$. The commutativity of the bottom square is because of the fact that j is epic and because

of $c'd'j = c'g = g''b' = g''k'j$.
By applying the dual of 1.9 to $DCB''C''$, we find out that $DCB''C''$ is a pull-back. For this reason d' and $g = d'j$ are admissible epics.
By the dual of 1.10, the unique morphism $d : A'' \to D$ such that $k'd = f''$ and $d'd = 0$ is a kernel of d'. Given the diagram

$$\begin{array}{ccccc} A' & = & A' & & \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle i} & & \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow{\scriptstyle a'} & & \downarrow{\scriptstyle j} & & \| \\ A'' & \xrightarrow{d} & D & \xrightarrow{d'} & C \end{array}$$

we have its commutativity because of the pull-back property of $DCB''C''$ and because $k'(da') = f''a' = b'f = k'(jf)$ and $d'(da') = 0 = gf = d'(jf)$ because of the hypotheses.
By applying the dual of 1.9, $ABA''D$ is bicartesian, so by 1.12 f is a kernel of g.

1.1 Quasi-abelian Categories

Definition 1.19 *An additive category $\mathscr{A}$ is called quasi-abelian if:*

1. *Every morphism has a kernel and a cokernel.*

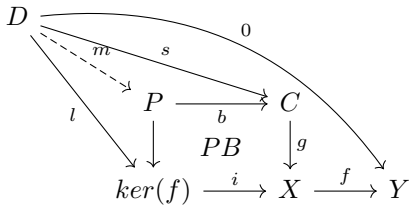
2. *The class of kernels is stable under push-out along arbitrary morphisms and the class of cokernels is stable under pull-back along arbitrary morphisms.*

Remark 1.20 *$\mathscr{A}$ is quasi-abelian if and only if $\mathscr{A}^{op}$ is quasi-abelian.*

Remark 1.21 *Let $\mathscr{A}$ be an preabelian category. If f is a kernel, it is also kernel of its cokernel and the dual statement is also true.*

Remark 1.22 *Let $\mathscr{A}$ be an additive category with kernels, then every pull-back of a kernel is a kernel.*

Proof:



We start by constructing the pull-back square of i and g. We need to show that b in the above diagram is a kernel. Suppose that i is the kernel of f and that we have $s : D \to C$ such that $(fg)s = 0 = f(gs)$. Then $gs = li$ with $l : D \to ker(f)$. By using the pull-back property with the maps $s : D \to C$ and $l : D \to ker(f)$, we have a unique morphism $m : D \to P$ such that $s = bm$.

Proposition 1.23 *The class $\mathscr{E}_{max}$ of all kernel-cokernel pairs in a quasi-abelian category is an exact structure.*

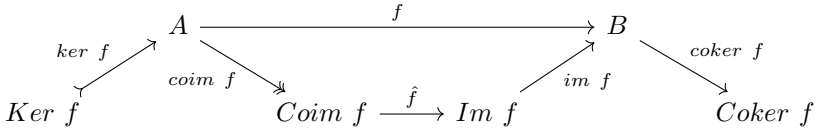
Proof: We start by recalling that if t is a kernel then it is also a kernel of its cokernel (and dually for cokernels).
$\mathscr{E}_{max}$ is closed under isomorphisms and since $i : 0 \to X$ and $j : X \to 0$ are respectively kernel and cokernel of 1_X for all X, ***E0*** and ***E0***^{op} hold.
(E2) and ***(E2)***^{op} are verified because of the definition. We have to prove only ***(E1)*** or ***(E1)***^{op} because of the duality. So assume we have $f : A \twoheadrightarrow B$ and $g : B \twoheadrightarrow C$ two cokernels and $h = gf$.

$$\begin{array}{ccccc} Ker\ f & \dashrightarrow^{u} & Ker\ h & \dashrightarrow^{v} & Ker\ g \\ \| & & \downarrow{\scriptstyle ker\ h} & & \downarrow{\scriptstyle ker\ g} \\ Ker\ f & \xrightarrow{ker\ f} & A & \xrightarrow{f} & B \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\ & & C & = & C \end{array}$$

The diagram is completed commutatively by two unique morphisms u and v.
Since the upper right-hand square is a pull-back because $h = gf$, v is a cokernel and u is its kernel.
By duality and because of the shape of the diagram, we have that the upper right-hand square is also a push-out and since h is epic, we have that h is a cokernel of $ker\ h$. This is because if we have $m : A \to X$ such that $m(ker\ h) = 0$ then we may apply the push-out property to m and to the zero map from $Ker\ g$ to X. We have $n : B \to X$ such that $nf = m$ and $n(ker\ g) = 0$.
Since g is a cokernel of $ker\ g$, there is a unique r such that $n = rg$. It follows that $rgf = m = rh$.

Remark 1.24 *We may observe that the proof of 1.23 is a particular case of 1.17.*

Definition 1.25 *The coimage of a morphism f in a category with all kernels and cokernels is defined as Coker(ker f); the image is defined to be Ker(coker f). We call analysis of f the following diagram*

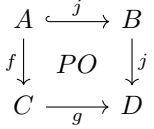


where $\hat{f}$ is uniquely determined to have a commutative diagram.

Remark 1.26 *It's important to remember that a category $\mathscr{A}$ with kernels and cokernels is abelian if and only if $\hat{f}$ is an isomorphism for all f.*

Proposition 1.27 *If $\mathscr{A}$ is an abelian category then it is also a quasi-abelian category.*

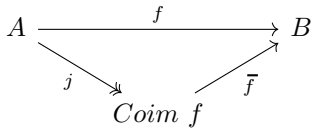
Proof: By definition of abelian category, every morphism of $\mathscr{A}$ has a kernel and a cokernel so (1) of 1.20 is verified.
So we just need to prove (2).
We can prove this by using Mitchell's embedding theorem:
By using this theorem and applying it to the small category spanned by



where j is a kernel and f is an appropriate morphism, we have to show that g is an injection in $R-Mod$ for some ring R.
By recalling that $D \cong \frac{B \oplus C}{\{j(x), f(x) | x \in A\}}$, g is the injection of C into $B \oplus C$ composed to the projection so $g = \pi i_C$ Suppose we have an $y \in C$ such that $g(y) = 0$. Because $g(y) = \pi i_C(y) = \pi((0, y))$, this implies that $(0, y) = (j(x'), f(x'))$. But since j is a monomorphism, it implies that $x' = 0$ and so $y = 0$. Hence g is a monomorphism and since $R - Mod$ is an abelian category, it is a kernel.

Proposition 1.28 *Let $\mathscr{A}$ be a quasi-abelian category. Given f a morphism, the morphism $\hat{f}$ which appears in the analysis of f is monic and epic.*

Proof: By duality, we may only check the $\bar{f}$ is monic in the diagram



To prove this, suppose we have $x : X \to Coim\ f$ such that $\bar{f}x = 0$. By pull-back of x along j we find $y : Y \to A$ that satisfies $fy = 0$. $jy = 0$ since y can be factored through $Ker\ f$. Since $Y \twoheadrightarrow X \to Coim\ f$ is zero, because of the commutativity of the diagram, $x = 0$.

Remark 1.29 *We may construct two epic-monic factorizations for every morphism f, one over $Coim\ f$ and one over Imf. In the abelian case, since $\bar{f}$ is an isomorphism for every f, the factorizations always coincide.*

A particular example of quasi-abelian category are Banach spaces. Let **Ban** be the category of Banach spaces with as morphisms the linear continuous maps.
To prove the quasi-abelianity of **Ban**, we can use 1.19.

Remark 1.30 *Let $f : X \to Y$ be a morphism of **Ban**. Then f has a kernel and cokernel.*

Proof: the kernel of f in **Ban** coincides with the kernel of f in $\mathbb{R} - Mod$.
We can define the cokernel of f as $\pi : Y \to Y/\overline{Im(f)}) = c(f)$. The norm of $c(f)$ is defined as $|\overline{y}|_{c(f)} = inf_{z \in \overline{Im(f)}} |y + z|_Y$. π is bounded since $|\overline{y}|_{c(f)} = |\pi(y)|_{c(f)} \leq |y|_Y$.
Let $g : Y \to Z$ be a morphism such that $gf = 0$. Then we need to find a unique morphism $j : c(f) \to Z$ as in the diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \longrightarrow & c(f) \\ & & \downarrow{\scriptstyle g} & \swarrow{\scriptstyle j} & \\ & & Z & & \end{array} .$$

We can do this by putting $j(\overline{x}) = g(x)$.
This is well defined since if $\overline{x} = \overline{x}'$ then there exists $l \in \overline{Im(f)}$ such that $x - x' = l$.
Then there exists a sequence y_i in Y such that $|y_i - l|_Y$ goes to 0. Then $g(x - x') = g(l)$ but $|g(l)| = 0$ since $|g(l)| = |g(y_i - l)| \leq C_g |y_i - l|_Y$ goes to 0.
The continuity of j follows from the observation that $j(\overline{x}) = g(x + l) \leq C_g |x + l|_Y$ for all $l \in \overline{Im(f)}$ by the above observation.
The uniqueness follows from the surjectivity of π.
This proves (1) of 1.19.
To prove (2) we have to define push-out and pull-backs of **Ban**.

Remark 1.31 *Let $f : X \to Y$ and $g : X \to Z$ then the push-out of f and g is*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle r} \\ Z & \xrightarrow[t]{} & P \end{array}$$

with $P = (Y \oplus Z)/S$ where S is the closure of $\{(f(x), -g(x) | x \in X\}$ and the norm of P is defined as $|\overline{(y,z)}|_P = inf_{(m,n) \in S} |(y,z) + (m,n)|_{Y \oplus Z}$.
$r : Y \to P$ is defined as $r(y) = \pi(y, 0)$ and $t : Z \to P$ as $t(z) = \pi(0, z)$.

Proof: t and r are continuous with a similar reasoning to the proof of the above remark.
Suppose we have $k : Z \to D$ and $l : Y \to D$ two morphisms such that $kg = lf$. This implies that $kg - lf = 0$. Now let $u : Y \oplus Z \to D$ such that $u(y, z) = l(y) - k(z)$. We observe that u is continuous and when restricted to $B = \{(f(x), -g(x) | x \in X\}$ is 0 since $u(f(x), -g(x)) = lf(x) - kg(x) = 0$.
Let $s \in S$ then there exists a sequence $a_i \in B$ such that $|a_i - s|_{Y \oplus Z}$ goes to 0.
From this, $|u(s)| = |u(a_i - s)| \leq C_u |a_i - s|$ that goes to 0 so all of S is in the kernel of u.
By repeating the reasoning of 1.30, we find a unique $h : P \to D$ that makes the diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & & \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle r} & \searrow{\scriptstyle l} & \\ Z & \xrightarrow[t]{} & P & \overset{h}{\dashrightarrow} & D \\ & \searrow{\scriptstyle k} & & \nearrow & \end{array}$$

commute. This proves that the starting diagram is actually a push-out.

Remark 1.32 *Suppose we are in the same conditions as the previous remark. Then if f is a kernel so is t.*

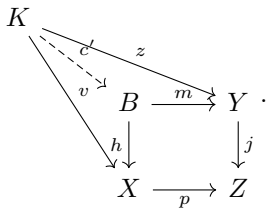
Proof: Suppose we have $c \in Z$ such that $t(c) = 0$. This means that $(0, c) \in S$ hence there exists a sequence $(f(a_i), -g(a_i))$ that converges to $(0, c)$. This implies that $f(a_i)$ is converging to 0. But since f is a kernel, this means that a_i is converging to 0. But $g(a_i)$ is converging to 0 and also to c so this means that $c = 0$ by the uniqueness of the limit.

Remark 1.33 *Let $p : X \to Z$ and $j : Y \to Z$ be two morphisms. Then the pull-back of p and j is the diagram*

$$\begin{array}{ccc} B & \xrightarrow{m} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow[p]{} & Z \end{array}$$

where $B = \{(x, y) \in (X, Y) | p(x) = j(x)\}$, $h : B \to X$ and $m : B \to Y$ are the two projections on X and Y composed with inclusions. The norm of B is the one induced by $X \oplus Y$.

Proof: The continuity of h and m is straightforward since the norm is induced by $X \oplus Y$.
Let $v : K \to X$ and $z : K \to Y$ two morphisms such that $pv = jz$. It follows that there is a function $c : K \to B$ such that $c(k) = (v(k), z(k))$. $|c(k)|_B = |(v(k), z(k))|_B = |v(k)|_X + |z(k)|_Y \leq (C_v + C_z)|k|_K$ so c is continuous hence it is a morphism. It commutes with the diagram, we only need to show the uniqueness of c. Let $c' : K \to B$ be another morphism that makes the following diagram commutative



We have that $h(c' - c) = v - v = 0$ and the same is true when composing with m. Since $(h, m) : B \to X \oplus Y$ is a monomorphism, $c' = c$.

Remark 1.34 *Suppose we are in the conditions as the previous remark. Then if p is a cokernel then so is m.*

Proof: We start by observing that m is surjective as a module morphism since p is surjective. Let B/E be the quotient of B by E where E is $\{(x, 0) | x \in X\}$ in B with the subspace norm. We observe that E is closed.
Let $a : B/E \to Y$ be the map described by $a(\overline{(x, y)}) = y$. This is well defined since E is contained in the kernel of m.
a is continuous since $|y|_Y \leq |x|_X + |y|_Y = |(x, y)|_B$ for any $(x, y) \in B$ and from this $|y|_Y = a(\overline{(x, y)}) \leq inf_{(z,0)\in E} |(x, y) + (z, 0)|_B = |\overline{(x, y)}|_{B/E}$.
a is surjective since for every y there is $x \in X$ such that $p(x) = j(y)$.
Moreover, it is injective since we are quotienting by E. At this point m is isomorphic as a morphism to the projection and therefore m is a cokernel.

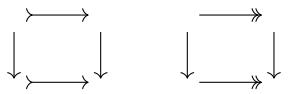
Corollary 1.35 ***Ban*** *is an exact category.*

1.2 Exact functors and Idempotent completion

Definition 1.36 *Let ($\mathscr{A}$,$\mathscr{E}$) and ($\mathscr{A}'$,$\mathscr{E}'$) be exact categories. An additive functor F: $\mathscr{A} \to \mathscr{A}'$ is called exact if $F(\mathscr{E}) \subseteq \mathscr{E}'$. The functor reflects exactness if for every object σ of $\mathbf{Fun}(\cdot \to \cdot \to \cdot, \mathscr{A})$, $F(\sigma) \in \mathscr{E}'$ implies $\sigma \in \mathscr{E}$.*

Proposition 1.37 *An exact functor preserves push-outs along admissible monics and pull-backs along admissible epics.*

Proof:Exact functors preserve admissible monics and admissible epics, the following diagrams



are preserved and thanks to 1.9 we have the push-out property.

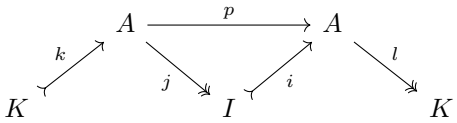
Definition 1.38 *An additive category $\mathscr{A}$ is idempotente complete if for every idempotent $p : A \to A$, there is a decomposition $A \cong K \oplus I$ such that $p \cong \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.*

Proposition 1.39 *An additive category $\mathscr{A}$ is idempotente complete if and only if every idempotent has a kernel.*

Proof: Let p be an idempotent and suppose that every idempotent has a kernel. We observe that $(1-p)(1-p) = (1-p-p+p^2) = (1-p)$ so we can find $k : K \to A$ kernel of p and $i : I \to A$ kernel of $(1-p)$. Since $p(1-p) = 0$, we have $(1-p) = kl$ for a unique $l : A \to K$ since k is the kernel of p. We can also observe that $(1-p)p = 0$, for this reason we can find a unique $j : A \to I$ such that $p = ij$. Since k is monic and kli=$(1-p)i = 0$, $li = 0$. Also, since $klk = (1-p)k = pk + (1-p)k = k$ we have $lk = 1_k$. Reasoning in a similar way we get $jk = 0$ and $ji = 1_I$.
We can conclude that $\begin{pmatrix} k & i \end{pmatrix} : K \oplus I \to A$ and $\begin{pmatrix} l \\ j \end{pmatrix} : A \to K \oplus I$ are inverse to each other. Also we obtain

$$\begin{pmatrix} l \\ j \end{pmatrix} p \begin{pmatrix} k & i \end{pmatrix} = \begin{pmatrix} l \\ j \end{pmatrix} ij \begin{pmatrix} k & i \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1_I \end{pmatrix}.$$

In particular we have this analysis of p



where $k : K \to A$ is a kernel of p and $i : I \to A$ is an image of p.
If $\mathscr{A}$ is an idempotent category and p an idempotent, $i : K \to K \oplus I \cong A$ is a kernel of p.

Proposition 1.40 *Every additive category $\mathscr{A}$ can be fully faithfully embedded into an idempotent complete additive category $\bar{\mathscr{A}}$.*

Proof: The objects of $\bar{\mathscr{A}}$ are the pairs composed by an object A of $\mathscr{A}$ and an idempotent morphism $p : A \to A$. The set of morphisms from (A, p) to (B, q) are

$$Hom_{\bar{\mathscr{A}}}((A,p),(B,q)) = q \circ Hom_{\mathscr{A}}(A,B) \circ p.$$

The composition and the group operation are induced by $\mathscr{A}$.
The biproduct $(A, p) \oplus (A', p')$ is $(A \oplus A', p \oplus p')$ and so $\bar{\mathscr{A}}$ is an additive category.
The embedding is given by the functor $i_{\mathscr{A}} : \mathscr{A} \to \bar{\mathscr{A}}$ defined by $i_{\mathscr{A}}(A) = (A, 1_A)$ and $i_{\mathscr{A}}(f) = f$ which is fully faithful. We now need to prove that $\bar{\mathscr{A}}$ is idempotent complete.
Let pfp be an idempotent of (A, p) in $\bar{\mathscr{A}}$.
We can find an isomorphism between (A, p) and $(A, p - pfp) \oplus (A, pfp)$ through the morphisms $\begin{pmatrix} p-pfp \\ pfp \end{pmatrix}$ and $\begin{pmatrix} p-pfp & pfp \end{pmatrix}$. The equality

$$\begin{pmatrix} p-pfp \\ pfp \end{pmatrix} pfp \begin{pmatrix} p-pfp & pfp \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & pfp \end{pmatrix}$$

proves that $\bar{\mathscr{A}}$ is idempotent complete.

Definition 1.41 *The pair $(\bar{\mathscr{A}}, i_{\mathscr{A}})$ is called the idempotent completion of $\mathscr{A}$*

Remark 1.42 *If $\mathscr{A}$ is idempotent complete, it is equivalent to $\bar{\mathscr{A}}$ through $i_{\mathscr{A}}$.*

Proof: We need to construct a quasi-inverse functor of $i_{\mathscr{A}}$. By recalling the construction of 1.39, for each idempotent $p : A \to A$ we can choose a kernel K_p, an image I_p and morphisms i_p, j_p, k_p, l_p. We can then map (A, p) to I_p. A morphism from (A, p) to (B, q) can be written as qfp and we can map it to $j_q qfpi_p = j_q f i_p$. We observe that $(I_p, 1)$ is isomorphic to (A, p) with isomorphisms given by j_p and i_p since p is the identity map of (A, p)

Remark 1.43 *Given the additive categories $\mathscr{A}$ and $\mathscr{B}$, a functor $F : \mathscr{A} \to \mathscr{B}$ can be uniquely extended to a functor $\bar{F} : \bar{\mathscr{A}} \to \bar{\mathscr{B}}$, by setting $\bar{F}(A, p) = (F(A), F(p))$ and $\bar{F}(qfp) = F(q)F(f)F(p)$. We also have $\bar{F} i_{\mathscr{A}} = i_{\mathscr{B}} F$ and the extension of the composition is the composition of the extensions.*

Remark 1.44 *A natural transformation $\alpha : F \Rightarrow G$ of functors $\mathscr{A} \to \mathscr{B}$ can be uniquely extended to $\bar{\alpha} : \bar{F} \Rightarrow \bar{G}$.*

Proof: First we have to observe that (A, p) is a retract of $(A, 1_A)$, using $s : (A, p) \to (A, 1_A)$ and $r : (A, 1_A) \to (A, p)$ given by $p \in Hom_{\mathscr{A}}(A, A)$. When we consider a natural transformation $\alpha' : F' \Rightarrow G'$ it must be true that

$$\alpha'_{(A,p)} = \alpha'_{(A,p)} F'(r) F'(s) = G'(r) \alpha'_{(A,1_A)} F'(s)$$

then α' is determined by the values on $i_{\mathscr{A}}(\mathscr{A})$. We can then define

$$\alpha_{(\bar{A},p)} = \bar{G}(r) \alpha_{(\bar{A},1_A)} \bar{F}(s).$$

By remembering how r and s are defined, we can see that $\bar{\alpha}_{(A,p)} = G(p)\alpha_A F(p)$ which belongs to

$$Hom_{\bar{\mathscr{B}}}(\bar{F}(A, p), \bar{G}(A, p)) = G(p) \circ Hom_{\mathscr{B}}(F(A), G(A)) \circ F(p).$$

We have to check $\bar{\alpha}$ yields a natural transformation $\bar{F} \Rightarrow \bar{G}$. Let $qfp : (A, p) \to (B, q)$ be a morphism in $\bar{\mathscr{A}}$. The first step is to observe that since p is an endomorphism of A in $\mathscr{A}$, given $\alpha : F \Rightarrow G$, we have $\alpha_A F(p) = G(p)\alpha_A$. In general, given $f : A \to B$, we have $\alpha_B F(f) = G(f)\alpha_A$.

$$\bar{G}(qfp)\bar{\alpha}_{(A,p)} = G(q)G(f)G(p)G(p)\alpha_A F(p) = G(q)G(f)\alpha_A F(p)F(p) =$$

$$= G(q)G(q)\alpha_B F(f)F(p) = G(q)\alpha_B F(q)F(f)F(p) = \bar{\alpha}_{(B,q)} \bar{F}(qfp)$$

proves that $\bar{\alpha}$ is a natural transformation.

Remark 1.45 *The assignment $\alpha \to \bar{\alpha}$ is compatible with vertical and horizontal compositions. Given F, G, $H : \mathscr{A} \to \mathscr{B}$ and given $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ then the extension to the idempotent category of $\beta \circ \alpha$ is the vertical composition of the extensions. Given $H, J : \mathscr{A} \to \mathscr{B}$ with $\alpha : H \to J$ and $K, L : \mathscr{B} \to \mathscr{C}$ with $\beta : K \to L$, the extension to the idempotent category of $\beta \star \alpha$ is the horizontal composition of the extensions.*

Remark 1.46 *A functor F: $\bar{\mathscr{A}} \to \mathscr{I}$, with $\mathscr{I}$ idempotent complete category, is completely determined by the values on $i_{\mathscr{A}}(\mathscr{A})$ up to isomorphism. A natural transformation $\alpha : F \Rightarrow G$ with $F, G : \bar{\mathscr{A}} \to \mathscr{I}$ is completely determined by its values on $I_{\mathscr{A}}(\mathscr{A})$.*

Proof: (A, p) is a retract of $(A, 1_A)$. Since p is an idempotent of $(A, 1_A)$, $F(p)$ is an idempotent of $F(A, 1_A)$. Since $\mathscr{I}$ is idempotent complete, we can define $I_{F(p)}$ as the image of $F(p)$. By using 1.42, we can observe that since $\bar{\mathscr{I}}$ is equivalent to $\mathscr{I}$ and by recalling that (A, p) is isomorphic to $(I_p, 1_{I_p})$ we have that $F(A, p)$ is isomorphic to $(F(A), F(p)) \cong (I_{F(p)}, 1)$ in $\bar{\mathscr{I}}$, which is equivalent to $\mathscr{I}$. So by this observation, $F(A, p)$ must go to $I_{F(p)}$. The natural transformation can be proved in a similar way to 1.44.

Proposition 1.47

1. *For every functor $F : \mathscr{A} \to \mathscr{I}$, where $\mathscr{I}$ is an idempotent complete category, there exists $\hat{F} : \bar{\mathscr{A}} \to \mathscr{I}$ and a natural isomorphism $\alpha : F \Rightarrow \hat{F} i_{\mathscr{A}}$.*

2. *Let $G : \bar{\mathscr{A}} \to \mathscr{I}$ be a functor and let $\gamma : F \Rightarrow G i_{\mathscr{A}}$ be a natural transformation, we may find a unique natural transformation $\beta : \hat{F} \Rightarrow G$ such that $\gamma = (\beta \star 1_{i_{\mathscr{A}}}) \cdot \alpha$.*

Sketch of proof:
(1) Given a functor $\mathscr{A} \to \mathscr{I}$, 1.46 helps us construct $\hat{F} : \bar{\mathscr{A}} \to \mathscr{I}$. We can also construct $\alpha : F \Rightarrow \hat{F} i_{\mathscr{A}}$ such that α is an isomorphism: let A be an object of $\mathscr{A}$. $i_{\mathscr{I}} \hat{F}(i_{\mathscr{A}}(A)) = \hat{F}(A, 1_A) = (F(A), 1_{F(A)})$ and by applying the equivalence map introduced in 1.46 for idempotent categories, we can define α.
(2): Once we defined $\hat{F}$ and α, $\bar{\beta} = \gamma \star \alpha^{-1}$ is a natural transformation $\hat{F} i_{\mathscr{A}} \Rightarrow G i_{\mathscr{A}}$. We can extend $\bar{\beta}$ to $\beta : \hat{F} \Rightarrow G$ by repeating the steps of 1.46.

Corollary 1.48 *Let $\mathscr{A}$ be a small additive category and $\mathscr{I}$ be an idempotent complete category. The functor $i_{\mathscr{A}} : \mathscr{A} \to \bar{\mathscr{A}}$ gives an equivalence of functor categories*

$$(i_{\mathscr{A}})^\star : Hom(\bar{\mathscr{A}}, \mathscr{I}) \xrightarrow{\cong} Hom(\mathscr{A}, \mathscr{I}).$$

Proof: Essential surjectivity is because, given $F : \mathscr{A} \to \mathscr{I}$, by using 1.47, we have a natural isomorphism between F and $(i_{\mathscr{A}})^\star(\bar{F})$.
$(i_{\mathscr{A}})^\star$ is fully faithful by using 1.47: the injectivity is due to the uniqueness of (2), the surjectivity by 1.44.

Suppose we have an exact category $(\mathscr{A}, \mathscr{E})$. We can define an exact structure on $\bar{\mathscr{A}}$ by calling an exact sequence in $\bar{\mathscr{A}}$ a sequence that is a direct summand of a sequence in $\mathscr{E}$. We denote by $\bar{\mathscr{E}}$ the class of short exact sequences in $\bar{\mathscr{A}}$.

Remark 1.49 *The class $\bar{\mathscr{E}}$ is an exact structure on $\bar{\mathscr{A}}$.*
The inclusion functor $i_{\mathscr{A}} : (\mathscr{A}, \mathscr{E}) \to (\bar{\mathscr{A}}, \bar{\mathscr{E}})$ preserves and reflect exactness.

- *For every exact functor $F : \mathscr{A} \to \mathscr{I}$, where $\mathscr{I}$ is an idempotent complete exact category, there exists an exact functor $\hat{F} : \bar{\mathscr{A}} \to \mathscr{I}$ and a natural isomorphism $\alpha : F \Rightarrow \hat{F} i_{\mathscr{A}}$.*

- *Let $G : \bar{\mathscr{A}} \to \mathscr{I}$ be another exact functor and let $\gamma : F \to G i_{\mathscr{A}}$ be a natural transformation, we may find a unique natural transformation $\beta : \hat{F} \Rightarrow G$ such that $\gamma = \beta \star \alpha$.*

Corollary 1.50 *Let $(\mathscr{A}, \mathscr{E})$ be a small exact category and $\mathscr{I}$ be an idempotent complete exact category. The exact functor $i_{\mathscr{A}} : \mathscr{A} \to \bar{\mathscr{A}}$ gives an equivalence of functor categories*

$$(i_{\mathscr{A}})^\star : Hom_{Ex}((\bar{\mathscr{A}}, \bar{\mathscr{E}}), \mathscr{I}) \xrightarrow{\cong} Hom_{Ex}((\mathscr{A}, \mathscr{E})\mathscr{I}).$$

1.3 Weak idempotent completeness

Remark 1.51 *In a category $\mathscr{C}$ a morphism $r : B \to C$ is called a retraction if there exists a section $s : C \to B$ such that $rs = 1_C$. $c : A \to B$ is called a coretraction if it admites a section $s : B \to A$ such that $sc = 1_A$. In particular, retractions are epics and coretractions are monics. We also observve that a section of a retraction is a coretraction and a section of a coretraction is a retraction.*

Remark 1.52 *: Let $r : B \to C$ be a retraction with section $s : C \to B$. Since $srsr = s(1_C)r = sr$, sr is an idempotent. Suppose r admits a kernel $k : A \to B$. $r(1_B - sr) = 0$ implies the existence of a unique morphism $t : B \to A$ such that $kt = 1_B - sr$. k is a coretraction since k is monic and $ktk = (1_B - sr)k = k$. Also, $kts = (1_B - sr)s = s - s = 0$ and since k is monic, $ts = 0$.*
Because of this,
$\begin{pmatrix} k & s \end{pmatrix} : A \oplus C \to B$ *and* $\begin{pmatrix} t \\ r \end{pmatrix} : B \to A \oplus C$ *are inverse to each other.*

Lemma 1.53 *In an additive category $\mathscr{A}$ the following conditions are equivalent:*

1. *Every coretraction has a cokernel.*

2. *Every retraction has a kernel.*

Proof: Since the two statementes are dual, it's sufficient to prove (*2* ⇒ *1*).
Given $c : C \to B$ coretraction with section s, we have that s has a kernel $k : A \to B$ because of (*2*).
By 1.52, k is a coretraction with section $t : B \to A$ and t is the cokernel of c because of the dual of the discussion in 1.52.

Definition 1.54 *If $\mathscr{A}$ satisfies the conditions of 1.53, $\mathscr{A}$ is said to be weakly idempotent complete.*

Remark 1.55 *Let $\mathscr{A}$ be an idempotent complete category. Then $\mathscr{A}$ is a weakly idempotent complete.*

Proof: Let $r : B \to C$ be a retraction and let $s : C \to B$ be a section of r. Since $srsr = sr$, sr is idempotent. Since A is idempotent complete, $B \cong K \oplus I$ where K is a kernel of sr. Since s is monic, K is a kernel of r.

Corollary 1.56 *Let $(\mathscr{A}, \mathscr{E})$ be an exact category. The following conditions are equivalent:*

1. *$\mathscr{A}$ is weakly idempotent complete.*

2. *Every coretraction is an admissible monic.*

3. *Every retraction is an admissible epic.*

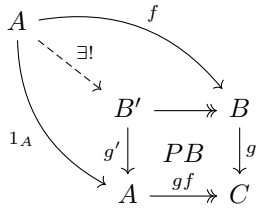
Proof: To prove (*1* ⇒ *3*), we can use the isomorphism of 1.52 since, if $A \to B$ is a coretraction with cokernel $B \to C$, $A \to B \to C$ is isomorphic to $A \rightarrowtail A \oplus C \twoheadrightarrow C$. Recalling 1.6, we have that r is an admissible monic.
By duality, we also have (*1* ⇒ *2*).
To prove (*2* ⇒ *1*), since every coretraction is an admissible monic, it has a cokernel.
(*3* ⇒ *1*) is given by a similar reasoning.

Proposition 1.57 *Suppose we have an exact category $(\mathscr{A}, \mathscr{E})$.*
The following conditions are equivalent:

1. *$\mathscr{A}$ is weakly idempotent complete.*

2. *Given $g : B \to C$ and $f : A \to B$, if their composition $gf : A \twoheadrightarrow C$ is an admissible epic and f is an admissible epic then g is an admissible epic.*

Proof:
(*2* ⇒ *1*): Given a retraction $r : B \to C$, with section $s : C \to B$, since $rs = 1_C$ is an admissible epic, r is an admissible epic.
(*1* ⇒ *2*):

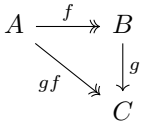


By considering the diagram, we observe that g' is a retraction. By hypotheses g' has a kernel $k : K' \to B'$. Since the square is a pull back, $K' \to B' \to B$ is a kernel of g.
By applying the dual of 1.13, g is an admissible epic.

Corollary 1.58 *Let $(\mathscr{A}, \mathscr{E})$ be an exact category. The following conditions are equivalent:*

1. *$\mathscr{A}$ is weakly idempotent complete*

2. *all morphisms $g : B \to C$ for which there is a commutative diagram*

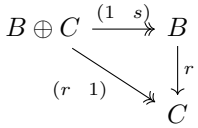


are admissible epics.

Proof:
(*1* $\Rightarrow$ *2*) can be proved through 1.57.
(*2* $\Rightarrow$ *1*): Let $r : B \to C$ and $s : C \to B$ be two morphisms such that $rs = 1_C$.
Given the sequences

$$B \overset{\begin{pmatrix} 1 \\ -r \end{pmatrix}}{\rightarrowtail} B \oplus C \overset{(r\ 1)}{\twoheadrightarrow} C, \; C \overset{\begin{pmatrix} -s \\ 1 \end{pmatrix}}{\rightarrowtail} B \oplus C \overset{(1\ s)}{\twoheadrightarrow} B,$$

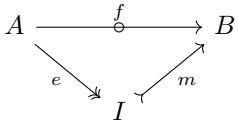
they are split exact so $(r \;\; 1)$ and $(1 \;\; s)$ are admissible epics.
Given the commutative diagram



by hypotheses, r is an admissible epic.

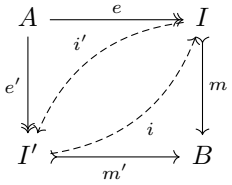
1.4 Admissible morphisms

Definition 1.59 *A morphism f:A $\to$ B in an exact category ($\mathscr{A}$,$\mathscr{E}$) is called admissible if it factors as a composition of an admissible monic with an admissible epic.*



Remark 1.60 *Given f an admissible morphism, m an admissible monic and e an admissible epic, mfe is an admissible morphism.*

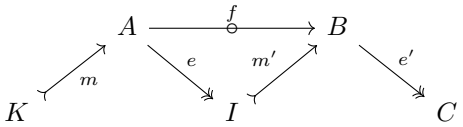
Lemma 1.61 *The factorization of an admissible morphism is unique up to unique isomorphism. In the diagram*



there exist unique morphisms i and i' that make the diagram commutative. Moreover, i and i' are mutually inverse.

Proof: Since e is an admissible epic, we can find k kernel of e. We observe $m'e'k = mek = 0$ and since m' is monic, $e'k = 0$. Since e is the cokernel of k, there exists a unique morphism $i : I \; I'$ such that $e' = ie$. Since e epic and $m'ie = m'e' = me$, $m'i = m$. We can repeat the reasoning for i'.

Remark 1.62 *For every admissible morphism f, we can construct the well-defined analysis*



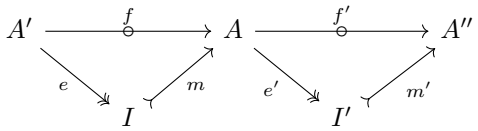
Lemma 1.63 *Admissible morphisms are stable under push-out along admissible monics and pull-back along admissible epics.*

Proof: Let $A \twoheadrightarrow I \rightarrowtail B$ be an admissible morphism. The diagram

$$\begin{array}{ccccc} A & \twoheadrightarrow & I & \rightarrowtail & B \\ \downarrowtail & PO & \downarrow & PO & \downarrowtail \\ A' & \twoheadrightarrow & I' & \rightarrowtail & B' \end{array}$$

depicts the push-out along an admissible monic. By using 1.12, we obtain that $A' \to I'$ is an admissible epic. By duality we prove that stability of admissible morphisms under pull-back along admissible epics.

Definition 1.64 *Given a sequence of admissible morphisms*



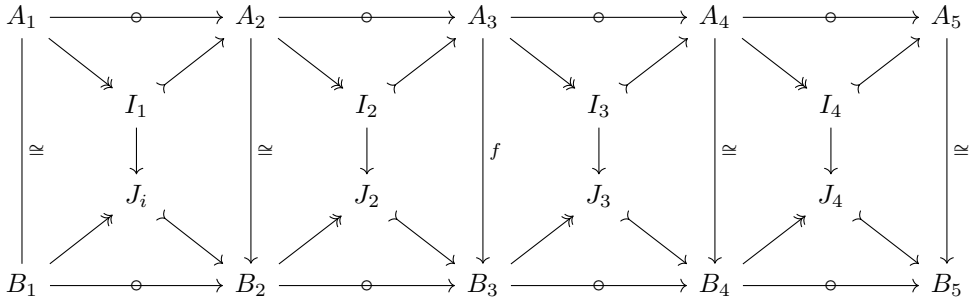
it is exact if $I \rightarrowtail A \twoheadrightarrow I'$ is short exact. Longer sequences of admissible morphisms are exact if any two consecutive morphisms sequence is exact. We will use also the term "acyclic".

Lemma 1.65 *Suppose we have the following diagram*

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow\cong & & \downarrow\cong & & \downarrow f & & \downarrow\cong & & \downarrow\cong \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

If it has exact rows, f is an isomorphism

Proof: The first step is to construct the diagram



To proceed we have to prove a corollary of 1.16:
Given a morphism of short exact sequences (diagramma), if any two of a, b, c are isomorphisms so it's third. We can consider the case in which b is an isomorphism since we already proved 1.16. The two cases are dual. Given a, b isomorphisms, we first observe that c is uniquely determined since $B' \twoheadrightarrow C'$ is a cokernel of $A' \rightarrowtail B'$. By the uniqueness and by repeating the reasoning with a^{-1} and b^{-1},c is an isomorphism.
By applying the first part, all the morphisms $I_i \to J_i$ are isomorphisms. Then f is an isomorphism.

Proposition 1.66 *(Ker-Coker-sequence) Let $(\mathscr{A}, \mathscr{E})$ be a weakly idempotent complete exact category. Suppose we have $f : A \to B$ and $g : B \to C$ two admissible morphisms such that $h = gf : A \to C$ is admissible as well. Then we obtain the exact sequence*

$$Ker\ f \rightarrowtail Ker\ h \longrightarrow Ker\ g \longrightarrow Coker\ f \longrightarrow Coker\ h \twoheadrightarrow Coker\ g$$

depending naturally on the diagram $h = gf$.

Proof: We can factor $Ker\ f \rightarrowtail A$ over $Ker\ h \rightarrowtail A$ through a unique morphism $Ker\ f \to Ker\ h$. This morphism is an admissible monic by the dual of 1.57. Let $Ker\ h \twoheadrightarrow X$ be a cokernel of $Ker\ f \rightarrowtail Ker\ h$. Dually we have an admissible epic $Coker\ h \twoheadrightarrow Coker\ g$. We can find $Z \rightarrowtail Coker\ h$ kernel of it. By using 1.17, we can construct these two commutative diagrams with exact rows and columns.

$$\begin{array}{ccccc} Ker\ f & \rightarrowtail & Ker\ h & \twoheadrightarrow & X \\ \| & & \downarrowtail & & \downarrowtail \\ Ker\ f & \rightarrowtail & A & \twoheadrightarrow & Im\ f \\ & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ & & Im\ h & = & Im\ h \end{array} \qquad \begin{array}{ccccc} Im\ h & \rightarrowtail & Im\ g & \twoheadrightarrow & Z \\ \| & & \downarrowtail & & \downarrowtail \\ Im\ h & \rightarrowtail & C & \twoheadrightarrow & Coker\ h \\ & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ & & Coker\ g & = & Coker\ g \end{array}$$

By composing the morphism $X \rightarrowtail Im\ f \rightarrowtail B$ with g, we get the zero morphism.
To proof this, we first observe that $(Im\ f \rightarrowtail B \twoheadrightarrow Im\ g) = (Im\ f \twoheadrightarrow Im\ h \rightarrowtail Im\ g)$ since
$(A \twoheadrightarrow Im\ f \rightarrowtail B \twoheadrightarrow Im\ g \rightarrowtail C) = (A \twoheadrightarrow Im\ f \twoheadrightarrow Im\ h \rightarrowtail Im\ g \rightarrowtail C) = (A \twoheadrightarrow Im\ h \rightarrowtail C)$.
Since we have that $(A \twoheadrightarrow Im\ f \twoheadrightarrow Im\ h) = (A \twoheadrightarrow Im\ h)$ and $(Im\ h \rightarrowtail Im\ g \rightarrowtail C) = Im\ h \rightarrowtail C$ are an epic and monic respectively, it follows that $(X \rightarrowtail Im\ f \rightarrowtail B \twoheadrightarrow Im\ g \rightarrowtail C) = (X \rightarrowtail Im\ f \twoheadrightarrow Im\ h \rightarrowtail Im\ g \rightarrowtail C) = 0$.
We can construct $X \rightarrowtail Ker\ g$ that such that $(X \rightarrowtail Ker\ g \rightarrowtail B) = (X \rightarrowtail Im\ f \rightarrowtail B)$ since it is a composition of admissible monics, it is an admissible monic. By denoting its cokernel by $Ker\ g \twoheadrightarrow Y$, we get the following diagram with exact rows and columns by 1.18

$$\begin{array}{ccccc} X & \rightarrowtail & Ker\ g & \twoheadrightarrow & Y \\ \downarrowtail & & \downarrowtail & & \downarrowtail \\ Im\ f & \rightarrowtail & B & \twoheadrightarrow & Coker\ f \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ Im\ h & \rightarrowtail & Im\ g & \twoheadrightarrow & Z \end{array}$$

By splicing the three diagrams

$$\begin{array}{ccccccccc} Ker\ f & \rightarrowtail & Ker\ h & \twoheadrightarrow & X & \rightarrowtail & Ker\ g & \twoheadrightarrow & Y \\ \| & & \downarrowtail & & \downarrowtail & & \downarrowtail & & \downarrowtail \\ Ker\ f & \rightarrowtail & A & \twoheadrightarrow & Im\ f & \rightarrowtail & B & \twoheadrightarrow & Coker\ f \\ & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ & & Im\ h & = & Im\ h & \rightarrowtail & Im\ g & \twoheadrightarrow & Z \\ & & & & \| & & \downarrowtail & & \downarrowtail \\ & & & & Im\ h & \rightarrowtail & C & \twoheadrightarrow & Coker\ h \\ & & & & & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ & & & & & & Coker\ g & = & Coker\ g \end{array}$$

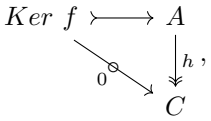
we can find the sequence starting from $Ker\ f \rightarrowtail Ker\ h$ and by following the outer edge.

Lemma 1.67 *Let $\mathscr{A}$ be an exact category in which each commutative triangle of admissible morphisms yields an exact Ker-Coker sequence. Then A is weakly idempotent complete.*

Proof: We want to verify the second condition of 1.57.
Given a commutative diagram

$$\begin{array}{ccc} A & \overset{f}{\twoheadrightarrow} & B \\ & \searrow^{h} & \downarrow g \\ & & C \end{array}$$

we have to check g is an admissible epic. By studying the associated $Ker - Coker - sequence$ of the diagram

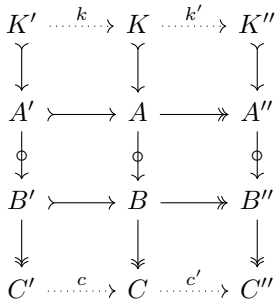


we obtain the following the sequence

$$0 \longrightarrow Ker\ f \longrightarrow Ker\ h \longrightarrow B \xrightarrow{g} C \longrightarrow 0\ .$$

Since it is an exact sequence, g is an admissible epic.

