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An SU(2)-gauge invariant (1+1) quantum cellular automaton for massless fermions

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Abstract

Simplicity is the principle that guided, among others, Feynman and Wheeler in conceiving of physics as information processing. Richard Feynman argued that the idea of a discrete universe is the only way it can be simulated by its own constituents, namely by quantum computers composed of quantum cellular automata (*QCA*). As well as enabling us to test whether a fundamentally discrete quantum theory is tenable, a rigorous definition of *SU(N)-gauge invariant QCA* would be of great interest for at least two further reasons: the quantum simulation of fundamental particles and the mathematical classification of these structures, as simple as they are fruitful. This thesis is an attempt to translate the physical idea of $SU(2)$-gauge invariance into the mathematical language of *QCA*. We look for the minimal way of satisfying that property and endeavour to highlight all possible physical interpretations of this framework. Finally, we provide the definition of an $SU(2)$-gauge invariant $(1+1)$ quantum cellular automaton for single massless fermions and a glimpse into the two-particle sector.

Introduction

> Entia non sunt multiplicanda praeter necessitatem
>
> William of Ockham

Simplicity is, both in life and in physics, as elegant as it is effective. This is the principle that guided, among others, Feynman and Wheeler in conceiving of physics as information processing [3, 4, 7]. Richard Feynman argued that the idea of a discrete universe is the only way it can be simulated by its own constituents, namely by a quantum computer. The architecture he had in mind for the latter was precisely that of a quantum cellular automaton (QCA). These structures, as simple as they are fruitful, can be rigorously defined starting, as we shall see in the next chapter, from elementary assumptions: homogeneity, causality, unitarity.

The idea of viewing QCA as the microscopic mechanism for an emerging Quantum Field Theory (QFT) has recently been proposed in Refs.[10, 13, 14], and also as a framework capable of reconciling a hypothetical Planck scale with the usual Fermi scale used in high-energy physics. We tend to dismiss discrete theories on the basis of a presumed mathematical convenience of continuous theories. However, the continuum produces mathematically unsolved problems when we consider the infinitely small, problems which, by contrast, do not seem to arise in the discrete. Consider, for example, UV divergences or the problem of particle localization [15, 6, 8, 9].

It should be noted that Einstein himself considered discrete spacetime as a possibility, but abandoned the idea due to the lack of a suitable mathematical theory to describe it [5]. Unfortunately, a fundamentally discrete quantum theory presents numerous unresolved challenges: the construction of a QCA that approaches the Dirac equation in the continuum, compatibility with the symmetries of Einstein's special relativity, or the resolution of the problem known as fermion doubling, appear to have already been resolved in Refs.[17, 16], [27] and [18] respectively. The problem we aim to analyze in this thesis consists in formulating a gauge theory for *QCA*.

As well as enabling us to test whether a fundamentally discrete quantum theory is tenable, a rigorous definition of *SU(N)-gauge invariant QCA* would be of great interest for at least two further reasons:

1. The quantum simulation of fundamental particles. This project aims to develop quantum algorithms that enable the efficient simulation of interactions between fundamental particles, using a quantum computer. This is a necessary step towards subsequently simulating atoms, molecules, etc. Being able to simulate means being able to design molecules, materials, etc. in silico for specific purposes, with end applications in chemistry, biochemistry and electronics.

2. Mathematical classification of a class of structures that could play a central role in many future theories and models. For example, error resistance can be expressed as a commutation property. Similarly, various physical theories are based on $SU(N)$ symmetries, such as in condensed matter, quantum gravity or grand unification theories.

In this thesis, we present some interesting results concerning the simple $SU(2)$ case. To simplify matters, we will construct the theory starting from the *QCA* for massless fermions in $(1+1)$ dimensions. The generalization to massive particles and $(3+1)$ dimensions, which we do not discuss here, is left for future work.

In particular, we will proceed as follows: In *Ch.(1)*, we will introduce the framework of *QCA* and how we can speak of *QFT* concepts, such as the Gauge Principle, in that mathematical setup. *Ch.(2)* will be devoted to the construction of an $SU(2)$-gauge invariant unitary evolution for massless single particles. In order to do so, we will spend some time on the Clebsch-Gordan theory and its generalization for some particular cases of infinite dimensional representations of $SU(2)$. Finally, we will put everything together to provide the definitions of *SU(2)-gauge invariant massless (1+1) right/left quantum walk* in *Ch.(3)*. Moreover, we will give some hints on how we intend to proceed for the multi-particle sector, *i.e.* the complete *SU(2)-gauge invariant massless (1+1) QCA*.