Corollary 1.68 *(Snake lemma) Let $\mathscr{A}$ be an exact weakly idempotent complete category. Given a morphism of short exact sequences from $A' \rightarrowtail A \twoheadrightarrow A''$ to $B' \rightarrowtail B \twoheadrightarrow B''$ with admissible components, we can construct the following commutative diagram:*

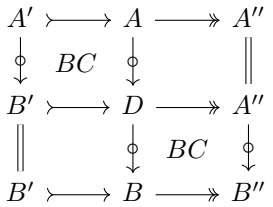


with exact rows and columns and a connecting morphism $\delta : K'' \to C'$ forming an exact sequence

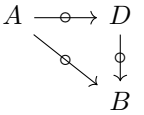
$$K' \xrightarrowtail{k} K \xrightarrow{k'} K'' \xrightarrow{\delta} C' \xrightarrow{c} C \xrightarrow{c'} C''$$

depending naturally on the morphism of short exact sequences.

Proof: By using 1.15 and 1.63, we can construct the diagram



We can then apply the Ker-Coker sequence to the commutative triangle of admissible morphisms



to end the proof.

2 Homotopy category of an additive category

2.1 Exact triangles (distinguished)

Definition 2.1 *Let $\mathscr{A}$ be an additive category. Then $tr_1(\mathscr{A})$ denotes the category such that*

- *The object are morphisms $X \xrightarrow{f} Y$ where X and Y are cochain complexes of $\mathscr{A}$.*

- *A morphism from $X \xrightarrow{f} Y$ to $X' \xrightarrow{f'} Y'$ is a triplet (g, h, a) where $X \xrightarrow{g} X'$, $Y \xrightarrow{h} Y'$ are morphisms and $a : f'g \rightsquigarrow hf$ is a homotopy between $f'g$ and hf (with $a^i : X^{i+1} \to Y^i$).*

- *The composition $(g, h, a) : (X \xrightarrow{f} Y) \to (X' \xrightarrow{f'} Y')$ and $(g', h', a') : (X' \xrightarrow{f'} Y') \to (X'' \xrightarrow{f''} Y'')$ is $(g'', h'', a'') : (X \xrightarrow{f} Y) \to (X'' \xrightarrow{f''} Y'')$ where $g'' = g'g$, $h'' = h'h$, $a'' = a' \star g + h' \star a$ where, given the morphism of cochain complexes $u : Y \to Y'$ and the homotopy $a : X \to Y$, $(u \star a)^i = u^i a^i$ (respectively for $v : X' \to X$, $(a \star v)^i = a^i v^{i+1}$) .*

We will represent $(g, h, a) : (X \xrightarrow{f} Y) \to (X' \xrightarrow{f'} Y')$ by

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle g} & a \rightsquigarrow & \downarrow{\scriptstyle h} \\ X' & \xrightarrow{f'} & Y' \end{array}$$

Remark 2.2 *$tr_1(\mathscr{A})$ is additive since the direct sum of $X \xrightarrow{f} Y$ and $X' \xrightarrow{f'} Y'$ is*

$$X \oplus X' \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}} Y \oplus Y'.$$

Definition 2.3 *Let $f : X \to Y$ be an object of $tr_1(\mathscr{A})$. c(f)(also Cone(f)) is the cochain complex such that*

$$c(f)^i = Y^i \oplus X^{i+1},$$

$$d^i_{c(f)} = \begin{pmatrix} d^i_Y & f^{i+1} \\ 0 & -d^{i+1}_X \end{pmatrix} : Y^i \oplus X^{i+1} \to Y^{i+1} \oplus X^{i+2}.$$

We can then define a functor $c : tr_1(\mathscr{A}) \to C(\mathscr{A})$ in this way:

$$c(X \xrightarrow{f} Y) = c(f),$$

$$c((g, h, a))^i = \begin{pmatrix} h^i & a^i \\ 0 & g^{i+1} \end{pmatrix} : Y^i \oplus X^{i+1} \to Y'^i \oplus X'^{i+1}$$

where $(g, h, a) : (X \xrightarrow{f} Y) \to (X' \xrightarrow{f'} Y')$ and by definition $c((g, h, a)) : c(f) \to c(f')$.
We can also construct a functor $s : C(\mathscr{A}) \to tr_1(\mathscr{A})$ such that $s(W) = (0 \to W)$ and $s(f) = (0, f, 0)$ where $f : W \to C$ is a morphism in $C(\mathscr{A})$.
Fixed f an object of $tr_1(\mathscr{A})$, we can define $\phi(f) : (X \xrightarrow{f} Y) \to sc(f)$ the morphism of $tr_1(\mathscr{A})$ defined by $\phi(f) = (0, p(f), a(f))$ which is represented by the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & a(f) \rightsquigarrow & \downarrow{\scriptstyle p(f)} \\ 0 & \longrightarrow & c(f) \end{array}$$

where

$$p(f)^i = \begin{pmatrix} id_{Y^i} \\ 0 \end{pmatrix} : Y^i \to Y^i \oplus X^{i+1},$$

$$a(f)^i = \begin{pmatrix} 0 \\ id_{X^{i+1}} \end{pmatrix} X^{i+1} \to Y^i \oplus X^{i+1}.$$

Proposition 2.4 *The functor c is the left adjoint of the functor s through ϕ.*

$$\alpha : Hom_{C(\mathscr{A})}(c(f), W) \to Hom_{tr_1(\mathscr{A})}((X \xrightarrow{f} Y), sW)$$

defined by ϕ is an isomorphism.

Proof: Let $c : c(f) \to W$ be a morphism of $C(\mathscr{A})$, by applying s we get

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & a(f) \rightsquigarrow & \downarrow{\scriptstyle p(f)} \\ 0 & \longrightarrow & c(f) \\ \downarrow & & \downarrow{\scriptstyle c} \\ 0 & \longrightarrow & W \end{array},$$

by composing the morphisms we get

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & c \star a(f) \rightsquigarrow & \downarrow{\scriptstyle cp(f)} \\ 0 & \longrightarrow & W \end{array}.$$

We can define $\alpha(c) = (0, cp(f), c \star a(f))$. By observing that $p(f)$ is a monomorphism, $cp(f) = bp(f) \Rightarrow c = b$. Because of this, we have that α is injective.

To prove surjectivity, suppose we have $(g, h, a) : (X \xrightarrow{f} Y) \to s(W)$. Then $g = 0$ since we are considering a morphism to $s(W)$. Thanks to $a^i : X^{i+1} \to W^i$ and $h^i : Y^i \to W^i$, we have a unique map $r^i : c(f)^i \to W^i$ such that $r^i \circ i_{Y^i} = h^i$ and $r^i \circ i_{X^{i+1}} = a^i$. By construction, $\alpha(r) = (g, h, a)$ since $\alpha(r)^i = (0, ri_{Y^i}, r \star i_{X^{i+1}}) = (0, h^i, a^i)$.

Definition 2.5 *Homotopy in $tr_1(\mathscr{A})$ from (g, h, a) to $(g', h', a') : (X \xrightarrow{f} Y) \rightrightarrows (X' \xrightarrow{f'} Y')$ is a triplet (b, b', α) such that $b : g \rightsquigarrow g'$, $b' : h \rightsquigarrow h'$ and $\alpha : f' \star b + a' \rightsquigarrow b' \star f + a$ is a homotopy of second order.*

Definition 2.6 *$Tr_1(\mathscr{A})$ is the category where*

- *The objects are the objects of $tr_1(\mathscr{A})$.*
- *The morphisms of $Tr_1(\mathscr{A})$ are the homotopy classes of morphisms of $tr_1(\mathscr{A})$.*

Proposition 2.7 *Let $D, D' : (X \xrightarrow{f} Y) \rightrightarrows (X' \xrightarrow{f'} Y')$ be two morphisms of $tr_1(\mathscr{A})$. If there is a homotopy between D and D' in $tr_1(\mathscr{A})$, then there is a homotopy between $c(D)$ and $c(D')$ in $C(\mathscr{A})$.*

Proof: Let $D = (g, h, a)$ and $D' = (g', h', a')$ two morphisms. Suppose (b, b', α) is a homotopy between D and D'. This implies that

$$g' - g = bd + db,$$
$$h' - h = b'd + db',$$
$$\alpha d - d\alpha = b'f + a - (f'b + a').$$

The homotopy between $c(D)$ and $c(D')$ is given by $\begin{pmatrix} b' & \alpha \\ 0 & -b \end{pmatrix}$ since

$$\begin{pmatrix} b'^{i+1} & \alpha^{i+1} \\ 0 & -b^{i+1} \end{pmatrix} \begin{pmatrix} d_Y^i & f^{i+1} \\ 0 & -d_X^{i+1} \end{pmatrix} + \begin{pmatrix} d_{Y'}^i & f'^{i+1} \\ 0 & -d_X^{i+1} \end{pmatrix} \begin{pmatrix} b'^i & \alpha^i \\ 0 & -b^i \end{pmatrix} = \begin{pmatrix} h'^i - h^i & a'^i - a^i \\ 0 & g'^{i+1} - g^{i+1} \end{pmatrix}$$

holds by the above equalities.

Corollary 2.8 *The two functors*

$$c : tr_1(\mathscr{A}) \to C(\mathscr{A}), \qquad s : C(\mathscr{A}) \to tr_1(\mathscr{A})$$

are compatible with homotopies, hence we can pass to the quotient

$$c : Tr_1(\mathscr{A}) \to K(\mathscr{A}), \qquad s : K(\mathscr{A}) \to Tr_1(\mathscr{A}).$$

$c : Tr_1(\mathscr{A}) \to K(\mathscr{A})$ *is the left adjoint of* $s : K(\mathscr{A}) \to Tr_1(\mathscr{A})$.

Remark 2.9 *Let* s' *be the functor* $C(\mathscr{A}) \to tr_1(\mathscr{A})$ *defined by* $s'(X) = (X \to 0)$. *The functor* $cs' : C(\mathscr{A}) \to C(\mathscr{A})$ *is the translation functor of* $C(\mathscr{A})$. *By passing to the quotient,* $cs' : K(\mathscr{A}) \to K(\mathscr{A})$ *is the translation functor of* $K(\mathscr{A})$.

We want now to introduce the notion of exact triangles.

Definition 2.10 *Let* $\mathscr{B}$ *be an additive category and* $[1]$ *be an autoequivalence of* $\mathscr{B}$. *A triangle* T *is a diagram of the form*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1].$$

We will denote T *by* (X, Y, Z, u, v, w) *and by the diagram*

$$\begin{array}{ccccc} & & Z & & \\ & {\scriptstyle w}\swarrow & & \nwarrow{\scriptstyle v} & \\ X & & \xrightarrow{\quad u \quad} & & Y \end{array}$$

Definition 2.11 *Let* (X, Y, Z, u, v, w) *and* (X', Y', Z', u', v', w') *be two triangles. A morphism from* (X, Y, Z, u, v, w) *to* (X', Y', Z', u', v', w') *is commutative diagram of the form*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f[1]} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}$$

Remark 2.12 *The triangles of* $\mathscr{B}$ *form a category.*

Let $X \xrightarrow{f} Y$ be an object of $tr_1(\mathscr{A})$. Given the morphism

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \| & & \downarrow \\ X & \longrightarrow & 0 \end{array},$$

by applying the functor c: we find $q(f) : c(f) \to X[1]$.
By applying c to

$$\begin{array}{ccc} 0 & \longrightarrow & Y \\ \downarrow & & \| \\ X & \xrightarrow{f} & Y \end{array}$$

we obtain $p(f) : Y \to c(f)$.
So for any object $X \xrightarrow{f} Y$ of $tr_1(\mathscr{A})$ there is a triangle in $C(\mathscr{A})$

$$X \xrightarrow{f} Y \xrightarrow{p(f)} c(f) \xrightarrow{q(f)} X[1].$$

Let $X' \xrightarrow{f'} Y'$ be another object and $D = (g, h, a) : (X \xrightarrow{f} Y) \to (X' \xrightarrow{f'} Y')$ be a morphism of $tr_1(\mathscr{A})$.
We construct the following diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{p(f)} & c(f) & \xrightarrow{q(f)} & X[1] \\ {\scriptstyle g}\downarrow & a \rightsquigarrow & {\scriptstyle h}\downarrow & & {\scriptstyle c(D)}\downarrow & & \downarrow{\scriptstyle g[1]} \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[p(f')]{} & c(f') & \xrightarrow[q(f')]{} & X'[1] \end{array}$$

The first square is commutative when passing to the quotient. The second and third are commutative because $p(f)$ and $q(f)$ are functors on f. By passing to the quotient, for every object $X \xrightarrow{f} Y$, there is a triangle of $K(\mathscr{A})$:

$$X \xrightarrow{\overline{f}} Y \xrightarrow{\overline{p}(f)} c(f) \xrightarrow{\overline{q}(f)} X[1]$$

where $\overline{p}(f)$ is the homotopy class of $p(f)$ and $\overline{q}(f)$ is the homotopy class of $q(f)$.

Remark 2.13 *The translation functor can be defined on $tr_1(\mathscr{A})$ by*

$$T(X \xrightarrow{f} Y) = X[1] \xrightarrow{f[1]} Y[1],\ T((g,h,a)) = (g[1], h[1], -a[1]).$$

Moreover, $\psi(f) : c(f([1])) \xrightarrow{\cong} c(f)[1]$:

$$\psi(f)^i = \begin{pmatrix} id_{Y^{i+1}} & 0 \\ 0 & -id_{X^{i+2}} \end{pmatrix} : Y^{i+1} \oplus X^{i+2} \to Y^{i+1} \oplus X^{i+2}$$

is an isomorphism.

Given $X \xrightarrow{f} Y$ an object of $tr_1(\mathscr{A})$ we can also associate the triangle of $C(\mathscr{A})$

$$Y \xrightarrow{p(f)} c(f) \xrightarrow{q(f)} X[1] \xrightarrow{f[1]} Y[1].$$

$q(f)p(f) : Y \to X[1]$ is the cone functor image of the morphism

$$\begin{array}{ccc} 0 & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & 0 \end{array}$$

We can also introduce $J : c(p(f)) \to X[1]$ such that J^i is the projection on X^{i+1}.
In particular

$$\begin{array}{ccc} c(f) & \xrightarrow{p(p(f))} & c(p(f)) \\ \| & & \downarrow J \\ c(f) & \xrightarrow{q(f)} & X[1] \end{array} .$$

is a commutative diagram and $Ja(p(f)) = 0$.

Proposition 2.14 *Let $f : B \to C$ be a map of cochain complexes. Then $c(p(f))$ is isomorphic to $B[1]$ in $K(\mathscr{A})$*

Proof: By definition $c(p(f))^i = C^{i+1} \oplus B^{i+1} \oplus C^i$ and the differential of $c(p(f))$ is given by

$$d^i_{c(p(f))} = \begin{pmatrix} -d^{i+1}_C & 0 & 0 \\ 0 & -d^{i+1}_B & 0 \\ id^{i+1}_C & f^{i+1} & d^i_C \end{pmatrix}.$$

Now let $\alpha^i : B[1]^i \to c(p(f))^i$ be defined by $\begin{pmatrix} -f^{i+1} \\ id^i_{B[1]} \\ 0 \end{pmatrix}$ and $\beta^i : c(p(f))^i \to B[1]^i$ be the projection.
We can observe that α and β are two cochain morphisms, moreover we can observe $\beta\alpha = id_{B[1]}$.
Let $s^i : c(p(f))^i \to c(p(f))^{i-1}$ be defined by

$$\begin{pmatrix} 0 & 0 & id^i_C \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then we have that s defines a chain homotopy from $id_{c(p(f))}$ to $\alpha\beta$. It follows from

$$\begin{pmatrix} -d^i_C & 0 & 0 \\ 0 & -d^i_B & 0 \\ id^i_C & f^i & d^{i-1}_C \end{pmatrix} s^i + s^{i+1} \begin{pmatrix} -d^{i+1}_C & 0 & 0 \\ 0 & -d^{i+1}_B & 0 \\ id^{i+1}_C & f^{i+1} & d^i_C \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & -d_C^i \\ 0 & 0 & 0 \\ 0 & 0 & id_C^i \end{pmatrix} + \begin{pmatrix} id_C^{i+1} & f^{i+1} & d_C^i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} id_C^{i+1} & f^{i+1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & id_C^i \end{pmatrix}.$$

By computing

$$id^i_{c(p(f))} - (\alpha\beta)^i = \begin{pmatrix} id_C^{i+1} & 0 & 0 \\ 0 & id_B^{i+1} & 0 \\ 0 & 0 & id_C^i \end{pmatrix} - \begin{pmatrix} 0 & -f^{i+1} & 0 \\ 0 & id_B^{i+1} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} id_C^{i+1} & f^{i+1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & id_C^i \end{pmatrix},$$

we prove the proposition.

Remark 2.15 *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be two morphisms of $C(\mathscr{A})$. Given $gf = h$ and constructed the two commutative diagrams*

$$D = \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \| & & \downarrow g \\ X & \xrightarrow{h} & Z \end{array}, \qquad D' = \begin{array}{ccc} X & \xrightarrow{h} & Z \\ \downarrow f & & \| \\ Y & \xrightarrow{g} & Z \end{array},$$

we can apply the cone functor to obtain

$$c(f) \xrightarrow{c(D)} c(h) \xrightarrow{c(D')} c(g).$$

By taking the commutative diagram

$$I = \begin{array}{ccc} Y & \xrightarrow{g} & Z \\ {\scriptstyle p(f)}\downarrow & & \downarrow{\scriptstyle p(h)} \\ c(f) & \xrightarrow{c(D)} & c(h) \end{array}$$

and applying the cone functor, there is a morphism

$$c(I) : c(g) \to c(c(D)).$$

Proposition 2.16 *1. The morphism $c(I) : c(g) \to c(c(D))$ is a homotopism.*

2. The diagram

$$\begin{array}{ccc} & & c(g) \\ & \overset{c(D')}{\nearrow} & \downarrow{\scriptstyle c(I)} \\ c(h) & \xrightarrow{p(c(D))} & c(c(D)) \end{array}$$

is commutative in $K(\mathscr{A})$.

3. The diagram

$$\begin{array}{ccc} c(g) & \xrightarrow{q(g)} & Y[1] \\ {\scriptstyle c(I)}\downarrow & & \downarrow{\scriptstyle p(f)[1]} \\ c(c(D)) & \xrightarrow{q(c(D))} & c(f)[1] \end{array}$$

is commutative.

Proof: $c(I)$ has components such that

$$c(I)^i = \begin{pmatrix} id & 0 \\ 0 & 0 \\ 0 & id \\ 0 & 0 \end{pmatrix} : Z^i \oplus Y^{i+1} \to Z^i \oplus X^{i+1} \oplus Y^{i+1} \oplus X^{i+2}.$$

The morphism $u : c(c(D)) \to c(g)$ with components

$$u^i = \begin{pmatrix} id & 0 & 0 & 0 \\ 0 & f^{i+1} & id & 0 \end{pmatrix} : Z^i \oplus X^{i+1} \oplus Y^{i+1} \oplus X^{i+2} \to Z^i \oplus Y^{i+1}$$

is the inverse of $c(I)$ in $K(\mathscr{A})$. We can observe $uc(I) = id_{Z^i \oplus Y^{i+1}}$. To prove that $c(I)u = id$ in $K(\mathscr{A})$, we can take the homotopy s such that

$$s^i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -id_{X^{i+1}} & 0 & 0 \end{pmatrix}.$$

To prove (2), it is sufficient to observe that the diagram

$$\begin{array}{ccc} & & c(g) \\ & \overset{c(D')}{\nearrow} & \uparrow u \\ c(h) & \xrightarrow[p(c(D))]{} & c(c(D)) \end{array}$$

is commutative in $C(\mathscr{A})$ and to recall that u is the inverse of $c(I)$ in $K(\mathscr{A})$. The third point can be shown by calculation.

Definition 2.17 *A triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ in $K(\mathscr{A})$ is an exact triangle if there is an object $X' \xrightarrow{f'} Y'$ in $Tr_1(\mathscr{A})$ and an isomorphism of triangles such that*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ {\scriptstyle f}\downarrow{\scriptstyle \cong} & & {\scriptstyle g}\downarrow{\scriptstyle \cong} & & {\scriptstyle h}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle f[1]} \\ X' & \xrightarrow{\overline{f}'} & Y' & \xrightarrow{\overline{p}(f')} & c(f') & \xrightarrow{\overline{q}(f')} & X'[1] \end{array}$$

Proposition 2.18 *For every object X of $K(\mathscr{A})$, the triangle $X \xrightarrow{id_X} X \to 0 \to X[1]$ is exact. Every morphism $u : X \to Y$ in $K(\mathscr{A})$ is contained in an exact triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$.*

Proof: The first part is proved by showing that $c(id_X)$ is homotopic to zero.
In fact, $s : X^{n+1} \oplus X^n \to X^n \oplus X^{n-1}$ $(x, x') \to (x', 0)$ gives a homotopy from $id_{c(id_X)}$ to 0.
Let $u : X \to Y$ be a morphism in $K(\mathscr{A})$, then there exists f in $C(\mathscr{A})$ such that its class in $K(\mathscr{A})$ is u. Then, by definition,

$$X \xrightarrow{u} Y \xrightarrow{\overline{p}(f)} c(f) \xrightarrow{\overline{q}(f)} X[1]$$

is an exact triangle that contains u.

Proposition 2.19 *A triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ in $K(\mathscr{A})$ is exact if and only if $Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$ is exact.*

Proof: Suppose $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is exact. Then it's isomorphic to a triangle of the form:

$$X' \xrightarrow{f} Y' \xrightarrow{\overline{p}(f)} c(f) \xrightarrow{\overline{q}(f)} X'[1],$$

where $X' \xrightarrow{f} Y'$ is an object of $Tr_1(\mathscr{A})$. The diagram

$$\begin{array}{ccccccc} Y' & \xrightarrow{\overline{p}(f)} & c(f) & \xrightarrow{\overline{p}(p(f))} & c(p(f)) & \xrightarrow{\overline{q}(p(f))} & Y'[1] \\ \| & & \| & & {\scriptstyle \overline{f}}\downarrow & & \| \\ Y' & \xrightarrow{\overline{p}(f)} & c(f) & \xrightarrow{\overline{q}(f)} & X'[1] & \xrightarrow{-\overline{f}[1]} & Y'[1] \end{array},$$

is an isomorphism of triangles in $K(\mathscr{A})$ by 2.14, so $Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$ is an exact triangle. Suppose now $K(\mathscr{A})$, so $Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$ is exact. By recalling 2.13, we deduce that

$$Y[-1] \xrightarrow{v[-1]} Z[-1] \xrightarrow{w[-1]} X \xrightarrow{u} Y$$

is an exact triangle. By applying the first step of the proof,

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

is an exact triangle.

Proposition 2.20 *Let*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1], \qquad X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} X'[1]$$

be two exact triangles of $K(\mathscr{A})$ and let the diagram

$$\begin{array}{ccc} X' & \xrightarrow{u'} & Y' \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle n} \\ X & \xrightarrow{u} & Y \end{array}$$

be commutative in $K(\mathscr{A})$. Then there exists a morphism $r : Z' \to Z$ such that

$$\begin{array}{ccccccc} X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \\ {\scriptstyle m}\downarrow & & {\scriptstyle n}\downarrow & & {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle m[1]} \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \end{array}$$

is a morphism of triangles.

Proof: Since the triangles are exact, they are isomorphic to some triangles of the form

$$V \xrightarrow{\overline{f}} W \xrightarrow{\overline{p}(f)} c(f) \xrightarrow{\overline{q}(f)} V[1],$$

$$V' \xrightarrow{\overline{f}'} W' \xrightarrow{\overline{p}(f')} c(f') \xrightarrow{\overline{q}(f')} V'[1].$$

By isomomorphism, the commutative diagram of the hypotheses is isomorphic to

$$\begin{array}{ccc} V' & \xrightarrow{\overline{f}'} & W' \\ {\scriptstyle \overline{g}}\downarrow & & \downarrow{\scriptstyle \overline{g}'} \\ V & \xrightarrow{\overline{f}} & W \end{array}$$

in $K(\mathscr{A})$.
By choosing $g : V' \to V$ and $g' : W' \to W$, the diagram, in $C(\mathscr{A})$,

$$\begin{array}{ccc} V' & \xrightarrow{f'} & W' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ V & \xrightarrow{f} & W \end{array}$$

is commutative when passing to the quotient so there is $a : fg \rightsquigarrow g'f'$. We recall, by 2.12, that the diagram

$$\begin{array}{ccccccc} V' & \xrightarrow{\overline{f}'} & W' & \xrightarrow{\overline{p}(f')} & c(f') & \xrightarrow{\overline{q}(f')} & V'[1] \\ {\scriptstyle \overline{g}}\downarrow & & \downarrow{\scriptstyle \overline{g}'} & & \downarrow{\scriptstyle \overline{c}(D)} & & \downarrow{\scriptstyle \overline{g}[1]} \\ V & \xrightarrow{\overline{f}} & W & \xrightarrow{\overline{p}(f)} & c(f) & \xrightarrow{\overline{q}(f)} & V[1] \end{array}$$

is a morphism of triangles in $K(\mathscr{A})$, hence we found $r : Z' \to Z$.

Proposition 2.21 *Suppose*

$$\begin{array}{ccccc} & & X_2 & & \\ & \overset{u_3}{\nearrow} & & \overset{u_1}{\searrow} & \\ X_1 & & \xrightarrow{\quad u_2 \quad} & & X_3 \end{array}$$

is a commutative diagram in $K(\mathscr{A})$. Then there exist three exact triangles

$$X_1 \xrightarrow{u_3} X_2 \xrightarrow{v_3} Z_3 \xrightarrow{w_3} X_1[1],$$
$$X_2 \xrightarrow{u_1} X_3 \xrightarrow{v_1} Z_1 \xrightarrow{w_1} X_2[1],$$
$$X_1 \xrightarrow{u_2} X_3 \xrightarrow{v_2} Z_2 \xrightarrow{w_2} X_1[1]$$

and two morphisms:

$$m_1 : Z_3 \to Z_2,$$
$$m_3 : Z_2 \to Z_1$$

such that

1. *The diagrams*

$$\begin{array}{ccccccc} X_1 & \xrightarrow{u_3} & X_2 & \xrightarrow{v_3} & Z_3 & \xrightarrow{w_3} & X_1[1] \\ \| & & \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle m_1} & & \| \\ X_1 & \xrightarrow{u_2} & X_3 & \xrightarrow{v_2} & Z_2 & \xrightarrow{w_2} & X_1[1] \end{array} \quad ,$$

$$\begin{array}{ccccccc} X_1 & \xrightarrow{u_2} & X_3 & \xrightarrow{v_2} & Z_2 & \xrightarrow{w_2} & X_1[1] \\ \downarrow{\scriptstyle u_3} & & \| & & \downarrow{\scriptstyle m_3} & & \downarrow{\scriptstyle u_3[1]} \\ X_2 & \xrightarrow{u_1} & X_3 & \xrightarrow{v_1} & Z_1 & \xrightarrow{w_1} & X_2[1] \end{array}$$

   are morphisms of triangles

2. *The triangle*

$$Z_3 \xrightarrow{m_1} Z_2 \xrightarrow{m_3} Z_1 \xrightarrow{v_3[1]w_1} Z_3[1]$$

   is exact.

Proof: Let $f : X_1 \to X_2$ and $g : X_2 \to X_3$ two morphisms of $C(\mathscr{A})$ such that their classes are respectively u_3 and u_1. Let h be $gf : X_1 \to X_3$. To prove the proposition, we can take as exact triangles

$$X_1 \xrightarrow{\overline{f}} X_2 \xrightarrow{\overline{p}(f)} c(f) \xrightarrow{\overline{q}(f)} X_1[1],$$
$$X_2 \xrightarrow{\overline{g}} X_3 \xrightarrow{\overline{p}(g)} c(g) \xrightarrow{\overline{q}(g)} X_2[1],$$
$$X_1 \xrightarrow{\overline{h}} X_3 \xrightarrow{\overline{p}(h)} c(h) \xrightarrow{\overline{q}(h)} X_2[1].$$

The two morphisms we take are

$$c(D) : c(f) \to c(h), \qquad c(D') : c(h) \to c(g),$$

where D and D' are the diagrams:

$$D = \begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ \| & & \downarrow{\scriptstyle g} \\ X_1 & \xrightarrow{h} & X_3 \end{array} \quad , \quad D' = \begin{array}{ccc} X_1 & \xrightarrow{h} & X_3 \\ \downarrow{\scriptstyle f} & & \| \\ X_2 & \xrightarrow{g} & X_3 \end{array}$$

By recalling 2.12, we prove the first point of the proposition. The diagram

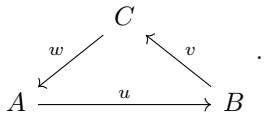
$$\begin{array}{ccccccc} c(f) & \xrightarrow{\overline{c}(D)} & c(h) & \xrightarrow{\overline{c}(D')} & c(g) & \xrightarrow{\overline{p}(f)[1]\overline{q}(g)} & c(f)[1] \\ \| & & \| & & \downarrow{\scriptstyle \overline{c}(I)} & & \| \\ c(f) & \xrightarrow{\overline{c}(D)} & c(h) & \xrightarrow{\overline{p}(c(D))} & c(c(D)) & \xrightarrow{\overline{q}(c(D))} & c(f)[1] \end{array}$$

thanks to 2.16 is an isomorphism of triangles and this proves (2).

3 Triangulated categories

3.1 Triangulated categories

Definition 3.1 *Let $\boldsymbol{K}$ be a category with an automorphism T. A triangle on an ordered triple (A, B, C) of objects of $\boldsymbol{K}$ is a triple (u, v, w) of morphism $u : A \to B$, $v : B \to C$ and $w : C \to T(A)$. It will be displayed this way*



Remark 3.2 *Let X be an object of $\boldsymbol{K}$. We will denote $T(X)$ by $X[1]$. $X[n]$ will denote the application of T to X n times.*

Definition 3.3 *A morphism of triangles $(u, v, w) \to (u', v', w')$ is a triple (f, g, h) such that the diagram*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f[1]} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}$$

is commutative in $\boldsymbol{K}$.

Definition 3.4 *An additive category K is called a triangulated category if it is equipped with an automorphism $T : \boldsymbol{K} \to \boldsymbol{K}$ and a distingued family of triangles (u, v, w), called exact, that respects the following four axioms:*

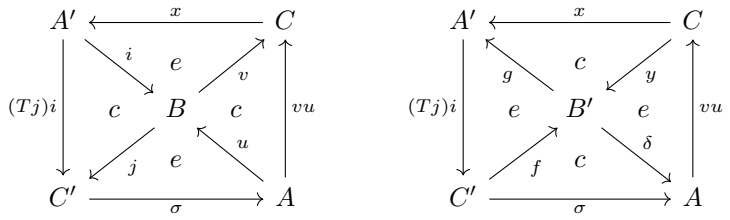
1. *Every morphism $u : A \to B$ can be embedded in an exact triangle (u, v, w). The triangle $(id_A, 0, 0)$ on $(A, A, 0)$ is exact. If a triangle (u, v, w) is isomorphic to an exact triangle (u', v', w'), then (u, v, w) is also exact.*

2. *If (u, v, w) is an exact triangle on (A, B, C) then $(v, w, -Tu)$ and $(-T^{-1}, u, v)$ are exact triangles on (B, C, TA) and $(T^{-1}C, A, B)$ respectively.*
   This is called the Rotation axiom.

3. *Let (u, v, w) and (u', v', w') be two exact triangles on (A, B, C) and (A', B', C'), respectively. If $f : A \to A'$, $g : B \to B'$ are two morphisms such that $gu = u'f$, then there exists $h : C \to C'$ such that (f, g, h) is a morphism of triangles*

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & A[1] \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \vdots{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\ A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \xrightarrow{w'} & A'[1] \end{array}.$$

4. *Given objects A, B, C, A', B', C' in $\boldsymbol{K}$, suppose there are three exact triangles (u, j, σ) on (A, B, C'), (v, x, i) on (B, C, A') and vu, y, δ on (A, C, B'). Then $(f, g, (Tj)i)$ on (C', B', A') is also an exact triangle. Moreover $\sigma = \delta f : C' \to B' \to TA$, $x = gy : C \to B' \to A'$, $yv = fj : B \to B'$, $u\delta = ig : B' \to B$.*



are the diagrams such that the first diagram is the lower cap of the octahedron of vertices A, B, C, A', B', C', the second diagram is the upper cap and in the diagrams e indicates an exact triangle while c a commutative diagram.

Remark 3.5 *If (u, v, w) is an exact triangle, vu, wv and $(Tu)w$ are zero in $\boldsymbol{K}$.*

Proof: This is proven thanks to the third axiom that gives us the commutative diagram

$$\begin{array}{ccccc} A & = & A & \longrightarrow & 0 \\ \| & & \downarrow{\scriptstyle u} & & \downarrow \\ A & \xrightarrow{u} & B & \xrightarrow{v} & C \end{array}.$$

By a similar reasoning and by using the second axiom, we prove that wv and $(Tu)w$ are also zero.

Proposition 3.6 *If $\mathscr{A}$ is an additive category, $K(\mathscr{A})$ is a triangulated category.*

Proof: The axioms of triangulated categories are all derived from the propositions of the previous section.
(1) is proved in the proposition 2.18, the same is true for (2), (3), (4), which are proven in 2.19, 2.20, 2.21 respectively.

Definition 3.7 *Let $\mathscr{D}$ and $\mathscr{D}'$ be two triangulated categories.*
A pair of a functor $F : \mathscr{D} \to \mathscr{D}'$ together with a given functorial isomorphisms $\sigma_X : F(X[1]) \to F(X)[1]$ is called an exact functor (in the sense of triangulated categories) if for every exact triangle (X, Y, Z, f, g, h) of $\mathscr{D}$ the triangle $(F(X), F(Y), F(Z), F(f), F(g), \sigma_X F(h))$ is an exact triangle of $\mathscr{D}'$.

Definition 3.8 *Let $\mathscr{D}$ be a triangulated category. A triangle*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

is called antiexact if the triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{-w} X[1]$$

is exact.
An additive functor from a triangulated category $\mathscr{D}'$ to $\mathscr{D}$ is called antiexact if it sends exact triangles of $\mathscr{D}'$ to antiexact of $\mathscr{D}$.

Remark 3.9 *Let $\mathscr{D}$ be a triangulated category. Then $\mathscr{D}^{op}$ is a triangulated category with the following class of exact triangles: let $((w))$ denote the morphism w of $\mathscr{D}$ in $\mathscr{D}^{op}$ and the translation functor of $\mathscr{D}^{op}$*

$$X \to X[[n]],\ n \in \mathbf{Z}$$

where

$$X[-n] = X[[n]].$$

A triangle of $\mathscr{D}^{op}$

$$X[[-1]] \xrightarrow{((w))} Z \xrightarrow{((v))} Y \xrightarrow{((u))} X$$

is exact if and only if

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

is antiexact.

3.2 Cohomological functors

Definition 3.10 *Let $\mathscr{D}$ be a triangulated category and $\mathscr{A}$ an abelian category. A functor $H : \mathscr{D} \to \mathscr{A}$ is called cohomological if for any exact triangle*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

the diagram

$$HX \xrightarrow{Hu} HY \xrightarrow{Hv} HZ \xrightarrow{Hw} H(X[1])$$

is an exact sequence in $\mathscr{A}$.

Remark 3.11 *Let $H : \mathscr{D} \to \mathscr{A}$ be a cohomological functor. We will use the notation $H^0(X) = H(X)$, $H^n(X) = H(X[n])$. By using the rotation axiom on an exact triangle*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

we have that the long sequence

$$\cdots \to H^{-1}(Z) \xrightarrow{H^{-1}(w)} H^0(X) \xrightarrow{H^0(u)} H^0(Y) \xrightarrow{H^0(v)} H^0(Z) \xrightarrow{H^0(w)} \cdots$$

is exact.

Proposition 3.12 *Let $\mathscr{D}$ be a triangulated category. For every object X of $\mathscr{D}$, the functor $Hom_{\mathscr{D}}(X, .)$ and $Hom_{\mathscr{D}}(., X)$ are cohomological functor respectively from $\mathscr{D}$ and from $\mathscr{D}^{op}$ into **Ab**.*

Proof: It is enough to prove that $Hom_{\mathscr{D}}(X, .)$ is cohomological since the other assertion is obtained by duality.
Let $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} X[1]$ be an exact triangle. We have to show that

$$Hom_{\mathscr{D}}(X, X') \xrightarrow{Hom_{\mathscr{D}}(X,u')} Hom_{\mathscr{D}}(X, Y') \xrightarrow{Hom_{\mathscr{D}}(X,v')} Hom_{\mathscr{D}}(X, Z') \xrightarrow{Hom_{\mathscr{D}}(X,w')} Hom_{\mathscr{D}}(X, X'[1])$$

is an exact sequence. Since $Hom_{\mathscr{D}}(X, .)$ is additive and $v'u' = 0$ by 3.5, it follows that $Hom_{\mathscr{D}}(X, v')Hom_{\mathscr{D}'}(X, u') = 0$ Suppose now to have $g : X \to Y'$ such that $v'g = 0$. By applying the first three axioms to the middle square of the above diagram, we can find $f : X \to X'$ such that the diagram is commutative. We deduce $g = u'f$ and so we have the exactness in $Hom_{\mathscr{D}}(X, Y')$. To prove the exactness in $Hom_{\mathscr{D}}(X, Z')$, we can use the second axiom and repeat the proof.

Corollary 3.13 *Let (X, Y, Z, u, v, w) and (X', Y', Z', u', v', w') be two exact triangles of $\mathscr{D}$ and (f_1, f_2, f_3) a morphism of triangles. If two of f_i are isomorphims so it is the third.*

Proof: Given W in $\mathscr{D}$ and $f_1 : X \to X'$ and $f_2 : Y \to Y'$ two isomorphisms, the commutative diagram

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & Hom_{\mathscr{D}}(W, X) & \longrightarrow & Hom_{\mathscr{D}}(W, Y) & \longrightarrow & Hom_{\mathscr{D}}(W, Z) & \longrightarrow & Hom_{\mathscr{D}}(W, X[1]) \longrightarrow \ldots \\
 & & \downarrow{\scriptstyle Hom_{\mathscr{D}}(W,f_1)} & & \downarrow{\scriptstyle Hom_{\mathscr{D}}(W,f_2)} & & \downarrow{\scriptstyle Hom_{\mathscr{D}}(W,f_3)} & & \downarrow{\scriptstyle Hom_{\mathscr{D}}(W,f_1[1])} \\
\ldots & \longrightarrow & Hom_{\mathscr{D}}(W, X') & \longrightarrow & Hom_{\mathscr{D}}(W, Y') & \longrightarrow & Hom_{\mathscr{D}}(W, Z') & \longrightarrow & Hom_{\mathscr{D}}(W, X'[1]) \longrightarrow \ldots
\end{array}$$

has exact rows. Thanks to the *five lemma*, $Hom_{\mathscr{D}}(W, f_3)$ is an isomorphism and so f_3 is an isomorphism. The other two possible variations are obtained by using the Rotation Axiom.

Corollary 3.14 *Let $X \xrightarrow{u} Y$ be a morphism in $\mathscr{D}$, then all exact triangles containing u are isomorphic.*

Proof: Suppose we have (X, Y, Z, u, v, w) and (X, Y, Z', u, v', w') two exact triangles. We can find $f : Z \to Z'$ such that

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\
\| & & \| & & \downarrow{\scriptstyle f} & & \| \\
X & \xrightarrow{u} & Y & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X[1]
\end{array}$$

is commutative. By using 3.13, f is an isomorphism.

Corollary 3.15 *The direct sum of exact triangles is an exact triangle.*

Proof: Let

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

and

$$X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} X'[1]$$

be two exact triangles. Given $X \oplus X' \xrightarrow{u \oplus u'} Y \oplus Y'$, there is an exact triangle

$$X \oplus X' \xrightarrow{u \oplus u'} Y \oplus Y' \xrightarrow{m} L \xrightarrow{n} (X \oplus X')[1].$$

Thanks to the third axiom, we can construct $f : Z \to L$ and $f' : Z' \to L$ such that

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ \downarrow{\scriptstyle i_X} & & \downarrow{\scriptstyle i_Y} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle i_{X[1]}} \\ X \oplus X' & \xrightarrow{u \oplus u'} & Y \oplus Y' & \xrightarrow{m} & L & \xrightarrow{n} & (X \oplus X')[1] \end{array},$$

$$\begin{array}{ccccccc} X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z & \xrightarrow{w'} & X'[1] \\ \downarrow{\scriptstyle i_{X'}} & & \downarrow{\scriptstyle i_{Y'}} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle i_{X'[1]}} \\ X \oplus X' & \xrightarrow{u \oplus u'} & Y \oplus Y' & \xrightarrow{m} & L & \xrightarrow{n} & (X \oplus X')[1] \end{array}$$

commute. We can construct the commutative diagram

$$\begin{array}{ccccccc} X \oplus X' & \xrightarrow{u \oplus u'} & Y \oplus Y' & \xrightarrow{v \oplus v'} & Z \oplus Z' & \xrightarrow{w \oplus w'} & (X \oplus X')[1] \\ \| & & \| & & \downarrow{\scriptstyle (f,f')} & & \| \\ X \oplus X' & \xrightarrow{u \oplus u'} & Y \oplus Y' & \xrightarrow{m} & L & \xrightarrow{n} & (X \oplus X')[1] \end{array}.$$

To prove that (f, f') is an isomorphism is sufficient to apply the $Hom_{\mathscr{D}}(M, .)$ to the diagram: thanks to the *five lemma*, $Hom_{\mathscr{D}}(M, (f, f'))$ is an isomorphism.

Corollary 3.16 *Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ be an exact triangle in $\mathscr{D}$ such that $w = 0$. Then there exists a section $s : Z \to Y$ of v. Every section $s : Z \to Y$ define an isomorphism:*

$$X \oplus Z \xrightarrow[\cong]{(u,s)} Y$$

Morevoer,

$$X \xrightarrow{\begin{pmatrix} id \\ 0 \end{pmatrix}} X \oplus Z \xrightarrow{\begin{pmatrix} 0 & id \end{pmatrix}} Z \xrightarrow{0} X[1]$$

is an exact triangle.

Proof: The third assertion is proven, thanks to 3.15, by taking the direct sum of

$$X \xrightarrow{id} X \to 0 \to X[1]$$

and

$$0 \to Z \xrightarrow{id} Z \to 0,$$

which are exact triangles by the first and second axioms.
By hypotheses,

$$0 \to Hom_{\mathscr{D}}(Z, X) \to Hom_{\mathscr{D}}(Z, Y) \to Hom_{\mathscr{D}}(Z, Z) \to 0$$

is exact. Then there exists a section $s : Z \to Y$ by taking a preimage of $id_Z \in Hom_{\mathscr{D}}(Z, Z)$.
Then by applying the functor $Hom_{\mathscr{D}}(W, .)$ to the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{i_X} & X \oplus Z & \xrightarrow{\pi_Z} & Z \\ \| & & \downarrow{\scriptstyle (u,s)} & & \| \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z \end{array},$$

by the *five lemma*,

$$Hom_{\mathscr{D}}(W, (u, s)) : Hom_{\mathscr{D}}(W, X \oplus Z) \to Hom_{\mathscr{D}}(W, Y)$$

is an isomorphism for every W, hence (u, s) is an isomorphism.

Corollary 3.17 *Let D and D' be two triangulated categories. Then every exact functor $\mathscr{D} \xrightarrow{F} \mathscr{D}'$ is additive.*

Proof:

$$0 \to 0 \to 0 \to 0$$

is an exact triangle. By hypotheses,

$$F(0) \xrightarrow{id_{F(0)}} F(0) \xrightarrow{id_{F(0)}} F(0) \to F(0)[1]$$

is an exact triangle. It follows that $F(0) = 0$. If a functor preserves finite products then it is additive. We can apply F to

$$X \xrightarrow{\begin{pmatrix} id \\ 0 \end{pmatrix}} X \oplus Z \xrightarrow{\begin{pmatrix} 0 & id \end{pmatrix}} Z \xrightarrow{0} X[1],$$

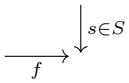
and by recalling 3.16, we find that $F(X \oplus Z) \cong F(X) \oplus F(Z)$ and so F is additive. By a similar reasoning, we can prove that if $\mathscr{D} \xrightarrow{F} \mathbf{Ab}$ is cohomological then F is an additive functor.

4 Multiplicative systems

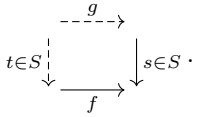
Definition 4.1 *Let $\mathscr{C}$ be a category. A set of morphisms S of $\mathscr{C}$ is called a multiplicative system if the following properties hold:*

1. *For every object X of $\mathscr{C}$, 1_X is in S. The composition of two morphisms in S is in S.*

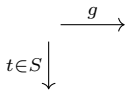
2. *Given a diagram of the form*



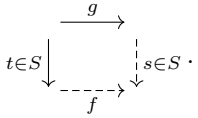
it can be completed into



Given a diagram of the form



it can be completed into



3. *Given a couple of morphisms $f, g : X \rightrightarrows Y$, the following properties are equivalent:*

   - *There exists a morphism s in S from Y such that $sf=sg$*

   - *There exists a morphism t in S with target X such that $ft=gt$.*

The multiplicative system is said to be saturated if it also has the following property:

- *if f, g and h are composable morphisms such that fg and gh are in S, then g is in s.*

Definition 4.2 *Let $\mathscr{D}$ be a triangulated category. A multiplicative system S of $\mathscr{D}$ is said to be compatible with the triangulation if the following properties hold:*

1. *For every element s of S and for every $n \in \mathbb{Z}$, $s[n]$ is still in S.*

2. *Given any couple of exact triangles:*

$$(X, Y, Z, u, v, w),\ (X', Y', Z', u', v', z')$$

and any commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s'} \\ X' & \xrightarrow{u'} & Y' \end{array}$$

such that s and s' are in S, then there exists $s'' : Z \to Z'$ in S such that (s, s', s'') is a morphism of triangles.

Remark 4.3 *Let $\mathscr{D}$ be a triangulated category and S a multiplicative system of $\mathscr{D}$ which is compatible with the triangulation. The set S is a multiplicative system of the opposite category of $\mathscr{D}$.*

Proposition 4.4 *Let $\mathscr{D}$ be a triangulated category and S a set of morphisms of $\mathscr{D}$ that satisfies (1) of 4.1 and (1) and (2) of 4.2. Then S satisfies (2) of 4.1.*

Proof: We have to show that given a diagram

$$\begin{array}{ccc} & & X' \\ & & \downarrow{\scriptstyle s\in S} \\ Z & \xrightarrow{f} & X \end{array}$$

it can be completed. By duality we obtain the other diagram completion.
By using the first axiom of triangulated categories, there exists an exact triangle $(X, Y, Z[1], u, v, f[1])$ and an exact triangle (X', Y, Z', us, v', w'). By the second axiom, since

$$\begin{array}{ccc} X' & \xrightarrow{us} & Y \\ {\scriptstyle s}\downarrow & & \| \\ X & \xrightarrow{u} & Y \end{array}$$

is a commutative diagram and by (2) of 4.2, we get $s' : Z' \to Z[1]$ in S such that $(s, 1_Y, s')$ is a morphism of triangles which gives us in particular the commutative diagram

$$\begin{array}{ccc} Z' & \xrightarrow{w'} & X'[1] \\ {\scriptstyle s'}\downarrow & & \downarrow{\scriptstyle s[1]} \\ Z[1] & \xrightarrow{f[1]} & X[1] \end{array}$$

that proves the proposition.

Remark 4.5 *For the rest of the chapter, the multiplicative systems in a triangulated category are always supposed to be compatible with the triangulation.*

Remark 4.6 *Let $\mathscr{D}$ be a triangulated category. A full triangulated subcategory of $\mathscr{D}$ is a full subcategory stable under translations with a triangulated structure such that the inclusion functor in $\mathscr{D}$ is exact. $\mathscr{D}'$ is a full triangulated subcategory of $\mathscr{D}$ if and only if it is stable under translations and for every morphism $X \xrightarrow{f} Y$ in $\mathscr{D}'$ there exists an exact triangle in $\mathscr{D}$ (X, Y, Z, f, g, h) with Z object in $\mathscr{D}'$.*

Definition 4.7 *Let $\mathscr{D}'$ be a full triangulated subcategory of the triangulated category $\mathscr{D}$. $\mathscr{D}'$ is saturated if the following property holds: Every direct factor in $\mathscr{D}$ of an object in $\mathscr{D}'$ is in $\mathscr{D}'$.*

Remark 4.8 *Let $\mathscr{D}'$ be a full triangulated subcategory of $\mathscr{D}$. $S(\mathscr{D}')$ is the set of morphisms $X \xrightarrow{f} Y$ of $\mathscr{D}$ such that there is an exact triangle (X, Y, Z, f, g, h) with Z in $\mathscr{D}'$.*

Proposition 4.9 *The set $S(\mathscr{D}')$ is a multiplicative system.*

Proof: By 4.4, it is sufficient to show (*1*) and (*3*) of 4.1 and (*1*) and (*2*) of 4.2.
Proof of (1) of 4.1: Since the null objects of $\mathscr{D}'$ are the null objects of $\mathscr{D}$, the identities are in $S(\mathscr{D}')$. Given $s_1 : A \to B, s_2 : B \to C$ two composable morphisms in $S(\mathscr{D}')$, by using the fourth axiom of triangulated categories, we have that $s_2 s_1$ is in $S(\mathscr{D}')$. In particular, given the exact triangles (A, B, C', s_1, u, v), (B, C, A', s_2, h, k) and $(A, C, B', s_2 s_1, r, y)$ we have that there exists an exact triangle (C', B', A', q, w, e). Since C' and A' are in $\mathscr{D}'$, we have that also B' is in $\mathscr{D}'$.
Proof of (3) of 4.1: Suppose we have $X \xrightarrow{s} Y \xrightarrow{f} Z$ such that $s \in S(\mathscr{D}')$ and $fs = 0$. Since $\mathscr{D}$ is an additive category, this is sufficient to be in the hypotheses of (*3*) of 4.1. By hypotheses, there exist an exact triangle (X, Y, N, s, h, u) with N object of $\mathscr{D}'$. By 3.12, there exists $g : N \to Z$ such that $f = gh$. Let (N, Z, W, g, t, v) be an exact triangle. Then $tg = 0$ since they are consecutives morphisms in an exact triangle, it follows that $tf = tgh = 0$ with $t \in S(\mathscr{D}')$ since N is in $\mathscr{D}'$. The other case is proved dually.
Proof of (1) of 4.2: Since $\mathscr{D}$ is stable under translation and by applying the second axiom of the triangulated categories, we prove (*1*) of 4.2.
Proof of (2) of 4.2: Let (X, Y, Z, u, v, w) and (X', Y', Z', u', v', w') be two exact triangles. We have to prove that given a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s'} \\ X' & \xrightarrow{u'} & Y' \end{array}, \quad s,\ s' \in S(\mathscr{D}')$$

we can find $s'' : Z \to Z'$ in $S(D')$ such that (s, s', s'') is a morphism of triangles.
We can start by studying these two cases:

- $X = X'$ and $s = 1_X$
- $Y = Y'$ and $s' = 1_Y$

since every morphism can be decomposed in

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ \| & & \downarrow{\scriptstyle s'} \\ X & \xrightarrow{s'u} & Y' \\ {\scriptstyle s}\downarrow & & \| \\ X' & \xrightarrow{u'} & Y' \end{array} .$$

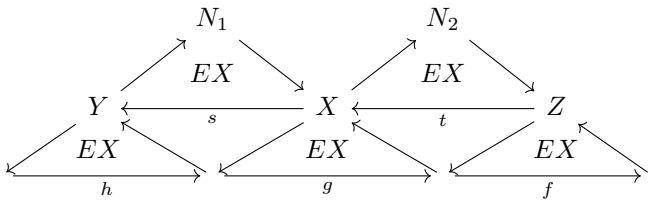
The second case is the dual of the first case so it is sufficient to prove the first case.
To prove the first case, we can apply the fourth axiom to the above diagram.
Let (X, Y, Z, u, f, g), (X, Y', Z', su, q, w) and (Y, Y', Z'', s', p, o) be exact triangles. By fourth axiom (Z, Z'', Z', s'', y, h) is an exact triangle and $(1_X, s', s'')$ is a morphism of triangles and since Z' is in $\mathscr{D}'$, s'' is in $S(\mathscr{D}')$.

Proposition 4.10 *$S(\mathscr{D}')$ is saturated if only if $\mathscr{D}'$ is saturated.*

Proof: Suppose $\mathscr{D}'$ is saturated and f, g, h are three composable morphisms such that fg and gh are in $S(\mathscr{D}')$. By applying the fourth axiom to

$$\begin{array}{ccc} & {\scriptstyle fg} & \\ {\scriptstyle g}\downarrow & \nearrow & \\ & {\scriptstyle f} & \end{array} \qquad \begin{array}{ccc} & {\scriptstyle gh} & \\ {\scriptstyle h}\downarrow & \nearrow & \\ & {\scriptstyle g} & \end{array} ,$$

we can construct the diagram



where N_1 and N_2 are objects in $\mathscr{D}'$ and the triangles with EX are exact triangles and the remaining ones are commutative. $s[1]t = 0$ since they are consecutive morphisms. Since s and t are in exact triangles having as third object N_1 and N_2 respectiely they are in $S(\mathscr{D}')$. There exists an exact triangle $(Z, Y[2], N, 0, u, v)$ where N is an object of $\mathscr{D}'$. Since $\mathscr{D}'$ is saturated and by applying 3.16 $Y[2]$ is a direct factor of an object of $\mathscr{D}'$, $Y[2]$ is isomorphic to an object of $\mathscr{D}'$. Since $\mathscr{D}'$ is a full triangulated subcategory, X is isomorphic to an object of $\mathscr{D}'$. We can conclude g is in $S(\mathscr{D}')$. Suppose now $S(\mathscr{D}')$ is saturated and X is a direct factor of an object of $\mathscr{D}'$. By corollary 3.16, there exists an exact triangle $(X, Y, N, 0, u, v)$ such that N is an object of $\mathscr{D}'$. This implies that the null morphism $X \to Y$ is a morphism of $S(\mathscr{D}')$. By recalling the definition of saturated and by applying it to $0 \to X \to 0 \to Y$, $X \to 0$ is an element of $S(\mathscr{D}')$.

Then there exists an exact triangle $(X, 0, N, 0, 0, i)$ with N object of $\mathscr{D}'$. It follows that $N \xrightarrow{i} X[1]$ is an isomorphism and it follows that X is isomorphic to $N[-1]$ which is an object of $\mathscr{D}'$.

Definition 4.11 *Let $\mathscr{D}'$ be a full triangulated subcategory of a triangulated category $\mathscr{D}$. $S(\mathscr{D}')$ is called the multiplicative system of the subcategory $\mathscr{D}'$.*

Proposition 4.12 *Let $F : \mathscr{D}_1 \to \mathscr{D}_2$ be an exact functor between two triangulated categories. Let $S(F)$ be the set of morphisms of $\mathscr{D}_1$ such that their images are isomorphisms of $\mathscr{D}_2$. Let $\mathscr{D}'(F)$ the full subcategory of $\mathscr{D}_1$ such that their images are null objects. Then*

1. *$\mathscr{D}'(F)$ is a saturated strictly full triangulated subcategory (strict subcategory means that every object of $\mathscr{D}_1$ that is isomorphic to an object of $\mathscr{D}'(F)$ is an object of $\mathscr{D}'(F)$).*

2. *$S(F)$ is a saturated multiplicative system of $\mathscr{D}_1$.*

3. *$S(F)$ is the multiplicative system associated to the subcategory $\mathscr{D}'(F)$.*

Proof *of (1)*: Suppose $X \xrightarrow{f} Y$ is a morphism of the full subcategory spanned by $\mathscr{D}'(F)$. Let (X, Y, Z, f, g, h) be the exact triangle in $\mathscr{D}_1$.
By applying F, $(F(X), F(Y), F(Z), F(f), F(g), F(h)) = (0, 0, F(Z), 0, 0, 0)$ is an exact triangle in $\mathscr{D}_2$. It follows $F(Z)$ is isomorphic to the null object. It is strict and full by definition.
Suppose we have X which is a direct factor of $X \oplus Y$, an element of $\mathscr{D}'(F)$. By using 3.17, $F(X)$ is the null object.
Proof *of (2)*: By 4.4 we don't need to check (2) of 4.1.
The composition of two isomorphisms is an isomorphism so *(1)* of 4.1 is verified.
As for (3) of 4.1, let $s : X \to Y$ be a morphism of $S(F)$ and $f : Y \to Z$ a morphism of D_1 such that $sf = 0$.
Then, since $F(s)F(f) = 0$ and $F(s)$ is an isomorphism, $F(f) = 0$. Moreover, there exists an exact triangle (X, Y, N, s, h, u). Since $fs = 0$, there exists $g : N \to Z$ such that $f = gh$.
Let (N, Z, W, g, t, v) be an exact triangle containing g. We now observe that $F(N) = 0$, since $F(s)$ is an isomorphism. Hence $F(t)$ is also an isomorphism, and so t is in $S(F)$.
We also have $tf = 0$ since $tgh = 0$.
(1) of 4.2 holds because exact functors commute with shifts.
(2) of 4.2 is proved by applying the *five lemma* to the image of

$$\begin{array}{ccccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] & \xrightarrow{-u[1]} & Y[1] \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s'} & & \downarrow{\scriptstyle s''} & & \downarrow{\scriptstyle s[1]} & & \downarrow{\scriptstyle s'[1]} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] & \xrightarrow{-u'[1]} & Y'[1] \end{array}$$

through F, supposing s and s' are transformed into isomorphisms and the two rows are exact triangles. The saturation property is a consequence of the third point by 4.10.
Proof *of (3)*: $S(F) \subseteq S(\mathscr{D}'(F))$ since given s in $S(F)$ and an exact triangle (X, Y, Z, s, g, h), $(F(X), F(Y), F(Z), F(s), F(g), F(h))$ is isomorphic to $(F(X), F(Y), 0, F(s), 0, 0)$ by a similar reasoning to the first two parts of proof.
It follows that $F(Z) \cong 0$ so s is in $S(\mathscr{D}'(F))$.
$S(\mathscr{D}'(F)) \subseteq S(F)$: let l be a morphism such that (X', Y', Z', l, k, j) is an exact triangle and $F(Z') = 0$. This implies that $F(k) = F(j) = 0$ and that l is in $S(F)$. It follows that since $\mathscr{D}'(F)$ is saturated, $S(F)$ is saturated.

Proposition 4.13 *Let $H : \mathscr{D} \to \mathscr{A}$ be a cohomological functor from a triangulated category to an abelian category. Let $S(H)$ be the set of morphisms of $\mathscr{D}$ such that their images and the images of their translations are isomorphisms of $\mathscr{A}$. Let $\mathscr{D}'(H)$ the full subcategory of $\mathscr{D}$ such that their images and the images of their translations are null objects. Then*

1. *$\mathscr{D}'(H)$ is a saturated strictly full triangulated subcategory.*

2. *$S(H)$ is a saturated multiplicative system of $\mathscr{D}$.*

3. *$S(H)$ is the multiplicative system associated to the subcategory $\mathscr{D}'(H)$.*

Proof: Since H is a cohomological functor to an abelian category, we have that

$$\cdots \to H^{-1}(Z) \xrightarrow{H^{-1}(-w)} H^0(X) \xrightarrow{H^0(u)} H^0(Y) \xrightarrow{H^0(v)} H^0(Z) \xrightarrow{H^0(w)} \cdots$$

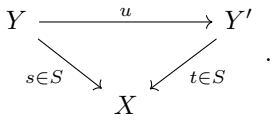
is an exact sequence for every exact triangle (X, Y, Z, u, v, w). Then if Z is in $\mathscr{D}'(F)$, then $H^i(X) \cong H^i(Y)$ for every i. After this observation, the proof is obtained by repeating the reasoning of the previous proposition.

4.1 Construction of the localized category

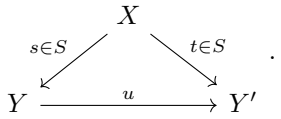
Let $\mathscr{C}$ a category and S a multiplicative system of $\mathscr{C}$.
For every object X, we define S/X the category

- whose objects are the morphisms of S with target X,

- the morphisms are the commutative diagrams of the form



Remark 4.14 *Similarly, we can define X/S as the category whose objects are morphisms from X in S and whose morphisms are diagrams of the form*



Definition 4.15 *$\mathscr{C}(S^{-1})$ is the category*

- *whose objects are the objects of $\mathscr{C}$,*

- *$Hom_{\mathscr{C}(S^{-1})}(X, Y) = colim_{(S/X)^{op}} Hom_{\mathscr{C}}(., Y)$*

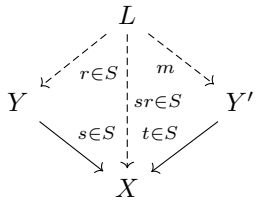
Remark 4.16 *$(S/X)^{OP}$ is a filtered category thanks to 4.1. In fact, $(S/X)^{OP}$*

1. *is not empty,*

2. *for every two objects j and j' in $(S/X)^{OP}$, there exists an object k and two arrows $f : j \to k$ and $f' : j' \to k$ in $(S/X)^{OP}$.*

3. *for every two parallel arrows $u, v : i \to j$ in $(S/X)^{OP}$, there exists an object k and an arrow $w : j \to k$ such that $wu = wv$.*

Proof: Since 1_X is in $(S/X)^{OP}$, it is not empty so (*1*) is proven.
To prove (*2*), suppose we have two object of $(S/X)^{op}$ $j = (Y \xrightarrow{s\in S} X)$ and $j' = (Y' \xrightarrow{t\in S} X)$. Then by the second property of multiplicative systems, we can construct the following diagram



and we can take $k = (L \to X)$.
(*3*) follows from the definition of multiplicative system since it is (*3*) of 4.1.

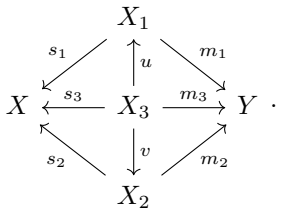
Since $(S/X)^{OP}$ is a filtered category, an element of $Hom_{\mathscr{C}(S^{-1})}(X, Y)$ is given by a diagram of the form

$$D = X \xleftarrow{s} X' \xrightarrow{m} Y,\ s \in S.$$

Two diagrams $D_1 = X \xleftarrow{s_1} X_1 \xrightarrow{m_1} Y$ and $D_2 = X \xleftarrow{s_2} X_2 \xrightarrow{m_2} Y$ $s_1,\ s_2 \in S$, are equivalent if and only if there exists a diagram

$$D_3 = X \xleftarrow{s_3} X_3 \xrightarrow{m_3} Y,\ s_3 \in S$$

such that there exists a commutative diagram



For every $f \in Hom_{\mathscr{C}}(X, Y)$, we call $\mathscr{Q}(f)$ the class of the diagram

$$X \xleftarrow{1_X} X \xrightarrow{f} Y.$$

For every $s : Y \to X,\ s \in S$, we define $\mathscr{Q}(s)^{-1}$ to be the class diagram of

$$X \xleftarrow{s} Y \xrightarrow{1_Y} Y.$$

$Mor(\mathscr{C}(S^{-1}))$ is the

$$\coprod_{(X,Y)\in Ob(\mathscr{C}\times\mathscr{C})} Hom_{\mathscr{C}(S^{-1})}(X, Y).$$

s and b are the applications from $Mor(\mathscr{C}(S^{-1}))$ into $Ob(\mathscr{C})$ such that

$$m \in Mor(\mathscr{C}(S^{-1})), s(m) = X,\ b(m) = Y$$

and $id : Ob(\mathscr{C}) \to Mor(\mathscr{C}(S^{-1}))$ such that $1_X \to \mathscr{Q}(1_X)$.

Theorem 4.17 *1. $\mathscr{C}(S^{-1})$ is a category.*

2. *$\mathscr{Q}(s)$ is an isomorphism and $\mathscr{Q}(s)^{-1}$ is its inverse for every s in S.*

3. *For every diagram $D = X \xleftarrow{s} X' \xrightarrow{m} Y$, $\bar{D} = \mathscr{Q}(m)\mathscr{Q}(s)^{-1}$ where $\bar{D}$ is the class of D in $Mor(\mathscr{C}(S^{-1}))$.*

4. *$\mathscr{Q}$ is a functor.*

5. *Every morphism of $\mathscr{C}(S^{-1})$ can be written as $\mathscr{Q}(t)^{-1}\mathscr{Q}(m)$, with $m \in Mor(\mathscr{C})$, $t \in S$.*

Sketch of Proof: To prove that $\mathscr{C}(S^{-1})$ is a category, we have to define composition, it can be done using (2) of 4.1:

$$\begin{array}{ccccc} W & \overset{m''}{\dashrightarrow} & Z' & \xrightarrow{m'} & Z \\ \downarrow{\scriptstyle s''\in S} & & \downarrow{\scriptstyle s'\in S} & & \\ X & \xleftarrow{s\in S} & X' & \xrightarrow{m} & Y \end{array} .$$

Now by composing $\mathscr{Q}(s)$, which has form $X = X \xrightarrow{s} Y$, with $Y \xleftarrow{s} X = X$ we prove (2).
Given the diagram

$$D = X \xleftarrow{s} X' \xrightarrow{m} Y,$$

we can observe

$$\bar{D} = (X' \xleftarrow{1} X' \xrightarrow{m} Y) \circ (X \xleftarrow{s} X' \xrightarrow{1} X') = \mathscr{Q}(m)\mathscr{Q}(s)^{-1}.$$

$\mathscr{Q}$ is a functor by the rest of the proof.
(5) is proved by using (2) of 4.1.

Remark 4.18 *$\mathscr{Q} : \mathscr{C} \to \mathscr{C}(S^{-1})$ is the solution of the following universal problem: every functor from $\mathscr{C}$ to a category $\mathscr{A}$ that transforms $s \in S$ into isomorphisms can be factored uniquely through $\mathscr{Q}$.*

Corollary 4.19 *Let $\mathscr{C}$ be a category and S be a multiplicative system of $\mathscr{C}$ Then*

1. *S is a multiplicative system of $\mathscr{C}^{op}$. Moreover, $(\mathscr{C}^{op}(S^{-1}))^{op} \xrightarrow{\cong} \mathscr{C}(S^{-1})$*

2. *If $\mathscr{C}$ is an additive category, $\mathscr{C}(S^{-1})$ is additive and $\mathscr{Q}$ is additive.*

3. *Let $\hat{S}$ be the set of morphisms of $\mathscr{C}$ which are transformed into isomorphisms of $\mathscr{C}(S^{-1})$ through $\mathscr{Q}$. $\hat{S}$ is a saturated multiplicative system of $\mathscr{C}$ that contains S. Every saturated multiplicative system of $\mathscr{C}$ that contains S contains $\hat{S}$.*

Proof: The first proposition is proved by the universal problem of 4.18 since $(\mathscr{C}^{op}(S^{-1}))^{op}$ is also a solution.
To define addition in $\mathscr{C}(S^{-1})$, we can observe that given two morphisms of the form

$$X \xleftarrow{s_1} X_1 \xrightarrow{f_1} Y,\ X \xleftarrow{s_2} X_2 \xrightarrow{f_2} Y,$$

by using *(2)* of 4.1 we can construct a diagram

$$\begin{array}{ccc} X_1 & \xleftarrow{a} & W \\ {\scriptstyle s_1}\downarrow & & \downarrow{\scriptstyle s\in S} \\ X & \xleftarrow[s_2]{} & X_2 \end{array} .$$

At this point

$$X \xleftarrow{s_2 s} W \xrightarrow{s} X_2 \xrightarrow{f_2} Y$$

and

$$X \xleftarrow{s_2 s} W \xrightarrow{a} X_1 \xrightarrow{f_1} Y$$

can be added in natural way.
It follows that $\mathscr{Q}$ is an additive functor.
(*3*): We notice that a morphism f is transformed into an isomorphism by $\mathscr{Q}$ if and only if there exist two morphisms g and h such that gf and fh are in S. In fact, if f is an isomorphism in $\mathscr{C}(S^{-1})$ then there is some right inverse $\xleftarrow{m}\xrightarrow{t}$. Hence $ft = m \in S$. Similarly for the left inverse. In the converse direction we can take as left inverse $\xrightarrow{g}\xleftarrow{gf}$, in $\mathscr{C}(S^{-1})$, and as right inverse $\xleftarrow{fh}\xrightarrow{h}$ in $\mathscr{C}(S^{-1})$. Composition of isomorphisms is an isomorphism so (1) of 4.1 is verified.
To prove (3) of 4.1, suppose we have two morphisms $f, g : X \to Y$ and suppose there exist $r : Y \to Z$ such that $\mathscr{Q}(r)$ is an isomorphism and $rf = rg$. Then there exist w such that wr is in S. Then since S is a multiplicative system we have that there exists a morphism $y \in S$ (hence in $\hat{S}$) such that $fy = gy$, by (3) of 4.1.
To prove (2) of 4.1, we can use a similar reasoning. Suppose f is transformed in an isomorphism by $\mathscr{Q}$. Then there exists g such that $gf \in S$ then by (2) of 4.1 we have

$$\begin{array}{ccc} T & \overset{t}{\dashrightarrow} & X \\ \vdots & & \downarrow{\scriptstyle g} \\ {\scriptstyle y}\vdots & & Y \\ & & \downarrow{\scriptstyle f} \\ R & \xrightarrow{m} & Z \end{array} \quad = \quad \begin{array}{ccc} T & \overset{gt}{\dashrightarrow} & Y \\ \vdots & & \\ {\scriptstyle y}\vdots & & \downarrow{\scriptstyle f} \\ R & \xrightarrow{m} & Z \end{array}$$

and $y \in S$ so it is transformed into an isomorphism. The other completion follows a similar reasoning.
$\hat{\mathscr{S}}$ is a saturated multiplicative system by the observation regarding isomorphisms in $\mathscr{Q}(\mathscr{C})$.

Remark 4.20 *Let $\mathscr{C}$ be a category and S a multiplicative system of $\mathscr{C}$. $\mathscr{C}(S^{-1})$ is called the localization of $\mathscr{C}$ by S. $\hat{S}$ is called the saturation of S.*

We now return to the context of triangulated categories to study the triangulated structure on the localization.

Theorem 4.21 *Let $\mathscr{D}$ be a triangulated category and S a multiplicative system of $\mathscr{D}$. Let $\mathscr{D}(S^{-1})$ be the localization of $\mathscr{D}$ and $\mathscr{Q} : \mathscr{D} \to \mathscr{D}(S^{-1})$. Then*

1. *$\mathscr{D}(S^{-1})$ admits a unique additive structure such that $\mathscr{Q}$ is additive.*

2. *There exists a unique triangulated structure on $\mathscr{D}(S^{-1})$ such that $\mathscr{Q}$ is an exact funtor. The exact triangles are the triangles that are isomorphic to the images of $\mathscr{Q}$ of exact triangles.*

3. *Every exact functor (respectively cohomological) F: $\mathscr{D} \to \mathscr{D}'$,where $\mathscr{D}'$ is a triangulated category (respectively abelian), that transforms the morphisms of S in isomorphisms factors uniquely through $\mathscr{Q}$.*
   Let $F_1, F_2 : \mathscr{D} \rightrightarrows \mathscr{D}'$ be two exact functors that transform the morphism of S into isomorphisms. Let $G_1,\ G_2 : \mathscr{D}(S^{-1}) \rightrightarrows \mathscr{D}'$ be the two respective factorizations and let $m : F_1 \to F_2$ a natural transformation. Then there exists a unique natural transformation $p : G_1 \to G_2$ such that $m = p \star \mathscr{Q}$

Proof: (*1*) follows from 4.19. The proof of (*3*) will follow from 4.18 and from (*2*) so it remains to prove (*2*).
The first two axioms of triangulated categories follow from the definition of exact triangles of $\mathscr{D}(S^{-1})$.
The third axiom and the fourth require a bit of work:

Lemma 4.22 *Let $u : X \to Y$ and $u' : X' \to Y'$ two morphisms of $\mathscr{D}$ and*

$$\begin{array}{ccc} Q(X) & \xrightarrow{Q(u)} & Q(Y) \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle p} \\ Q(X') & \xrightarrow{Q(u')} & Q(Y') \end{array}$$

a commutative diagram of $\mathscr{D}(S^{-1})$. Then there exists a morphism $u'' : X'' \to Y''$ and two commutative diagrams of $\mathscr{D}$

$$\begin{array}{ccc} X'' & \xrightarrow{u''} & Y'' \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ X & \xrightarrow{u} & Y \end{array} \qquad \begin{array}{ccc} X'' & \xrightarrow{u''} & Y'' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{u'} & Y \end{array} \quad , \quad s,t \in S$$

such that

$$m = \mathscr{Q}(f)\mathscr{Q}(s)^{-1},\ p = \mathscr{Q}(g)\mathscr{Q}(t)^{-1}.$$

Let us assume we already proved the lemma in order to prove the third axiom; given the commutative diagram of the hypotheses, by using the lemma, we can reduce to two cases:
(1): $m = \mathscr{Q}(s)^{-1},\ p = \mathscr{Q}(t)^{-1}$ with $s,t \in S$ and with the commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{u'} & Y' \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ X & \xrightarrow{u} & Y \end{array} .$$

(2) : $m = \mathscr{Q}(f),\ p = \mathscr{Q}(g)$ and with the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow{u'} & Y' \end{array} .$$

By applying the third axiom and thanks to the *(2)* in 4.2, we conclude the proof.
Proof of the lemma: By using 4.17, there exists a morphism $t : Y'' \to Y,\ t \in S$ such that $p\mathscr{Q}(t) = \mathscr{Q}(g)$ and with $g : Y'' \to Y$.
Thanks to the multiplicative system we can construct the diagram

$$\begin{array}{ccc} X'' & & \\ {\scriptstyle s_3}\downarrow & \searrow^{u''} & \\ X''_2 & & \\ {\scriptstyle s_2}\downarrow & \swarrow_{u''_2} & \\ X''_1 & \xrightarrow[u''_1]{} & Y'' \\ {\scriptstyle s_1}\downarrow & & \downarrow{\scriptstyle t} \\ X & \xrightarrow{u} & Y \end{array}$$

in which

1. $us_1 = tu''_1$ and $s_1 \in S$ by using *(2)* of 4.1.

2. By using 4.17, $m\mathscr{Q}(s_1s_2) = \mathscr{Q}(f')$ where $f' : X''_2 \to X'$ is a morphism of $\mathscr{D}$, $u''_2 = u''_1 s_2$ and $s_2 \in S$.

3. $u'f's_3 = gu''_2 s_3$ and $u'' = u''_2 s_3$ with $s_3 \in S$.

By taking $f = f's_3,\ s = s_1s_2s_3$, we conclude the proof of the lemma.

Definition 4.23 *Let $\mathscr{D}$ be a triangulated category and $\mathscr{B}$ a (strictly) full triangulated subcategory, $S(\mathscr{B})$ be the associated multiplicative system.*
$\mathscr{D}(S(\mathscr{B})^{-1})$ will be denoted by $\mathscr{D}/\mathscr{B}$ and is called the quotient category of $\mathscr{D}$ by the subcategory $\mathscr{B}$. The canonical functor $\mathscr{Q} : \mathscr{D} \to \mathscr{D}/\mathscr{B}$ is the quotient functor.
Let F: $\mathscr{D} \to \mathscr{D}'$ be an exact (respectively cohomological) functor into a triangulated category (respectively abelian). Ker(F) is called kernel of F and is the saturated full triangulated subcategory whose objects (resp. with all their translations) are sent into the null object by F.

Corollary 4.24 *Let $\mathscr{D}$ be a triangulated category and $\mathscr{B} \subset \mathscr{D}$ be a full triangulated subcategory. Then*

1. *The kernel of the canonical functor $\mathscr{Q} : \mathscr{D} \to \mathscr{D}/\mathscr{B}$ is the smallest strictly full saturated triangulated subcategory that contains $\mathscr{B}$.*

2. *The application $\mathscr{B} \to S(\mathscr{B})$ that associates a strictly full saturated triangulated subcategory to its saturated multiplicative system is an isomorphism of partially ordered sets between the sets of strictly full saturated triangulated subcategories, ordered by inclusion, and saturated multiplicative systems, ordered by inclusion.*
   The inverse application sends a saturated multiplicative system to the kernel of the localization functor.

3. *An exact functor (respectively cohomological) $F : \mathscr{D} \to \mathscr{D}'$ whose kernel contains $\mathscr{B}$ factors in a unique way through $\mathscr{Q} : \mathscr{D} \to \mathscr{D}/\mathscr{B}$ and a functor $\mathscr{D}/\mathscr{B} \to \mathscr{D}'$ which is exact (resp. cohomological). A natural transformation between exact functors (respectively cohomological) whose kernels contain $\mathscr{B}$ can be factored uniquely through $\mathscr{Q}$.*

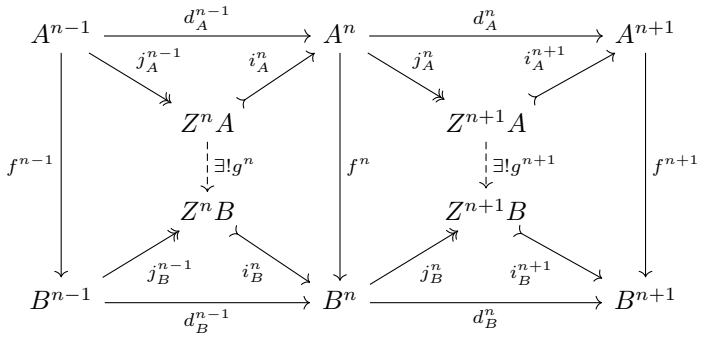
5 Derived categories of exact categories

5.1 The homotopy category of acyclic complexes

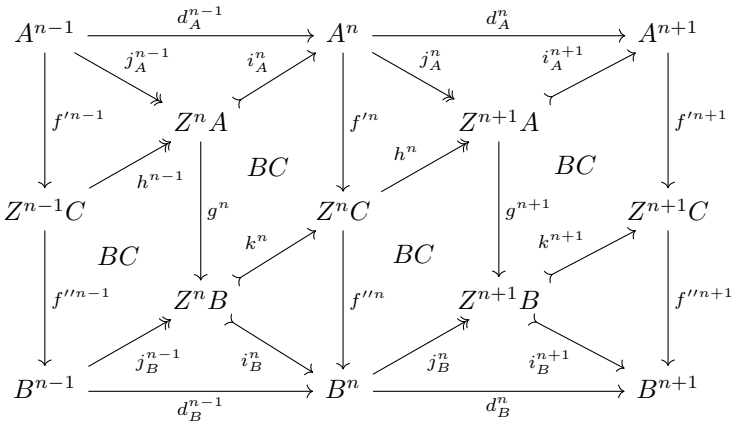
We want to combine the multiplicative system results we proved in the previous section with the exact categories.

Lemma 5.1 *In an exact category the mapping cone of a chain map $f : A \to B$ between acyclic complexes is acyclic.*

Proof: We can construct the diagram



made commutative by g^n.
By using 1.15, we can find Z^nC fitting into a diagram commutative



where $f^n = f''^n f'^n$ and where BC indicates a bicartesian quadriterals. Since Z^nC are obtained by push-outs under i_A^n and g^n,

$$Z^nB \rightarrowtail Z^nC \twoheadrightarrow Z^{n+1}A$$

is short exact.
By using 1.11 for every n the sequence

$$Z^{n-1}C \xrightarrow{\begin{pmatrix} -i_A^n h^{n-1} \\ f''^{n-1} \end{pmatrix}} A^n \oplus B^{n-1} \xrightarrow{\begin{pmatrix} f'^n & k^n j_B^{n-1} \end{pmatrix}} Z^nC$$

is short exact. We can observe that

$$(A^n \oplus B^{n-1} \xrightarrow{\begin{pmatrix} -d_A^n & 0 \\ f^n & d_B^{n-1} \end{pmatrix}} A^{n+1} \oplus B^n) = (A^n \oplus B^{n-1} \xrightarrow{\begin{pmatrix} f'^n & k^n j_B^{n-1} \end{pmatrix}} Z^nC \xrightarrow{\begin{pmatrix} -i_A^{n+1} h^n \\ f''^n \end{pmatrix}} A^{n+1} \oplus B^n)$$

proves that $cone(f)$ is acyclic.

Corollary 5.2 *Acyclic complexes of $\mathscr{A}$ are a full triangulated subcategory of $K(\mathscr{A})$.*

Remark 5.3 *Acyclic complexes of $\mathscr{A}$ are not necessarily a strictly full triangulated subcategory.*

Lemma 5.4 *Let $(\mathscr{A}, \mathscr{E})$ be an idempotent complete exact category. Every retract in $K(\mathscr{A})$ of an acyclic complex A is acyclic.*

Proof: Let the chain map $f : X \to A$ be a coretraction in $K(\mathscr{A})$. This means there exists a chain map $s : A \to X$ such that

$$s^n f^n - 1_{X^n} = d_X^{n-1} h^n + h^{n+1} d_X^n$$

for some morphisms $h^n : X^n \to X^{n-1}$.
The complex IX with components $(IX)^n = X^n \oplus X^{n+1}$ and with differential $i_{X^{n+1}} \pi_{X^{n+1}}$ is acyclic.
We can construct a chain map $i_X : X \to IX$ by

$$i_X^n = \begin{pmatrix} 1_{X^n} \\ d_X^n \end{pmatrix} : X^n \to X^n \oplus X^{n+1}$$

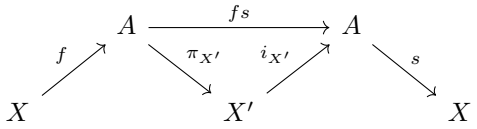
and a chain map

$$\begin{pmatrix} f \\ i_X \end{pmatrix} : X \to A \oplus IX$$

that has as a left inverse

$$\begin{pmatrix} s^n & -d_X^{n-1}h^n & -h^{n+1} \end{pmatrix} : A^n \oplus X^n \oplus X^{n+1} \to X^n.$$

We can replace the acyclic complex A by the acyclic complex $A \oplus IX$ so that we may assume $f : X \to A$ has s as a left inverse in $Ch(\mathscr{A})$. It follows that $e = fs : A \to A$ is an idempotent in $Ch(\mathscr{A})$. By the proof of 5.1, it also induces an idempotent on the exact sequence $Z^nA \rightarrowtail A^n \twoheadrightarrow Z^{n+1}A$.
By recalling 1.14, the exact sequence decomposes as direct sum of two short exact sequence since $\mathscr{A}$ is idempotent complete.
We can decompose the acyclic complex $A = X' \oplus Y'$ where X' and Y' are respectively the decomposition of A in the direct sum of the kernel of e and of the image of e. Moreover, X' and Y' are acyclic complexes.
Now by studying the diagram



and by recalling that $sf = id_X$, we observe that $\pi_{X'}f$ and $si_{X'}$ are mutual inverse and this implies that $X \cong X'$.

Proposition 5.5 *Let $\mathscr{A}$ be an exact category. Then the following are equivalent:*

1. *Every null-homotopic complex in $Ch(\mathscr{A})$ is acyclic.*

2. *The category $\mathscr{A}$ is idempotent complete.*

3. *The class of acyclic complexes is closed under isomorphism in $K(\mathscr{A})$.*

Proof: (*1* $\Rightarrow$ *2*): Let $e : A \to A$ be an idempotent of $\mathscr{A}$.
Then the complex

$$\ldots \xrightarrow{1-e} A \xrightarrow{e} A \xrightarrow{1-e} A \xrightarrow{e} \ldots$$

is null-homotopic since $(1-e)(1-e) + ee = 1 - e + e = 1$. By (1) the complex is acyclic and this implies that e has a kernel. By definition $\mathscr{A}$ is idempotent complete.
(*2* $\Rightarrow$ *3*): Let X in $K(\mathscr{A})$ be isomorphic to an acyclic complex A. In particular X is a retract of A in $K(\mathscr{A})$ so by 5.4 X is acyclic.
(*3* $\Rightarrow$ *1*): a null-homotopic complex X is isomorphic in $K(\mathscr{A})$ to the zero complex, which is acyclic.

Definition 5.6 *A complex A is left bounded if $A^n = 0$ for $n << 0$.*
A complex A is right bounded if $A^n = 0$ for $n >> 0$ and it is bounded if it is both left and right bounded.

Definition 5.7 *Let $K(\mathscr{A})$ be the homotopy category of an additive category $\mathscr{A}$. Then $K^+(\mathscr{A}), K^-(\mathscr{A}), K^b(\mathscr{A})$ are respectively the full subcategories of $K(\mathscr{A})$ generated by left bounded, right bounded and bounded complexes.*

Remark 5.8 *In general $K^*(\mathscr{A})$ are not strictly full subcategories where $* = \{+, -, b\}$.*

Definition 5.9 *For $* = \{+, -, b\}$, we define $Ac^*(\mathscr{A}) = K^*(\mathscr{A}) \cap Ac(\mathscr{A})$.*

Proposition 5.10 *The following assertions are equivalent:*

1. *The subcategories $Ac^+(\mathscr{A})$ of $K^+(\mathscr{A})$ and $Ac^-(\mathscr{A})$ of $K^-(\mathscr{A})$ are saturated.*

2. *The subcategory $Ac^b(\mathscr{A})$ of $K^b(\mathscr{A})$ is saturated.*

3. $\mathscr{A}$ is weakly idempotent complete.