Before proceeding, we want to clarify one thing: this thesis is an attempt to translate the physical idea of $SU(2)$-gauge invariance into the mathematical language of *QCA*. We will look for the minimal way of satisfying that property and endeavour to highlight all possible physical interpretations of this framework. However, many passages should still be regarded as formal mathematical constructions whose full physical justification remains to be established. The ultimate objective of this approach is to rigorously recover many concepts linked to the QFT.

Chapter 1

QCA and QFT

We want to understand whether *QCA* are fundamental structures or whether they can at least simulate *QFT*. Unfortunately, we are still a long way from being able to test the first hypothesis. One of the main problems is how to rigorously define a continuous limit. However, proving the second hypothesis would already be an incredible step forward: being able to simulate fundamental interactions would, in principle, allow us to simulate any physical system starting from first principles; the applications are therefore considerable. Recently, the first quantum algorithm for simulating the electromagnetic interaction between electrons and photons in $(3 + 1)$ dimensions has been developed in Ref.[25]. This is a discrete-time, discrete-space numerical scheme: a *U(1)-gauge invariant QCA*. This thesis aims to generalize this result and pave the way for a rigorous definition of *SU(N)-gauge invariant QCA*. The objective is therefore to formulate the non-abelian gauge symmetries underlying the standard model in a fundamentally discrete manner. To reduce the complexity of the problem, we will focus on the $SU(2)$ case in $(1 + 1)$ dimensions with massless fermions.

Although non-abelian cases are more complex, the strategy will be the same as that adopted for the $U(1)$ case [24]: to provide a mathematically rigorous definition of $SU(2)$-gauge invariance and to find the simplest way to satisfy it by extending the *QCA* for the free propagation of massless fermions in $(1 + 1)$ dimensions. By studying the $U(1)$ case, it is easy to see that the gauge invariance requirement cannot be satisfied without introducing a gauge field into the *QCA*. This result, which emerges mathematically, naturally reveals a deeper physical truth consistent with the Gauge Principle: requiring gauge invariance with respect to the groups of the Standard Model forces us to enrich the state space and the dynamics by introducing and interacting with a gauge field. However, before moving on to the possible formulation of

QFT via *QCA*, we feel it is appropriate to summarize what the latter are.

1.1 *QCA*

Heuristically [1], we can imagine a *QCA* as an infinite n-dimensional *array* of d-dimensional quantum systems evolving respecting some properties that we will outline in this section. In other words, each cell is a *qudit*, *i.e.* a normalized vector in the Hilbert space $\mathbb{C}^d$. Since an infinite n-dimensional *array* has $\mathbb{Z}^n$ cells [2], the Hilbert space in which the possible states of the system live should be something like $\mathcal{H} \doteq \bigotimes_{\mathbb{Z}^n} \mathbb{C}^d$. Unfortunately, the latter is not a Hilbert space because it exhibits some pathological properties related to the inner product.

However, by adopting the appropriate precautions, it is possible to overcome the issue and rigorously construct a *QCA* starting from the following preliminary definitions:

Definition 1.1 (Configurations): let Σ be a finite set, called *alphabet*, and 0 a distinguished element of Σ, called *empty state*. A *configuration* c on Σ is a function:

$$c: \ \mathbb{Z}^n \to \Sigma \quad , \quad (i_1, \ ..., \ i_n) \mapsto c_{i_1, \ ..., \ i_n} \tag{1.1}$$

s.t. the subset of $(i_1, \ ..., \ i_n) \in \mathbb{Z}^n$ whose image $c_{i_1, \ ..., \ i_n} \neq 0$ is finite. We will call the set of every configuration $\mathcal{C}$

We note that $\mathcal{C}$ is countable. Therefore, we can finally define the Hilbert space that contains all configurations and their superpositions:

Definition 1.2 (State space): The configurations Hilbert space $\mathcal{H}_\mathcal{C}$ is the one generated by the following basis: $|c\rangle_{c\in\mathcal{C}}$. The orthogonal relations are $\langle c|c'|c|c'\rangle = \delta_c^{c'}$

[1] we draw inspiration from Ref.[23] for this part

[2] We chose $\mathbb{Z}^n$ instead of $\mathbb{N}^n$ because, by doing so, it is possible to identify a central "zero"

The global evolution of a *QCA* is required to be translation-invariant, meaning that it acts everywhere in the same way, *i.e.*:

Definition 1.3 (Translation invariance): Let $\hat{\tau}_k$ denote the translation operator along the k^{th} dimension, *i.e.* the linear operator over $\mathcal{H}_{\mathcal{C}}$ which maps $|c\rangle$ into $|c'\rangle$, where $|c'\rangle$ is *s.t.* $\forall (i_1,\ ...,\ i_n)$, $c'_{i_1,\ ...,\ i_k,\ ...,\ i_n} = c_{i_1,\ ...,i_{k+1},\ ...,i_n}$. A linear operator $\hat{G}$ over $\mathcal{H}_{\mathcal{C}}$ is said to be *translation-invariant* if and only if

$$[\hat{G}, \hat{\tau}_k] = 0 \quad \forall k \tag{1.2}$$

Moreover, the global evolution of a *QCA* is required to be causal, which means that information propagates at a bounded speed. We can formalize this property as follows:

Definition 1.4 (Causality): A linear operator $\hat{G}$ over $\mathcal{H}_{\mathcal{C}}$ is said to be *causal* with neighborhood $\mathcal{N} \subset \mathbb{Z}^n$ if and only if $\forall x \in \mathbb{Z}^n$ and $\forall \hat{\rho} = \sum_i p_i |c_i\rangle\langle c_i|$ there exist a function f *s.t.*

$$Tr_{\bar{x}}\hat{\rho}' = \hat{\rho}'_x = f(\hat{\rho}_{x+\mathcal{N}}) \quad , \quad where \;\; \hat{\rho}' = \hat{G}\hat{\rho}\hat{G}^{\dagger} \tag{1.3}$$

It is important to emphasize that causality for quantum systems is a far more delicate matter than it may appear.

Finally, we can outline a rigorous definition of *QCA*:

Definition 1.5 (*QCA*): A *QCA* is a linear operator over $\mathcal{H}_{\mathcal{C}}$ which is translation-invariant, causal and unitary

From the above axiomatic definition of *QCA*, it has been deduced in Ref.[12] that every *QCA* can be directly simulated by a finite depth quantum circuit of *local unitary gates*, infinitely repeating across space. Therefore, unitarity and causality imply *localizability*

1.2 *Free Dirac (1+1) QCA*

We present the framework developed in Refs.[17, 16] limiting ourselves to $(1 + 1)$ dimensions. The authors of these papers show how it is possible to recover the Dirac equation as the continuum limit of a simple *QCA*. Natively discretizing space and time, we end up with a *countable* infinite number of spatial degrees of freedom for the Dirac free fields instead of *uncountable* as in the continuum. Moreover, we emphasize that the free particles described by a *Free Dirac QCA for fermions* [3] are *indistinguishable.*

We recall the following subtlety: particles are *indistinguishable* if there are no observables capable of enumerating them. Fermions in a *QCA* are indistinguishable, therefore the occupation mode basis in Heisenberg picture is the most suitable option to describe globally a fermionic *QCA*. However, we exploit one of the main advantages of *QCA* discussed in *Sec.(1.1)*: localizability.

Remark 1.6: As long as our analysis is local, we can use different kinds of bases, *e.g.* the first-quantization basis. Moreover, we will adopt the Schrödinger picture, which is clearer at this stage, as the states evolve explicitly under the action of local unitary gates. When necessary, we will encode the canonical anticommutation relations (*CAR*) by antisymmetrizing the states by hand in the first-quantization basis, *e.g.* in *Subsec.(3.3.2)*. These choices are convenient also because the Clebsch-Gordan theory, which we will employ, was developed primarily in this framework.

We can now proceed to present the elements and notation required to carry out a local analysis of the *Free Dirac (1+1) QCA*.