Proof: $(1 \Rightarrow 2)$ follows from the definition of $Ac^b(\mathscr{A})$.
$(2 \Rightarrow 3)$: Let $s : B \to A$ and $t : A \to B$ be morphisms such that $ts = 1_B$.
Let X be the complex defined by

$$... \to 0 \to B \xrightarrow{s} A \xrightarrow{1-st} A \xrightarrow{t} B \to 0 \to ...$$

which is a direct summand of $X \oplus X[1]$ which is acyclic. This is proved by studying the commutative diagram

$$\begin{array}{ccccccccc}
B & \xrightarrow{i_B} & B \oplus A & \xrightarrow{\left(\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right)} & A \oplus A & \xrightarrow{\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)} & A \oplus B & \xrightarrow{\pi_B} & B \\
\| & & \left(\begin{smallmatrix}0 & -t\\ s & 1-st\end{smallmatrix}\right)\downarrow & & \left(\begin{smallmatrix}-1+st & st\\ st & 1-st\end{smallmatrix}\right)\downarrow & & \left(\begin{smallmatrix}1-st & -s\\ t & 0\end{smallmatrix}\right)\downarrow & & \| \\
B & \xrightarrow[\left(\begin{smallmatrix}0\\ s\end{smallmatrix}\right)]{} & B \oplus A & \xrightarrow[\left(\begin{smallmatrix}-s & 0\\ 0 & 1-st\end{smallmatrix}\right)]{} & A \oplus A & \xrightarrow[\left(\begin{smallmatrix}-1+st & 0\\ 0 & t\end{smallmatrix}\right)]{} & A \oplus B & \xrightarrow[(-t \;\; 0)]{} & B
\end{array}.$$

The first row is acyclic and the second row is $X \oplus X[1]$ and the map is an isomorphism. Since $Ac^b(\mathscr{A})$ is saturated, X is acyclic and this implies that s has a cokernel and t has a kernel and this implies that $\mathscr{A}$ is weakly indempotent complete.
$(3 \Rightarrow 1)$: Suppose that X is a direct summand in $K^+(\mathscr{A})$ of a complex $A \in Ac^+(\mathscr{A})$. This implies that there exists a chain map $f : X \to A$ such that there exist a chain map $s : A \to X$ and morphisms $h^n : X^n \to X^{n-1}$ such that $s^n f^n - 1 = d_X^{n-1} h^n + h^{n+1} d_X^n$.
By replacing A by $A \oplus IX$ as in 5.4 we may assume that s is a left inverse of f in $Ch^+(\mathscr{A})$. Since $\mathscr{A}$ is weakly idempotent complete, 1.57 implies f^n is an admissible monic and s^n is an admissible epic for all n. Since both X and A are left bounded we may assume that $A^n = 0 = X^n$ when $n < 0$. Hence $d_A^0 : A^0 \rightarrowtail A^1$ is an admissible monic since A is acyclic.
Since $d_A^0 f^0 = f^1 d_X^0$ is an admissible monic, by 1.57, d_X^0 is an admissible monic.
Let $e_X^1 : X^1 \twoheadrightarrow Z^2X$ be a cokernel of d_X^0 and $e_A^1 : A^1 \twoheadrightarrow Z^2A$ be a cokernel of d_A^0. Then we can find g^2 and t^2 uniquely such that the diagram

$$\begin{array}{ccccc}
X^0 & \xrightarrowtail{d_X^0} & X^1 & \xrightarrow{e_X^1}\!\!\!\!\rightarrow & Z^2X \\
\downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle g^2} \\
A^0 & \xrightarrowtail{d_A^0} & A^1 & \xrightarrow{e_A^1}\!\!\!\!\rightarrow & Z^2A \\
\downarrow{\scriptstyle s^0} & & \downarrow{\scriptstyle s^1} & & \downarrow{\scriptstyle t^2} \\
X^0 & \xrightarrowtail{d_X^0} & X^1 & \xrightarrow{e_X^1}\!\!\!\!\rightarrow & Z^2X
\end{array}$$

is commutative. Since $s^i f^i = 1$ for $i = 0, 1$, it follows that $t^2 g^2 = 1$ and this implies that t^2 and g^2 are an admissible epic and an admissible monic respectively by 1.57.
A and X are complexes so there are unique maps $m_X^2 : Z^2X \to X^2$ and $m_A^2 : Z^2A \to A^2$ such that $d_X^1 = m_X^2 e_X^1$ and $d_A^1 = m_A^2 e_A^1$. m_A^2 is an admissible monic since A is acyclic.

$$\begin{array}{ccc}
Z^2X & \xrightarrow{m_X^2} & X^2 \\
\downarrow{\scriptstyle g^2} & & \downarrow{\scriptstyle f^2} \\
Z^2A & \xrightarrowtail{m_A^2} & A^2 \\
\downarrow{\scriptstyle t^2} & & \downarrow{\scriptstyle s^2} \\
Z^2X & \xrightarrow{m_X^2} & X^2
\end{array}$$

is a commutative diagram: since $(m_A^2 g^2 - f^2 m_X^2) e_X^1 = d_A^1 f^1 - f^2 d_X^1 = 0$ and since e_X^1 is epic, we have the commutativity in the upper square. By a similar reasoning and since e_A^1 is epic, also the lower square is commutative.
From the upper square and by 1.57, m_X^2 is an admissible monic. By induction X is acyclic.

Corollary 5.11 *The triangulated subcategory $Ac(\mathscr{A})$ of $K(\mathscr{A})$ is saturated if and only if $\mathscr{A}$ is idempotent complete.*

Definition 5.12 *Let $(\mathscr{A}, \mathscr{E})$ be an exact category. $D(\mathscr{A}) = K(\mathscr{A})/Ac(\mathscr{A})$ is called the derived category of $\mathscr{A}$.*

5.2 Serre subcategories

Definition 5.13 *Let $\mathscr{A}$ be an abelian category. A full additive subcategory $\mathscr{C} \subseteq \mathscr{A}$ is a Serre subcategory if it is closed under subobjects, quotients and extensions.*
Equivalently, for every exact sequence $0 \to A \to B \to C \to 0$ in $\mathscr{A}$, $B \in \mathscr{C}$ if and only if A, C belong to $\mathscr{C}$

An interesting property of Serre subcategories is that they induce a natural multiplicative system.
Let $\mathscr{C}$ be a Serre subcategory of $\mathscr{A}$. Then we can define

$$S(\mathscr{C}) = \{\sigma \in Mor(\mathscr{A}) | Ker(\sigma),\ Coker(\sigma) \in \mathscr{C}\}$$

and

$$\mathscr{C}^{\perp} = \{Y \in \mathscr{A} | \mathscr{A}(X, Y) = 0 = Ext^1_{\mathscr{A}}(X, Y)\ \forall X \in \mathscr{C}\}.$$

Given a pair of subobjects $X' \subseteq X$ and $Y' \subseteq Y$, we have an induced map $\mathscr{A}(X, Y) \to \mathscr{A}(X', Y/Y')$. Moreover the pairs (X', Y') such that both X/X' and Y' are in $\mathscr{C}$ form a directed set so we can define an abelian group $\mathscr{A}/\mathscr{C}(X, Y) = colim_{(X',Y')}\mathscr{A}(X', Y/Y')$.

Definition 5.14 *The abelian quotient category $\mathscr{A}/\mathscr{C}$ has the same objects of $\mathscr{A}$ and the morphisms are defined by $\mathscr{A}/\mathscr{C}(X, Y) = colim_{(X',Y')}\mathscr{A}(X', Y/Y')$ with composition induced by composition in $\mathscr{A}$.*

Definition 5.15 *Let $\mathscr{E}$ be a category and $T \subseteq Mor(\mathscr{E})$ multiplicative system.*
An object Y in $\mathscr{E}$ is said to be $S-local$ if the map $\mathscr{E}(\sigma, Y)$ is bijective for all $\sigma \in S$.

Proposition 5.16 *An object Y in $\mathscr{E}$ is $T-local$ if and only if the canonical map*

$$p_{X,Y} : \mathscr{E}(X, Y) \to \mathscr{E}(T^{-1})(X, Y)$$

is bijective for all $X \in \mathscr{E}$.

Proof: If Y is $T-local$, the functor $\mathscr{E}(-, Y) : \mathscr{E}^{op} \to Set$ induces a functor still denoted $\mathscr{E}(-, Y) : \mathscr{E}(T^{-1})^{op} \to Set$.
By Yoneda's lemma, we get a natural transformation

$$\mathscr{E}(T^{-1})(-, Y) \to \mathscr{E}(-, Y)$$

corresponding to id_Y that is an inverse of the canonical morphism $\mathscr{E}(-, Y) \to \mathscr{E}(T^{-1})(-, Y)$ because of the definition of Yoneda's Lemma.
Suppose now that $p_{X,Y}$ is bijective for all $X \in \mathscr{E}$.
Since $\mathscr{E}(T^{-1})(\sigma, Y)$ is bijective, we have that $\mathscr{E}(\sigma, Y)$ is bijective for all $\sigma \in T$.

Lemma 5.17 *Given a Serre subcategory $\mathscr{C} \subseteq \mathscr{A}$, the following holds.*

1. *$S(\mathscr{C})$ is a multiplicative system.*

2. *An object in $\mathscr{A}$ is $S(\mathscr{C}) - local$ if and only if it belongs to $\mathscr{C}^{\perp}$.*

3. *The canonical functor $\mathscr{A} \to \mathscr{A}/\mathscr{C}$ induces an isomorphism $\mathscr{A}[S(\mathscr{C})^{-1}] \xrightarrow{\cong} \mathscr{A}/\mathscr{C}$.*

Proof:
(1) We have to prove the properties of 4.1.
(1) of 4.1: Let f and g be two composable morphisms of $S(\mathscr{C})$.
Then we can recall 1.66. Since Serre subcategories are closed under subobjects, quotients and extensions, we have a short exact sequence

$$0 \to Ker(f) \to Ker(h) \to T \to 0$$

with $T \subseteq Ker(g)$ we conclude that $Ker(h) \in \mathscr{C}$. We can do a similar reasoning for cokernels.
(2) of 4.1: By the embedding theorem for small abelian categories, we can suppose we are in $R-Mod$

for some ring R.
Suppose we have a diagram of this form

$$\begin{array}{ccc} & & C \\ & & \downarrow f \\ B & \xrightarrow{s} & D \end{array}.$$

Since we are in an abelian category, through the pull back, we can complete it to the square

$$\begin{array}{ccc} X & \xrightarrow{r} & C \\ g\downarrow & & \downarrow f \\ B & \xrightarrow{s} & D \end{array}$$

with $X = \{(b,c) \in B \times C | s(b) = f(c)\}$.
Now the kernel of $r : X \to C$ are the couples $(b, 0)$ such that $s(b) = f(0) = 0$ hence it is isomorphic to $ker(s)$ which belongs to $\mathscr{C}$.
Let π_s and π_r be the projections to $coker(s)$ and $coker(r)$ respectively. By the fact that $\pi_s fr = \pi_s sg = 0$ because of the definition of X, we have $\bar{f} : coker(r) \to coker(s)$ such that $\pi_s(f) = \bar{f}\pi_r$.
Let $\bar{c}$ be an element such that $\bar{f}(\bar{c}) = \bar{0}$. But we have $\bar{f}(\bar{c}) = \pi_s(f(c)) = 0$. Hence $f(c) \in Im(s)$ and this implies that $f(c) = s(b)$ for some b so $c \in im(r)$.
It follows that $\bar{f}$ is a monomorphism and so $coker(r)$ is a subobject of $coker(s)$ which is in $\mathscr{C}$.
Since $\mathscr{C}$ is a Serre subcategory if and only if it is a Serre subcategory in the opposite category, the other completion is proved dually.
(3) of 4.1: suppose we have two morphisms $f, g : A \to B$ such that there is an $s : B \to C \in S(\mathscr{C})$ and $sf = gs$.
This implies that $s(f - g) = 0$ hence $im(f - g)$ is contained in $ker(s) \in \mathscr{C}$. We take as $t \in S(\mathscr{C})$, such that $(f - g)t = 0$,the inclusion of $ker(f - g)$ in A.
In fact, $ft = gt$ and moreover since t is a monomorphism its kernel is 0 which is in $\mathscr{C}$. We also have that the coker of the inclusion is $A/(ker(f - g))$ which is isomorphic to $im(f - g)$. It follows that t is in $S(\mathscr{C})$.
The other implication follows by duality.
(2): Suppose Y is $S(\mathscr{C}) - local$. We observe that in $\mathscr{A}[S(\mathscr{C})^{-1}]$ all objects in $\mathscr{C}$ are null objects. Let X be an object in $\mathscr{C}$. By recalling 5.16, we have that $\mathscr{A}(X, Y) \cong \mathscr{A}[S(\mathscr{C})^{-1}](X, Y) \cong \{0\}$.
It remains to show that $Ext^1_{\mathscr{A}}(X, Y) = 0$. Let $\delta = [0 \to Y \xrightarrow{y} S \to X \to 0]$ be an element of $Ext^1_{\mathscr{A}}(X, Y)$. Since y is in $S(\mathscr{C})$, we have that $\mathscr{A}(s, Y)$ is an isomorphism hence the extension δ is split. This proves that Y is in $\mathscr{C}^{\perp}$.
Suppose now that Y is in $\mathscr{C}^{\perp}$. Let $\sigma : X \to Z \in S$ be a morphism. Since $\mathscr{A}$ is an abelian category, we have the following commutative diagram

$$\begin{array}{ccccccc} & & X & & \xrightarrow{\quad s \quad} & & Z & & \\ & \overset{i}{\nearrow} & & \overset{s'}{\searrow} & & \overset{s''}{\nearrow} & & \overset{j}{\searrow} & \\ Ker(s) & & & & Im(s) & & & & Coker(s) \end{array}.$$

Now we have two short exact sequences:

$$0 \to Ker(s) \xrightarrow{i} X \xrightarrow{s'} Im(s) \to 0,$$

$$0 \to Im(s) \xrightarrow{s''} Z \xrightarrow{j} Coker(s) \to 0.$$

The long exact sequences linked to the functor $\mathscr{A}(-, Y)$ are respectively

$$... \to Ext^1_{\mathscr{A}}(Ker(s), Y) \to \mathscr{A}(Im(s), Y) \xrightarrow{\mathscr{A}(s',Y)} \mathscr{A}(X, Y) \xrightarrow{\mathscr{A}(i,Y))} \mathscr{A}(Ker(s), Y) \to 0,$$

$$... \to Ext^1_{\mathscr{A}}(Im(s), Y) \to \mathscr{A}(Coker(s), Y) \xrightarrow{\mathscr{A}(j,Y)} \mathscr{A}(Z, Y) \xrightarrow{\mathscr{A}(s'',Y))} \mathscr{A}(Im(s), Y) \to 0.$$

Since $Y \in \mathscr{C}^{\perp}$ and $Coker(s)$, $Ker(s)$ belong to $\mathscr{C}$, we have that $\mathscr{A}(s', Y)$ and $\mathscr{A}(s'', Y)$ are isomorphisms. Hence we have that $\mathscr{A}(s'', Y)\mathscr{A}(s', Y) = \mathscr{A}(s, Y)$ is an isomorphism and therefore Y is $S(\mathscr{C}) - local$. (3): By recalling 4.15, we have that for every pair of objects X, Y we have

$$\mathscr{A}/\mathscr{C}(X, Y) = colim_{X', Y'} \mathscr{A}(X', Y/Y') \cong colim_{\sigma, \tau} \mathscr{A}(\hat{X}, \hat{Y}) \cong \mathscr{A}[S^{-1}](X, Y)$$

where $\sigma : \hat{X} \to X$ and $\tau : Y \to \hat{Y}$ vary through all morphisms in $S(\mathscr{C})$.

This helps us describe a morphism $\phi : X \to Y$ in $\mathscr{A}/\mathscr{C}$ through a diagram

$$\begin{array}{ccc} X' & \longrightarrow & Y/Y' \\ \downarrow & & \uparrow \\ X & \xrightarrow{\phi} & Y \end{array} ,$$

where the other three morphisms come from $\mathscr{A}$ and moreover the vertical morphisms are isomorphisms in $\mathscr{A}/\mathscr{C}$.

Lemma 5.18 *Let $\mathscr{C} \subseteq \mathscr{A}$ be a Serre subcategory and $Y \in \mathscr{A}$. Then $Y \in \mathscr{C}^{\perp}$ if and only if the canonical map*

$$\mathscr{A}(X, Y) \to \mathscr{A}/\mathscr{C}(X, Y)$$

is a bijection for all $X \in \mathscr{A}$.

Proof: This is proven by 5.17 and 5.16.

Proposition 5.19 *Let $\mathscr{A}$ be an abelian category and $\mathscr{C} \subseteq \mathscr{A}$ a Serre subcategory. Then*

1. *The category $\mathscr{A}/\mathscr{C}$ is abelian and the canonical functor $Q : \mathscr{A} \to \mathscr{A}/\mathscr{C}$ is an exact functor and $Q(\mathscr{C}) = 0$.*

2. *Let $\mathscr{B}$ be an abelian category and let $F : \mathscr{A} \to \mathscr{B}$ be an exact functor such that $F(\mathscr{C}) = 0$. Then there exists a unique exact functor $\bar{F} : \mathscr{A}/\mathscr{C} \to \mathscr{B}$ such that $F = \bar{F}Q$.*

Proof: By using 5.17, $\mathscr{A}/\mathscr{C} = \mathscr{A}[S^{-1}]$ for $S = S(\mathscr{C})$.
From 4.21, $\mathscr{A}/\mathscr{C}$ is additive.
By the above observation a morphism $X \to Y$ in $\mathscr{A}/\mathscr{C}$ is isomorphic to a morphism of the form $Q\phi$ for some $\phi : X \to Y$ in $\mathscr{A}$.
Since $\mathscr{A}$ is abelian, we can choose a cokernel $\psi : Y \to Z$ of ϕ. Then for every $A \in \mathscr{A}$ we have an exact sequence

$$0 \to \mathscr{A}(Z, A) \to \mathscr{A}(Y, A) \to \mathscr{A}(X, A)$$

and hence an exact sequence

$$0 \to colim_{S(A)} \mathscr{A}(Z, A') \to colim_{S(A)} \mathscr{A}(Y, A') \to colim_{S(A)} \mathscr{A}(X, A')$$

where $S(A)$ are all the morphisms in S starting from A and therefore the sequence

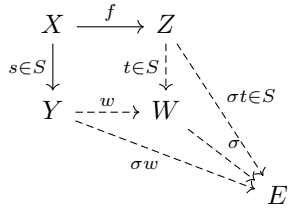
$$0 \to \mathscr{A}/\mathscr{C}(Z, A) \to \mathscr{A}/\mathscr{C}(Y, A) \to \mathscr{A}/\mathscr{C}(X, A)$$

is exact by 5.17 and this implies $Q\psi$ is a cokernel of $Q\phi$ and dually we prove that every morphism in $\mathscr{A}/\mathscr{C}$ admits a kernel.
Moreover, Q preserves kernel and cokernels so $\mathscr{A}/\mathscr{C}$ is abelian and Q is exact.
(2): Let $F : \mathscr{A} \to \mathscr{B}$ be an exact functor such that $F(\mathscr{C}) = 0$, hence F inverts all morphisms in S. This implies that F factors through $Q : \mathscr{A} \to \mathscr{A}/\mathscr{C}$ via a unique functor $\bar{F} : \mathscr{A}/\mathscr{C} \to \mathscr{B}$.
The functor $\bar{F}$ is exact since any exact sequence in $\mathscr{A}/\mathscr{C}$ is isomorphic to the image of an exact sequence in $\mathscr{A}$.

Definition 5.20 *Let $\mathscr{C}$ be a category and $S \subseteq Mor(\mathscr{C})$. A full subcategory $\mathscr{D}$ of $\mathscr{C}$ is left cofinal with respect to S if for every morphism $\sigma : X \to Y$ in S with $X \in \mathscr{D}$ there is a morphism $\tau : Y \to Z$ with $\tau\sigma$ in $S \cap \mathscr{D}$. Dually for right cofinal.*

Lemma 5.21 *Let S be a multiplicative system and $\mathscr{D} \subseteq \mathscr{C}$ be left and right cofinal with respect to S. Then $S \cap \mathscr{D}$ is a multiplicative system and the induced functor $\mathscr{D}[(S \cap \mathscr{D})^{-1}] \to \mathscr{C}[S^{-1}]$ is fully faithful.*

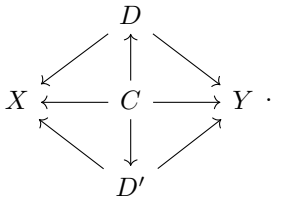
Proof: (1) of 4.1 is verified since $\mathscr{D}$ is a full subcategory.
Let X, Y, Z be objects of $\mathscr{D}$, let $s : X \to Y$ be a morphism in S and $f : X \to Z$ be a morphism in $\mathscr{D}$. By (2) of 4.1 and by left cofinality of $\mathscr{D}$, we have the following commutative diagram



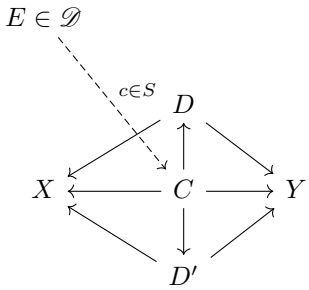
with $E \in \mathscr{D}$ and dually for the other condition of (2) of 4.1.
Suppose now we have $f, g : X \to Y$ such that there exists $t \in S \cap \mathscr{D}$ that satisfies $ft = gt$. Then by (3) of 4.1 there exists $s \in S$ such that $sf = sg$. By the left cofinality of $\mathscr{D}$, there exists r such that $rs \in S \cap \mathscr{D}$ hence we verified (3) of 4.1 for $S \cap \mathscr{D}$.
Let X, Y be objects in $\mathscr{D}$.
We have an induced map

$$f : \mathscr{D}[(S \cap \mathscr{D})^{-1}](X, Y) \to \mathscr{C}[S^{-1}](X, Y)$$

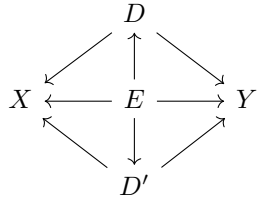
defined by sending the equivalence class of a fraction to its equivalence class in $\mathscr{C}[S^{-1}]$.
Let $\overset{s}{\leftarrow}\overset{\alpha}{\rightarrow}$ be a morphism of $\mathscr{C}[S^{-1}](X, Y)$, with $s \in S$, and τ is a morphism such that τs in $S \cap \mathscr{D}$, then $\overset{\tau s}{\leftarrow}\overset{\tau\alpha}{\rightarrow} = \overset{s}{\leftarrow}\overset{\alpha}{\rightarrow}$.
In particular $f(\overset{\tau s}{\leftarrow}\overset{\tau\alpha}{\rightarrow}) = \overset{s}{\leftarrow}\overset{\alpha}{\rightarrow}$ hence f is surjective.
Suppose now two morphisms in $\mathscr{C}[S^{-1}](X, Y)$ are equivalent hence there exists a commutative diagram



By the cofinality of $\mathscr{D}$ we have a morphism in S $c : E \to C$



hence we have a commutative diagram



and since all of the objects are in $\mathscr{D}$, we have that two morphisms equivalent in $\mathscr{C}[S^{-1}](X, Y)$ are also equivalent in $\mathscr{D}[(S \cap \mathscr{D})^{-1}]$ hence f is injective.

Proposition 5.22 *Let $\mathscr{C} \subseteq \mathscr{B} \subseteq \mathscr{A}$ be Serre subcategories of an abelian category $\mathscr{A}$. Then $\mathscr{B}/\mathscr{C}$ corresponds with a Serre subcategory of $\mathscr{A}/\mathscr{C}$.*
Moreover every Serre subcategory of $\mathscr{A}/\mathscr{C}$ is of this form.
In addition, the canonical functor $\mathscr{A} \to \mathscr{A}/\mathscr{C}$ induces an isomorphism $\mathscr{A}/\mathscr{B} \xrightarrow{\cong} (\mathscr{A}/\mathscr{C})(\mathscr{B}/\mathscr{C})$.

Proof: The inclusion $\mathscr{B} \to \mathscr{A}$ induces a fully faithful functor $\mathscr{B}/\mathscr{C} \to \mathscr{A}/\mathscr{C}$ by recalling 5.21 and by observing that $\mathscr{B}$ is left and right cofinal with respect to $S(\mathscr{C})$. Moreover, $\mathscr{B}/\mathscr{C}$ is a Serre subcategory of $\mathscr{A}/\mathscr{C}$ since $Q : \mathscr{A} \to \mathscr{A}/\mathscr{C}$ is an exact functor.
Let $\mathscr{D} \subseteq \mathscr{A}/\mathscr{C}$ be a Serre subcategory, put $\mathscr{B} := Q^{-1}(\mathscr{D})$, by the exactness of Q we have $\mathscr{B}/\mathscr{C} \xrightarrow{\cong} \mathscr{D}$.
The last assertion follows from the observation that $\mathscr{B}$ is the kernel of the composite $\mathscr{A} \to \mathscr{A}/\mathscr{C} \to (\mathscr{A}/\mathscr{C})/(\mathscr{B}/\mathscr{C})$.

Remark 5.23 5.22 *can be visualized by the diagram*

$$\begin{array}{ccccc}
\mathscr{C} & \rightarrowtail & \mathscr{B} & \twoheadrightarrow & \mathscr{B}/\mathscr{C} \\
\| & & \downarrow & & \downarrow \\
\mathscr{C} & \rightarrowtail & \mathscr{A} & \twoheadrightarrow & \mathscr{A}/\mathscr{C} \\
 & & \downarrow & & \downarrow \\
 & & \mathscr{A}/\mathscr{B} & \xrightarrow{\cong} & (\mathscr{A}/\mathscr{C})/(\mathscr{B}/\mathscr{C})
\end{array} .$$

Suppose now we have two functors $F : \mathscr{G} \to \mathscr{H}$ and $G : \mathscr{H} \to \mathscr{G}$ such that F is left adjoint to G.
We set $S = S(F) = \{\sigma \in Mor(\mathscr{G}) | F\sigma \text{ is invertible}\}$. Moreover we have the following diagram

$$\begin{array}{ccc}
 & & \mathscr{G} \\
 & \overset{Q}{\swarrow} & \uparrow G \\
\mathscr{G}[S^{-1}] & & \downarrow F \\
 & \underset{\hat{F}}{\searrow} & \\
 & & \mathscr{H}
\end{array} .$$

Proposition 5.24 *In the above conditions, the following facts are equivalent.*

1. *The functor G is fully faithful.*
2. *The counit $FG(X) \to X$ is invertible for every object $X \in \mathscr{H}$.*
3. *The functor F induces an equivalence $\bar{F} : \mathscr{G}(S^{-1}) \xrightarrow{\cong} \mathscr{H}$.*

In this case G induces an equivalence $\mathscr{H} \xrightarrow{\cong} S^{\perp}$ (where $S^{\perp}$ denothes the full subcategory of S-local objects in $\mathscr{G}$) with quasi-inverse $S^{\perp} \hookrightarrow \mathscr{G} \xrightarrow{F} \mathscr{H}$.

Proof: Let $\eta : id_{\mathscr{G}} \to GF$ be the unit and $\epsilon : FG \to id_{\mathscr{H}}$ the counit of the adjunction.
We can observe that the composite $F \xrightarrow{F\eta} FGF \xrightarrow{\epsilon F} F$ is equal to id_F and $G \xrightarrow{\eta G} GFG \xrightarrow{G\epsilon} G$ equals id_G, this is equivalent to the fact that (F, G) is an adjoint pair.
$(1 \iff 2)$: The counit $\epsilon_X : FG(X) \to X$ induces for all $Y \in \mathscr{H}$ a natural map

$$\mathscr{H}(X, Y) \to \mathscr{G}(GX, GY) \xrightarrow{\cong} \mathscr{H}(FG(X), Y)$$

and, by Yoneda's Lemma, it is a bijective if and only if ϵ_X is an isomorphism.
$(2 \Rightarrow 3)$: We want to prove QG is quasi-inverse of $\bar{F}$. We observe that $\bar{F}(QG) = FG \cong id_{\mathscr{H}}$. We also have that $F\eta$ is invertible because of the invertibility of ϵF.
It follows by definition of S that $Q\eta : Q \to QGF$ is invertible, and therefore

$$(QG)\bar{F}Q = QGF \cong Q \cong id_{\mathscr{G}(S^{-1})}Q.$$

By the universal property of Q, $(QG)\bar{F} \cong id_{\mathscr{G}(S^{-1})}$.
$(3 \Rightarrow 2)$: We start by observing that for every category $\mathscr{X}$ the functor

$$Fun(\mathscr{G}(S^{-1}), \mathscr{X}) \to Fun(\mathscr{G}, \mathscr{X}) : \quad T \to TQ$$

is fully faithful and identifies $Fun(\mathscr{G}(S^{-1}), \mathscr{X})$ with the full subcategory of functors in $Fun(\mathscr{G}, \mathscr{X})$ that send $s \in S$ into isomorphisms.
If $\bar{F}$ is an equivalence, then by the above fact, the composition with F induces a fully faithful functor $Fun(\mathscr{H}, \mathscr{X}) \to Fun(\mathscr{G}, \mathscr{X})$ for any category $\mathscr{X}$. By choosing $\mathscr{X} = \mathscr{H}$, it follows that there exists $\eta' : id_{\mathscr{H}} \to FG$ such that $F\eta = \eta' F$.
We have that $F \xrightarrow{F\eta} FGF \xrightarrow{\epsilon F} F$ is equal to id_F, this implies that $(\epsilon\eta')F = \epsilon F \circ \eta' F = id_F$. By the fully faithfulness of the composition with F, we have that $\epsilon\eta' = id_{id_{\mathscr{H}}}$.
We have that $G \xrightarrow{\eta G} GFG \xrightarrow{G\epsilon} G$ equals id_G. By applying F, it follows that $FG\epsilon \circ \eta' FG = FG\epsilon \circ F\eta G = id_{FG}$.
Hence $(id_{\mathscr{H}}, FG)$ is an adjoint pair and therefore ϵ is an isomorphism since it's the counit of the adjoint pair.

Suppose now we are the in previous equivalent conditions. It remains to show that G induces an equivalence $\mathscr{D} \xrightarrow{\cong} S^{\perp}$, therefore we need to prove that the essential image of G equals $S^{\perp}$.
Observe that

$$\mathscr{G}(S^{-1})(Q(X), QG(Y)) \cong \mathscr{H}(\bar{F}Q(X), Y) = \mathscr{H}(F(X), Y) \cong \mathscr{G}(X, G(Y)),$$

hence $Im(G) \subseteq S^{\perp}$.
Suppose $X \in S^{\perp}$ then $\mathscr{G}(\eta_X, X)$ is bijective because $\eta_X \in S$, therefore η_X has an inverse so $X \cong GF(X)$.

Lemma 5.25 *Let $\mathscr{A}$ be an abelian category and $\mathscr{C} \subseteq \mathscr{A}$ a Serre subcategory. Suppose there exists a right adjoint $Q_\rho : \mathscr{A}/\mathscr{C} \to \mathscr{A}$ to the canonical functor $Q : \mathscr{A} \to \mathscr{A}/\mathscr{C}$. Then*

1. *The functor Q_ρ is fully faithful and induces an equivalence*

$$\mathscr{A}/\mathscr{C} \xrightarrow{\cong} \mathscr{C}^{\perp} \text{ with inverse } \mathscr{C}^{\perp} \hookrightarrow \mathscr{A} \xrightarrow{Q} \mathscr{A}/\mathscr{C}.$$

2. *By the adjunction for every X in $\mathscr{A}$ we have a natural exact sequence*

$$0 \to X' \to X \xrightarrow{\eta} Q_\rho Q(X) \to X'' \to 0$$

with X' and X'' in $\mathscr{C}$.

3. *The assignment $X \to X'$ gives a right adjoint of the inclusion $\mathscr{C} \to \mathscr{A}$*

Proof: (1) follows from 5.24 and 5.17.
(2) is proved by taking $X' = ker(\eta)$ and $X'' = coker(\eta)$ and by observing that $Q(\eta)$ is invertible as seen in the proof of 5.24.
(3): $\mathscr{A}(C, X') \to \mathscr{A}(C, X)$ is bijective for $C \in \mathscr{C}$ because $Q_\rho Q(X)$ is in $\mathscr{C}^{\perp}$.
The following diagram describes the situation

$$\mathscr{C} \underset{I_\rho}{\overset{I}{\rightleftarrows}} \mathscr{A} \underset{Q_\rho}{\overset{Q}{\rightleftarrows}} \mathscr{A}/\mathscr{C} \ .$$

We may observe that the assignment $X \to X'$ is functorial since it's the *ker* of the unit hence each object $X \in \mathscr{A}$ fits into a functorial exact sequence

$$0 \to I_\rho(X) \to X \to Q_\rho Q(X).$$

Proposition 5.26 *Let (F, G) be an adjoint pair of functors*

$$\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$$

between abelian categories. Suppose that F is exact and let $\mathscr{C}$ be $Ker\ F$. Then G is fully faithful if and only if F induces an equivalence $\mathscr{A}/\mathscr{C} \xrightarrow{\cong} \mathscr{B}$

Proof : Let $S = \{\sigma \in Mor(\mathscr{A}) | F\sigma \text{ is invertible}\}$. G is fully faithful if and only if F induces an equivalence $\mathscr{A}[S^{-1}] \to \mathscr{B}$ by 5.24 and $\mathscr{A}[S^{-1}] \to \mathscr{A}/\mathscr{C}$ is an equivalence by 5.17.

6 Extriangulated categories

We now introduce extriangulated categories: these categories are a generalization of exact and triangulated categories.

Definition 6.1 *Let $\mathscr{C}$ be an additive category and $\mathbb{E} : \mathscr{C}^{op} \times \mathscr{C} \to \boldsymbol{Ab}$ be a biaddictive functor. For any pair of objects $A, C \in \mathscr{C}$, an element $\delta \in \mathbb{E}(C, A)$ is called an $\mathbb{E}-$extension. Hence we can associate to an $\mathbb{E}-$extension a triplet (A, δ, C).*

Remark 6.2 *Let (A, δ, C) be any $\mathbb{E}-$extension. Then for any $a \in \mathscr{C}(A, A')$ and $c \in \mathscr{C}(C', C)$, since $\mathbb{E}$ is a bifunctor,*

$$\mathbb{E}(C, a)(\delta) \in \mathbb{E}(C, A'),\ \mathbb{E}(c, A)(\delta) \in \mathbb{E}(C', A)$$

are $\mathbb{E}-$extensions. We will denote these by $a_*\delta$ and $c^*\delta$. In this notation we have

$$\mathbb{E}(c, a)(\delta) = c^* a_* \delta = a_* c^* \delta \in \mathbb{E}(C', A').$$

Definition 6.3 *Let (A, δ, C), (A', δ', C') be any pair of $\mathbb{E}-$extensions.*
A morphism $(a, c) : (A, \delta, C) \to (A', \delta', C')$ of $\mathbb{E}-$extensions is a pair of morphisms $a \in \mathscr{C}(A, A')$ and $c \in \mathscr{C}(C, C')$ such that

$$a_*\delta = c^*\delta'.$$

We will denote this morphism by $(a, c) : \delta \to \delta'$.
We can then construct the category $\mathbb{E} - Ext(\mathscr{C})$ of $\mathbb{E}-$extensions with composition and identities induced by $\mathscr{C}$.

Remark 6.4 *Let (A, δ, C) be any $\mathbb{E}-$extension. Then*

1. *Given a morphism $a \in \mathscr{C}(A, A')$, we have the morphism of $\mathbb{E}-$extensions*

$$(a, 1_C) : \delta \to a_*\delta.$$

2. *Given a morphism $c \in \mathscr{C}(C', C)$, we have the morphism of $\mathbb{E}-$extensions*

$$(1_A, c) : c^*\delta \to \delta.$$

Definition 6.5 *For any $A, C \in \mathscr{C}$, the zero element $0 \in \mathbb{E}(C, A)$ is called the split $\mathbb{E}-$extension.*

Definition 6.6 *Let (A, δ, C) and (A', δ', C') be any pair of $\mathbb{E}-$extensions. Let*

$$C \xrightarrow{i_C} C \oplus C' \xleftarrow{i_{C'}} C', \quad A \xleftarrow{p_A} A \oplus A' \xrightarrow{p_{A'}} A'$$

be a coproduct and product in $\mathscr{C}$ respectively.
By the biadditivity of $\mathbb{E}$, there is an isomorphism

$$\mathbb{E}(C \oplus C', A \oplus A') \cong \mathbb{E}(C, A) \oplus \mathbb{E}(C', A) \oplus \mathbb{E}(C, A') \oplus \mathbb{E}(C', A').$$

Let $\delta \oplus \delta' \in \mathbb{E}(C \oplus C', A \oplus A')$ be the image throught the isomorphism of $(\delta, 0, 0, \delta')$.
This is the unique element such that

$$\mathbb{E}(i_C, p_A)(\delta \oplus \delta') = \delta,\ \mathbb{E}(i_C, p_{A'})(\delta \oplus \delta') = 0,$$

$$\mathbb{E}(i_{C'}, p_A)(\delta \oplus \delta') = 0, \mathbb{E}(i_{C'}, p_{A'})(\delta \oplus \delta') = \delta'.$$

If $A = A'$ and $C = C'$, then the sum $\delta + \delta' \in \mathbb{E}(C, A)$ is defined by

$$\delta + \delta' = \mathbb{E}(\triangle_C, \triangledown_A)(\delta \oplus \delta'),$$

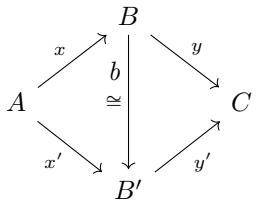
where $\triangledown_A = \begin{pmatrix} 1 & 1 \end{pmatrix} : A \oplus A \to A,\ \triangle_C = \begin{pmatrix} 1 \\ 1 \end{pmatrix} : C \to C \oplus C.$

6.1 Realizations

Definition 6.7 *Let $A, C \in \mathscr{C}$ be a pair of objects. Two sequence of morphisms*

$$A \xrightarrow{x} B \xrightarrow{y} C,\ A \xrightarrow{x'} B' \xrightarrow{y'} C$$

are equivalent if there exists an isomorphism $b \in \mathscr{C}(B, B')$ such that



is commutative. The equivalence class of $A \xrightarrow{x} B \xrightarrow{y} C$ will be denoted by $[A \xrightarrow{x} B \xrightarrow{y} C]$

Definition 6.8 *1. For any $A, C \in \mathscr{C}$, we denote as*

$$0 = [A \xrightarrow{i_A} A \oplus C \xrightarrow{\pi_C} C].$$

2. For any $[A \xrightarrow{x} B \xrightarrow{y} C]$ and $[A' \xrightarrow{x'} B' \xrightarrow{y'} C']$,

$$[A \xrightarrow{x} B \xrightarrow{y} C] \oplus [A' \xrightarrow{x'} B' \xrightarrow{y'} C'] = [A \oplus A' \xrightarrow{x \oplus x'} B \oplus B' \xrightarrow{y \oplus y'} C \oplus C'].$$

Definition 6.9 *Let $\boldsymbol{s}$ be a correspondence which associates to any $\mathbb{E}-$ extension $\delta \in \mathbb{E}(C, A)$ an equivalence class $\boldsymbol{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$.*
$\boldsymbol{s}$ is called a realization of $\mathbb{E}$ if it satisfies the following condition.

- *Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of $\mathbb{E}-$extensions such that $\boldsymbol{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$, $\boldsymbol{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$. Then for any morphism $(a, c) \in \mathbb{E} - Ext(\mathscr{C})(\delta, \delta')$, there exists $b \in \mathscr{C}(B, B')$ such that*

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ A' & \xrightarrow[x']{} & B' & \xrightarrow[y']{} & C' \end{array}$$

  is commutative.

We say that $A \xrightarrow{x} B \xrightarrow{y} C$ realizes δ if $\boldsymbol{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ and that (a, b, c) realizes (a, c).
The existence of b does not depend on the choices of representatives.

Definition 6.10 *A realization of $\mathbb{E}$ is said to be additive if*

1. *For any $A, C \in \mathscr{C}$, the split $\mathbb{E} - extension$ $0 \in \mathbb{E}(C, A)$ satisfies*

$$\boldsymbol{s}(0) = [A \xrightarrow{i_A} A \oplus B \xrightarrow{\pi_B} B].$$

2. *For any pair of* $\mathbb{E}-$*extensions* $\delta \in \mathbb{E}(C,A)$ *and* $\delta' \in \mathbb{E}(C',A')$,

$$\boldsymbol{s}(\delta \oplus \delta') = \boldsymbol{s}(\delta) \oplus \boldsymbol{s}(\delta')$$

holds.

Remark 6.11 *If* $\boldsymbol{s}$ *is an additive realization of* $\mathbb{E}$ *then*

1. *For any pair of elements* $A, C \in \mathscr{C}$, *if* $0 \in \mathbb{E}(C,A)$ *is realized by* $A \xrightarrow{x} B \xrightarrow{y} C$, *since it is equivalent to* $A \xrightarrow{i_A} A \oplus C \xrightarrow{\pi_C} C$, *there exists a retraction* $r \in \mathscr{C}(B,A)$ *of* x *and a section* $s \in \mathscr{C}(C,B)$ *of* y *such that*

$$\begin{pmatrix} r \\ y \end{pmatrix} : B \xrightarrow{\cong} A \oplus C.$$

2. *For any* $f \in \mathscr{C}(A,B)$, *the sequence*

$$A \xrightarrow{\begin{pmatrix} 1 \\ -f \end{pmatrix}} A \oplus B \xrightarrow{\begin{pmatrix} f & 1 \end{pmatrix}} B$$

is a realization of $0 \in \mathbb{E}(B,A)$.

We can now introduce the main definition:

Definition 6.12 *A pair* $(\mathbb{E}, \boldsymbol{s})$ *is an external triangulation of* $\mathscr{C}$ *if it satifies the following conditions:*

- *(ET1)* $\mathbb{E} : \mathscr{C}^{op} \times \mathscr{C} \to \boldsymbol{Ab}$ *is a biadditive functor.*

- *(ET2)* $\boldsymbol{s}$ *is an additive realization of* $\mathbb{E}$.

- *(ET3) Let* (A,δ,C) *and* (A',δ',C') *be any pair of* $\mathbb{E}-$*extensions realized by*

$$\boldsymbol{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C], \ \boldsymbol{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C'].$$

Then for any commutative square

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} & & \\ A' & \xrightarrow[x']{} & B' & \xrightarrow[y']{} & C' \end{array}$$

in $\mathscr{C}$, *there exists a morphism* $(a,c) : \delta \to \delta'$ *which is realized by the morphism of* $\mathbb{E}-$*extensions* (a,b,c).

- *(ET3)*^{op} *Let* (A,δ,C) *and* (A',δ',C') *be any pair of* $\mathbb{E}-$*extensions realized by*

$$\boldsymbol{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C], \ \boldsymbol{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C'].$$

Then for any commutative square

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ & & {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ A' & \xrightarrow[x']{} & B' & \xrightarrow[y']{} & C' \end{array}$$

in $\mathscr{C}$, *there exists a morphism* $(a,c) : \delta \to \delta'$ *which is realized by the morphism of* $\mathbb{E}-$*extensions* (a,b,c).