We arrange the *QCA* as shown in *Fig.(1.1)*. Each space-time cell (x, t) of the one-dimensional lattice with step ϵ hosts at most two fermionic modes, a left mover $(-)$ and a right mover $(+)$. However, for the purposes of our analysis, it is more useful to define the state of each segment of the *QCA*, *i.e.* those entering and exiting the unitary gates $\hat{W}$ which we define as:

$$\left|\phi^+(x,t)\right\rangle \otimes \left|\phi^-(x+\epsilon,t)\right\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \tag{1.4}$$

where each 2-dimensional Hilbert space accounts for the presence or absence of a right mover in x and of a left mover in $x + \epsilon$.

Notation: We will adopt the following simplified notation for the *segment states* of the *QCA*

$$\left|\phi^+(x,t)\right\rangle \otimes \left|\phi^-(x+\epsilon,t)\right\rangle \doteq |\psi_{in}\rangle \tag{1.5}$$

[3] From now on, unless needed, we will simply write *Free Dirac QCA*

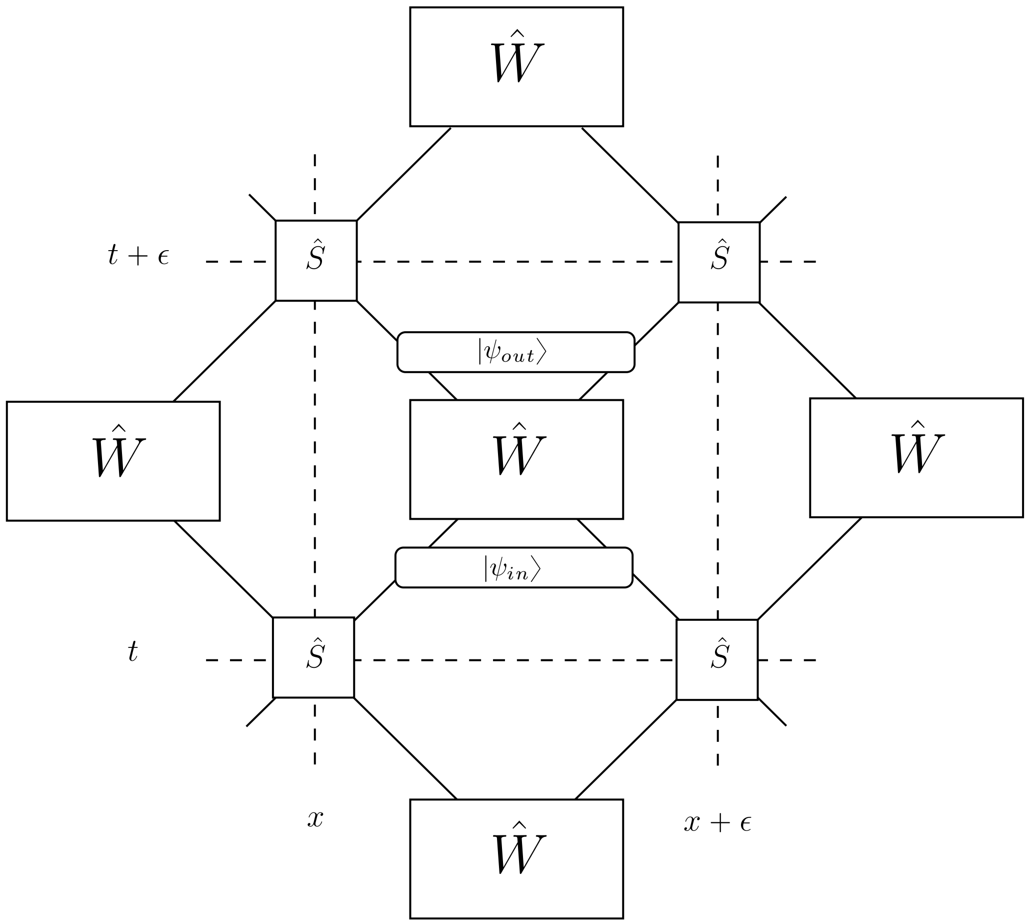

Figure 1.1: A diagramatic representation of the *Free-Dirac (1+1) QCA*

and call the Hilbert space of the segment states $\mathcal{H}_{loc}$

Therefore, the unitary evolution of a segment state of the *Free Dirac (1+1)-QCA* is described by the following formula:

$$\hat{W}\,|\psi_{in}\rangle = |\psi_{out}\rangle \tag{1.6}$$

Exploiting the isomorphism $\mathcal{H}_{loc} \doteq \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$, we write the segment states and $\hat{W}$ in the following basis:

$$|0\rangle_+ \otimes |0\rangle_- \;\cong\; \vec{e}_1 \doteq \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad , \quad |1\rangle_+ \otimes |0\rangle_- \;\cong\; \vec{e}_2 \doteq \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad ,$$

$$|0\rangle_+ \otimes |1\rangle_- \;\cong\; \vec{e}_3 \doteq \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad , \quad |1\rangle_+ \otimes |1\rangle_- \;\cong\; \vec{e}_4 \doteq \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \tag{1.7}$$

Then,

Definition 1.7 (*Free Dirac (1+1) QCA*): The *Free Dirac (1+1) QCA* is a *(1+1) QCA* whose gate $\hat{W} : \mathcal{H}_{loc} \rightarrow \mathcal{H}_{loc}$ in the basis of segment states reads:

$$\hat{W} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -is & c & 0\\ 0 & c & -is & 0\\ 0 & 0 & 0 & -1\end{pmatrix} \tag{1.8}$$

With $c = cos(m\epsilon)$ and $s = sin(m\epsilon)$. This gate allows the fermions to change direction with an amplitude that depends on their mass. We emphasize that the components are ordered so that, when the mass is zero, the particles do not change direction. This is what we obtain when $m = 0$ and, therefore, $c = 1$ and $s = 0$. Antisymmetrizing amounts to adding the minus sign of the bottom-right entry of the matrix, *i.e.* the case of two fermions crossing each other. For the same reason, each crossing of wires also has a gate

$$\hat{S} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix} \tag{1.9}$$

It would be useful and interesting to delve into rigorous treatments of the continuum limit to see that this *QCA* is a discretization of non-interacting Dirac particles, *e.g.* Refs.[17, 16]. However, since this is not the focus of the present thesis, we present a rough justification of the continuum limit given in Ref.[24]

Let us consider the one-particle sector, *i.e.* the *quantum walk* (QW), by restricting the QCA to the subspace spanned by states with a single fermionic mode occupied. In other words, we consider states with all but one qubit in the zero state. Define $|x+\rangle$ to be the state with the qubit corresponding to a right mover at position x being in the 1 state and zeroes everywhere else. Analogously, we define $|x-\rangle$ for left movers. Then we consider the state at time t given by

$$|\psi(t)\rangle = \sum_x \left[\psi^+(t,x)\,|x+\rangle + \psi^-(t,x)\,|x-\rangle\right] \tag{1.10}$$

with $|\psi(t)\rangle$ normalized to one. Then, the unitary dynamics gives us the update rule after one timestep to be

$$\begin{aligned} \psi^+(t+\epsilon,x) &= c\psi^+(t,x-\epsilon) - is\psi^-(t,x) \\ \psi^-(t+\epsilon,x) &= c\psi^-(t,x+\epsilon) - is\psi^+(t,x) \end{aligned} \tag{1.11}$$

We can now simply expand to first order in ϵ:

$$\begin{aligned} \epsilon\partial_t\psi^+ &= -\epsilon\partial_x\psi^+ - im\epsilon\psi^- \\ \epsilon\partial_t\psi^- &= +\epsilon\partial_x\psi^- - im\psi^+ \end{aligned} \tag{1.12}$$

Dividing across by ϵ allows us to write this in terms of Pauli matrices to get

$$\partial_t\psi = -\sigma_3\partial_x\psi - im\sigma_1\psi \tag{1.13}$$

which is the Dirac equation, where $\psi = (\psi^+,\psi^-)^T$.

1.3 Gauge principle for QCA

> The laws of nature are invariant under local transformations of a symmetry group G. Every local symmetry requires the introduction of a gauge field that mediates the corresponding interaction.

The Gauge Principle is a fundamental guiding principle of standard *QFT*. By "standard *QFT*," we mean conventional quantum field theories where Lagrangians are asked to be *G-gauge invariant* depending on the interactions we want to deal with. A Lagrangian is G-gauge invariant, if it is invariant under *local linear actions of* G on the quantum fields.