- *(ET4) Let (A, δ, D) and (B, δ', F) be $\mathbb{E}-$extensions realized by*

$$A \xrightarrow{f} B \xrightarrow{f'} D,\ B \xrightarrow{g} C \xrightarrow{g'} F.$$

Then there exist an object $E \in \mathscr{C}$, an $\mathbb{E}-$extensions $\delta'' \in \mathbb{E}(E, A)$ realized by $A \xrightarrow{h} C \xrightarrow{h'} E$ such that

1. *there is a commutative diagram*

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{f'} & D \\ \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle d} \\ A & \xrightarrow[h]{} & C & \xrightarrow[h']{} & E \\ & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle e} \\ & & F & = & F \end{array}$$

2. *$D \xrightarrow{d} E \xrightarrow{f} F$ realizes $\mathbb{E}(F, f')(\delta')$,*

3. *$\mathbb{E}(d, A)(\delta'') = \delta$,*

4. *$\mathbb{E}(E, f)(\delta'') = \mathbb{E}(e, B)(\delta')$.*

It follows that $(f, e) : \delta'' \to \delta$ is a morphism of $\mathbb{E}-$extensions realized by

$$(f, 1_C, e) : [A \xrightarrow{h} C \xrightarrow{h'} E] \to [B \xrightarrow{g} C \xrightarrow{g'} F].$$

5. *$(ET4)^{op}$ Dual of (ET4).*

Definition 6.13 *A triplet $(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ satisfying 6.12 is called an extriangulated category.*

Extriangulated categories are a generalization of exact and triangulated categories. While the proof that a triangulated category is an extriangulated category requires a bit of work, the proof that an exact category (with some condition concerning the smallness) is an extriangulated category runs smoothly.

Proposition 6.14 *An exact category $\mathscr{C}$ such that for any A, C the equivalence classes of the short exact sequence of the form*

$$A \rightarrowtail^{x} Y \twoheadrightarrow^{y} C$$

are a small set is an extriangulated category.

Proof: the first difficulty is to define the biadditive functor $\mathbb{E}$.
We put

$$Ext^1(C, A) = \{A \xrightarrow{x} B \xrightarrow{y} C | (x, y) \text{ is a short exact sequence in } \mathscr{C}\}/\sim$$

where $\sim$ is the equivalence relation defined in 6.7. We will take as $\mathbb{E}(C, A) = Ext^1(C, A)$ and define $\mathbb{E}$ on morphisms as follows:

- For any $\delta = [A \xrightarrow{x} B \xrightarrow{y} C] \in Ext^1(C, A)$ and any $a \in \mathscr{C}(A, A')$, since we are in an exact category, we can take the pushout and obtain the morphism of short exact sequences

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ {\scriptstyle a}\downarrow & PO & \downarrow & & \| \\ A' & \xrightarrow[m]{} & M & \xrightarrow[e]{} & C \end{array}.$$

$[A' \xrightarrow{m} M \xrightarrow{e} C]$ is an exact sequence by 1.10 and this gives $Ext^1(C,a)(\delta) = a_*\delta = [A' \xrightarrow{m} M \xrightarrow{e} C]$.

- Dually, given $c \in \mathscr{C}(C', C)$, the map $Ext^1(c, A) = c^* : Ext^1(C, A) \to Ext^1(C', A)$ can be defined using pullbacks.

- We now need to define the commutative operation. We start by taking as zero element in $Ext^1(C, A)$ the split short exact sequence

$$0 = [A \xrightarrow{i_A} A \oplus C \xrightarrow{\pi_C} C].$$

Given δ_1 and δ_2 elements of $Ext^1(C, A)$, the sum $\delta_1 + \delta_2$ can be defined by $\triangle_C^*(\nabla_A)_*(\delta_1 \oplus \delta_2)$.

This gives us the biadditive functor $\mathbb{E}$. We can define the realization $\mathbf{s}$ by defining $\mathbf{s}(\delta) = \delta$ itself where $\delta = [A \xrightarrow{x} B \xrightarrow{y} C]$.
To prove the conditions $(ET3)$ we can recall 1.15.
This implies that $a_*[A \xrightarrow{x} B \xrightarrow{y} C] = c^*[A' \xrightarrow{x'} B' \xrightarrow{y'} C']$ for every morphism of short exact sequences

$$\begin{array}{ccccc} A & \overset{x}{\rightarrowtail} & B & \overset{y}{\twoheadrightarrow} & C \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & {\scriptstyle c}\downarrow \\ A' & \overset{x'}{\rightarrowtail} & B' & \overset{y'}{\twoheadrightarrow} & C' \end{array}.$$

$(ET3)$ follows by the universality of the cokernel.
$(ET4)$ follows instead from 1.17.
$(ET3)^{op}$ and $(ET4)^{op}$ are proved dually since axioms of exact categories are self-dual.

Definition 6.15 *Let $(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ be a triplet satisfying $(ET1)$ and $(ET2)$.*

- *We call a sequence $A \xrightarrow{x} B \xrightarrow{y} C$ a conflation if it realizes some extension $\delta \in \mathbb{E}(C, A)$.*

- *A morphism $f \in \mathscr{C}(A, B)$ is called an inflation if it admits some conflation $A \xrightarrow{f} B \to C$.*

- *A morphism $f \in \mathscr{C}(A, B)$ is called a deflation if it admits some conflation $K \to A \xrightarrow{f} B$.*

Remark 6.16 *$(ET4)$ implies that inflations are closed under composition. Dually, $(ET4)^{op}$ implies that deflations are closed under composition.*

Definition 6.17 *Let $\mathscr{D} \subseteq \mathscr{C}$ be a strictly full additive subcategory. $\mathscr{D}$ is extension-closed if for any conflation $A \to B \to C$ such that $A, C \in \mathscr{D}$ implies $B \in \mathscr{D}$.*

Remark 6.18 *Let $(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ be an extriangulated category and let $\mathscr{D} \subseteq \mathscr{C}$ be an extension-closed subcategory. By calling $\mathbb{E}_D$ the restriction of $\mathbb{E}$ onto $\mathscr{D}^{op} \times \mathscr{D}$ and $\boldsymbol{s}_D$ the restriction of $\boldsymbol{s}$, then $(\mathscr{D}, \mathbb{E}_{\mathscr{D}}, \boldsymbol{s}_{\mathscr{D}})$ becomes an extriangulated category.*

Definition 6.19 *Let $(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ be a triplet satisfying $(ET1)$ and $(ET2)$.*

1. *If a conflation $A \xrightarrow{x} B \xrightarrow{y} C$ realizes $\delta \in \mathbb{E}(C, A)$, the pair $(A \xrightarrow{x} B \xrightarrow{y} C, \delta)$ is called an $\mathbb{E}-$triangle and will be denoted by $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta}$*

2. *Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta}$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \xrightarrow{\delta'}$ be any pair of $\mathbb{E}-$triangles. If a triplet (a, b, c) realizes the morphism $(a, c) : \delta \to \delta'$ with the definition of 6.9 then we represent it by*

$$\begin{array}{ccccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C & \xrightarrow{\delta} & \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ A' & \xrightarrow[m]{} & B' & \xrightarrow[e]{} & C' & \xrightarrow[\delta']{} & \end{array}$$

and call (a, b, c) a morphism of $\mathbb{E}-$triangles.

Remark 6.20 $(ET3)$ *means that any diagram of the form*

$$\begin{array}{ccccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C & \xrightarrow{\delta} & \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} & & & & \\ A' & \xrightarrow[m]{} & B' & \xrightarrow[e]{} & C' & \xrightarrow[\delta']{} & \end{array}$$

can be completed into a morphism of $\mathbb{E}-$*triangles and dually for* $(ET3)^{op}$.
From the condition of 6.9*, it follows that any morphism of* $\mathbb{E}-$*extensions can be be realized by a morphism of* $\mathbb{E}-$*triangles.*

Remark 6.21 *We can paraphrase* $(ET4)^{op}$ *in this way:*
let $D \xrightarrow{f'} A \xrightarrow{f} B \xrightarrow{\delta}$ *and* $F \xrightarrow{g'} B \xrightarrow{g} C \xrightarrow{\delta'}$ *be two* $\mathbb{E}-$*triangles. Then there exist an* $\mathbb{E}-$*triangle* $E \xrightarrow{h'} A \xrightarrow{h} C \xrightarrow{\delta''}$ *such that*

1. *the diagram*

$$\begin{array}{ccccc} D & \xrightarrow{d} & E & \xrightarrow{e} & F \\ \| & & {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle g'} \\ D & \xrightarrow[f']{} & A & \xrightarrow[f]{} & B \\ & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ & & C & = & C \end{array}$$

   is commutative.

2. $D \xrightarrow{d} E \xrightarrow{e} F \xrightarrow{g'^*\delta}$ *is an* $\mathbb{E}-$*triangle.*

3. $\delta' = e_*\delta''$

4. $d_*\delta = g^*\delta''$.

6.2 Associated exact sequence

Definition 6.22 *Let* $\mathscr{C}$ *be an additive category and* $\mathbb{E}$ *be a biadditive functor.*
By Yoneda's lemma, given an $\mathbb{E}-$*extension* $\delta \in \mathbb{E}(C, A)$ *there are the natural transformations*

$$\delta_{\#} : \mathscr{C}(-, C) \Rightarrow \mathbb{E}(-, A),\ \ \delta^{\#} : \mathscr{C}(A, -) \Rightarrow \mathbb{E}(C, -)$$

where for any $X \in \mathscr{C}$

- $(\delta_{\#})_X : \mathscr{C}(X, C) \to \mathbb{E}(X, A) : f \to f^*\delta$.

- $(\delta^{\#})_X : \mathscr{C}(A, X) \to \mathbb{E}(C, X) : f \to f_*\delta$.

Lemma 6.23 *Assume* $(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ *satisfies* $(ET1), (ET2), (ET3), (ET3)^{op}$.
Then for any $\mathbb{E}-$*triangle* $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta}$

1. $yx = 0$,

2. $x_*\delta = \delta^{\#}x = 0$,

3. $y^*\delta = \delta_{\#}y = 0$.

Proof: To prove the first point, we can observe by $(ET2)$ that $A \xrightarrow{1_A} A \to 0$ realizes $0 \in \mathbb{E}(0, A)$. By applying $(ET3)$ to

$$\begin{array}{ccccc} A & = & A & \longrightarrow & 0 \\ \| & & \downarrow{\scriptstyle x} & & \\ A & \xrightarrow[x]{} & B & \xrightarrow[y]{} & C \end{array},$$

we obtain a morphism of $\mathbb{E}-$*triangles* $(1_A, x, 0)$ which implies $yx = 0$.
To prove the second point, with a similar reasoning, by applying $(ET3)$ to

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ {\scriptstyle x}\downarrow & & \| & & \\ B & = & B & \longrightarrow & 0 \end{array},$$

we obtain a morphism of $\mathbb{E}-$*extensions* $(x, 0) : \delta \to 0$ that implies $x_*\delta = 0$.
The third point is dual to the second.

Proposition 6.24 *Assume $(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ satisfies $(ET1), (ET2)$. Then the following facts are equivalent.*

1. *$(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ satisfies $(ET3)$ and $(ET3)^{op}$.*

2. *Let $Mod(\mathscr{C})$ denote the abelian category of additive functors from $\mathscr{C}$ to $\boldsymbol{Ab}$.*
   For any $\mathbb{E}-$triangle $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta}$, the following sequences of natural transformations are exact.

   (a) $\mathscr{C}(C,-) \xrightarrow{\mathscr{C}(y,-)} \mathscr{C}(B,-) \xrightarrow{\mathscr{C}(x,-)} \mathscr{C}(A,-) \xrightarrow{\delta^\#} \mathbb{E}(C,-) \xrightarrow{\mathbb{E}(y,-)} \mathbb{E}(B,-)$ is exact in $Mod(\mathscr{C})$.

   (b) $\mathscr{C}(-,A) \xrightarrow{\mathscr{C}(-,x)} \mathscr{C}(-,B) \xrightarrow{\mathscr{C}(-,y)} \mathscr{C}(-,C) \xrightarrow{\delta_\#} \mathbb{E}(-,A) \xrightarrow{\mathbb{E}(-,x)} \mathbb{E}(-,B)$ is exact in $Mod(\mathscr{C}^{op})$.

   The exactness means that for every $X \in \mathscr{C}$

$$\mathscr{C}(C,X) \xrightarrow{\mathscr{C}(y,X)} \mathscr{C}(B,X) \xrightarrow{\mathscr{C}(x,X)} \mathscr{C}(A,X) \xrightarrow{\delta_X^\#} \mathbb{E}(C,X) \xrightarrow{\mathbb{E}(y,X)} \mathbb{E}(B,X)$$

   is exact in $\boldsymbol{Ab}$ and similarly for the other condition.

Proof: The exactness *(1 → 2)*. Since the two conditions of *(2)* are dual, we only have to show the first one. By 6.23, composition of consecutive morphisms of the first sequence is equal to 0.
Given $X \in \mathscr{C}$, to show the exactness of

$$\mathscr{C}(C,X) \xrightarrow{\mathscr{C}(y,X)} \mathscr{C}(B,X) \xrightarrow{\mathscr{C}(x,X)} \mathscr{C}(A,X) \xrightarrow{\delta_X^\#} \mathbb{E}(C,X) \xrightarrow{\mathbb{E}(y,X)} \mathbb{E}(B,X),$$

we have to study the exactness in the middle elements.

Exactness at $\mathscr{C}(B,X)$:
Let $b \in \mathscr{C}(B,X)$ such that $\mathscr{C}(x,X)(b) = bx = 0$.
By applying $(ET3)$ to

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \downarrow & & \downarrow{\scriptstyle b} & & \\ 0 & \longrightarrow & X & = & X \end{array}$$

we can find $c \in \mathscr{C}(C,X)$ such that $b = cy = \mathscr{C}(y,X)(c)$.
Exactness at $\mathscr{C}(A,X)$:
Let $a \in \mathscr{C}(A,X)$ be any morphism satisfying $\delta_X^\#(a) = a_*\delta = 0$. It follows that $(a, 0) : \delta \to 0$ is a

morphism of $\mathbb{E}-$*extensions*.
Since $\mathbf{s}$ is a realization, there exists $b \in \mathscr{C}(B, X)$ such that

$$\begin{array}{ccccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C & \xrightarrow{\delta} & \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} & & \downarrow & & \\ X & = & X & \longrightarrow & 0 & \xrightarrow[0]{} & \end{array}$$

is a morphism of $\mathbb{E}-$*triangles*.
This implies that $a = bx = \mathscr{C}(x, X)(b)$.
Exactness at $\mathbb{E}(C, X)$:
Let $\theta \in \mathbb{E}(C, X)$ be any $\mathbb{E}-$*extension* such that $\mathbb{E}(y, X)(\theta) = y^*\theta = 0$. Let their realizations be

$$X \xrightarrow{f} Y \xrightarrow{g} C \xrightarrow{\theta}, \ X \xrightarrow{m} Z \xrightarrow{e} B \xrightarrow{y^*\theta} .$$

Then the morphism $(1_X, y) : y^*\theta \to \theta$ can be realized by

$$\begin{array}{ccccccc} X & \xrightarrow{m} & Z & \xrightarrow{e} & B & \xrightarrow{y^*\theta} & \\ \| & & \downarrow{\scriptstyle e'} & & \downarrow{\scriptstyle y} & & \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & C & \xrightarrow[\theta]{} & \end{array}$$

for some $e' \in \mathscr{C}(Z, Y)$.
Since $y^*\theta = 0$, it splits and this implies that e has a section s.
Applying $(ET3)^{op}$ to

$$\begin{array}{ccccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C & \xrightarrow{\delta} & \\ & {\scriptstyle e's}\downarrow & & & \| & & \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & C & \xrightarrow[\theta]{} & \end{array} ,$$

we find $a \in \mathscr{C}(A, X)$ which gives a morphism $(a, 1_C) : \delta \to \theta$ that implies $\theta = a_*\delta = \delta^{\#}a$.
(*2* $\to$ *1*): Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta}$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \xrightarrow{\delta'}$ be a pair of $\mathbb{E}-$*triangles*. We have to prove that, given the commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} & & \\ A' & \xrightarrow[x']{} & B' & \xrightarrow[y']{} & C' \end{array} ,$$

we can find $c \in \mathscr{C}(C, C')$ such that (a, c) is a morphism of $\mathbb{E}-$*triangles*.
Recall that $\mathbb{E}(-, x)\delta_{\#} = 0$ is equivalent to $x_*\delta = 0$.
By the exactness of $\mathscr{C}(C, C') \xrightarrow{(\delta'_{\#})_C} \mathbb{E}(C, A') \xrightarrow{\mathbb{E}(C, x')} \mathbb{E}(C, B')$ and

$$\mathbb{E}(C, x')(a_*\delta) = x'_*a_*\delta = b_*x_*\delta = 0,$$

there is $c' \in \mathscr{C}(C, C')$ such that $a_*\delta = \delta'_{\#}c' = c'^*\delta'$ and hence $(a, c') : \delta \to \delta'$ is a morphism of $\mathbb{E}-$*extensions*. We can take the realization of (a, c'), which has form

$$\begin{array}{ccccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C & \xrightarrow{\delta} & \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b'} & & \downarrow{\scriptstyle c'} & & \\ A' & \xrightarrow[x']{} & B' & \xrightarrow[y']{} & C' & \xrightarrow[\delta']{} & \end{array} .$$

By the exactness of

$$\mathscr{C}(C, B') \xrightarrow{\mathscr{C}(y, B')} \mathscr{C}(B, B') \xrightarrow{\mathscr{C}(x, B')} \mathscr{C}(A, B')$$

and by

$$(b-b')x = x'a - x'a = 0$$

there exists $c'' \in \mathscr{C}(C, B')$ such that $c''y = b - b'$.
The c we need to prove $(ET3)$ is defined by $c = c' + y'c''$ since

$$cy = c'y + y'c''y = y'b' + (y'b - y'b') = y'b,$$

$$c^*\delta' = c'^*\delta' + c''^*y'^*\delta' = a_*\delta.$$

By duality, we can also prove $(ET3)^{op}$.

Corollary 6.25 *Assume $(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ satisfies $(ET1), (ET2), (ET3), (ET3)^{op}$.
Let*

$$\begin{array}{ccccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C & \xrightarrow{\delta} & \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ A' & \xrightarrow[x']{} & B' & \xrightarrow[y']{} & C' & \xrightarrow[\delta']{} & \end{array}$$

be a morphism of $\mathbb{E}-$triangles. Then the following facts are equivalent:

1. *a factors through x.*

2. *$a_*\delta = c^*\delta' = 0$.*

3. *c factors through y'.*

Proof: Since $\delta^{\#}(a) = a_*\delta$ by definition, the equivalence of the first two facts follows from the exactness of

$$\mathscr{C}(B, A') \xrightarrow{\mathscr{C}(x, A')} \mathscr{C}(A, A') \xrightarrow{\delta^{\#}_{A'}} \mathbb{E}(C, A').$$

The equivalences of the latter two follows by duality.

Remark 6.26 *In particular, 6.25 implies that if $\delta = \delta'$ and $(a, b, c) = (1, 1, 1)$ it follows that x has a retraction $\iff$ δ splits $\iff$ y has a section.*

Corollary 6.27 *Assume $(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ satisfies $(ET1), (ET2), (ET3), (ET3)^{op}$.
Then if (a, b, c) is a morphism of $\mathbb{E}-$triangles then*

1. *if a and c are isomorphisms in $\mathscr{C}$, then so is b.*

2. *if a and b are isomorphisms in $\mathscr{C}$, then so is c.*

3. *if b and c are isomorphisms in $\mathscr{C}$, then so is a.*

Proof:
(1) :By recalling 6.24, we have the following morphisms of exact sequences of abelian groups

$$\begin{array}{ccccccc} \mathscr{C}(B', A) & \xrightarrow{\mathscr{C}(B', x)} & \mathscr{C}(B', B) & \xrightarrow{\mathscr{C}(B', y)} & \mathscr{C}(B', C) & \xrightarrow{(\delta_{\#})_{B'}} & \mathbb{E}(B', A) \\ \cong\downarrow{\scriptstyle \mathscr{C}(B', a)} & & \downarrow{\scriptstyle \mathscr{C}(B', b)} & & {\scriptstyle \mathscr{C}(B', c)}\downarrow\cong & & \cong\downarrow{\scriptstyle \mathbb{E}(B', a)}, \\ \mathscr{C}(B', A') & \xrightarrow[\mathscr{C}(B', x')]{} & \mathscr{C}(B', B') & \xrightarrow[\mathscr{C}(B', y')]{} & \mathscr{C}(B', C') & \xrightarrow[(\delta'_{\#})_{B'}]{} & \mathbb{E}(B', A') \end{array}$$

$$\begin{array}{ccccccc} \mathscr{C}(C', B) & \xrightarrow{\mathscr{C}(y', B)} & \mathscr{C}(B', B) & \xrightarrow{\mathscr{C}(x', B)} & \mathscr{C}(A', B) & \xrightarrow{\delta'^{\#}_{B}} & \mathbb{E}(C', B) \\ \cong\downarrow{\scriptstyle \mathscr{C}(c, B)} & & \downarrow{\scriptstyle \mathscr{C}(b, B)} & & \cong\downarrow{\scriptstyle \mathscr{C}(a, B)} & & \cong\downarrow{\scriptstyle \mathbb{E}(c, B)}. \\ \mathscr{C}(C, B) & \xrightarrow[\mathscr{C}(y, B)]{} & \mathscr{C}(B, B) & \xrightarrow[\mathscr{C}(x, B)]{} & \mathscr{C}(A, B) & \xrightarrow[(\delta^{\#})_{B}]{} & \mathbb{E}(C, B) \end{array}$$

The first diagram, by applying the *four lemma* shows the surjectivity of $\mathscr{C}(B',b)$ and hence b has a right inverse in $\mathscr{C}$.
By a similar reasoning, the second diagram implies the surjectivity of $\mathscr{C}(b,B)$ and by this that b has a left inverse.
(*2*): By *Yoneda's lemma* and the five lemma applied to the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\mathscr{C}(-,A) & \Longrightarrow & \mathscr{C}(-,B) & \Longrightarrow & \mathscr{C}(-,C) & \Longrightarrow & \mathbb{E}(-,A) & \Longrightarrow & \mathbb{E}(-,B) \\
\cong\Downarrow & & \cong\Downarrow & & \Downarrow{\scriptstyle \mathscr{C}(-,c)} & & \cong\Downarrow & & \Downarrow\cong \\
\mathscr{C}(-,A') & \Longrightarrow & \mathscr{C}(-,B') & \Longrightarrow & \mathscr{C}(-,C') & \Longrightarrow & \mathbb{E}(-,A') & \Longrightarrow & \mathbb{E}(-,B')
\end{array}.$$

This, by duality, also proves (*3*).

Proposition 6.28 *Assume $(\mathscr{C},\mathbb{E},\boldsymbol{s})$ satisfies $(ET1),(ET2),(ET3),(ET3)^{op}$. Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta}$ be any $\mathbb{E}-$triangle and let $a \in \mathscr{C}(A,A')$ and $c \in \mathscr{C}(C',C)$ be two isomorphims.*
Then

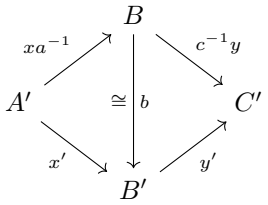
$$A' \xrightarrow{xa^{-1}} B \xrightarrow{c^{-1}y} C' \xrightarrow{a^*c_*\delta}$$

is an $\mathbb{E}-$triangle.

Proof: Let the realization of $a_*c^*\delta$ be $[A' \xrightarrow{x'} B' \xrightarrow{y'} C']$.
Since

$$(c^{-1})^*(a_*c^*\delta) = (cc^{-1})^*a_*\delta = a_*\delta,$$

$(a,c^{-1}) : \delta \to c^*a_*\delta$ is a morphism of $\mathbb{E}-$*extensions*.
By taking the morphism (a,b,c^{-1}) realizing (a,c^{-1}), we have that b is an isomorphism by 6.27.
Since the diagram



is commutative, $[A' \xrightarrow{x'} B' \xrightarrow{y'} C'] = [A' \xrightarrow{xa^{-1}} B \xrightarrow{c^{-1}y} C']$.

Corollary 6.29 *Assume $(\mathscr{C},\mathbb{E},\boldsymbol{s})$ satisfies $(ET1),(ET2),(ET3),(ET3)^{op}$. Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta}$ be any $\mathbb{E}-$triangle. Then for any $\delta' \in \mathbb{E}(C,A)$, the following facts are equivalent:*

1. $\boldsymbol{s}(\delta) = \boldsymbol{s}(\delta')$
2. $\delta' = a_*\delta$ *where $a \in \mathscr{C}(A,A)$ is an automorphism such that $xa = x$.*
3. $\delta' = c^*\delta$ *where $c \in \mathscr{C}(C,C)$ is an automorphism such that $cy = y$.*
4. $\delta' = a_*c^*\delta$ *where $a \in \mathscr{C}(A,A)$, $c \in \mathscr{C}(C,C)$ are automorphisms such that $cy = y$, $xa = x$.*

Proof: $(2 \to 4)$ and $(3 \to 4)$ are trivial. By 6.28, we prove $(4 \to 1)$.
We just need to show $(1 \to 2)$.
By hypotheses $\mathbf{s}(\delta) = \mathbf{s}(\delta') = [A \xrightarrow{x} B \xrightarrow{y} C]$.
We can use $(ET3)^{op}$ to

$$\begin{array}{ccccccc}
A & \xrightarrow{x} & B & \xrightarrow{y} & C & \xrightarrow{\delta} & \\
 & & \| & & \| & & \\
A & \xrightarrow[x]{} & B & \xrightarrow[y]{} & C & \xrightarrow[\delta']{} &
\end{array}$$

which gives us $a \in \mathscr{C}(A, A)$ such that $(a, 1_B, 1_C)$ is a morphism of $\mathbb{E} - triangles$.
By 6.27, a is an isomorphism.
$(1 \to 3)$ can be shown with a similar reasoning.

Definition 6.30 *Assume $(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ satisfies $(ET1), (ET2), (ET3), (ET3)^{op}$.*

1. *For an inflation $f \in \mathscr{C}(A, B)$, we can find a conflation $A \xrightarrow{f} B \to C$. C will be denoted by $Cone(f)$.*

2. *For a deflation $f \in \mathscr{C}(A, B)$, we can find a deflation $K \to A \xrightarrow{f} B$. K will be denoted by $CoCone(f)$.*

Remark 6.31 *Let $f \in \mathscr{C}(A, B)$ be an inflation. Let*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\delta}, \ A \xrightarrow{f} B \xrightarrow{g'} C' \xrightarrow{\delta'}$$

be a pair of $\mathbb{E} - triangles$.
By applying $(ET3)$ to

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{\delta} & \\ \| & & \| & & & & \\ A & \xrightarrow[f]{} & B & \xrightarrow[g']{} & C' & \xrightarrow[\delta']{} & \end{array},$$

there exists $c \in \mathscr{C}(C, C')$ such that $(1_A, 1_B, c)$ is a morphism of $\mathbb{E} - triangles$. Because of 6.27, c is an isomorphism.

Remark 6.32 *$Cone(f)$ is determined uniquely up to isomorphism but it's not functorial, since extriangulated categories will be shown to be a generalization of triangulated categories.*

Proposition 6.33 *Assume $(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ satisfies $(ET1), (ET2), (ET3), (ET3)^{op}$. Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta}$ be any $\mathbb{E} - triangle$. Then*

1. *If $(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ satisfies $(ET4)$, then*

$$\mathbb{E}(-, A) \xrightarrow{\mathbb{E}(-,x)} \mathbb{E}(-, B) \xrightarrow{\mathbb{E}(-,y)} \mathbb{E}(-, C)$$

   is exact.

2. *If $(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ satisfies $(ET4)^{op}$, then*

$$\mathbb{E}(C, -) \xrightarrow{\mathbb{E}(y,-)} \mathbb{E}(B, -) \xrightarrow{\mathbb{E}(x,-)} \mathbb{E}(A, -)$$

   is exact.

Proof:
(1): By 6.23, $yx = 0$ and so $\mathbb{E}(-, y)\mathbb{E}(-, x) = 0$. Let $X \in \mathscr{C}$ be any object.
Let $\theta \in \mathbb{E}(X, B)$ be any $\mathbb{E} - extension$ and let $B \xrightarrow{f} Y \xrightarrow{g} X \xrightarrow{\theta}$ be its realization.
By using $(ET4)$, there exist $E \in \mathscr{C}, \theta' \in \mathbb{E}(E, A)$ such that

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle d} \\ A & \xrightarrow[h]{} & Y & \xrightarrow[h']{} & E \\ & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle e} \\ & & X & = & X \end{array}$$

is commutative and

1. $\mathbf{s}(y_*\theta) = [C \xrightarrow{d} E \xrightarrow{e} X]$,

2. $\mathbf{s}(\theta') = [A \xrightarrow{h} Y \xrightarrow{h'} E]$

3. $x_*\theta' = e^*\theta$.

Then if $\mathbb{E}(X,y)(\theta) = y_*\theta = 0$, e has a section $s \in \mathscr{C}(X,E)$.
By calling $\rho = s^*\theta'$ then

$$\mathbb{E}(X,x)(\rho) = x_*s^*\theta' = s^*x_*\theta' = s^*e^*\theta = \theta.$$

2 is the dual of *1*.

Corollary 6.34 *Let $(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ be an extriangulated category.*
Given any $\mathbb{E}-$triangle $A \xrightarrow{x} Y \xrightarrow{y} C \xrightarrow{\delta}$ then the following sequences are exact

$$\mathscr{C}(C,-) \xrightarrow{\mathscr{C}(y,-)} \mathscr{C}(B,-) \xrightarrow{\mathscr{C}(x,-)} \mathscr{C}(A,-) \xrightarrow{\delta^\#} \mathbb{E}(C,-) \xrightarrow{\mathbb{E}(y,-)} \mathbb{E}(B,-) \xrightarrow{\mathbb{E}(x,-)} \mathbb{E}(A,-),$$

$$\mathscr{C}(-,A) \xrightarrow{\mathscr{C}(-,x)} \mathscr{C}(-,B) \xrightarrow{\mathscr{C}(-,y)} \mathscr{C}(-,C) \xrightarrow{\delta_\#} \mathbb{E}(A,-) \xrightarrow{\mathbb{E}(-,x)} \mathbb{E}(-,B) \xrightarrow{\mathbb{E}(-,y)} \mathbb{E}(-,C).$$

Proof: This follows from 6.33 and 6.24.

Proposition 6.35 *Let*

$$\begin{array}{ccccc}
A & \xrightarrow{f} & B & \xrightarrow{f'} & D \\
\| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle d} \\
A & \xrightarrow[h]{} & C & \xrightarrow[h']{} & E \\
 & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle e} \\
 & & F & = & F
\end{array}$$

be a commutative diagram in $\mathscr{C}$ and let

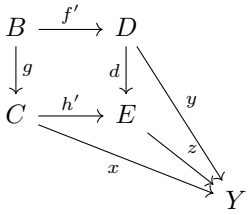
$$A \xrightarrow{f} B \xrightarrow{f'} D \xrightarrow{d^*\delta''}, \quad B \xrightarrow{g} C \xrightarrow{g'} F \xrightarrow{\delta},$$

$$A \xrightarrow{h} C \xrightarrow{h'} E \xrightarrow{\delta''}, \quad D \xrightarrow{d} E \xrightarrow{e} F \xrightarrow{f'_*\delta'}$$

be $\mathbb{E}-$triangles such that $e^*\delta' = f_*\delta''$.
Suppose we have a commutative diagram

$$\begin{array}{ccc}
B & \xrightarrow{f'} & D \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle y} \\
C & \xrightarrow[x]{} & Y
\end{array}$$

in $\mathscr{C}$. Then there exists $z \in \mathscr{C}(E,Y)$ such that



is commutative.

Proof: Because $(f'_*\delta')^{\#}(y) = y_*f'_*\delta' = x_*g_*\delta' = 0$ and because of the exactness of

$$\mathscr{C}(E,Y) \xrightarrow{\mathscr{C}(d,Y)} \mathscr{C}(D,Y) \xrightarrow{(f'_*\delta'^{\#}_Y)} \mathbb{E}(F,Y),$$

there exists $z_1 \in \mathscr{C}(E,Y)$ such that $z_1 d = y$. It follows by

$$(x - z_1 h')g = yf' - z_1 df' = 0$$

and the exactness of

$$\mathscr{C}(F,Y) \xrightarrow{\mathscr{C}(g',Y)} \mathscr{C}(C,Y) \xrightarrow{\mathscr{C}(g,Y)} \mathscr{C}(B,Y),$$

that we can find $z_2 \in \mathscr{C}(F,Y)$ such that $z_2 g' = x - z_1 h'$. By taking $z = z_1 + z_2 e$ we have the commutativity.

Lemma 6.36 *Let $(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ be an extriangulated category.*
Let

$$A \xrightarrow{f} B \xrightarrow{f'} D \xrightarrow{\delta_f},$$
$$B \xrightarrow{g} C \xrightarrow{g'} F \xrightarrow{\delta_g},$$
$$A \xrightarrow{h} C \xrightarrow{h_0} E_0 \xrightarrow{\delta_h}$$

be three $\mathbb{E}-$triangles such that $h = gf$.
Then there exist morphisms $d_0,\ e_0$ in $\mathscr{C}$ such that the diagram

$$\begin{array}{ccccc}
A & \xrightarrow{f} & B & \xrightarrow{f'} & D \\
\| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle d_0} \\
A & \xrightarrow[h]{} & C & \xrightarrow[h_0]{} & E_0 \\
 & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle e_0} \\
 & & F & = & F
\end{array}$$

is commutative and such that

1. *$D \xrightarrow{d_0} E_0 \xrightarrow{e_0} F \xrightarrow{f'_*(\delta_g)}$ is an $\mathbb{E}-$triangle.*

2. *$d_0^*(\delta'') = \delta_f$,*

3. *$f_*(\delta'') = e_0^*(\delta_g)$.*

Proof: By using $(ET4)$, we can find $E \in \mathscr{C}$, a commutative diagram

$$\begin{array}{ccccc}
A & \xrightarrow{f} & B & \xrightarrow{f'} & D \\
\| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle d} \\
A & \xrightarrow[h]{} & C & \xrightarrow[h']{} & E \\
 & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle e} \\
 & & F & = & F
\end{array}$$

in $\mathscr{C}$, and an $\mathbb{E}-$*triangle* $A \xrightarrow{h} C \xrightarrow{h'} E \xrightarrow{\delta''}$ such that

1. $D \xrightarrow{d} E \xrightarrow{e} F \xrightarrow{f'_*(\delta_g)}$ is an $\mathbb{E}-$*triangle.*

2. $d^*(\delta'') = \delta_f$,

3. $f_*(\delta'') = e^*(\delta_g)$.

By using 6.31, we can find a morphism of $\mathbb{E}-$*triangles*

$$\begin{array}{ccccccc} A & \xrightarrow{h} & C & \xrightarrow{h'} & E & \xrightarrow{\delta''} & \\ \| & & \| & & \downarrow{\scriptstyle u} & & \\ A & \xrightarrow[h]{} & C & \xrightarrow[h_0]{} & E_0 & \xrightarrow[\delta_h]{} & \end{array}$$

in which u is an isomorphism. It follows that $\delta'' = u^*(\delta_h)$.
By taking as $d_0 = ud$ and $e_0 = eu^{-1}$ then the diagram commutativity of the lemma follows from the commutativity of the diagram above.
It also follows that $[D \xrightarrow{d_0} E_0 \xrightarrow{e_0} F] = [D \xrightarrow{d} E \xrightarrow{e} F]$. The other conditions follow by $\delta'' = u^*(\delta_h)$.

Proposition 6.37 *Let $(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ be an extriangulated category. Then*

1. *Let C be any object and let*

$$A_1 \xrightarrow{x_1} B_1 \xrightarrow{y_1} C \xrightarrow{\delta_1}, \ A_2 \xrightarrow{x_2} B_2 \xrightarrow{y_2} C \xrightarrow{\delta_2}$$

be a pair of $\mathbb{E}-$triangles. Then there is a commutative diagram

$$\begin{array}{ccccc} & & A_2 & = & A_2 \\ & & {\scriptstyle m_2}\downarrow & & \downarrow{\scriptstyle x_2} \\ A_1 & \xrightarrow{m_1} & M & \xrightarrow{e_1} & B_2 \\ \| & & \downarrow{\scriptstyle e_2} & & \downarrow{\scriptstyle y_2} \\ A_1 & \xrightarrow[x_1]{} & B_1 & \xrightarrow[y_1]{} & C \end{array}$$

in $\mathscr{C}$ such that

$$\boldsymbol{s}(y_2^*\delta_1) = [A_1 \xrightarrow{m_1} M \xrightarrow{e_1} B_2],$$
$$\boldsymbol{s}(y_1^*\delta_2) = [A_2 \xrightarrow{m_2} M \xrightarrow{e_2} B_1],$$
$$m_{1*}\delta_1 + m_{2*}\delta_2 = 0.$$

2. *Dual of (1).*

Proof: Since $\mathbf{s}$ is additive, it follows that

$$\mathbf{s}(\delta_1 \oplus \delta_2) = [A_1 \oplus A_2 \xrightarrow{x_1 \oplus x_2} B_1 \oplus B_2 \xrightarrow{y_1 \oplus y_2} C \oplus C].$$

Let $\mu = (\triangle_C)^*(\delta_1 \oplus \delta_2)$ and let $\mathbf{s}(\mu) = [A_1 \oplus A_2 \xrightarrow{j} M \xrightarrow{k} C]$.
Then it follows that

$$p_{1*}\mu = \delta_1, \ p_{2*}\mu = \delta_2.$$

By applying $(ET4)$ to $\mathbf{s}(0) = [A_1 \xrightarrow{i_1} A_1 \oplus A_2 \xrightarrow{p_2} A_2]$ and $\mathbf{s}(\mu) = [A_1 \oplus A_2 \xrightarrow{j} M \xrightarrow{k} C]$ we can find $B_2' \in \mathscr{C}$, $\theta_1 \in \mathbb{E}(B_2', A_1)$ and a commutative diagram

$$\begin{array}{ccccc} A_1 & \xrightarrow{i_1} & A_1 \oplus A_2 & \xrightarrow{p_2} & A_2 \\ \| & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle x_2'} \\ A_1 & \xrightarrow[m_1]{} & M & \xrightarrow[e_1']{} & B_2' \\ & & {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle y_2'} \\ & & C & = & C \end{array}$$

such that

1. $\mathbf{s}(p_{2*}\mu) = [A_2 \xrightarrow{x_2'} B_2 \xrightarrow{y_2'} C]$,

2. $x_2^*\theta_1 = 0$,

3. $\mathbf{s}(\theta_1) = [A_1 \xrightarrow{m_1} M \xrightarrow{e_1'} B_2']$,

4. $(i_1, 1_M, y_2')$ is a morphism of $\mathbb{E}-$*triangles*. In particular $y\prime_2^*\mu = i_{1*}\theta_1$ and

$$[A_2 \xrightarrow{x_2'} B_2 \xrightarrow{y_2'} C] = \mathbf{s}(\delta_2) = [A_2 \xrightarrow{x_2} B_2 \xrightarrow{y_2} C].$$

By 6.29, these exists an isomorphism $b_2 \in \mathscr{C}(B_2, B_2')$ such that $b_2x_2 = x_2'$ and $y_2'b_2 = y_2$.
By calling $e_1 = b_2^{-1}e_1'$ then

$$\mathbf{s}(b_2^*\theta_1) = [A_1 \xrightarrow{m_1} M \xrightarrow{e_1} B_2].$$

Thus we obtain the commutative diagram

$$\begin{array}{ccccc} A_1 & \xrightarrow{i_1} & A_1 \oplus A_2 & \xrightarrow{p_2} & A_2 \\ \| & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle x_2} \\ A_1 & \xrightarrow[m_1]{} & M & \xrightarrow[e_1]{} & B_2 \\ & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle y_2} \\ & & C & = & C \end{array}$$

which satisfies

$$y_2^*\delta_1 = b_2^*y\prime_2^*p_{1*}\mu = b_2^*p_{1*}y\prime_2^*\mu = b_2^*p_{1*}i_{1*}\theta_1 = b_2^*\theta_1.$$

It follows that $\mathbf{s}(y_2^*\delta_1) = [A_1 \xrightarrow{m_1} M \xrightarrow{e_1} B_2]$.
With a similar reasoning, by $\mathbf{s}(0) = [A_2 \xrightarrow{i_2} A_1 \oplus A_2 \xrightarrow{p_1} A_1]$ and $\mathbf{s}(\mu) = [A_1 \oplus A_2 \xrightarrow{j} M \xrightarrow{k} C]$, we can find a commutative diagram

$$\begin{array}{ccccc} A_2 & \xrightarrow{i_2} & A_1 \oplus A_2 & \xrightarrow{p_1} & A_1 \\ \| & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle x_1} \\ A_2 & \xrightarrow[m_2]{} & M & \xrightarrow[e_2]{} & B_1 \\ & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle y_1} \\ & & C & = & C \end{array}$$

such that

$$\mathbf{s}(y_1^*\delta_2) = [A_2 \xrightarrow{m_2} M \xrightarrow{e_2} B_1].$$

Since

$$e_2m_1 = e_2ji_1 = x_1p_1i_1 = x_1,$$
$$e_1m_2 = e_1ji_2 = x_2p_2i_2 = x_2,$$
$$y_2e_1 = k = y_1e_2,$$

the diagram of the thesis is commutative.
In addition, by the first part of the proof and by 6.23,

$$m_{1*}\delta_1 + m_{2*}\delta_2 = j_*(i_{1*}\delta_1 + i_{2*}\delta_2) = j_*((i_1p_1)_* + (i_2p_2)_*)(\mu) = j_*\mu = 0.$$

Corollary 6.38 *Let $x \in \mathscr{C}(A, B)$, $f \in \mathscr{C}(A, D)$ be two morphisms.*
Then if x is an inflation, so is $\begin{pmatrix} f \\ x \end{pmatrix} \in \mathscr{C}(A, D \oplus B)$. Dually for deflations.

Proof: Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta}$ be an $\mathbb{E}-triangle$. We can realize $f_*\delta$ by the $\mathbb{E}-triangle$

$$D \xrightarrow{d} E \xrightarrow{e} C \xrightarrow{f_*\delta}.$$

By using the first point of 6.37, we obtain a commutative diagram

$$\begin{array}{ccccccc}
 & & A & = & A & & \\
 & & {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle x} & & \\
D & \xrightarrow{k} & M & \xrightarrow{l} & B & \xrightarrow{y^*f_*\delta} & \\
\| & & \downarrow{\scriptstyle e} & & {\scriptstyle y}\downarrow & & \\
D & \xrightarrow{d} & E & \xrightarrow{c} & C & \xrightarrow{f_*\delta} & \\
 & & \downarrow{\scriptstyle e^*\delta} & & \downarrow{\scriptstyle \delta} & &
\end{array}$$

composed by $\mathbb{E}-triangles$ such that $m_*\delta + k_*f_*\delta = 0$.
Since $y^*f_*\delta = f_*y^*\delta = 0$, we can take $M = D \oplus B$ and suppose that $k = i_D$ and $l = \pi_B$, moreover we will call $i = i_B$ and $p = \pi_D$.
By the exactness of

$$\mathscr{C}(B, M) \xrightarrow{\mathscr{C}(x,M)} \mathscr{C}(A, M) \xrightarrow{\delta^\#} \mathbb{E}(C, M)$$

and the equality $\delta^\#(m + kf) = m_*\delta + k_*f_*\delta = 0$, we can find $b \in \mathscr{C}(B, M)$ such that $bx = m + kf$.
By modifying $A \xrightarrow{m} M \xrightarrow{e} E$ through the automorphism

$$n = \begin{pmatrix} -1 & pb \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} (1_M - kpbl) : M \xrightarrow{\cong} M,$$

we obtain a conflation

$$A \xrightarrow{nm} D \oplus B \xrightarrow{en^{-1}} E.$$

We can observe that $pnm = f$ since

$$pnm = -p(1_M - kpbl)m = pkpblm - pm =$$

$$= pbx - pm = pkf = f$$

and that $lnm = x$ since

$$lnm = l(1_M - kpbl)m = lm = x.$$

It follows that $nm = \begin{pmatrix} f \\ x \end{pmatrix}$.

Proposition 6.39 *Let*

$$D \xrightarrow{f} A \xrightarrow{f'} C \xrightarrow{\delta_f},$$

$$A \xrightarrow{g} B \xrightarrow{g'} F \xrightarrow{\delta_g},$$

$$E \xrightarrow{h} B \xrightarrow{h'} C \xrightarrow{\delta_h}$$

be $\mathbb{E}-triangles$ such that $h'g = f'$.
Then there is an $\mathbb{E}-triangle$

$$D \xrightarrow{d} E \xrightarrow{e} F \xrightarrow{\theta}$$

such that the diagram

$$\begin{array}{ccccc}
D & \xrightarrow{f} & A & \xrightarrow{f'} & C \\
{\scriptstyle d}\downarrow & & \downarrow{\scriptstyle g} & & \| \\
E & \xrightarrow{h} & B & \xrightarrow{h'} & C \\
{\scriptstyle e}\downarrow & & \downarrow{\scriptstyle g'} & & \\
F & = & F & &
\end{array}$$

is commutative and such that

1. $d_*(\delta_f) = \delta_h$.

2. $f_*(\theta) = \delta_g$.

3. $g'^*(\theta) + h'^*(\delta_f) = 0$.

Proof: By using $(ET4)$, we have the $\mathbb{E}-triangles$

$$D \xrightarrow{gf} B \xrightarrow{a} G \xrightarrow{\mu}, \ C \xrightarrow{b} G \xrightarrow{c} F \xrightarrow{\nu}$$

such that the diagram

$$\begin{array}{ccccc} D & \xrightarrow{f} & A & \xrightarrow{f'} & C \\ \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle b} \\ D & \xrightarrow{gf} & B & \xrightarrow{a} & G \\ & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle c} \\ & & F & = & F \end{array}$$

is commutative and such that

$$f'_*(\delta_g) = \nu, \ b^*\mu = \delta_f, \ c^*(\delta_g) = f_*\mu.$$

By using 6.23, $\nu = f'_*(\delta_g) = h'_*g_*(\delta_g) = 0$.
We may then assume that $G = C \oplus F, \ b = i_C, \ c = \pi_F$.
Then $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} : B \to G = C \oplus F$ such that $a_1 g = f', \ a_2 = g'$ because of the commutativity of the above diagram.
Since $h' - a_1 \in \mathscr{C}(B, C)$ is such that $(h' - a_1)g = 0$, we can find $z \in \mathscr{C}(F, C)$ satisfying $zg' = h' - a_1$.
Let's call $z' = \begin{pmatrix} -z \\ 1 \end{pmatrix}$.
By applying the dual of 6.36 to the diagram

$$\begin{array}{ccccccc} & & E & & F & & \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle z'} & & \\ D & \xrightarrow{gf} & B & \xrightarrow{a} & G & \xrightarrow{\mu} & \\ & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle (1 \ z)} & & \\ & & C & = & C & & \\ & & \downarrow{\scriptstyle \delta_h} & & \downarrow{\scriptstyle 0} & & \end{array}$$

composed of $\mathbb{E}-triangles$, we obtain an $\mathbb{E}-triangle$

$$D \xrightarrow{d} E \xrightarrow{e} F \xrightarrow{\theta}$$

such that the following diagram

$$\begin{array}{ccccc} D & \xrightarrow{d} & E & \xrightarrow{e} & F \\ \| & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle z'} \\ D & \xrightarrow{gf} & B & \xrightarrow{a} & G \\ & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle (1 \ z)} \\ & & C & = & C \end{array}$$

is commutative and satisfies $\theta = z'^*\mu, \ d_*\mu = \begin{pmatrix} 1 & z \end{pmatrix}^* (\delta_h)$.
The commutativity of the diagram of the thesis is shown by the previous part.

We have now to prove the equalities:
(1) is proved by

$$d_*(\delta_f) = d_* b^* \mu = b^* \begin{pmatrix} 1 & z \end{pmatrix}^* (\delta_h) = (\begin{pmatrix} 1 & z \end{pmatrix} b)^* (\delta_h) = \delta_h.$$

(2) is given by the injectivity of $\mathbb{E}(c, A) = c^*$ and because

$$c^* f_*(\theta) = c^* f_* z'^* \mu = f_*(z'c)^* \mu =$$

$$= f_* \begin{pmatrix} 0 & -z \\ 0 & 1 \end{pmatrix}^* \mu = f_*(1 - b \begin{pmatrix} 1 & z \end{pmatrix})^* \mu =$$

$$= f_* \mu - f_* \begin{pmatrix} 1 & z \end{pmatrix}^* (\delta_f) = f_* \mu - \begin{pmatrix} 1 & z \end{pmatrix}^* f_*(\delta_f) = f_* \mu = c^*(\delta_g).$$

(3) follows from

$$g'^*(\theta) + h'^*(\delta_f) = g'^* z'^* \mu + h'^* b^* \mu =$$

$$\begin{pmatrix} -zg' + h' \\ g' \end{pmatrix}^* \mu = \begin{pmatrix} a_1 \\ g' \end{pmatrix}^* \mu = a^* \mu = 0.$$

6.3 Weakly extriangulated categories

Definition 6.40 *Let $(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ be a triplet of an additive category $\mathscr{C}$, a biadditive functor $\mathbb{E} : \mathscr{C}^{op} \times \mathscr{C} \to Ab$, and a map $\boldsymbol{s}$ which assigns an equivalence $s(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ for each $\delta \in \mathbb{E}(C, A)$.*
$(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ is a weakly extriangulated category if it satisfies $(ET1)$ with $\mathbb{E}$, $(ET2)$ with $\boldsymbol{s}$ and moreover

1. *For any $\mathbb{E}-$triangle $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta}$, the following sequences of natural transformations are exact.*

   (a) $\mathscr{C}(C, -) \xrightarrow{\mathscr{C}(y,-)} \mathscr{C}(B, -) \xrightarrow{\mathscr{C}(x,-)} \mathscr{C}(A, -) \xrightarrow{\delta^\#} \mathbb{E}(C, -)$ is exact in $Mod(\mathscr{C})$.

   (b) $\mathscr{C}(-, A) \xrightarrow{\mathscr{C}(-,x)} \mathscr{C}(-, B) \xrightarrow{\mathscr{C}(-,y)} \mathscr{C}(-, C) \xrightarrow{\delta_\#} \mathbb{E}(A, -)$ is exact in $Mod(\mathscr{C}^{op})$.

   The exactness means that for every $X \in \mathscr{C}$

   $$\mathscr{C}(C, X) \xrightarrow{\mathscr{C}(y,X)} \mathscr{C}(B, X) \xrightarrow{\mathscr{C}(x,X)} \mathscr{C}(A, X) \xrightarrow{\delta_X^\#} \mathbb{E}(C, X)$$

   is exact in $\boldsymbol{Ab}$ and similarly for the other condition.

2. *$w(s)$ Given an $\mathbb{E}-$triangle $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta}$, a morphism $a \in \mathscr{C}(A, A')$ and any $\mathbb{E}-$triangle $A' \xrightarrow{x'} B' \xrightarrow{y'} C \xrightarrow{a_*\delta}$, there exists $b \in \mathscr{C}(B, B')$ such that*

   $$\begin{array}{ccccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C & \xrightarrow{\delta} & \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} & & \| & & \\ A' & \xrightarrow[x']{} & B' & \xrightarrow[y']{} & C & \xrightarrow{a_*\delta} & \end{array}$$

   is a morphism of $\mathbb{E}-$triangles and such that

   $$A \xrightarrow{\begin{pmatrix} x \\ a \end{pmatrix}} B \oplus A' \xrightarrow{\begin{pmatrix} b & -x' \end{pmatrix}} B' \xrightarrow{y'^*\delta}$$

   is an $\mathbb{E}-$triangle.

3. $w(s)^{op}$ *Given an* $\mathbb{E}-$*triangle* $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta}$, *a morphism* $c \in \mathscr{C}(C', C)$ *and any* $\mathbb{E}-$*triangle* $A \xrightarrow{x'} B' \xrightarrow{y'} C' \xrightarrow{c^*\delta}$, *there exists* $b \in \mathscr{C}(B', B)$ *such that*

$$\begin{array}{ccccccc} A & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' & \xrightarrow{c^*\delta} & \\ \| & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ A & \xrightarrow{x} & B & \xrightarrow[y]{} & C & \xrightarrow{\delta} & \end{array}$$

is a morphism of $\mathbb{E}-$*triangles and such that*

$$B' \xrightarrow{\begin{pmatrix} -y' \\ b \end{pmatrix}} C' \oplus B \xrightarrow{\begin{pmatrix} c & y \end{pmatrix}} C \xrightarrow{x'_*\delta}$$

is an $\mathbb{E}-$*triangle.*

Definition 6.41 *Let* $(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$, $(\mathscr{C}', \mathbb{E}', \boldsymbol{s}')$ *and* $(\mathscr{C}'', \mathbb{E}'', \boldsymbol{s}'')$ *be weakly extriangulated categories.*

1. *An exact functor* $(F, \phi) : (\mathscr{C}, \mathbb{E}, \boldsymbol{s}) \to (\mathscr{C}', \mathbb{E}', \boldsymbol{s}')$ *is a pair of an additive functor* $F : \mathscr{C} \to \mathscr{C}'$ *and a natural transformation* $\phi : \mathbb{E} \Rightarrow \mathbb{E}' \circ (F^{op} \times F)$ *such that*

$$\boldsymbol{s}'(\phi_{C,A}(\delta)) = [F(A) \xrightarrow{F(x)} F(B) \xrightarrow{F(y)} F(C)]$$

for any $\mathbb{E}-$*triangle* $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta}$ *in* $\mathscr{C}$.

2. *If* $(F, \phi) : (\mathscr{C}, \mathbb{E}, \boldsymbol{s}) \to (\mathscr{C}', \mathbb{E}', \boldsymbol{s}')$ *and* $(F', \phi') : (\mathscr{C}', \mathbb{E}', \boldsymbol{s}') \to (\mathscr{C}'', \mathbb{E}'', \boldsymbol{s}'')$ *are exact functors, then their composition* $(F'', \phi'') = (F', \phi') \circ (F, \phi)$ *is defined by* $F'' = F' \circ F$ *and* $\phi'' = (\phi' \circ (F^{op} \times F))\phi$.

3. *Let* $(F, \phi), (G, \psi) : (\mathscr{C}, \mathbb{E}, \boldsymbol{s}) \to (\mathscr{C}', \mathbb{E}', \boldsymbol{s}')$ *be exact functors. Then a natural transfomation* $\eta : (F, \phi) \Rightarrow (G, \psi)$ *of exact functors is a natural transformation* $\eta : F \Rightarrow G$ *of additive functors such that*
$$(\eta_A)_* \phi_{C,A}(\delta) = (\eta_C)^* \psi_{C,A}(\delta)$$
for any $\delta \in \mathbb{E}(C, A)$.
Horizontal and vertical compositions are defined by the compositions of natural transformations of additive functors.

Definition 6.42 *Let* $(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ *be a (weakly) extriangulated category.*
Let $X^0 \xrightarrow{d^0_X} X^1 \xrightarrow{d^1_X} X^2 \xrightarrow{\delta}$ *and* $X^0 = Y^0 \xrightarrow{d^0_Y} Y^1 \xrightarrow{d^1_Y} Y^2 \xrightarrow{\rho}$ *be two* $\mathbb{E}-$*triangles and* $f = (1_{X^0}, f^1, f^2)$ *be an* $\mathbb{E}-$*triangle morphism from* δ *to* ρ.
Then the mapping cone M_f *is defined to be the complex*

$$X^1 \xrightarrow{\begin{pmatrix} -d^1_X \\ f^1 \end{pmatrix}} X^2 \oplus Y^1 \xrightarrow{\begin{pmatrix} f^2 & d^1_Y \end{pmatrix}} Y^2.$$

Definition 6.43 *Let* $\mathscr{C}$ *be an additive category.*
Let $X^0 \xrightarrow{d^0_X} X^1 \xrightarrow{d^1_X} X^2$ *and* $Y^0 \xrightarrow{d^0_Y} Y^1 \xrightarrow{d^1_Y} Y^2$ *be two complexes of* $\mathscr{C}$ *and let* f^* *and* g^* *be two morphisms of complexes from* X^* *to* Y^*.
f^* *and* g^* *are said to be homotopic if there exists a pair of morphisms* $\phi^* = (\phi^1, \phi^2)$ *with* $\phi^i : X^i \to Y^{i-1}$ *such that*

$$g^0 - f^0 = \phi^1 d^0_X,$$
$$g^1 - f^1 = d^0_Y \phi^1 + \phi^2 d^1_X,$$
$$g^2 - f^2 = d^1_Y \phi^2.$$

Remark 6.44 *Let X^* and Y^* be two triangles of $\mathscr{C}$ and let $f^*, g^* : X^* \to Y^*$ be two homotopic morphisms through ϕ^*.*
Then if $f^0 = g^0$ and $\mathscr{C}(X^2, A) \xrightarrow{(d_X^1)^*} \mathscr{C}(X^1, A) \xrightarrow{(d_X^0)^*} \mathscr{C}(X^0, A)$ is exact for any $A \in Ob(\mathscr{C})$, then we can modify ϕ^* in order to satisfy $\phi^1 = 0$.

Proof : Since we have $\phi^1 d_X^0 = g^0 - f^0 = 0$ and by the exactness of

$$\mathscr{C}(X^2, Y^0) \xrightarrow{(d_X^1)^*} \mathscr{C}(X^1, Y^0) \xrightarrow{(d_X^0)^*} \mathscr{C}(X^0, Y^0),$$

there is $h \in \mathscr{C}(X^2, Y^0)$ such that $h d_X^1 = \phi^1$. Then $(0, \phi^2 + d_Y^0 h)$ is the homotopy we are searching.

Proposition 6.45 *Let $(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ be a weakly extriangulated category.*
Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} =: R$ and let $A \xrightarrow{m} P \xrightarrow{n} D \xrightarrow{\delta'} =: S$ be two $\mathbb{E}-$triangles. Let $f^*, g^* : R \to S$ be any pair of morphisms of $\mathbb{E}-$triangles such that $f^0 = g^0 = 1_A$.
If g^* is homotopic to f^* by ϕ^*, then the mapping cones of f^* and g^* are isomorphic as complexes in $\mathscr{C}$.

Proof : By 6.44 we can assume that $\phi^1 = 0$.
Under this circumstance, we can observe that

$$F^* = (1_B, \begin{pmatrix} 1 & 0 \\ \phi^2 & 1 \end{pmatrix}, 1_D) : M_{f^*} \to M_{g^*}$$

is a morphism of complexes of $\mathscr{C}$ and moreover it is an isomorphism.

Proposition 6.46 *Let $(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ be a weakly extriangulated category. Let $\delta \in \mathbb{E}(C, A)$ be an $\mathbb{E}-$extension realized by $A \xrightarrow{x} B \xrightarrow{y} C$. Then the following holds*

1. *$\mathscr{C}(-, C) \xRightarrow{\delta_\#} \mathbb{E}(-, A) \xRightarrow{x_*} \mathbb{E}(-, B)$ is exact.*

2. *$\mathscr{C}(A, -) \xRightarrow{\delta^\#} \mathbb{E}(C, -) \xRightarrow{y^*} \mathbb{E}(B, -)$ is exact.*

Proof: Since (2) is dual to (1), it's sufficient to prove the exactness of

$$\mathscr{C}(D, C) \xRightarrow{\delta_\#} \mathbb{E}(D, A) \xRightarrow{x_*} \mathbb{E}(D, B)$$

for any $D \in \mathscr{C}$.
Let $\theta \in \mathbb{E}(D, A)$ be an $\mathbb{E}-$*extension* satisfying $x_*\theta = 0$.
Let $A \xrightarrow{m} G \xrightarrow{n} D \xrightarrow{\theta}$ be the realization of θ through $\mathbf{s}$.
Given $\triangle_A = \begin{pmatrix} 1 & 1 \end{pmatrix} : A \oplus A \to A$, let $\mu = (\triangle_A)_*(\delta \oplus \theta)$. Let $A \xrightarrow{r} V \xrightarrow{d} C \oplus D \xrightarrow{\mu}$ be its realization.
Since $d : V \to C \oplus D$ can be written as $\begin{pmatrix} k \\ l \end{pmatrix} : V \to C \oplus D$ and $d^*\mu = 0$, we have

$$k^*\delta + l^*\theta = 0.$$

Since $i_C : C \to C \oplus D$ satisfies

$$(i_C)^*\mu = (\triangle_A)_*(i_C)^*(\delta \oplus \theta) = \delta,$$

we have a morphism $(1_A, i_C) : \delta \to \mu$.
By $w(s)^{op}$, there exists a morphism of $\mathbb{E}-$*triangles* $v = (1_A, s, i_C)$ from $\mathbf{s}(\delta)$ to $\mathbf{s}(\mu)$.
Moreover, by $w(s)^{op}$, we have that the mapping cone of v is an $\mathbb{E}-$*triangle*, realization of $x_*\mu$, whose deflation is

$$z : C \oplus V \xrightarrow{\begin{pmatrix} 1 & k \\ 0 & l \end{pmatrix}} C \oplus D.$$

Since $x_*\theta = 0$ and $x_*\delta = 0$ we have

$$x_*\mu = x_*(\triangle_A)_*(\delta \oplus \theta) = \begin{pmatrix} x & x \end{pmatrix}_* (\delta \oplus \theta) = 0.$$

Hence z has a section $h = \begin{pmatrix} a & q \\ b & f \end{pmatrix} : C \oplus D \to C \oplus V$ and this implies that $q + kf = 0$ and $lf = 1_D$.
Hence $q \in \mathscr{C}(D, C)$ satisfies

$$\delta_\#(q) = q^*(\delta) = -(kf)^*\delta = f^*(-k^*\delta) = f^*l^*\theta = \theta$$

and this proves the exactness.

Corollary 6.47 *Let $(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ be a weakly extriangulated category. Then it satisfies $(ET3)$ and $(ET3)^{op}$.*

Proof: It follows from 6.24 and 6.46.

Proposition 6.48 *Let $(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ be an extriangulated category.*
Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta}$ be any $\mathbb{E}-$triangle. Let $f : A \to D$ be a morphism and let $D \xrightarrow{d} E \xrightarrow{e} C \xrightarrow{f_*\delta}$ be any $\mathbb{E}-$triangle realizing $f_*\delta$.
By the definition of the realization there is a morphism g such that $(f, g, 1)$ is a morphism of $\mathbb{E}-$triangles.
Then

$$A \xrightarrow{\begin{pmatrix} -f \\ x \end{pmatrix}} D \oplus B \xrightarrow{\begin{pmatrix} d & g \end{pmatrix}} E \xrightarrow{e^*\delta}$$

is an $\mathbb{E}-$triangle.

Proof: By 6.37, the diagram made of $\mathbb{E}-$*triangles*

$$\begin{array}{ccccccc}
 & & A & = & A & & \\
 & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \downarrow \begin{pmatrix} h \\ x \end{pmatrix} & & \downarrow x & & \\
D & \longrightarrow & D \oplus B & \xrightarrow{(0\ 1)} & B & \xrightarrow{0} & \\
\| & & \downarrow{\scriptstyle (d\ g')} & & \downarrow y & & \\
D & \xrightarrow{d} & E & \xrightarrow[e]{} & C & \xrightarrow{f_*\delta} & \\
 & & {\scriptstyle e^*\delta}\downarrow & & \downarrow{\scriptstyle \delta} & & \\
\end{array}$$

is commutative and $\begin{pmatrix} h \\ x \end{pmatrix}_* \delta + \begin{pmatrix} 1 \\ 0 \end{pmatrix}_* f_*\delta = 0$, where the middle row is split since $y^* f_*\delta = 0$.
It follows that

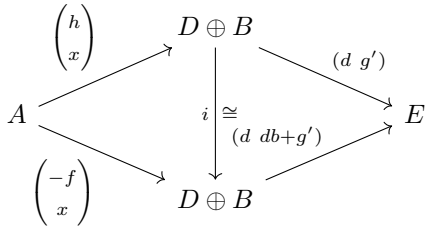
$$A \xrightarrow{\begin{pmatrix} h \\ x \end{pmatrix}} D \oplus B \xrightarrow{\begin{pmatrix} d & g' \end{pmatrix}} E \xrightarrow{e^*\delta}$$

is an $\mathbb{E}-$*triangle*.
Moreover we have that

$$(f + h)_*\delta = \begin{pmatrix} 1 & 0 \end{pmatrix}_* (\begin{pmatrix} h \\ x \end{pmatrix}_* \delta + \begin{pmatrix} 1 \\ 0 \end{pmatrix}_* f_*\delta) = 0$$

in $\mathbb{E}(C, D)$.
Hence by the exactness of $\mathscr{C}(B, D) \xrightarrow{-\circ x} \mathscr{C}(A, D) \xrightarrow{\delta^\#} \mathbb{E}(C, D)$, there is $b \in \mathscr{C}(B, D)$ such that $f + h = bx$.

By the isomorphism $i = \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} : D \oplus B \xrightarrow{\cong} D \oplus B$, the diagram



is commutative.
Therefore by taking $g = db + g'$,

$$A \xrightarrow{\begin{pmatrix} -f \\ x \end{pmatrix}} D \oplus B \xrightarrow{\begin{pmatrix} d & g \end{pmatrix}} E \xrightarrow{e^*\delta}$$

is an $\mathbb{E}-$*triangle*. $(f, g, 1)$ is a morphism of $\mathbb{E}-$*triangles* since $gx = (db+g')x = df + (dh+g'x) = df$ and $eg = e(db + g') = edb + eg' = eg' = y$.

Proposition 6.49 *Let $(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ be an extriangulated category.*
Let $\rho \in \mathbb{E}(D, A)$ and $c \in \mathscr{C}(C, D)$ and let $s(c^*\rho)$ and $s(\rho)$ be two $\mathbb{E}-$triangles. Then the mapping cone of $h = (1, f, c)$ is an $\mathbb{E}-$triangle where f is the morphism we obtain by 6.9.

Proof: this is the dual of 6.48.

Remark 6.50 *By the previous propositions and by 6.24, all extriangulated categories are also weakly extriangulated categories (but we will prove a stronger result).*

Lemma 6.51 *Let $(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ be a weakly extriangulated category. Let*

$$X^0 \oplus A \xrightarrow{d} X^1 \oplus A \xrightarrow{\begin{pmatrix} d^1_X & w \end{pmatrix}} X^2 \xrightarrow{\theta}$$

be an $\mathbb{E}-$triangle where

$$d = \begin{pmatrix} x & u \\ v & 1 \end{pmatrix} \in \mathscr{C}(X^0 \oplus A, X^1 \oplus A).$$

Then

$$X^0 \xrightarrow{d^0_X} X^1 \xrightarrow{d^1_X} X^2 \xrightarrow{p_*\theta},$$

where $p = \pi_{X^0} : X^0 \oplus A \to X^0$ and $d^0_X = x - uv$, is an $\mathbb{E}-$triangle.

Proof: Let $q = \pi_A : X^0 \oplus A \to A$ be the projection on A. Put $\delta = p_*\theta$ and $\rho = q_*\theta$. It follows that $\begin{pmatrix} \delta \\ \rho \end{pmatrix}$ corresponds to θ through the natural isomorphism

$$\mathbb{E}(X^2, X^0 \oplus A) \cong \mathbb{E}(X^2, X^0) \oplus \mathbb{E}(X^2, A).$$

Moreover $d_*\theta = 0$ implies $v_*\delta + \rho = 0$.
Hence

$$(a = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}, b = \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix})$$

gives an isomorphism from the first $\mathbb{E}-$*triangle* of the proposition to

$$X^0 \oplus A \xrightarrow{d^0_X \oplus 1_A} X^1 \oplus A \xrightarrow{\begin{pmatrix} d^1_X & 0 \end{pmatrix}} X^2 \xrightarrow{a_*\theta}$$

where $a_*\theta$ corresponds to $\begin{pmatrix} \delta \\ 0 \end{pmatrix}$ through the previous isomorphism.
It follows that

$$X^0 \oplus A \xrightarrow{d^0_X \oplus 1_A} X^1 \oplus A \xrightarrow{\begin{pmatrix} d^1_X & 0 \end{pmatrix}} X^2 \xrightarrow{a_*\theta}$$

is the direct sum of

$$X^0 \xrightarrow{d^0_X} X^1 \xrightarrow{d^1_X} X^2 \xrightarrow{p_*\theta}$$

and $A \xrightarrow{1} A \xrightarrow{0} 0 \xrightarrow{0}$ and hence

$$X^0 \xrightarrow{d^0_X} X^1 \xrightarrow{d^1_X} X^2 \xrightarrow{p_*\theta}$$

is an $\mathbb{E}-$*triangle.*

Proposition 6.52 *Let $(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ be a weakly extriangulated category.*
Let $X^0 \xrightarrow{d^0_X} X^1 \xrightarrow{d^1_X} X^2 \xrightarrow{\delta}$ and $Y^0 \xrightarrow{d^0_Y} Y^1 \xrightarrow{d^1_Y} Y^2 \xrightarrow{\rho}$ be two $\mathbb{E}-$triangles. Suppose there is a commutative square

$$\begin{array}{ccc} X^0 & \xrightarrow{d^0_X} & X^1 \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ Y^0 & \xrightarrow[d^0_Y]{} & Y^1 \end{array}$$

in $\mathscr{C}$.
Then

1. *There is a morphism $f^* : X^* \to Y^*$ of $\mathbb{E}-$triangles such that $f^0 = a$ and $f^1 = b$.*

2. *Suppose that $X^0 = Y^0$ and $a = 1_{X^0}$, then the morphism f^* of the above point can be chosen in order to have an $\mathbb{E}-$triangle $(M^*_f, (d^0_X)_*\rho)$.*

Proof: By 6.24

$$\mathscr{C}(X_2, Y_2) \xrightarrow{\rho_\#} \mathbb{E}(X_2, Y_0) \xrightarrow{(d^0_Y)_*} \mathbb{E}(X_2, Y^1)$$

is exact.
Since $(d^0_Y)_*(a_*\delta) = b_*(d^0_X)_*\delta = 0$, there is $c \in \mathscr{C}(X_2, Y_2)$ such that $\rho_\#(c) = a_*\delta$ hence we have a morphism of $\mathbb{E}-$*extensions* $(a, c) : \delta \to \rho$.
By the definition of additive realization, we can find a morphism $g : X^1 \to Y^1$ such that (a, g, c) is a morphism of $\mathbb{E}-$*triangles*.
By the exactness of

$$\mathscr{C}(X^2, Y^1) \xrightarrow{(d^1_X)^*} \mathscr{C}(X^1, Y^1) \xrightarrow{(d^0_X)^*} \mathscr{C}(X^0, Y^1)$$

and by $(b - g^1)d^0_X = 0$, there is $m \in \mathscr{C}(X^2, Y^1)$ such that $md^1_X = b - g^1$.
By modifying g^* by the homotopy $\phi^* = (0, m)$, we have $f^* = (a, b, c)$ that satisfies (1).
To prove (2), the reasoning is similar: we can find $c \in \mathscr{C}(C, D)$ such that $(1_A, c) : \delta \to \rho$ is a morphism of $\mathbb{E}-$*extensions*.
By $w(s)$, there is $g^* = (1_A, g^1, c) : \mathbf{s}(\delta) \to \mathbf{s}(\rho)$ morphism of $\mathbb{E}-$*triangles*, moreover $(M^*_g, (d^0_X)_*\rho)$ is an $\mathbb{E}-$*triangle*.
By a similar reasoning to the first point, we can find a homotopy $\phi^* = (0, m)$ from g^* to a morphism f^* such that $f^0 = 1_A$ and $f^1 = b$.
By 6.45 $M(f^*) \cong M(g^*)$ hence $(M^*_f, (d^0_X)_*\rho)$ is an $\mathbb{E}-$*triangle*.

Lemma 6.53 *Suppose $(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ is a weakly extriangulated category.*
Let $A \xrightarrow{f} B \xrightarrow{f'} D \xrightarrow{\delta}$ and $B \xrightarrow{g} C \xrightarrow{g'} F \xrightarrow{\delta'}$ be two $\mathbb{E}-$triangles. Suppose $h = gf$ is an inflation and hence there exists an $\mathbb{E}-$triangle $A \xrightarrow{h} C \xrightarrow{h'} E \xrightarrow{\delta''}$.
Then there exist $d \in \mathscr{C}(D, E)$ and $e \in \mathscr{C}(E, F)$ such that

1. *$D \xrightarrow{d} E \xrightarrow{e} F \xrightarrow{f'_*\delta'}$ is an $\mathbb{E}-$triangle.*

2. *$d^*\delta'' = \delta$.*

3. *$f_*\delta'' = e^*\delta'$.*

Proof: By 6.52, there is $d \in \mathscr{C}(D,E)$ such that $df' = h'g, d^*\delta'' = \delta$ and moreover

$$B \xrightarrow{u} D \oplus C \xrightarrow{\begin{pmatrix} d & h' \end{pmatrix}} E \xrightarrow{f_*\delta''},$$

with $u = \begin{pmatrix} -f' \\ g \end{pmatrix}$, is an $\mathbb{E}-$*triangle* since it is the mapping cone of the above morphism of $\mathbb{E}-$*triangles*. We can apply 6.52 to the diagram

$$\begin{array}{ccccccc} B & \xrightarrow{u} & D \oplus C & \xrightarrow{(d\ h')} & E & \xrightarrow{f_*\delta''} & \\ \| & & \downarrow{\scriptstyle (0\ 1)} & & & & \\ B & \xrightarrow[g]{} & C & \xrightarrow[g']{} & F & \xrightarrow[\delta']{} & \end{array}$$

and we obtain $e \in \mathscr{C}(E,F)$ such that $e\begin{pmatrix} d & h' \end{pmatrix} = g'\begin{pmatrix} 0 & 1 \end{pmatrix}$, $e^*\delta' = f_*\delta''$. This makes

$$D \oplus C \xrightarrow{v} E \oplus C \xrightarrow{\begin{pmatrix} e & g' \end{pmatrix}} F \xrightarrow{u_*\delta'}$$

an $\mathbb{E}-$*triangle* with $v = \begin{pmatrix} -d & -h' \\ 0 & 1 \end{pmatrix}$.
By 6.51,

$$D \xrightarrow{-d} E \xrightarrow{e} F \xrightarrow{-f'_*\delta'}$$

is an $\mathbb{E}-$*triangle* and it is isomorphic to

$$D \xrightarrow{d} E \xrightarrow{e} F \xrightarrow{f'_*\delta'}$$

and hence we prove the thesis.

Corollary 6.54 *Let $(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ be a (weakly) extriangulated category. Let $X^0 \xrightarrow{d^0_X} X^1 \xrightarrow{d^1_X} X^2 \xrightarrow{\delta}$ and $X^0 = Y^0 \xrightarrow{d^0_Y} Y^1 \xrightarrow{d^1_Y} Y^2 \xrightarrow{\rho}$ be two $\mathbb{E}-$triangles and suppose we have a commutative square*

$$\begin{array}{ccc} X^0 & \xrightarrow{d^0_X} & X^1 \\ \| & & \downarrow{\scriptstyle u} \\ Y^0 & \xrightarrow[d^0_Y]{} & Y^1 \end{array}.$$

Let h the morphism of $\mathbb{E}-$triangles constructed by the completion by $(ET3)^{op}$/6.52. Then $M_h = \boldsymbol{s}((d^0_X)_*\rho)$.

Proof: in the case of weakly extriangulated category, this is 6.52. In the case of the extriangulated category, this is a consequence of 6.49.

Proposition 6.55 *A triplet $(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ is an extriangulated category if and only if it is a weakly extriangulated category such that the composition of inflations is an inflation and the composition of deflations is a deflation.*

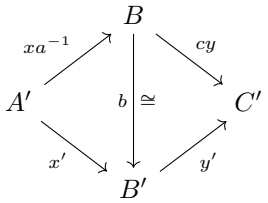
Proof: For the 'if' part 6.53 proves $(ET4)$ and its dual proves $(ET4)^{op}$. The 'only if' part follows from 6.16 and 6.50.

6.4 Relations between extriangulated categories and exact and triangulated categories

Relations with exact categories

Proposition 6.56 *Let $(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ be an extriangulated category such that any inflation is monomorphic and any deflation is epimorphic. Let $\mathbb{S}$ be the class of conflations given by the $\mathbb{E}-$triangles, then $(\mathscr{C}, \mathbb{S})$ is an exact category.*

Proof: Thanks to 6.24, given any conflation $A \xrightarrow{x} B \xrightarrow{y} C$ the pair (A, x) gives a weak kernel of y but since x is monomorphic, it is a kernel of y.
By working dually, (C, y) gives a cokernel of x and this shows that $A \xrightarrow{x} B \xrightarrow{y} C$ is a kernel-cokernel pair.
It follows that $\mathbb{S}$ is made up of kernel-cokernel pairs and that is closed under isomorphisms.
To show this, given an $\mathbb{E}-$*triangle* $A \xrightarrow{x} B \xrightarrow{y} C$ and the sequence of composable morphisms $A' \xrightarrow{x'} B' \xrightarrow{y'} C'$ such that there are three isomorphisms $a \in \mathscr{C}(A, A'),\ b \in \mathscr{C}(B, B'),\ c \in \mathscr{C}(C, C')$ such that $x'a = bx,\ y'b = cy$.
Thanks to 6.28, there is an $\mathbb{E}-$*triangle* $A' \xrightarrow{xa^{-1}} B \xrightarrow{cy} C' \xrightarrow{(c^{-1})^* a_* \delta}$.
Since the diagram



is commutative,

$$\mathbf{s}((c^{-1})^* a_* \delta) = [A' \xrightarrow{xa^{-1}} B \xrightarrow{cy} C'] = A' \xrightarrow{x'} B' \xrightarrow{y'} C'$$

that shows that $A' \xrightarrow{x'} B' \xrightarrow{y'} C'$ is in $\mathbb{S}$. We now want to verify 1.1 :
*[**E0**]*: For any object $A \in \mathscr{C}$, the split sequence $A \xrightarrow{1_A} A \to 0$ belongs to $\mathbb{S}$ by $(ET2)$.
*[**E1**]*: The class of inflations (which are the admissible monics of the exact category we want to define) is closed under composition by using 6.16.
*[**E2**]*: Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta}$ be any $\mathbb{E}-$*triangle* and let $a \in \mathscr{C}(A, A')$ be a morphism.
By using 6.38, there exist $B' \in \mathscr{C}$, $b \in \mathscr{C}(B, B')$ and $x' \in \mathscr{C}(A', B')$ such that

$$A \xrightarrow{s} B \oplus A' \xrightarrow{\begin{pmatrix} b & x' \end{pmatrix}} B'$$

is a conflation where $s = \begin{pmatrix} x \\ -a \end{pmatrix}$.
Since it is a kernel-cokernel pair

$$\begin{array}{ccc} A & \xrightarrow{x} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A' & \xrightarrow[x']{} & B' \end{array}$$

is a pushout square.
By using the dual of 6.39, we can find the commutative diagram

$$\begin{array}{ccccc} & & A' & = & A' \\ & & \downarrow{\scriptstyle i_{A'}} & & \downarrow{\scriptstyle x'} \\ A & \xrightarrow{s} & B \oplus A' & \xrightarrow{(b\ x')} & B' \\ \| & & \downarrow{\scriptstyle \pi_B} & & \downarrow{\scriptstyle y'} \\ A & \xrightarrow{x} & B & \xrightarrow{y} & C \end{array}$$

composed of conflations and this proves that x' is an inflation.
Relations with triangulated categories
Let $\mathscr{C}$ be an additive category equipped with an automorphism $[1]$ and let $\mathbf{E}^1 : \mathscr{C}^{op} \times \mathscr{C} \to \mathbf{Ab}$ be the bifunctor $\mathscr{C}(-, -[1])$.

We will use the same notation of the triangulated categories section so $[n]$ is the composition of $[1]$ n times.

Remark 6.57 *Using the same notation as above, given any $\delta = \boldsymbol{E}^1(C, A) = \mathscr{C}(C, A[1])$. Then*

1. $\delta_\# = \mathscr{C}(-, \delta) : \mathscr{C}(-, C) \Rightarrow \mathscr{C}(-, A[1])$.

2. $\delta^\#$ *is defined by*
$$\delta^\#_X : \mathscr{C}(A, X) \to \mathscr{C}(C, X[1]) : f \to (f[1])\delta$$
for any $X \in \mathscr{C}$.

Lemma 6.58 *Using the same notation as above, suppose $\boldsymbol{s}$ is an $\boldsymbol{E}^1-$triangulation. Then given any $A \in \mathscr{C}$, the $\boldsymbol{E}^1-$extension $1_{A[1]} \in \boldsymbol{E}^1(A[1], A) = \mathscr{C}(A[1], A[1])$ can be realized as*
$$\boldsymbol{s}(1_{A[1]}) = [A \to 0 \to A[1]].$$

Proof: Let $\mathbf{s}(1_A) = [A \xrightarrow{x} X \xrightarrow{y} A[1]]$.
By using 6.24
$$\mathscr{C}(-, A) \xRightarrow{\mathscr{C}(-,x)} \mathscr{C}(-, X) \xRightarrow{\mathscr{C}(-,y)} \mathscr{C}(-, A[1]) \xRightarrow{1_\#=1} \mathscr{C}(-, A[1]) \xRightarrow{\mathscr{C}(-,x[1])} \mathscr{C}(-, X[1])$$
is exact.
We notice that 1_X satisfies $y = (1_{A[1]})_\#)_X \circ \mathscr{C}(X, y)(1_X) = 0$.
Similarly $x[1] = 0$ which implies $x = 0$. Hence
$$0 \Rightarrow \mathscr{C}(-, X) \Rightarrow 0$$
is exact and this implies $X = 0$.

Proposition 6.59 *Let $\mathscr{C}$ be an additive category with an automorphism $[1]$ and denote $\boldsymbol{E}^1 = \mathscr{C}(-, -[1])$. Then*

1. *Suppose $\mathscr{C}$ is a triangulated category with shift functor $[1]$. Given any $\delta \in \boldsymbol{E}^1(C, A)$, we can take the exact triangle*
$$A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} A[1]$$
and define $\boldsymbol{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$.
Then $(\mathscr{C}, \boldsymbol{E}^1, \boldsymbol{s})$ is an extriangulated category.

2. *Suppose we are given an $\boldsymbol{E}^1-$triangulation $\boldsymbol{s}$ of $\mathscr{C}$.*
Then we can define that $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} A[1]$ is an exact triangle if and only if $\boldsymbol{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$.
With this class of exact triangles, $\mathscr{C}$ is a triangulated category.

Proof:(*1*) is proved by comparing the axioms of triangulated categories and the axioms of extriangulated categories.
(*2*): all the axioms of the triangulated categories except (2) can be proved by comparison.
Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta}$ be an $\mathbf{E}^1-$*triangle*. By applying 6.37 to δ and $A \to 0 \to A[1] \xrightarrow{1_{A[1]}}$, we can construct a commutative diagram
$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \downarrow & & \downarrow{\scriptstyle m'} & & \| \\ 0 & \longrightarrow & M & \xrightarrow{e} & C \\ \downarrow & & \downarrow{\scriptstyle e'} & & \\ A[1] & = & A[1] & & \end{array}$$
such that

1. $[0 \to M \xrightarrow{e} C] = 0_*\delta = 0$,

2. $\mathbf{s}(x[1]) = \mathbf{s}(x_*1_{A[1]}) = [B \xrightarrow{m'} M \xrightarrow{e'} A[1]]$,

3. $e^*\delta + e'^*1_{A[1]} = 0$.

By the first point, e is an isomorphism by 6.11. We also have that $\delta e + e' = 0$ in $\mathscr{C}(M, A[1])$ by the third point and in particular $e'e^{-1} = -\delta$.
By the second point

$$\mathbf{s}(x[1]) = [B \xrightarrow{y} C \xrightarrow{-\delta} A[1]]$$

and this shows that $B \xrightarrow{y} C \xrightarrow{-\delta} A[1] \xrightarrow{x[1]} B[1]$ is an exact triangle which is isomorphic to $B \xrightarrow{y} C \xrightarrow{\delta} A[1] \xrightarrow{-x[1]} B[1]$.

Remark 6.60 *The exact functors of extriangulated categories coincide with exact functors of exact categories and exact functors of triangulated categories in the respective cases.*

7 Localization of extriangulated categories

Let $(\mathscr{C}, \mathbb{E}, \mathbf{s})$ be an extriangulated category.
Let $\mathscr{S} \subseteq Mor(\mathscr{C})$ be a set of morphisms such that the following condition holds:

Definition 7.1 *$\mathscr{S}$ satisfies $(M0)$ if $\mathscr{S}$ contains all isomorphisms in $\mathscr{S}$ and is closed by compositions. Moreover, $\mathscr{S}$ is closed by taking finite direct sums: if $f_i \in \mathscr{S}(X_i, Y_i)$ for $i = 1, 2$, then $f_1 \oplus f_2 \in \mathscr{S}(X_1 \oplus X_2, Y_1 \oplus Y_2)$.*

Definition 7.2 *Let $\mathscr{S} \subseteq Mor(\mathscr{C})$ be a set of morphisms which satisfies $(M0)$. Let $N_{\mathscr{S}}$ be the full subcategory of $\mathscr{C}$ consisting of objects $N \in \mathscr{C}$ such that $N \to 0$ and $0 \to N$ belong to $\mathscr{S}$.*

Let $[N_{\mathscr{S}}]$ be the ideal of morphisms of $\mathscr{C}$ that factor through an element of $N_{\mathscr{S}}$.
For the rest of the section, $p : \mathscr{C} \to \mathscr{C}/[N_{\mathscr{S}}] = \bar{\mathscr{C}}$ will be ideal quotient functor and, given $f \in \mathscr{C}(X, Y)$ $\bar{f}$ will denote the image of f through p. Morevoer, $\bar{\mathscr{S}}$ will denote the smallest family of morphisms of $\bar{\mathscr{C}}$ closed under compositions and containing $p(\mathscr{S})$ and all the isomorphisms.

Lemma 7.3 *Let $\mathscr{S}$ and $\bar{\mathscr{C}}$ be as above. Then the following holds.*

1. *Let $f \in \mathscr{S}(A, B)$ and $i \in \mathscr{C}(A, N)$ be two morphisms and suppose $N \in N_{\mathscr{S}}$. Then $\begin{pmatrix} f \\ i \end{pmatrix} \in \mathscr{S}(A, B \oplus N)$.*

2. *Let $f, g \in \mathscr{C}(A, B)$ such that $\bar{f} = \bar{g}$ in $\bar{\mathscr{C}}$.*
   Then $f \in \mathscr{S}$ if and only if $g \in \mathscr{S}$.