In what follows, we will translate the same physical idea into the mathematical framework of *QCA*. The same topic has already been studied for classical *cellular automata* (*CA*) in Ref.([22]). We tackle the problem by looking heuristically at the simplest mathematical way of solving the puzzle. The possible physical interpretation of the objects we introduce will be given *a posteriori*.

Remark 1.8 We note that gauge symmetry, *i.e.* local symmetry, just constrains the dynamics and does not uniquely determine it. What uniquely determines it are the experiments and the theorized continuum limit we want to tend to. Therefore, future developments of this project should study why and which solutions to this mathematical problem are also physical. *E.g.* is the minimal mathematical solution already physical?

Our starting point is the *Free Dirac (1+1) QCA*. However, in order to speak about G-gauge invariance, we need to add the *G-charge* degree of freedom for the fermions through representation theory. Therefore, we exploit some useful notations, results and definitions of the representation theory for groups that are shown in *App.(A)*

Definition 1.9 (G-charge): Let Π be a finite-dimensional complex representation of a matrix Lie group G on V. The associated Hilbert space $\mathcal{H}_G = (V, (\cdot,\cdot))$ is the state space of the *G-charge* degree of freedom: its vectors label the possible charge states of the system, and G acts on them via the representation.

In *Sec.(1.2)* we explained how to describe the infinite countable spatial degrees of freedom for the *Free Dirac (1+1) QCA*, while here we introduced the G-charge degree of freedom. Now, we have to put them together in order to depict the *G (1+1) QCA for fermions* [4]

Each space-time cell (x,t) of the one-dimensional lattice with step ϵ can be occupied at most by $2g$ fermions with opposite direction or with different G-charges, where $g = dim\ \mathcal{H}_G$. For simplicity and since it will be the case of our interest, we pick $g = 2$.

The physical system located on each cell can be written as:

$$|\phi_l(x,t)\rangle = \left|\phi_l^-(x,t)\right\rangle \otimes \left|\phi_l^+(x,t)\right\rangle \in (\mathbb{C} \oplus \mathcal{H}_G) \otimes (\mathbb{C} \oplus \mathcal{H}_G) \tag{1.14}$$

where $l \in \{-1,+1\}$, $\mathcal{H}_G \cong \mathbb{C}^2$ and $\mathbb{C}$ is the one-dimensional Hilbert space accounting for the vacuum state

Remark 1.10: Note that the local space $\mathbb{C} \oplus \mathcal{H}_G \cong span\{|0\rangle, |+1\rangle, |-1\rangle\}$ is isomorphic to the local Fock space of a single fermionic mode; however, we treat it as a first-quantization space with explicit labels, consistently with the localizability of the QCA framework.

Moreover, the segment states in the new Hilbert space $\mathcal{H}_{loc} \doteq (\mathbb{C} \oplus \mathcal{H}_G) \otimes (\mathbb{C} \oplus \mathcal{H}_G)$ can be defined following the same procedure adopted in *Sec.(1.2)*.

$$\begin{gathered}
|0\rangle_+ \otimes |0\rangle_- \ \cong\ \vec{e}_1 \quad , \quad |-1\rangle_+ \otimes |0\rangle_- \ \cong\ \vec{e}_2 \quad , \\
|0\rangle_+ \otimes |-1\rangle_- \ \cong\ \vec{e}_3 \quad , \quad |+1\rangle_+ \otimes |0\rangle_- \ \cong\ \vec{e}_4 \quad , \\
|0\rangle_+ \otimes |+1\rangle_- \ \cong\ \vec{e}_5 \quad , \quad |-1\rangle_+ \otimes |-1\rangle_- \ \cong\ \vec{e}_6 \quad , \\
|+1\rangle_+ \otimes |-1\rangle_- \ \cong\ \vec{e}_7 \quad , \quad |-1\rangle_+ \otimes |+1\rangle_- \ \cong\ \vec{e}_8 \quad , \\
|+1\rangle_+ \otimes |+1\rangle_- \ \cong\ \vec{e}_9
\end{gathered} \tag{1.15}$$