3. *The following conditions are equivalent.*
   (a) *$p(\mathscr{S}) = \bar{\mathscr{S}}$*
   (b) *$p^{-1}(\bar{\mathscr{S}}) = \mathscr{S}$.*
   (c) *$p^{-1}(Iso(\bar{\mathscr{C}})) \subseteq \mathscr{S}$.*
   (d) *if $f \in \mathscr{C}(A, B)$ is a split monomorphism such that $\bar{f}$ is an isomorphism in $\bar{\mathscr{C}}$ then $f \in \mathscr{S}$.*

(e) if $f \in \mathscr{C}(A,B)$ is a split epimorphism such that $\bar{f}$ is an isomorphism in $\bar{\mathscr{C}}$ then $f \in \mathscr{S}$.

Proof:
(1): Since $\mathscr{S}$ is closed by finite direct sums, $i_A : A \to A \oplus N \in \mathscr{S}(A, A \oplus N)$.
$\begin{pmatrix} 1 \\ i \end{pmatrix} \in \mathscr{S}(A, A \oplus N)$ since we can observe that

$$\begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Hence $\begin{pmatrix} f \\ i \end{pmatrix} \in \mathscr{S}(A, B \oplus N)$ because $\begin{pmatrix} f \\ i \end{pmatrix} = (f \oplus 1_N) \begin{pmatrix} 1 \\ i \end{pmatrix}$.
(2): Suppose $f \in \mathscr{S}$. If $\bar{f} = \bar{g}$, there exist $N \in N_{\mathscr{S}}$ and $i \in \mathscr{C}(A,N)$, $j \in \mathscr{C}(N,B)$ such that $g = f + ji$.
By (1) and its dual, $\begin{pmatrix} f \\ i \end{pmatrix} \in \mathscr{S}(A, B \oplus N)$ and $\begin{pmatrix} 1 \\ j \end{pmatrix} \in \mathscr{S}(B \oplus N, B)$.
It follows that since $g = \begin{pmatrix} 1 & j \end{pmatrix} \begin{pmatrix} f \\ i \end{pmatrix}$, $g \in \mathscr{S}$.
(3): $(a \Rightarrow b)$ follows from (2)
$(b \Rightarrow c)$ follows from the fact that $Iso(\bar{\mathscr{C}})$ is contained in the closure of $p(\mathscr{S})$ since 1_A is in $\mathscr{S}$ for every $A \in \mathscr{C}$.
$(c \Rightarrow a)$: since $\bar{\mathscr{S}}$ is the closure with respect to isomorphisms in $\bar{\mathscr{C}}$ and by hypothesis and by fullness of p the representatives of these are in $\mathscr{S}$. By recalling that compositions of morphisms in $\mathscr{S}$ are still contained in $\mathscr{S}$, we have (a).
Since (d) and (e) are dual, we can show $(d \iff c)$.
$(c \Rightarrow d)$ is trivial so we need to prove $(d \Rightarrow c)$.
Suppose $\bar{f}$ is an isomorphism in $\bar{\mathscr{C}}$ with some representative $f \in \mathscr{C}(A,B)$.
It follows that there exists $\bar{g} \in \bar{\mathscr{C}}(B,A)$ such that $\bar{g}\bar{f} = i\bar{d}_A$ hence there exist $N \in N_{\mathscr{S}}$, $i \in \mathscr{C}(A,N)$ and $j \in \mathscr{C}(N,A)$ such that $gf - id_A = ji$.
Since $\begin{pmatrix} g & j \end{pmatrix} \begin{pmatrix} f \\ i \end{pmatrix} = id_A$, $\begin{pmatrix} f \\ i \end{pmatrix} \in \mathscr{C}(A, B \oplus N)$ is a split monomorphism. By hypothesis, $\begin{pmatrix} f \\ i \end{pmatrix} \in \mathscr{S}$ and since $f = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} f \\ i \end{pmatrix}$, $f \in \mathscr{S}$.

The rest of this section will be aimed at the proof of the main theorem regarding localization of extriangulated categories.

Theorem 7.4 *Suppose we have a set of morphisms $\mathscr{S} \subseteq Mor(\mathscr{C})$ satisfying 7.1. Then the following holds.*

1. *Suppose that $\bar{\mathscr{S}}$ satisfies the following conditions:*

   (a) $\bar{\mathscr{S}} \subseteq Mor(\bar{\mathscr{C}})$ satisfies $2-out-of-3$ with respect to compositions.

   (b) $\bar{\mathscr{S}}$ is a multiplicative system in $\bar{\mathscr{C}}$.

   (c) Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta}$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \xrightarrow{\delta'}$ be two $\mathbb{E}-$triangles and let $a \in \mathscr{C}(A,A')$, $c \in \mathscr{C}(C,C')$ be two morphisms such that $a_*\delta = c^*\delta'$ and such that $\bar{a}$ and $\bar{c}$ are in $\bar{\mathscr{S}}$. Then there exists $\boldsymbol{b} \in \bar{\mathscr{S}}(B,B')$ such that $\boldsymbol{b}\bar{x} = \bar{x}'\bar{a}$ and $\bar{c}\bar{y} = \boldsymbol{b}\bar{y}'$

   Then we obtain a weakly extriangulated category $(\hat{\mathscr{C}}, \hat{\mathbb{E}}, \hat{\boldsymbol{s}})$ and an exact functor $(Q,\mu) : (\mathscr{C}, \mathbb{E}, \boldsymbol{s}) \to (\hat{\mathscr{C}}, \hat{\mathbb{E}}, \hat{\boldsymbol{s}})$.

2. *The exact functor $(Q,\mu) : (\mathscr{C}, \mathbb{E}, \boldsymbol{s}) \to (\hat{\mathscr{C}}, \hat{\mathbb{E}}, \hat{\boldsymbol{s}})$ obtained above has the following universal properties:*

(a) *For any exact functor* $(F,\phi):(\mathscr{C},\mathbb{E},\boldsymbol{s})\to(\mathscr{D},\mathbb{F},\boldsymbol{t})$ *such that* $F(s)$ *is an isomorphism for any* $s\in\mathscr{S}$*, there exists a unique exact functor* $(\hat{F},\hat{\phi}):(\hat{\mathscr{C}},\hat{\mathbb{E}},\hat{\boldsymbol{s}})\to(\mathscr{D},\mathbb{F},\boldsymbol{t})$ *such that* $(F,\phi)=(\hat{F},\hat{\phi})(Q,\mu)$.

(b) *Let* $(F,\phi),(G,\psi):(\mathscr{C},\mathbb{E},\boldsymbol{s})\to(\mathscr{D},\mathbb{F},\boldsymbol{t})$ *be two exact functors which send any* $s\in\mathscr{S}$ *to isomorphisms. Let* $(\hat{F},\hat{\phi}),(\hat{G},\hat{\psi}):(\hat{\mathscr{C}},\hat{\mathbb{E}},\hat{\boldsymbol{s}})\to(\mathscr{D},\mathbb{F},\boldsymbol{t})$ *be the two functors obtained by the previous point. Then for any natural transformation* $\eta:(F,\phi)\Rightarrow(G,\psi)$ *of exact functors, there exists a unique natural transformation* $\hat{\eta}:(\hat{F},\hat{\phi})\Rightarrow(\hat{G},\hat{\psi})$ *of exact functors such that* $\eta=\hat{\eta}\circ(Q,\mu)$.

3. *Suppose* $\mathscr{S}$ *also satisfies* $(MR1)$ *where* $(MR1)$ *is*

- $\bar{M}_{inf}:=\{t\bar{x}s|x\ is\ an\ inflation,s,t\in\bar{\mathscr{S}}\}$ *is closed by composition in* $Mor(\bar{\mathscr{C}})$*. Dually,* $\bar{M}_{def}:=\{t\bar{x}s|x\ is\ a\ deflation,s,t\in\bar{\mathscr{S}}\}$ *is closed by composition in* $Mor(\bar{\mathscr{C}})$.

then $(\hat{\mathscr{C}},\hat{\mathbb{E}},\hat{\boldsymbol{s}})$ *is extriangulated.*

Corollary 7.5 *Suppose we have a set of morphisms* $\mathscr{S}\subseteq Mor(\mathscr{C})$ *satisfying* 7.1 *and such that* $p(\mathscr{S})=\bar{\mathscr{S}}$*. Then the following holds.*

1. *Suppose that* $\mathscr{S}$ *satisfies the following conditions:*

(a) $\mathscr{S}\subseteq Mor(\mathscr{C})$ *satisfies* $2-out-of-3$ *with respect to compositions.*

(b) $\mathscr{S}$ *is a multiplicative system in* $\mathscr{C}$.

(c) *Let* $A\xrightarrow{x}B\xrightarrow{y}C\xrightarrow{\delta}$ *and* $A'\xrightarrow{x'}B'\xrightarrow{y'}C'\xrightarrow{\delta'}$ *be two* $\mathbb{E}-triangles$ *and let* $a\in\mathscr{C}(A,A')$, $c\in\mathscr{C}(C,C')$ *be two morphisms in* $\mathscr{S}$ *such that* $a_*\delta=c^*\delta'$*. Then there exists* $b\in\mathscr{S}(B,B')$ *such that* (a,b,c) *is a morphism from* δ *to* δ'.

Then we obtain a weakly extriangulated category $(\hat{\mathscr{C}},\hat{\mathbb{E}},\hat{\boldsymbol{s}})$ *and an exact functor* $(Q,\mu):(\mathscr{C},\mathbb{E},\boldsymbol{s})\to(\hat{\mathscr{C}},\hat{\mathbb{E}},\hat{\boldsymbol{s}})$.

2. *The exact functor* $(Q,\mu):(\mathscr{C},\mathbb{E},\boldsymbol{s})\to\hat{\mathscr{C}},\hat{\mathbb{E}},\hat{\boldsymbol{s}}$ *obtained above has the following universal properties:*

(a) *For any exact functor* $(F,\phi):(\mathscr{C},\mathbb{E},\boldsymbol{s})\to(\mathscr{D},\mathbb{F},\boldsymbol{t})$ *such that* $F(s)$ *is an isomorphism for any* $s\in\mathscr{S}$*, there exists a unique exact functor* $(\hat{F},\hat{\phi}):(\hat{\mathscr{C}},\hat{\mathbb{E}},\hat{\boldsymbol{s}})\to(\mathscr{D},\mathbb{F},\boldsymbol{t})$ *such that* $(F,\phi)=(\hat{F},\hat{\phi})(Q,\mu)$.

(b) *Let* $(F,\phi),(G,\psi):(\mathscr{C},\mathbb{E},\boldsymbol{s})\to(\mathscr{D},\mathbb{F},\boldsymbol{t})$ *be two exact functors which send any* $s\in\mathscr{S}$ *to isomorphisms. Let* $(\hat{F},\hat{\phi}),(\hat{G},\hat{\psi}):(\hat{\mathscr{C}},\hat{\mathbb{E}},\hat{\boldsymbol{s}})\to(\mathscr{D},\mathbb{F},\boldsymbol{t})$ *be the two functors obtained by the previous point. Then for any natural transformation* $\eta:(F,\phi)\Rightarrow(G,\psi)$ *of exact functors, there exists a unique natural transformation* $\hat{\eta}:(\hat{F},\hat{\phi})\Rightarrow(\hat{G},\hat{\psi})$ *of exact functors such that* $\eta=\hat{\eta}\circ(Q,\mu)$.

3. *Suppose* $\mathscr{S}$ *also satisfies* $(M1)$ *where* $(M1)$ *is*

- $Mor(\mathscr{C})_{inf}:=\{txs|x\ is\ an\ inflation,s,t\in\mathscr{S}\}$ *is closed by composition in* $Mor(\mathscr{C})$*. Dually,* $Mor(\mathscr{C})_{def}:=\{txs|x\ is\ a\ deflation,s,t\in\mathscr{S}\}$ *is closed by composition in* $Mor(\mathscr{C})$.

then $(\hat{\mathscr{C}},\hat{\mathbb{E}},\hat{\boldsymbol{s}})$ *is extriangulated.*

7.4 implies 7.5 thanks to the following result.

Proposition 7.6 *Suppose that $\mathscr{S}$ satisfies 7.1 and $p(\mathscr{S}) = \bar{\mathscr{S}}$. Then the following holds*

1. *1(a) of 7.5 is equivalent to 1(a) of 7.4.*

2. *1(b) of 7.5 implies 1(b) of 7.4.*

3. *1(c) of 7.5 implies 1(c) of 7.4.*

4. *1(d) of 7.5 implies 1(d) of 7.4.*

Proof: (1) follows from $p(\mathscr{S}) = \bar{\mathscr{S}}$ and 7.3. (3) and (4) are clear.
Only (2) remains to be proved. Suppose that $\mathscr{S}$ satisfies 1(b) of 7.5. $\bar{\mathscr{S}}$ is closed by compositions by definition.
So we need to verify

1. For any $\bar{f} \in \bar{\mathscr{C}}(A, B)$, if there exists $\bar{s} \in \bar{\mathscr{S}}(A', A)$ such that $\bar{f}\bar{s} = 0$, then there exists $\bar{t} \in \bar{\mathscr{S}}(B, B')$ such that $\bar{t}\bar{f} = 0$.

2. For any $A' \xleftarrow{\bar{s}} A \xrightarrow{\bar{f}} B$ with $\bar{s} \in \bar{\mathscr{S}}$, there exists a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\bar{f}} & B \\ {\scriptstyle \bar{s}}\downarrow & & \downarrow{\scriptstyle \bar{s}'} \\ A' & \xrightarrow[\bar{f}']{} & B' \end{array}$$

   in $\bar{\mathscr{C}}$ such that $\bar{s}' \in \bar{\mathscr{S}}$.

The second point follows from 1(b) of 7.5.
We need to prove the first point: let $\bar{f} \in \bar{\mathscr{C}}(A, B)$ and $\bar{s} \in \bar{\mathscr{S}}(A', A)$ such that $\bar{f}\bar{s} = 0$. Then there exist $N \in N_{\mathscr{S}}$ and $i \in \mathscr{C}(A', N)$, $j \in \mathscr{C}(N, B)$ such that $fs = ji$. It follows that

$$\begin{pmatrix} f & -j \end{pmatrix} \begin{pmatrix} s \\ i \end{pmatrix} = 0.$$

By 7.3, $\begin{pmatrix} s \\ i \end{pmatrix} \in \mathscr{S}$, there exists $t \in \mathscr{S}(B, B')$ such that $t \begin{pmatrix} f & -j \end{pmatrix} = 0$ by 1(b) of 7.5. It follows that $tf = 0$.

Remark 7.7 *It could be useful to observe that replacing $\mathscr{S}$ by $p^{-1}(\bar{\mathscr{S}})$ preserves $\bar{\mathscr{S}}$ and does not affect the localization.*
Since all the conditions of 7.4 only care about $\bar{\mathscr{S}}$ and since the replacement preserves 7.1, $p^{-1}(\bar{\mathscr{S}})$ satisfies the hypothesis of 7.4 whenever $\mathscr{S}$ does. For this reason we can replace $\mathscr{S}$ by $p^{-1}(\bar{\mathscr{S}})$ and have $\mathscr{S} = p^{-1}(\bar{\mathscr{S}})$ already satisfied.

Remark 7.8 *Suppose that $\mathscr{S}$ satisfies 7.1 and 1(a), 1(b), 1(c) of 7.4. Since a localization $\mathscr{C} \to \hat{\mathscr{C}}$ factors uniquely through $p : \mathscr{C} \to \bar{\mathscr{C}}$, we may consider $\hat{\mathscr{C}}$ as a localization of $\bar{\mathscr{C}}$ by $\bar{\mathscr{S}}$.*

Definition 7.9 *We can define the localization of $\mathscr{C}$ by $\mathscr{S}$ by the localization functor $Q = \bar{Q} \circ p : \mathscr{C} \to \hat{\mathscr{C}}$ where $\bar{Q} : \bar{\mathscr{C}} \to \hat{\mathscr{C}}$ is the localization by the multiplicative system $\bar{\mathscr{S}}$.*

Lemma 7.10 *For any extension $\delta \in \mathbb{E}(C, A)$, the following facts are equivalent.*

1. *There exists $s \in \mathscr{S}(A, A')$ satisfying $s_*\delta = 0$.*

2. *There exists $t \in \mathscr{S}(C', C)$ satisfying $t^*\delta = 0$.*

Proof: By duality, we only need to prove $(1 \Rightarrow 2)$.
By 7.7, we have the equivalent conditions of 7.3 satisfied.
Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta}$ be an $\mathbb{E}-$*triangle* and let $s \in \mathscr{S}(A, A')$ be a morphism such that $s_*\delta = 0$.
By using $1(c)$ of 7.4, there exists $u \in \mathscr{S}(B, A' \oplus C)$ which makes

$$\begin{array}{ccccc} A & \xrightarrow{\overline{x}} & B & \xrightarrow{\overline{y}} & C \\ {\scriptstyle \overline{s}}\downarrow & & \downarrow{\scriptstyle \overline{u}} & & \| \\ A' & \xrightarrow[i_{A'}]{} & A' \oplus C & \xrightarrow[\pi_C]{} & C \end{array}$$

commutative in $\overline{\mathscr{C}}$. By using $1(b)$ of 7.4, we can find a commutative diagram

$$\begin{array}{ccc} B' & \xrightarrow{\overline{v}} & C \\ {\scriptstyle \overline{b}}\downarrow & & \downarrow{\scriptstyle i_C} \\ B & \xrightarrow[\overline{u}]{} & A' \oplus C \end{array}$$

in $\overline{\mathscr{C}}$ such that $v \in \mathscr{S}$. Then it follows that $\overline{v} = \overline{y}\overline{b}$ and this implies $yb \in \mathscr{S}$ by 7.3. Then $t = yb$ satisfies $t^*\delta = 0$.

Definition 7.11 *Let*

$$K(C, A) = \{\delta \in \mathbb{E}(C, A) | s_*\delta = 0 \text{ for some } s \in \mathscr{S}(A, A')\} =$$

$$= \{\delta \in \mathbb{E}(C, A) | t^*\delta = 0 \text{ for some } t \in \mathscr{S}(C', C)\}$$

be a subset of $\mathbb{E}(C, A)$ for all $A, C \in \mathscr{C}$. And by using 7.10, this defines a subfunctor $K \subseteq \mathbb{E}$.

Proposition 7.12 *K has the following properties.*

1. *$K \subseteq \mathbb{E}$ is an additive subfunctor.*

2. *$\mathbb{E}/K : \mathscr{C}^{op} \times \mathscr{C} \to Ab$ is a biadditive functor which satisfies $(\mathbb{E}/K)(N, -) = 0$ and $(\mathbb{E}/K)(-, N) = 0$ for any $N \in N_{\mathscr{S}}$.*

Proof: (1) For any $\delta \in K(C, A)$ and $\delta' \in K(C', A')$, $\delta \oplus \delta' \in K(C \oplus C', A \oplus A')$ which corresponds to $(\delta, 0, 0, \delta') \in \mathbb{E}(C, A) \oplus \mathbb{E}(C, A') \oplus \mathbb{E}(C', A) \oplus \mathbb{E}(C, A') \cong \mathbb{E}(C \oplus C', A \oplus A')$ and this shows the additivity of K.
(2) follows from (1) and by recalling the definition of $N_{\mathscr{S}}$.

Remark 7.13 *Given $\delta \in \mathbb{E}(X, Y)$, $\overline{\delta} \in (\mathbb{E}/K)(X, Y)$ will denote the element represented by δ.*

Definition 7.14 *By 7.12, we obtain a well-defined biadditive functor $\overline{E} : \overline{\mathscr{C}}^{op} \times \overline{\mathscr{C}} \to Ab$ given by*

1. *$\overline{\mathbb{E}}(C, A) = (\mathbb{E}/K)(C, A)$ for any $A, C \in \mathscr{C}$.*

2. *$\overline{a}_*\overline{\delta} = \overline{a_*\delta}$ for any $\overline{\delta} \in \overline{\mathbb{E}}(C, A)$ and $\overline{a} \in \overline{\mathscr{C}}(A, A')$.*

3. *$\overline{c}^*\overline{\delta} = \overline{c^*\delta}$ for any $\overline{\delta} \in \overline{\mathbb{E}}(C, A)$ and $\overline{c} \in \overline{\mathscr{C}}(C', C)$.*

Morevoer, we have a natural transformation $\pi : \mathbb{E} \Rightarrow \bar{\mathbb{E}}(p^{op} \times p)$ *defined by*

$$\pi_{C,A} : \mathbb{E}(C, A) \to \bar{\mathbb{E}}(C, A) : \delta \to \bar{\delta}$$

for each $A, C \in \mathscr{C}$.
We will denote an $\bar{\mathbb{E}}$ *– extension* $\bar{\delta} \in \mathbb{E}(C, A)$ *by* $C \overset{\bar{\delta}}{\to} A$.

Remark 7.15 *By definition, the following are equivalent for any* $\delta \in \mathbb{E}(C, A)$.

1. $\bar{\delta} = 0$ *holds in* $\bar{\mathbb{E}}(C, A)$.

2. $\bar{s}_* \bar{\delta} = 0$ *holds in* $\bar{\mathbb{E}}(C, A)$ *for some/any* $\bar{s} \in \bar{\mathscr{T}}(A, A')$.

3. $\bar{t}^* \bar{\delta} = 0$ *holds in* $\bar{\mathbb{E}}(C, A)$ *for some/any* $\bar{t} \in \bar{\mathscr{T}}(C', C)$.

Lemma 7.16 *Given a finite family of morphisms* $\{\bar{s}_i \in \bar{\mathscr{T}}(A, X_i)\}_{1 \le i \le n}$, *there exist* $X \in \mathscr{C}$, $\bar{s} \in \bar{\mathscr{T}}(A, X)$ *and* $\{\bar{u}_i \in \bar{\mathscr{T}}(X_i, X)\}_{1 \le i \le n}$ *such that* $\bar{s} = \bar{u}_i \bar{s}_i$ *for* $1 \le i \le n$. *Dually for a finite family of morphisms* $\{\bar{s}_i \in \bar{\mathscr{T}}(Z_i, C)\}_{1 \le i \le n}$.

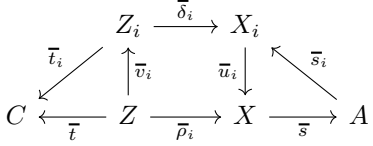
Proof: This proof is a similar reasoning to 4.19 by using $1(a)$, $1(b)$ of 7.4.

Proposition 7.17 *Let* $n \ge 2$ *be an integer. Let* $A, C \in \mathscr{C}$ *be a pair of objects and let*

$$(\bar{t}_i \setminus \bar{\delta}_i / \bar{s}_i) = (C \overset{\bar{t}_i}{\leftarrow} Z_i \overset{\bar{\delta}_i}{\to} X_i \overset{\bar{s}_i}{\leftarrow} A) \quad (1 \le i \le n)$$

be triplets with $\bar{s}_i \in \bar{\mathscr{T}}(A, X_i), \bar{t}_i \in \bar{\mathscr{T}}(Z_i, C), \bar{\delta}_i \in \bar{\mathbb{E}}(Z_i, X_i)$. *Then the following holds.*

1. *We can take a common denominator* $\bar{s}, \bar{t} \in \bar{\mathscr{T}}$. *This means there exist common* $X, Z \in \mathscr{C}$, $\bar{s} \in \bar{\mathscr{T}}(A, X)$ *and* $\bar{t} \in \bar{\mathscr{C}}(Z, C)$ *and* $\bar{u}_i \in \bar{\mathscr{T}}(X_i, X)$, $\bar{v}_i \in \bar{\mathscr{T}}(Z, Z_i)$ *such that* $\bar{s} = \bar{u}_i \bar{s}_i$ *and* $\bar{t} = \bar{t}_i \bar{v}_i$ *as in the diagram*



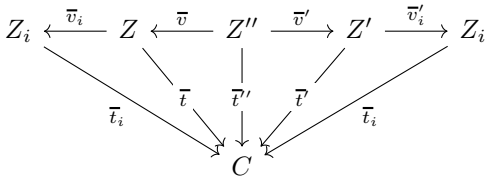
where $\bar{\rho}_i = \bar{v}_i^* \bar{u}_{i*} \bar{\delta}_i$.
In particular if $n = 2$, *we will say* $(\bar{t}_1 \setminus \bar{\delta}_1 / \bar{s}_1) \sim (\bar{t}_2 \setminus \bar{\delta}_2 / \bar{s}_2)$ *if they satisfy* $\bar{\rho}_1 = \bar{\rho}_2$.

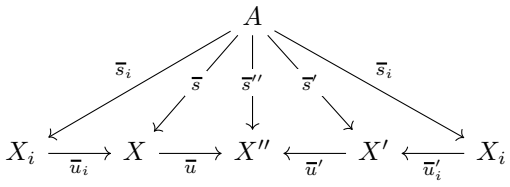
2. *The relation* $(\bar{t}_1 \setminus \bar{\delta}_1 / \bar{s}_1) \sim (\bar{t}_2 \setminus \bar{\delta}_2 / \bar{s}_2)$ *is independent of the choice of common denominators. In particular, for any other choice of*

$$(\bar{t}' \setminus \bar{\rho}'_i / \bar{s}') = (C \overset{\bar{t}'}{\leftarrow} Z' \overset{\bar{\rho}'_i}{\to} X' \overset{\bar{s}'}{\leftarrow} A)$$

and $\bar{u}'_i \in \bar{\mathscr{T}}(X_i, X')$, $\bar{v}'_i \in \bar{\mathscr{T}}(Z', Z_i), \bar{\rho}'_i \in \bar{\mathbb{E}}(Z', X')$ *such that* $\bar{s}' = \bar{u}'_i \bar{s}_i$, $\bar{t}' = \bar{t}_i \bar{v}'_i$ *and* $\bar{\rho}'_i = \bar{v}'^*_i \bar{u}'_{i*} \bar{\delta}_i$ *for* $i = 1, 2$ *then* $\bar{\rho}_1 = \bar{\rho}_2$ *if and only if* $\bar{\rho}'_1 = \bar{\rho}'_2$.

Proof: (1) follows from 7.16.
(2): Assume we have the diagram of the hypothesis with $\bar{\rho}_1 = \bar{\rho}_2$. We have to show that, given another common denominator $\bar{s}', \bar{t}' \in \bar{\mathscr{T}}$, we have $\bar{\rho}'_1 = \bar{\rho}'_2$. By 7.16 given two triplets $(\bar{t} \setminus \bar{\rho}_i / \bar{s})$ and $(\bar{t}' \setminus \bar{\rho}_i / \bar{s}')$, there exist a common denominator $\bar{s}'', \bar{t}'' \in \bar{\mathscr{T}}$ and two commutative diagrams





where all morphisms belong to $\bar{\mathscr{S}}$.
By compositions of some morphisms $Z''' \to Z''$ and $X'' \to X'''$ in $\bar{\mathscr{S}}$ if needed, we can suppose that $\overline{v_i v} = \overline{v_i' v'}$ and $\overline{uu_i} = \overline{u' u_1'}$ for $i = 1, 2$.
It follows that

$$\bar{v}^* \bar{u}_* \bar{\rho}_i = \bar{v}^* \bar{u}_* (\bar{v}_i^* \bar{u}_{i*} \bar{\delta}_i) = (\overline{v_i v})^* (\overline{uu_i})_* \bar{\delta}_i =$$
$$= (\overline{v_i' v'})^* (\overline{u' u_i'})_* \bar{\delta}_i = \bar{v}'^* \bar{u}_*' (\bar{v}_i'^* \bar{u}_{i*}' \bar{\delta}_i) = \bar{v}'^* \bar{u}_*' \bar{\rho}_i'$$

for $i = 1, 2$.
By hypothesis $\bar{v}'^* \bar{u}_*' \bar{\rho}_1' = \bar{v}'^* \bar{u}_*' \bar{\rho}_2'$ and it follows that $\bar{v}'^* \bar{u}_*' (\bar{\rho}_1' - \bar{\rho}_2') = 0$. By 7.15, $\bar{\rho}_1' = \bar{\rho}_2'$.

Corollary 7.18 *The above $\sim$ defines an equivalence relation on the set of triplets*

$$\{(\bar{t} \setminus \bar{\delta} / \bar{s}) = (C \xleftarrow{\bar{t}} Z \xrightarrow{\bar{\delta}} X \xleftarrow{\bar{s}} A) | X, Z \in \mathscr{C}, \bar{s}, \bar{t} \in \bar{\mathscr{S}}, \bar{\delta} \in \bar{\mathbb{E}}(Z, X)\} \quad \forall A, C \in \mathscr{C}.$$

Proof: We can show transitivity by using 7.17.

Definition 7.19 *Let $A, C \in \mathscr{C}$ be a pair of objects. We define $\hat{\mathbb{E}}(C, A)$ as the quotient set of 7.18 by the equivalence relation.*
The equivalence class of $(\bar{t} \setminus \bar{\delta} / \bar{s}) = (C \xleftarrow{\bar{t}} Z \xrightarrow{\bar{\delta}} X \xleftarrow{\bar{s}} A)$ will be denoted by $[\bar{t} \setminus \bar{\delta} / \bar{s}] = [C \xleftarrow{\bar{t}} Z \xrightarrow{\bar{\delta}} X \xleftarrow{\bar{s}} A] \in \hat{\mathbb{E}}(C, A)$.
In conclusion

$$\hat{\mathbb{E}}(C, A) = \{[\bar{t} \setminus \bar{\delta} / \bar{s}] | X, Z \in \mathscr{C}, \bar{s}, \bar{t} \in \bar{\mathscr{S}}, \bar{\delta} \in \bar{\mathbb{E}}(Z, X)\}$$

Remark 7.20 *By $1(b)$ of 7.4, a morphism $\alpha \in \hat{\mathscr{C}}(A, B)$ can be expressed as*

$$\alpha = \bar{Q}(\bar{s})^{-1} \bar{Q}(\bar{f}) = \bar{Q}(\bar{g}) \bar{Q}(\bar{t})^{-1}$$

by some pairs of morphisms $A \xrightarrow{\bar{f}} B' \xleftarrow{\bar{s}} B$ amd $A \xleftarrow{\bar{t}} A' \xrightarrow{\bar{g}} B$ where $\bar{s}, \bar{t} \in \bar{\mathscr{S}}$.

Definition 7.21 *Let $[\bar{t} \setminus \bar{\delta} / \bar{s}] = [C \xleftarrow{\bar{t}} Z \xrightarrow{\bar{\delta}} X \xleftarrow{\bar{s}} A]$ be any element in $\hat{\mathbb{E}}(C, A)$.*

1. *Given any morphism $\alpha \in \hat{\mathscr{C}}(A, A')$, expressed as $\alpha = \bar{Q}(\bar{u})^{-1} \bar{Q}(\bar{a})$ for some $\bar{a} \in \bar{\mathscr{C}}(A, D)$ and $\bar{u} \in \bar{\mathscr{S}}(A', D)$ and given a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{\bar{a}'} & X' \\ {\scriptstyle \bar{s}}\uparrow & & \uparrow{\scriptstyle \bar{s}'} \\ A & \xrightarrow[\bar{a}]{} & D \end{array}$$

in $\bar{\mathscr{C}}$ with $\bar{s}' \in \bar{\mathscr{S}}$, we put

$$\alpha_* [\bar{t} \setminus \bar{\delta} / \bar{s}] = [\bar{t} \setminus \overline{a_*' \delta} / \overline{s' u}].$$

2. *Given any morphism $\gamma \in \hat{\mathscr{C}}(C', C)$, expressed as $\alpha = \bar{Q}(\bar{c}) \bar{Q}(\bar{v})^{-1}$ for some $\bar{c} \in \bar{\mathscr{C}}(E, C)$ and $\bar{v} \in \bar{\mathscr{S}}(E, C')$ and given a commutative diagram*

$$\begin{array}{ccc} E' & \xrightarrow{\bar{c}'} & Z \\ {\scriptstyle \bar{v}'}\downarrow & & \downarrow{\scriptstyle \bar{t}} \\ E & \xrightarrow[\bar{c}]{} & C \end{array}$$

in $\bar{\mathscr{C}}$ with $\bar{t}' \in \bar{\mathscr{S}}$, we put

$$\gamma^* [\bar{t} \setminus \bar{\delta} / \bar{s}] = [\overline{vt'} \setminus \overline{c'^* \delta} / \bar{s}].$$

Lemma 7.22 *The definition above is well defined.*

Proof: (1) Suppose we have the triplet $(\bar{t}\setminus\bar{\delta}/\bar{s}) = (C \xleftarrow{\bar{t}} Z \xrightarrow{\bar{\delta}} X \xleftarrow{\bar{s}} A)$ with $\bar{s},\bar{t}\in\bar{\mathscr{S}}$ and $\bar{\delta}\in\bar{\mathbb{E}}(Z,X)$. Given an expression $\alpha = \bar{Q}(\bar{u})^{-1}\bar{Q}(\bar{a})$ where $\bar{a}\in\bar{\mathscr{C}}(A,D)$ and $\bar{u}\in\bar{\mathscr{S}}(A',D)$, the equivalence class $[\bar{t}\setminus\overline{a'_*\delta}/\overline{s'u}]$ does not depend on the choice of the commutative square in (1) of 7.21.
In fact, given another commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\bar{a}''} & X'' \\ {\scriptstyle\bar{s}}\uparrow & & \uparrow{\scriptstyle\bar{s}''} \\ A & \xrightarrow[\bar{a}]{} & D \end{array}$$

with $\bar{s}''\in\bar{\mathscr{S}}$, by using $1(b)$ of 7.4, we can find two morphisms $\bar{s}_1\in\bar{\mathscr{S}}(X',X''')$ and $\bar{s}_2\in\bar{\mathscr{S}}(X'',X''')$ such that $\bar{s}_1\bar{a}' = \bar{s}_2\bar{a}''$ and $\bar{s}_1\bar{s}' = \bar{s}_2\bar{s}''$. It follows that $[\bar{t}\setminus\overline{a'_*\delta}/\overline{s'u}] = [\bar{t}\setminus\overline{a''_*\delta}/\overline{s''u}]$.
We now have to prove that $[\bar{t}\setminus\overline{a'_*\delta}/\overline{s'u}]$ does not depend on a choice of expressions of $\alpha\in\hat{\mathscr{C}}(A,A')$. By using $1(b)$ of 7.4, we can assume that $\alpha = \bar{Q}(\bar{u}_1)^{-1}\bar{Q}(\bar{a}_1) = \bar{Q}(\bar{u})^{-1}\bar{Q}(\bar{a})$ admits a commutative diagram

$$\begin{array}{ccccc} & & D_1 & & \\ & {\scriptstyle\bar{a}_1}\nearrow & \downarrow{\scriptstyle\bar{u}_2} & \nwarrow{\scriptstyle\bar{u}_1} & \\ A & \xrightarrow[\bar{a}]{} & D & \xleftarrow[\bar{u}]{} & A' \end{array}$$

where $\bar{u}_1,\bar{u}_2,\bar{u}\in\bar{\mathscr{S}}$.
By using $1(b)$ of 7.4, the morphisms $(X \xleftarrow{\bar{s}} A \xrightarrow{\bar{a}_1} D_1 \xrightarrow{\bar{a}_2} D)$ yield the commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow[\bar{a}_1]{} & D_1 & \xrightarrow[\bar{u}_2]{} & D \\ {\scriptstyle\bar{s}}\downarrow & & \downarrow{\scriptstyle\bar{s}''} & & \downarrow{\scriptstyle\bar{s}'} \\ X & \xrightarrow[\bar{a}'_1]{} & X'' & \xrightarrow[\bar{u}'_2]{} & X' \end{array}$$

with $\bar{a}$ over $A\to D$ and $\bar{a}'$ over $X\to X'$.

where all the vertical arrows belong to $\bar{\mathscr{S}}$. By using $1(a)$ of 7.4, we have that $\bar{u}'_2\in\bar{\mathscr{S}}$.
Since $\bar{a}' = \bar{u}'_2\bar{a}'_1$ and $\bar{s}'\bar{u} = \bar{u}'_2\bar{s}''\bar{u}_1$, it follows that $[\bar{t}\setminus\overline{a'_*\delta}/\overline{s'u}] = [\bar{t}\setminus\overline{a'_{1*}\delta}/\overline{s''u_1}]$ in $\hat{\mathbb{E}}(C,A)$.
We now have to show the indepencence from the choice of representatives of $[\bar{t}\setminus\bar{\delta}/\bar{s}]$. Suppose that $(\bar{t}''\setminus\bar{\rho}/\bar{s}'')$ is another choice of representatives.
It is sufficient to show when we are given a commutative diagram

$$\begin{array}{ccccccc} & & Z & \xrightarrow{\bar{\delta}} & X & & \\ & {\scriptstyle\bar{t}}\swarrow & \uparrow{\scriptstyle\bar{v}''} & & {\scriptstyle\bar{u}''}\downarrow & \nwarrow{\scriptstyle\bar{s}} & \\ C & \xleftarrow[\bar{t}'']{} & Z'' & \xrightarrow[\bar{\rho}]{} & X'' & \xleftarrow[\bar{s}'']{} & A \end{array}$$

where $\bar{v}'',\bar{u}''\in\bar{\mathscr{S}}$ such that $v''^{*}\overline{u''_*\delta} = \bar{\rho}$. Suppose we have a commutative square as in (1) of 7.21. Given the morphisms $X''\xleftarrow{\bar{u}''}X\xrightarrow{\bar{a}'}X'$, by using $1(b)$ of 7.4, we get the commutative square

$$\begin{array}{ccc} X & \xrightarrow{\bar{a}'} & X' \\ {\scriptstyle\bar{u}''}\downarrow & & \downarrow{\scriptstyle\bar{w}} \\ X'' & \xrightarrow[\bar{a}'']{} & W \end{array}$$

where $\bar{w}\in\bar{\mathscr{S}}$.
We can then construct the diagram

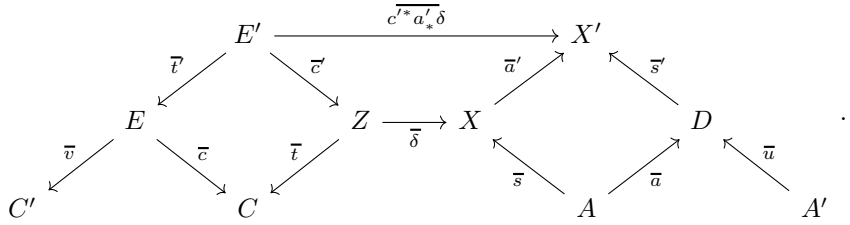
$$\begin{array}{ccccccc} & & Z & \xrightarrow{\bar{a}'_*\bar{\delta}} & X' & & \\ & {\scriptstyle\bar{t}}\swarrow & \downarrow{\scriptstyle\bar{v}''} & & {\scriptstyle\bar{w}}\downarrow & \nwarrow{\scriptstyle\overline{s'u}} & \\ C & \xleftarrow[\bar{t}'']{} & Z'' & \xrightarrow[\bar{a}''_*\bar{\rho}]{} & W & \xleftarrow[\overline{ws'u}]{} & A' \end{array}$$

where all morphisms except the two horizontal in the center belong to $\bar{\mathscr{I}}$. This shows $[\bar{t} \setminus \bar{a}'_* \bar{\delta} / \overline{s'u}] = [\bar{t}'' \setminus \bar{a}''_* \bar{\rho} / \overline{ws'u}]$.
(2) is dual to (1).

Remark 7.23 *Given* $[\bar{t} \setminus \bar{\delta}/\bar{s}] \in \hat{\mathbb{E}}(C, A)$, $\alpha = \bar{Q}(\bar{u})^{-1}\bar{Q}(\bar{a}) \in \hat{\mathscr{C}}(A, A')$ *and* $\gamma = \bar{Q}(\bar{c})\bar{Q}(\bar{v})^{-1} \in \hat{\mathscr{C}}(C', C)$ *as in* 7.21,

$$\gamma^* a_* [\bar{t} \setminus \bar{\delta}/\bar{s}] = \overline{[vt' \setminus c'^* \overline{a'_*} \delta / \overline{s'u}]}$$

is given by the diagram



Lemma 7.24 *Let* A, C *be a pair of objects of* $\mathscr{C}$. *Then*

1. *Given a pair of elements* $[\bar{t}_i \setminus \bar{\delta}_i/\bar{s}_i] \in \hat{\mathbb{E}}(C, A)$ $(i = 1, 2)$, *we define*

$$[\bar{t}_1 \setminus \bar{\delta}_1/\bar{s}_1] + [\bar{t}_2 \setminus \bar{\delta}_2/\bar{s}_2] = [\bar{t} \setminus \bar{\rho}_1 + \bar{\rho}_2/\bar{s}]$$

by using (1) *of* 7.17 *by taking common denominators such that* $[\bar{t}_i \setminus \bar{\delta}_i/\bar{s}_i] = [\bar{t} \setminus \bar{\rho}_i/\bar{s}]$ *for* $i = 1, 2$. *The sum is well-defined and therefore* $\hat{\mathbb{E}}(C, A)$ *is an abelian group with zero element* $[\bar{1} \setminus 0/\bar{1}]$.

2. *The map* $\bar{\mu}_{C,A} : \bar{\mathbb{E}}(C, A) \to \hat{\mathbb{E}}(C, A)$ *defined by*

$$\bar{\mu}_{C,A}(\bar{\delta}) = [\bar{1} \setminus \bar{\delta}/\bar{1}]$$

is a monomorphism of abelian groups.

Proof: The well-definedness of the sum follows from (2) of 7.17. Commutativity follows from the definition.
We now have to check the associativity, the zero element and the inverse.
Given $[\bar{t}_i \setminus \bar{\delta}_i/\bar{s}_i]$ $(i = 1, 2, 3)$, to show associativity we can take common denominators such that $[\bar{t}_i \setminus \bar{\delta}_i/\bar{s}_i] = [\bar{t} \setminus \bar{\rho}_i/\bar{s}]$ for $i = 1, 2, 3$ and then

$$([\bar{t}_1 \setminus \bar{\delta}_1/\bar{s}_1] + [\bar{t}_2 \setminus \bar{\delta}_2/\bar{s}_2]) + [\bar{t}_3 \setminus \bar{\delta}_3/\bar{s}_3] = [\bar{t} \setminus \bar{\rho}_1 + \bar{\rho}_2 + \bar{\rho}_3/\bar{s}] =$$

$$= [\bar{t}_1 \setminus \bar{\delta}_1/\bar{s}_1] + ([\bar{t}_2 \setminus \bar{\delta}_2/\bar{s}_2] + [\bar{t}_3 \setminus \bar{\delta}_3/\bar{s}_3]).$$

Given $[\bar{t} \setminus \bar{\delta}/\bar{s}] \in \hat{\mathbb{E}}(C, A)$, we observe that

$$[\bar{t} \setminus \bar{\delta}/\bar{s}] + [\bar{1} \setminus 0/\bar{1}] = [\bar{t} \setminus \bar{\delta}/\bar{s}] + [\bar{t} \setminus 0/\bar{s}] = [\bar{t} \setminus \bar{\delta}/\bar{s}].$$

Hence $[\bar{1} \setminus 0/\bar{1}]$ is the zero element.
It follows that the inverse element of an element $[\bar{t} \setminus \bar{\delta}/\bar{s}] \in \hat{\mathbb{E}}(C, A)$ is $[\bar{t} \setminus \overline{-\delta}/\bar{s}]$.
(2) Additivity follows from the definition and monomorphicity follows from 7.15.

Proposition 7.25 *The above definitions give a biadditive functor* $\hat{\mathbb{E}} : \hat{\mathscr{C}}^{op} \times \hat{\mathscr{C}} \to Ab$ *and a natural transformation* $\bar{\mu} : \bar{\mathbb{E}} \Rightarrow \hat{\mathbb{E}}(\bar{Q}^{op} \times \bar{Q})$.

Proof: The first step is to show that $\hat{\mathbb{E}}$ is a biadditive functor. By 7.23, it follows that $\gamma^* \alpha_* = \alpha_* \gamma^*$ for any morphisms α, γ in $\hat{\mathscr{C}}$. We first show that given a morphism $\alpha \in \hat{\mathscr{C}}(A, A')$ α_* is a group homomorphism. Let α be expressed by $\alpha = \bar{Q}(\bar{u})^{-1}\bar{Q}(\bar{a})$ where $\bar{a} \in \bar{\mathscr{C}}(A, D)$ and $\bar{u} \in \bar{\mathscr{I}}(A, D)$.
We have to show that

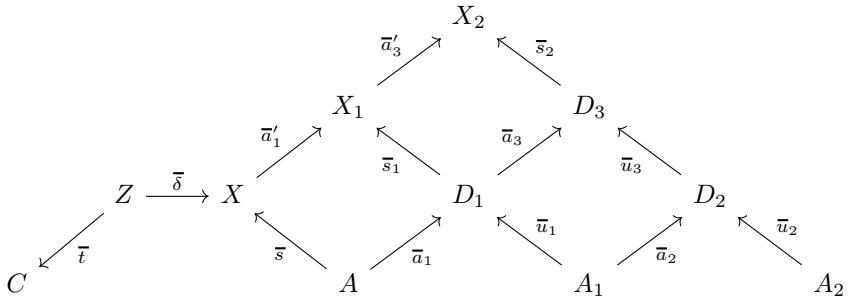
$$\alpha_*([\bar{t} \setminus \bar{\delta}_1/\bar{s}] + [\bar{t}' \setminus \bar{\delta}_2/\bar{s}']) = \alpha_*([\bar{t} \setminus \bar{\delta}_1/\bar{s}]) + \alpha_*([\bar{t}' \setminus \bar{\delta}_2/\bar{s}']).$$

By taking common denominators, we can assume that $t = t', s = s'$, $[\bar{t} \setminus \bar{\delta}_i/\bar{s}] = [C \xleftarrow{\bar{t}} Z \xrightarrow{\bar{\delta}_i} X \xleftarrow{\bar{s}} A] \in \hat{\mathbb{E}}(C, A)$ hold for $i = 1, 2$.
By using $1(b)$ of 7.4, there exist morphisms $\bar{a}' : X \to X'$ and $\bar{s}' : D \to X'$ such that $\overline{a's} = \overline{s'a}$. It follows that

$$\alpha_*([\bar{t} \setminus \bar{\delta}_1/\bar{s}] + [\bar{t} \setminus \bar{\delta}_2/\bar{s}]) = \alpha_*([\bar{t} \setminus \delta_1 \overline{+ \delta_2}/\bar{s}] = [[\bar{t} \setminus a'_*(\overline{\delta_1 + \delta_2})/\overline{s'u}] =$$

$$= [\bar{t} \setminus \overline{a'_*\delta_1}/\overline{s'u}] + [\bar{t} \setminus \overline{a'_*\delta_2}/\overline{s'u}] = \alpha_*([\bar{t} \setminus \bar{\delta}_1/\bar{s}]) + \alpha_*([\bar{t} \setminus \bar{\delta}_2/\bar{s}]).$$

By duality, γ^* is a group homomorphism.
We now that $(\beta\alpha)_* = \beta_*\alpha_*$ for any $\alpha \in \hat{\mathscr{C}}(A, A_1)$ and $\beta \in \hat{\mathscr{C}}(A_1, A_2)$. Let α and β be expressed by $\alpha = \bar{Q}(\bar{u}_1)^{-1}\bar{Q}(\bar{a}_1)$ and $\beta = \bar{Q}(\bar{u}_2)^{-1}\bar{Q}(\bar{a}_2)$.
Given $[\bar{t} \setminus \bar{\delta}/\bar{s}] = [C \xleftarrow{\bar{t}} Z \xrightarrow{\bar{\delta}} X \xleftarrow{\bar{s}} A] \in \hat{\mathbb{E}}(C, A)$ and by using $1(b)$ of 7.4, we obtain the following diagram



in which $\bar{u}_i, \bar{s}_i \in \bar{\mathscr{S}}$ for all i. Then it follows that

$$(\beta\alpha)_*([\bar{t} \setminus \bar{\delta}/\bar{s}]) = (\bar{Q}(\overline{u_3u_2})^{-1}\bar{Q}(\overline{a_3a_1}))_*([\bar{t} \setminus \bar{\delta}/\bar{s}]) =$$

$$= [\bar{t} \setminus (a'_3)_*\overline{(a'_1)_*\delta}/s_2\overline{u_3u_2}] =$$

$$= \beta_*([\bar{t} \setminus \overline{(a'_1)_*\delta}/\overline{s_1u_1}]) = \beta_*\alpha_*([\bar{t} \setminus \bar{\delta}/\bar{s}]).$$

Dually we show that $(\beta\alpha)^* = \alpha^*\beta^*$.
For any $\alpha_i = \bar{Q}(\bar{u}_i)^{-1}\bar{Q}(\bar{a}_i) \in \hat{\mathscr{C}}(A, A')$ for $i = 1, 2$, we prove that $(\alpha_1 + \alpha_2)_* = (\alpha_1)_* + (\alpha_2)_*$.
By taking common denominators, we can assume $\bar{u}_1 = \bar{u}_2$ where $u_1 : A' \to \underline{D}$. Given $[\bar{t} \setminus \bar{\delta}/\bar{s}] \in \hat{\mathbb{E}}(C, A)$, by $1(b)$ of 7.4, there exist morphisms $\bar{a}'_1 : X \to X', \bar{s}_i : D \to X'$ such that $\overline{a'_i s} = \overline{s_i a_i}$ for $i = 1, 2$.
By taking common denominators, we may assume that $s_1 = s_2$. It follows that

$$((\alpha_1)_* + (\alpha_2)_*)([\bar{t} \setminus \bar{\delta}/\bar{s}]) = (\alpha_1)_*([\bar{t} \setminus \bar{\delta}/\bar{s}]) + (\alpha_2)_*([\bar{t} \setminus \bar{\delta}/\bar{s}]) =$$

$$= [\bar{t} \setminus \overline{(a'_1)_*\delta}/\overline{s_1u_1}] + [\bar{t} \setminus \overline{(a'_2)_*\delta}/\overline{s_1u_1}] = [\bar{t} \setminus (a'_1 \overline{+ a'_2})_*\delta/\overline{s_1u_1}] =$$

$$= (\bar{Q}(\bar{u}_1)^{-1}\bar{Q}((a_1 \overline{+ a_2}))_*([\bar{t} \setminus \bar{\delta}/\bar{s}]) = (\alpha_1 + \alpha_2)_*([\bar{t} \setminus \bar{\delta}/\bar{s}]).$$

Dually, we obtain $(\alpha_1 + \alpha_2)^* = (\alpha_1)^* + (\alpha_2)^*$. Hence $\hat{\mathbb{E}}$ is a biadditive functor.
We now want to prove $\bar{\mu}$ is a natural transformation.
By 2 of 7.24, we only need to check the naturality of $\bar{\mu}$. Let $\bar{a} : A \to A'$ be an arbitrary morphism in $\bar{\mathscr{C}}$. Then for any object $C \in \mathscr{C}$ and any $\bar{\delta} \in \bar{\mathbb{E}}(C, A)$ we have

$$\hat{\mathbb{E}}(C, \bar{Q}(\bar{a})(\bar{\mu}_{C,A}(\bar{\delta})) = \hat{\mathbb{E}}(C, \bar{Q}(\bar{a})([\bar{1} \setminus \bar{\delta}/\bar{1}]) =$$

$$= [\bar{1} \setminus \overline{a_*\delta}/\bar{1}] = \bar{\mu}_{C,A'}(\overline{a_*\delta}) = \bar{\mu}_{C,A'}(\bar{\mathbb{E}}(C, \bar{a})(\bar{\delta})).$$

Dually, $\hat{\mathbb{E}}(\bar{Q}(\bar{c}), A)\bar{\mu}_{C,A} = \mu_{C',A}\bar{\mathbb{E}}(\bar{c}, A)$ is also true for any morphisms $\bar{c} : C' \to C$ in $\bar{\mathscr{C}}$. Hence $\bar{\mu}$ is a natural transformation.

We now construct $\hat{\mathbf{s}}$. Suppose $\mathscr{S}$ satisfies $(M0)$ and $1(a)$, $1(b)$, $1(c)$ of 7.4 and $\mathscr{S} = p^{-1}(\bar{\mathscr{S}})$.

Proposition 7.26 *Let A, C be a pair of objects of $\mathscr{C}$. Let $(\overline{t}_i \setminus \overline{\delta}_i/\overline{s}_i) = (C \xleftarrow{\overline{t}_i} Z_i \xrightarrow{\overline{\delta}_i} X_i \xleftarrow{\overline{s}_i} A)$ be a triplet for $i = 1, 2$ where $\overline{s}_i \in \overline{\mathscr{S}}(A, X_i)$, $\overline{t}_i \in \overline{\mathscr{S}}(Z_i, C)$, $\overline{\delta}_i \in \overline{\mathbb{E}}(Z_i, X_i)$ with $\boldsymbol{s}(\delta_i) = [X_i \xrightarrow{x_i} Y_i \xrightarrow{y_i} Z_i]$. If $[\overline{t}_1 \setminus \overline{\delta}_1/\overline{s}_1] = [\overline{t}_2 \setminus \overline{\delta}_2/\overline{s}_2]$ holds in $\hat{\mathbb{E}}(C, A)$, then*

$$[A \xrightarrow{Q(x_1 s_1)} Y_1 \xrightarrow{Q(t_1 y_1)} C] = [A \xrightarrow{Q(x_2 s_2)} Y_2 \xrightarrow{Q(t_2 y_2)} C]$$

holds as sequences in $\hat{\mathscr{C}}$.

Proof: By recalling how the equivalence relation is defined, it is sufficient to prove

$$[A \xrightarrow{Q(x_1 s_1)} Y_1 \xrightarrow{Q(t_1 y_1)} C] = [A \xrightarrow{Q(x_2 s_2)} Y_2 \xrightarrow{Q(t_2 y_2)} C]$$

in the case where there exist $\overline{u} \in \overline{\mathscr{S}}(X_1, X_2)$, $\overline{v} \in \overline{\mathscr{S}}(Z_2, Z_1)$ such that $\overline{s}_2 = \overline{u}\overline{s}_1$, $\overline{t}_2 = \overline{t}_1\overline{v}$ and $u_*\overline{v^*}\overline{\delta}_1 = \overline{\delta}_2$.
Recalling the definition of $\overline{\mathbb{E}}$, we can find $u' \in \mathscr{S}(X_2, X_2')$ such that $u'_*(u_* v^* \delta_1) = u'_* \delta_2$.
Therefore we can replace u, s_2, δ_2 by $u'u, u's_2, u'_*\delta_2$ and this let us assume that $u_* v^* \delta_1 = \delta_2$ holds from the beginning of the proof.
Under this condition, if $\mathbf{s}(u_*\delta_1) = [X_2 \xrightarrow{x_3} Y_3 \xrightarrow{y_3} Z_1]$, there exist $w_1, w_2 \in \bar{\mathscr{S}}$ such that the diagram

$$\begin{array}{ccccc}
X_1 & \xrightarrow{\overline{x}_1} & Y_1 & \xrightarrow{\overline{y}_1} & Z_1 \\
{\scriptstyle \overline{u}}\downarrow & & \downarrow{\scriptstyle \mathbf{w}_1} & & \| \\
X_2 & \xrightarrow{\overline{x}_3} & Y_3 & \xrightarrow{\overline{y}_3} & Z_1 \\
\| & & {\scriptstyle \mathbf{w}_2}\uparrow & & \uparrow{\scriptstyle \overline{v}} \\
X_2 & \xrightarrow[\overline{x}_2]{} & Y_2 & \xrightarrow[\overline{y}_2]{} & Z_2
\end{array}$$

is commutative in $\overline{\mathscr{C}}$ by $1(c)$ of 7.4.
Then by taking the isomorphism $\alpha = \bar{Q}(w_2)^{-1}\bar{Q}(w_1)$, we have the commutative diagram

$$\begin{array}{ccccc}
 & & Y_1 & & \\
 & {\scriptstyle Q(x_1 s_1)} \nearrow & \downarrow{\scriptstyle \alpha \cong} & \searrow {\scriptstyle Q(t_1 y_1)} & \\
A & & & & C \\
 & {\scriptstyle Q(x_2 s_2)} \searrow & & \swarrow {\scriptstyle Q(t_2 y_2)} & \\
 & & Y_2 & &
\end{array}$$

in $\hat{\mathscr{C}}$, which concludes the proof.

Definition 7.27 *Let $[\overline{t} \setminus \overline{\delta}/\overline{s}] = [C \xleftarrow{\overline{t}} Z \xrightarrow{\overline{\delta}} X \xleftarrow{\overline{s}} A] \in \hat{\mathbb{E}}(C, A)$ be any $\hat{\mathbb{E}}-$triangle. Suppose that $\boldsymbol{s}(\delta) = [X \xrightarrow{x} Y \xrightarrow{y} Z]$, then we define*

$$\hat{\boldsymbol{s}}([\overline{t} \setminus \overline{\delta}/\overline{s}]) = [A \xrightarrow{Q(xs)} Y \xrightarrow{Q(ty)} C]$$

in $\hat{\mathscr{C}}$.

Remark 7.28 *The previous definition is well defined by 7.26.*

Lemma 7.29 *Let $f \in \mathscr{C}(A, B)$ be a morphism. Then*

1. *If f is an $\boldsymbol{s}-$inflation and $Cone(f) \in N_{\mathscr{S}}$ then $f \in \mathscr{S}$.*

2. *If f is an $\boldsymbol{s}-$deflation with $CoCone(f) \in N_{\mathscr{S}}$ then $f \in \mathscr{S}$.*

Proof: (1) Recall that $A \xrightarrow{1} A \xrightarrow{0} 0 \xrightarrow{0}$ is an $\mathbf{s}-$*triangle*. Then given $A \xrightarrow{f} B \xrightarrow{g} N \xrightarrow{\delta}$, by $1(c)$ of 7.4, there exists $\overline{u} \in \bar{\mathscr{S}}(A, B)$ such that the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\overline{1}} & A & \longrightarrow & 0 \\ {\scriptstyle \overline{1}}\downarrow & & {\scriptstyle \overline{u}}\downarrow & & \downarrow \\ A & \xrightarrow[\overline{f}]{} & B & \xrightarrow[\overline{g}]{} & N \end{array}$$

is commutative.
This implies that $\bar{f} = \overline{u} \in \bar{\mathscr{S}}$, hence $f \in \mathscr{S}$ follows.

Proposition 7.30 *Let $[\overline{t} \setminus \overline{\delta}/\overline{s}] = [C \xleftarrow{\overline{t}} Z \xrightarrow{\overline{\delta}} X \xleftarrow{\overline{s}} A] \in \hat{\mathbb{E}}(C, A)$ be any $\hat{\mathbb{E}}-$triangle and choose its representative.*
Then, given $\boldsymbol{s}(\delta) = [X \xrightarrow{x} Y \xrightarrow{y} Z]$,

1. *The sequence*

$$\hat{\mathscr{C}}(-, A) \xrightarrow{Q(xs)\circ -} \hat{\mathscr{C}}(-, Y) \xrightarrow{Q(ty)\circ -} \hat{\mathscr{C}}(-, C) \xrightarrow{[\overline{t}\setminus\overline{\delta}/\overline{s}]_{\#}} \hat{\mathbb{E}}(-, A)$$

is exact.

2. *The sequence*

$$\hat{\mathscr{C}}(C, -) \xrightarrow{-\circ Q(ty)} \hat{\mathscr{C}}(Y, -) \xrightarrow{-\circ Q(xs)} \hat{\mathscr{C}}(A, -) \xrightarrow{[\overline{t}\setminus\overline{\delta}/\overline{s}]^{\#}} \hat{\mathbb{E}}(C, -)$$

is exact.

Proof:
(1): Given the commutative diagram

$$\begin{array}{ccccccc} \hat{\mathscr{C}}(-, A) & \xrightarrow{Q(xs)\circ -} & \hat{\mathscr{C}}(-, Y) & \xrightarrow{Q(ty)\circ -} & \hat{\mathscr{C}}(-, C) & \xrightarrow{[\overline{t}\setminus\overline{\delta}/\overline{s}]_{\#}} & \hat{\mathbb{E}}(-, A) \\ {\scriptstyle Q(s)\circ -}\downarrow\cong & & \| & & {\scriptstyle Q(t)\circ -}\downarrow\cong & & \cong\downarrow{\scriptstyle Q(s)_*} \\ \hat{\mathscr{C}}(-, X) & \xrightarrow[Q(x)\circ -]{} & \hat{\mathscr{C}}(-, Y) & \xrightarrow[Q(y)\circ -]{} & \hat{\mathscr{C}}(-, Z) & \xrightarrow[[\overline{1}\setminus\overline{\delta}/\overline{1}]_{\#}]{} & \hat{\mathbb{E}}(-, X) \end{array},$$

we need to show the exactness of the bottom row.
The first step is to show the exactness of

$$\hat{\mathscr{C}}(P, X) \xrightarrow{Q(x)\circ -} \hat{\mathscr{C}}(P, Y) \xrightarrow{Q(y)\circ -} \hat{\mathscr{C}}(P, Z)$$

for any $P \in \mathscr{C}$.
Given $\beta \in \hat{\mathscr{C}}(P, Y)$, we can express β as $\beta = \bar{Q}(\bar{f})\bar{Q}(\overline{s})^{-1}$ where $\bar{f} \in \bar{\mathscr{C}}(P', Y)$ and $\overline{s} \in \bar{\mathscr{S}}(P', P)$. If $Q(y)\beta = 0$ then $\bar{Q}(\overline{y}\bar{f}) = 0$, therefore there exists $\overline{s}' \in \bar{\mathscr{S}}(P'', P')$ such that $\overline{y}\bar{f}\overline{s}' = 0$ in $\bar{\mathscr{C}}$.
This implies that there exist $N \in N_{\mathscr{S}}$ and $i \in \mathscr{C}(P'', N),\ j \in \mathscr{C}(N, Z)$ such that $yfs' = ji$.
Now given $\mathbf{s}(j^*\delta) = [X \xrightarrow{x'} Y' \xrightarrow{y'} N]$, by $w(s)^{op}$ we have a morphism of $\mathbb{E}-$*triangles*

$$\begin{array}{ccccccc} X & \xrightarrow{x'} & Y' & \xrightarrow{y'} & N & \xrightarrow{j^*\delta} & \\ \| & & {\scriptstyle k}\downarrow & & {\scriptstyle j}\downarrow & & \\ X & \xrightarrow[x]{} & Y & \xrightarrow[y]{} & Z & \xrightarrow[\delta]{} & \end{array}$$

such that

$$Y' \xrightarrow{\begin{pmatrix} -y' \\ k \end{pmatrix}} N \oplus Y \xrightarrow{\begin{pmatrix} j & y \end{pmatrix}} Z \xrightarrow{x'_*\delta}$$

is an $\mathbb{E}-$*triangle*.
Since $\mathscr{C}(P'',Y')\to\mathscr{C}(P'',N\oplus Y)\to\mathscr{C}(P'',Z)$ is exact, there is $p\in\mathscr{C}(P'',Y')$ such that

$$\begin{pmatrix}-y'\\ k\end{pmatrix}p=\begin{pmatrix}-i\\ fs'\end{pmatrix}.$$

By using 7.29, we have $\overline{x}'\in\overline{\mathscr{I}}$. By taking $\alpha=\overline{Q}(\overline{x}')^{-1}Q(p)Q(ss')^{-1}$, it satisfies $\beta=Q(x)\alpha$.
The second step is to prove the exactness of

$$\hat{\mathscr{C}}(P,Y)\xrightarrow{Q(y)\circ-}\hat{\mathscr{C}}(P,Z)\xrightarrow{[\overline{1}\setminus\overline{\delta}/\overline{1}]_{\#}}\hat{\mathbb{E}}(P,X)$$

for any $P\in\mathscr{C}$.
Let $\beta\in\hat{\mathscr{C}}(P,Z)$ be a morphism such that $[\overline{1}\setminus\overline{\delta}/\overline{1}]_{\#}(\beta)=\beta^*[\overline{1}\setminus\overline{\delta}/\overline{1}]=0$.
Let $\overline{f}\in\overline{\mathscr{C}}(P',Z)$ and $\overline{s}\in\overline{\mathscr{I}}(P',P)$ two morphisms such that $\beta=\overline{Q}(\overline{f})\overline{Q}(\overline{s})^{-1}$.
Then $\beta^*[\overline{1}\setminus\overline{\delta}/\overline{1}]=[\overline{s}\setminus\overline{f^*\delta}/\overline{1}]$ holds by definition.
By recalling 7.15, $\overline{f^*\delta}=0$ and this implies $t^*(f^*\delta)=0$ for some $t\in\mathscr{I}(P'',P')$.
By the exact sequence $\mathscr{C}(P'',Y)\to\mathscr{C}(P'',Z)\to\mathbb{E}(P'',X)$, there exists $g\in\mathscr{C}(P'',Y)$ such that $yg=ft$.
By taking $\alpha=\overline{Q}(\overline{g})\overline{Q}(\overline{st})^{-1}\in\hat{\mathscr{C}}(P,Y)$, we have that $Q(y)\alpha=\beta$ in $\hat{\mathscr{C}}(P,Z)$ by construction.
(2) is shown with a dual argument to (1).

Proof of 7.4(1) : We want to verify the weakly extriangulated category axiom of 6.40.
By recalling 7.30 and 6.24, we verify $(ET3)$ and by its dual $(ET3)^{op}$.
$(ET2)$ and $(ET1)$ follow from definition.

We just need to check the $w(s)$. Let $[\overline{t}\setminus\overline{\delta}/\overline{s}]=[C\xleftarrow{\overline{t}}Z\xrightarrow{\overline{\delta}}X\xleftarrow{\overline{s}}A]\in\hat{\mathscr{E}}(C,A)$ be any $\mathbb{E}-$*triangle* and let $\alpha\in\hat{\mathscr{C}}(A,A')$ be a morphism. Let α be expressed by $\alpha=\overline{Q}(\overline{a})^{-1}\overline{Q}(\overline{a})$ where $\overline{a}\in\overline{\mathscr{C}}(A,D)$ and $\overline{u}\in\overline{\mathscr{I}}(A',D)$.
By definition $\alpha_*[\overline{t}\setminus\overline{\delta}/\overline{s}]=[\overline{t}\setminus\overline{a'_*\delta}/\overline{s'u}]$, given by the commutative diagram

$$\begin{array}{ccc}A&\xrightarrow{\overline{a}}&D\\ {\scriptstyle\overline{s}}\downarrow&&\downarrow{\scriptstyle\overline{s}'}\\ X&\xrightarrow[\overline{a}']{}&X'\end{array}$$

in $\overline{\mathscr{C}}$ with $\overline{s}'\in\overline{\mathscr{I}}(D,X')$.
By $w(s)$ for $(\mathscr{C},\mathbb{E},\mathbf{s})$ we obtain a morphism of $\mathbb{E}-$*triangles*

$$\begin{array}{ccccccc}X&\xrightarrow{x}&Y&\xrightarrow{y}&Z&\xrightarrow{\delta}&\\ {\scriptstyle a'}\downarrow&&{\scriptstyle b}\downarrow&&\|&&\\ X'&\xrightarrow[x']{}&Y'&\xrightarrow[y']{}&Z&\xrightarrow{a'_*\delta}&\end{array}$$

which makes

$$X\xrightarrow{\begin{pmatrix}x\\ a'\end{pmatrix}}Y\oplus X'\xrightarrow{\begin{pmatrix}b&-x'\end{pmatrix}}Y'\xrightarrow{y'^*\delta}$$

an $\mathbb{E}-$*triangle*.
By definition, $\hat{\mathbf{s}}([\overline{t}\setminus\overline{\delta}/\overline{s}])=[A\xrightarrow{Q(xs)}Y\xrightarrow{Q(ty)}C]$ and $\hat{\mathbf{s}}(\alpha_*(\overline{t}\setminus\overline{\delta}/\overline{s}])=[A'\xrightarrow{Q(x's'u)}Y'\xrightarrow{Q(ty')}C]$.
Let x'' be defined as $x''=x's'u$.
We have the following morphism of $\hat{\mathbb{E}}-$*triangles*

$$\begin{array}{ccccccc}A&\xrightarrow{Q(xs)}&Y&\xrightarrow{Q(ty)}&C&\xrightarrow{[\overline{t}\setminus\overline{\delta}/\overline{s}]}&\\ {\scriptstyle\alpha}\downarrow&&\downarrow{\scriptstyle Q(b)}&&\|&&\\ A'&\xrightarrow[Q(x'')]{}&Y'&\xrightarrow[Q(ty')]{}&C&\xrightarrow{\alpha_*[\overline{t}\setminus\overline{\delta}/\overline{s}]}&\end{array}.$$

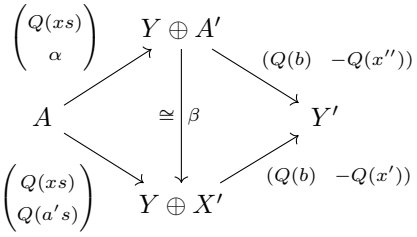
Since $Q(ty')^*([\overline{t} \setminus \overline{\delta}/\overline{s}]) = [\overline{1} \setminus \overline{y'^*\delta}/\overline{s}]$, to conclude we have to show

$$A \xrightarrow{\begin{pmatrix} Q(xs) \\ a \end{pmatrix}} Y \oplus A' \xrightarrow{\begin{pmatrix} Q(b) & -Q(x'') \end{pmatrix}} Y' \xrightarrow{[\overline{1}\setminus\overline{y'^*\delta}/\overline{s}]}.$$

Since $y'^*\delta$ is an $\mathbb{E}-triangle$, we have by definition

$$\hat{\mathbf{s}}([\overline{1} \setminus \overline{y'^*\delta}/\overline{s}) = [A \xrightarrow{Q(\begin{pmatrix} x \\ a' \end{pmatrix} s)} Y \oplus X' \xrightarrow{Q(\begin{pmatrix} b & -x' \end{pmatrix})} Y'.]$$

By the commutative diagram



where $\beta = 1_Y \oplus Q(s'u)$, we conclude the proof.
(2): We have to show the universal property of the functor $(Q, \mu) : (\mathscr{C}, \mathbb{E}, \mathbf{s}) \to (\hat{\mathscr{C}}, \hat{\mathbb{E}}, \hat{\mathbf{s}})$. Let $(\mathscr{D}, \mathbb{F}, \mathbf{t})$ be a weakly extriangulated category.
(a): Suppose we have an exact functor $(F, \phi) : (\mathscr{C}, \mathbb{E}, \mathbf{s}) \to (\mathscr{D}, \mathbb{F}, \mathbf{t})$ that sends any $s \in \mathscr{S}$ to an isomorphism. It follows that F uniquely factors through Q via an additive functor $\hat{F} : \hat{\mathscr{C}} \to \mathscr{D}$. To have an exact functor $\hat{F}$ in the extriangulated sense we have to define a natural transformation $\hat{\phi} : \mathbb{E} \to \mathbb{F}(\hat{F}^{op} \times \hat{F})$.
Let $[\overline{t} \setminus \overline{\delta}/\overline{s}] = [C \xleftarrow{\overline{t}} Z \xrightarrow{\overline{\delta}} X \xleftarrow{\overline{s}} A]$ be an element in $\hat{\mathbb{E}}(C, A)$ with $\mathbf{s}(\delta) = [X \xrightarrow{x} Y \xrightarrow{y} Z]$.
By hypothesis $F(s)$ and $F(t)$ are isomorphims so we may define $\hat{\phi}([\overline{t}\setminus\overline{\delta}/\overline{s}]) := (F(t)^{-1})^*(F(s)^{-1})_*(\phi(\delta))$.
The additivity and the well definition of $\hat{\phi}_{C,A} : \hat{\mathbb{E}}(C, A) \to \mathbb{F}(F(C), F(A))$ follow from 7.18.
We may also observe that

$$\mathbf{t}(\hat{\phi}([\overline{t} \setminus \overline{\delta}/\overline{s}])) = [FA \xrightarrow{F(xs)} FY \xrightarrow{F(ty)} FC] =$$

$$= [\hat{F}QA \xrightarrow{\hat{F}Q(xs)} \hat{F}QY \xrightarrow{\hat{F}Q(ty)} \hat{F}QC]$$

by 6.28.
We now need to check $\hat{\phi}$ is a natural transformation.
Let $\alpha : \bar{Q}(\overline{u})^{-1}\bar{Q}(\overline{a}) : A \to A'$ be a morphism and the same commutative square at the start of the proof of (1).

$$\hat{\phi}(\alpha_*[\overline{t} \setminus \overline{\delta}/\overline{s}]) = \hat{\phi}([\overline{t} \setminus \overline{a'_*\delta}/\overline{s'u}]) = (F(t)^{-1})^*(F(u)^{-1}F(s')^{-1})_*(\phi(a'_*\delta)) =$$

$$= (F(t)^{-1})^*(F(u)^{-1}F(s')^{-1}F(a')_*(\phi(\delta)) = (F(t)^{-1})^*(F(u)^{-1}F(a)F(s)^{-1})_*(\phi(\delta)) = \hat{F}(\alpha)_*\hat{\phi}([\overline{t}\setminus\overline{\delta}/\overline{t}]).$$

With a similar reasoning it follows that $\hat{\phi}(\beta^*[\overline{t} \setminus \overline{\delta}/\overline{s}]) = \hat{F}(\beta)^*\hat{\phi}([\overline{t} \setminus \overline{\delta}/\overline{s}])$ for any $\beta \in \hat{\mathscr{C}}(C', C)$.
We also have that, given $\delta \in \mathbb{E}(C, A)$,

$$\hat{\phi}_{C,A}(\mu_{C,A}(\delta)) = \hat{\phi}_{C,A}([\overline{1} \setminus \overline{\delta}/\overline{1}]) = \phi(\delta).$$

This implies $(\hat{\phi}(Q^{op} \times Q))\mu = \phi$ and from this $(\hat{F}, \hat{\phi})(Q, \mu) = (F, \phi)$. The above condition shows the uniqueness of $\hat{\phi}$.
(b) : Suppose we have $\eta : (F, \phi) \Rightarrow (G, \psi)$ be a natural transformation of exact functors between $(F, \phi), (G, \psi) : (\mathscr{C}, \mathbb{E}, \mathbf{s}) \to (\mathscr{D}, \mathbb{F}, \mathbf{t})$ which send $s \in \mathscr{S}$ to isomorphisms.
Thanks to the universal property of localization for additive categories, there exists a unique natural

transformation $\hat{\eta} : \hat{F} \Rightarrow \hat{G}$ of additive functors such that $\hat{\eta}Q = \eta$. It follows that $\hat{\eta}_C = \eta_C$ for any $C \in \mathscr{C}$.
Moreover by the naturality of η, we have

$$(\hat{\eta}_A)_* \hat{\phi}_{C,A}([\overline{t} \setminus \overline{\delta}/\overline{s}]) = (\hat{\eta}_C)^* \hat{\psi}_{C,A}([\overline{t} \setminus \overline{\delta}/\overline{s}])$$

for any $[\overline{t} \setminus \overline{\delta}/\overline{s}] \in \hat{\mathbb{E}}(C, A)$.
The proof of (3) requires another lemma.

Lemma 7.31 *Let α be a morphism in $\hat{\mathscr{C}}$. Then*

1. *α is an $\hat{\mathbb{E}} - inflation$ in $\hat{\mathbb{C}}$ if and only if $\alpha = \beta Q(f)\gamma$ for some $\mathbb{E} - inflation$ f in $\mathscr{C}$ and for some isomorphisms β, γ in $\hat{\mathscr{C}}$.*

2. *α is an $\hat{\mathbb{E}} - deflation$ in $\hat{\mathbb{C}}$ if and only if $\alpha = \beta Q(f)\gamma$ for some $\mathbb{E} - deflation$ f in $\mathscr{C}$ and for some isomorphisms β, γ in $\hat{\mathscr{C}}$.*

Proof:
(1) By recalling the definition of $\hat{\mathbf{s}}$, any $\hat{\mathbb{E}} - inflation$ can be expressed as $\beta Q(f)\gamma$ for some $\mathbb{E} - inflation$ f and isomorphisms β, γ in $\hat{\mathscr{C}}$.
We now need to show the converse.
Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\delta}$ be an $\mathbb{E} - triangle$ and isomorphisms $\gamma \in \hat{\mathscr{C}}(A, X), \beta \in \hat{\mathscr{C}}(Y, B)$ in $\hat{\mathscr{C}}$.
We can express γ^{-1} as $\gamma^{-1} = \bar{Q}(\overline{t})^{-1}\bar{Q}(\overline{e})$ with $\overline{e} \in \overline{\mathscr{C}}(X, X')$ and $\overline{t} \in \overline{\mathscr{F}}(A, X')$.
We observe that $\bar{Q}(\overline{e}) \in Iso(\hat{\mathscr{C}})$ and $\gamma = \bar{Q}(\overline{e})^{-1}\bar{Q}(\overline{t})$.
From $\mathbf{s}(e_*\delta) = [X' \xrightarrow{f'} Y' \xrightarrow{g'} Z]$, we obtain a morphism of $\mathbb{E} - triangles$

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{\delta} & \\ {\scriptstyle e}\downarrow & & {\scriptstyle d}\downarrow & & \| & & \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z & \xrightarrow{a'_*\delta} & \end{array}$$

where $d \in \mathscr{C}(Y, Y')$ since $\mathbf{s}$ is a realization of $(\mathscr{C}, \mathbb{E}, \mathbf{s})$.
(Q, μ) is an exact functor so

$$\begin{array}{ccccccc} X & \xrightarrow{Q(f)} & Y & \xrightarrow{Q(g)} & Z & \xrightarrow{\mu_{Z,X}(\delta)} & \\ {\scriptstyle Q(e)}\downarrow & & \downarrow{\scriptstyle Q(d)} & & \| & & \\ X' & \xrightarrow[Q(f')]{} & Y' & \xrightarrow[Q(g')]{} & Z & \xrightarrow[\mu_{Z,X'}(e_*\delta)]{} & \end{array}$$

is a morphism of $\hat{\mathbb{E}} - triangles$.
It follows that $Q(d) \in Iso(\hat{\mathscr{C}})$.
By the definiton of $\hat{\mathbf{s}}$, $\hat{\mathbf{s}}([\overline{1} \setminus \overline{e_*\delta}/\overline{t}]) = [A \xrightarrow{Q(f't)} Y' \xrightarrow{Q(g')} Z]$.
Given $\lambda = \beta Q(d)^{-1}$, we have a commutative diagram

$$\begin{array}{ccccc} & & Y' & & \\ & {\scriptstyle Q(f't)}\nearrow & \| & \searrow{\scriptstyle Q(g')} & \\ A & & {\scriptstyle \lambda\cong} & & Z \\ & {\scriptstyle \beta Q(f)\gamma}\searrow & \downarrow & \nearrow{\scriptstyle Q(g')\lambda^{-1}} & \\ & & B & & \end{array}$$

in $\hat{\mathscr{C}}$ and this implies that $\beta Q(f)\gamma$ is an $\hat{\mathbb{E}} - inflation$. (2) is proved dually.

We now prove the last point of 7.4:
Proof : Suppose now $\overline{\mathscr{F}}$ satisfies $(MR1)$ of 7.4.
Let $X_1 \xrightarrow{\alpha_1} X_2$ and $X_2 \xrightarrow{\alpha_2} X_3$ be $\hat{\mathbb{E}} - inflations$. By using 7.31 $(i = 1, 2)$, we can express α_i as

$\alpha_i = \beta_i Q(f_i)\gamma_i$ with $\beta_i, \gamma_i \in Iso(\hat{\mathscr{C}})$ and where f_i is an $\mathbb{E}-inflation$. To prove the theorem we have to show their composition $\alpha_2\alpha_1$ is again of this form.
We can express $(\gamma_2\beta_1)^{-1}$ as $(\gamma_2\beta_1)^{-1} = \bar{Q}(\bar{s})^{-1}\bar{Q}(\bar{e})$ with $\bar{e}$ in $\bar{\mathscr{C}}$ and $\bar{s} \in \bar{\mathscr{S}}$.
In particular, $\bar{Q}(\bar{e})$ is an isomorphism and $\gamma_2\beta_1 = \bar{Q}(\bar{e})^{-1}\bar{Q}(\bar{s})$. Suppose $f_2 \in \mathscr{C}(A,B)$ and $e \in \mathscr{C}(A,A')$.
f_2 is an $\mathbb{E}-inflation$ hence there exists an $\mathbb{E}-triangle$ $A \xrightarrow{f_2} B \xrightarrow{g} C \xrightarrow{\delta}$.
We can also associate to $e_*\delta$ the $\mathbb{E}-triangle$ $[A' \xrightarrow{f_2'} B \xrightarrow{g'} C \xrightarrow{e_*\delta}]$. By using the fact that $\mathbf{s}$ is a realization for $(\mathscr{C}, \mathbb{E}, \mathbf{s})$ we can find a morphism of $\mathbb{E}-triangles$

$$\begin{array}{ccccccc} A & \xrightarrow{f_2} & B & \xrightarrow{g} & C & \xrightarrow{\delta} & \\ {\scriptstyle e}\downarrow & & {\scriptstyle d}\downarrow & & \| & & \\ A' & \xrightarrow[f_2']{} & B' & \xrightarrow[g']{} & C & \xrightarrow[e_*\delta]{} & \end{array}$$

for some $d \in \mathscr{C}(B, B')$.
In particular, $Q(d) \in Iso(\hat{\mathscr{C}})$.
By applying (3) of 7.4 to $f_2' s f_1$, we find out that $\bar{f}_2'\bar{s}\bar{f}_1 = v\bar{x}u$ for some $u, v \in \bar{\mathscr{S}}$ and an $\mathbb{E}-inflation$ x.
It follows that

$$\alpha_2\alpha_1 = (\beta_2\bar{Q}(\bar{f}_2)\gamma_2)(\beta_1\bar{Q}(\bar{f}_1)\gamma_1) = (\beta_2 Q(d)^{-1}\bar{Q}(v))\bar{Q}(\bar{x})(\bar{Q}(u)\gamma_1).$$

Since $\bar{Q}(u)\gamma_1$ and $\beta_2 Q(d)^{-1}\bar{Q}(v)$ are both isomorphisms of $\hat{\mathscr{C}}$, $\alpha_2\alpha_1$ is an $\mathbb{E}-inflation$ by 7.31.
We obtain the case of $\hat{\mathbb{E}}-deflations$ by duality.
Now by using 6.55, we conclude the proof of the theorem.

7.1 Relation to localization of triangulated categories

We start by recalling some definitions and by adapting them to the extriangulated case.

Definition 7.32 *Let $(\mathscr{C}, \mathbb{E}, \boldsymbol{s})$ be an extriangulated category. A full additive subcategory $N \subseteq \mathscr{C}$ is said to be a thick subcategory in the extriangulated sense if it satisfies the following condition:*

1. *$N \subseteq \mathscr{C}$ is closed under isomorphisms and direct summands.*

2. *Given an $\mathbb{E}-conflation$ $A \xrightarrow{x} B \xrightarrow{y} C$, if any two objects belong to N then the third does as well.*

Remark 7.33 1. *A thick subcategory $N \subseteq \mathscr{C}$ is extension-closed, therefore it is an extriangulated category by 6.18.*

2. *Given $(F, \phi) : (\mathscr{C}, \mathbb{E}, \boldsymbol{s}) \to (\mathscr{D}, \boldsymbol{F}, \boldsymbol{t})$ an exact functor to a weakly extriangulated category, $Ker\, F \subseteq \mathscr{C}$ is a thick subcategory.*

Definition 7.34 *Let $N \subseteq \mathscr{C}$ be a thick subcategory then we can define*

$$\mathscr{L} = \{f \in Mor(\mathscr{C}) |\ f \text{ is an } \mathbb{E}-inflation \text{ with } Cone(f) \in N\},$$

$$\mathscr{R} = \{f \in Mor(\mathscr{C}) |\ f \text{ is an } \mathbb{E}-deflation \text{ with } CoCone(f) \in N\}.$$

Let $\mathscr{S}_N \subseteq Mor(\mathscr{C})$ be the smallest subset closed by compositions containing $\mathscr{L}$ and $\mathscr{R}$. It follows that $\mathscr{S}_N$ verifies 7.1.

Remark 7.35 *$\mathscr{S}_N$ is the set with all finite compositions of morphisms in $\mathscr{L}$ and $\mathscr{R}$.*

Lemma 7.36 *We have $N_{\mathscr{S}_N} = N$.*

Proof: By definition $N \subseteq N_{\mathscr{S}_N}$ so we need to prove the other inclusion.
Let X be an object in $N_{\mathscr{S}_N}$. By definition $0 \to X$ is in $\mathscr{S}_N$.
Then $0 \to X$ is a finite composition of morphisms in $\mathscr{L}$ and $\mathscr{R}$ by 7.35.
We also observe that given a morphism $f : C \to Y$ with $C \in N$, if f is in $\mathscr{L}$ or in $\mathscr{R}$, Y is in N by definition of thick subcategory.
Since $0 \in N$, it follows that X is in N.

Lemma 7.37 *$\mathscr{S}_N$ satisfies $1(c)$ of 7.5.*

Proof: Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta}$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \xrightarrow{\delta}$ be $\mathbb{E}-$*triangles*. Let $a \in \mathscr{S}_N(A, A')\, c \in \mathscr{S}_N(C, C')$ be two morphisms such that $a_*\delta = c^*\delta'$.
We need to show there exists $b \in \mathscr{S}_N$ such that

$$\begin{array}{ccccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C & \xrightarrow{\delta} & \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow & & \\ A' & \xrightarrow[x']{} & B' & \xrightarrow[y']{} & C' & \xrightarrow[\delta']{} & \end{array}$$

is a morphism of $\mathbb{E}-$*triangles*.
We can suppose that $c = 1$. By recalling 7.35, we may assume a belongs to either $\mathscr{R}$ or $\mathscr{L}$.
We can now use $(ET4)$ in case $a \in \mathscr{R}$ and 6.37 in case $a \in \mathscr{L}$.

Remark 7.38 *Suppose that $\mathscr{C}$ is a triangulated category. Let $\mathscr{D}$ be a thick triangulated subcategory. Then $\mathscr{D}$ is a thick subcategory in the sense of extriangulated categories when considering $\mathscr{C}$ as an extriangulated category.*
Moreover $S(\mathscr{D}) = \mathscr{S}_{\mathscr{D}} = \mathscr{L} = \mathscr{R}$ and the localization of $\mathscr{C}$ by $\mathscr{S}_{\mathscr{D}}$ is triangulated and corresponds to the localization in the triangulated sense.

Remark 7.39 *Let $\mathscr{A}$ be an abelian category and let $\mathscr{C} \subseteq \mathscr{A}$ be a Serre subcategory.*
$\mathscr{C}$ is a thick subcategory in the extriangulated sense defined in 7.32.
We may observe that N, defined as in 7.2, coincides with $\mathscr{C}$ and $\mathscr{S}_N$ is the multiplicative system induced by $\mathscr{C}$.

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