[4]From now on, unless needed, we will simply write *G (1+1) QCA*

Then,

Definition 1.11 (*G (1+1) QCA*): The *G (1+1) QCA* is a *(1+1) QCA* composed of gates $\hat{W} : \mathcal{H}_{loc} \to \mathcal{H}_{loc}$

Moreover,

Definition 1.12 (*Free Dirac-G (1+1) QCA*): The *Free Dirac-G (1+1) QCA* is a *G (1+1) QCA* whose gate $\hat{W}_G : \mathcal{H}_{loc} \to \mathcal{H}_{loc}$ in the basis of segment states read:

$$\hat{W}_G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -is & c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & -is & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -is & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -is & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.16}$$

With $c = cos(m\epsilon)$ and $s = sin(m\epsilon)$. This gate allows the fermions to change direction with an amplitude that depends on their mass. The components are ordered so that, when the mass is zero, the particles do not change direction. Antisymmetrizing by hand amounts to adding the minus sign of the bottom-right entries of the matrix, *i.e.* the case of two fermions crossing each other. For the same reason, each crossing of wires also has a gate

$$\hat{S}_G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.17}$$

We are ready to provide the definition of *local linear action of G* on a segment of a *G (1+1) QCA*:

Definition 1.14 (Local linear action of G): Given a G *(1+1) QCA*, let Π be a representation of G on $\mathcal{H}_G$. $\Pi(A, A') \in GL(\mathcal{H}_{loc})$ is a *local linear action* of G on $\mathcal{H}_{loc}$, with $A, A' \in G$. It is defined as

$$\Pi(A, A') = (\mathbb{I} \oplus \Pi(A)) \otimes (\mathbb{I} \oplus \Pi(A')) \tag{1.18}$$

Moreover, the property of G-gauge invariance for the evolution of a segment state of a G *(1+1) QCA* can be defined as

Definition 1.13 (G-gauge invariance): Given a G *(1+1) QCA*, we say that it is *G-gauge invariant* if the following equation holds:

$$\hat{W} \circ \Pi(A, A') = \Pi(A, A') \circ \hat{W} \quad \forall A, A' \in G \tag{1.19}$$

where $\Pi(A, A')$ is the local linear action of G as in *Def.(1.12)*.

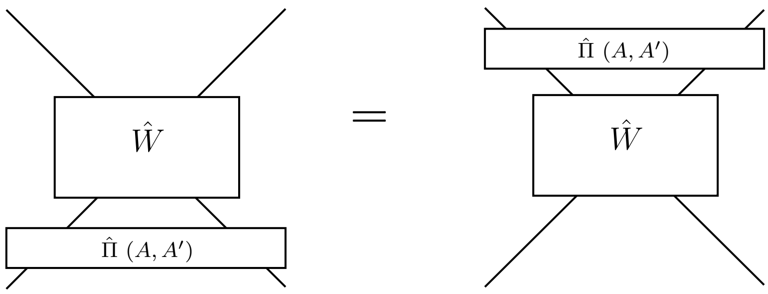

Figure 1.2: A diagramatic representation of G-gauge invariance for G *(1+1) QCA*

Adapting this definition, we will check whether the *Free Dirac-G (1+1) QCA* is G-gauge invariant with respect to the matrix Lie groups we are interested in. It has already been shown in Ref.[24] that the *Free Dirac-U(1) (1+1) QCA* is not $U(1)$-gauge invariant

Remark 1.15: This result should not surprise us. If we restrict ourselves to the standard model, it is consistent with the standard QFT: G-gauge invariance cannot arise between fermions without the bosons who carry the interaction. The extra terms involving them allow the Lagrangian to be G-gauge invariant. As pointed out in the introduction to this chapter, mathematics naturally reveals a deeper physical truth consistent with the Gauge

Principle, also for QCA.

Therefore, we proceed by introducing the fewest new degrees of freedom to recover G-gauge invariance for the new QCA: the *G-gauge invariant (1+1) QCA*. Extra G-charges, as we will prove, are enough and we will refer to them as the *gauge field*

Definition 1.16 (Gauge field): The gauge field, in the spirit of Ref.[24], are G-charges located in the spatial links between cells. It enrich every segment state with a additional degrees of freedom that evolves according to the QCA. The states of the gauge field depend on G

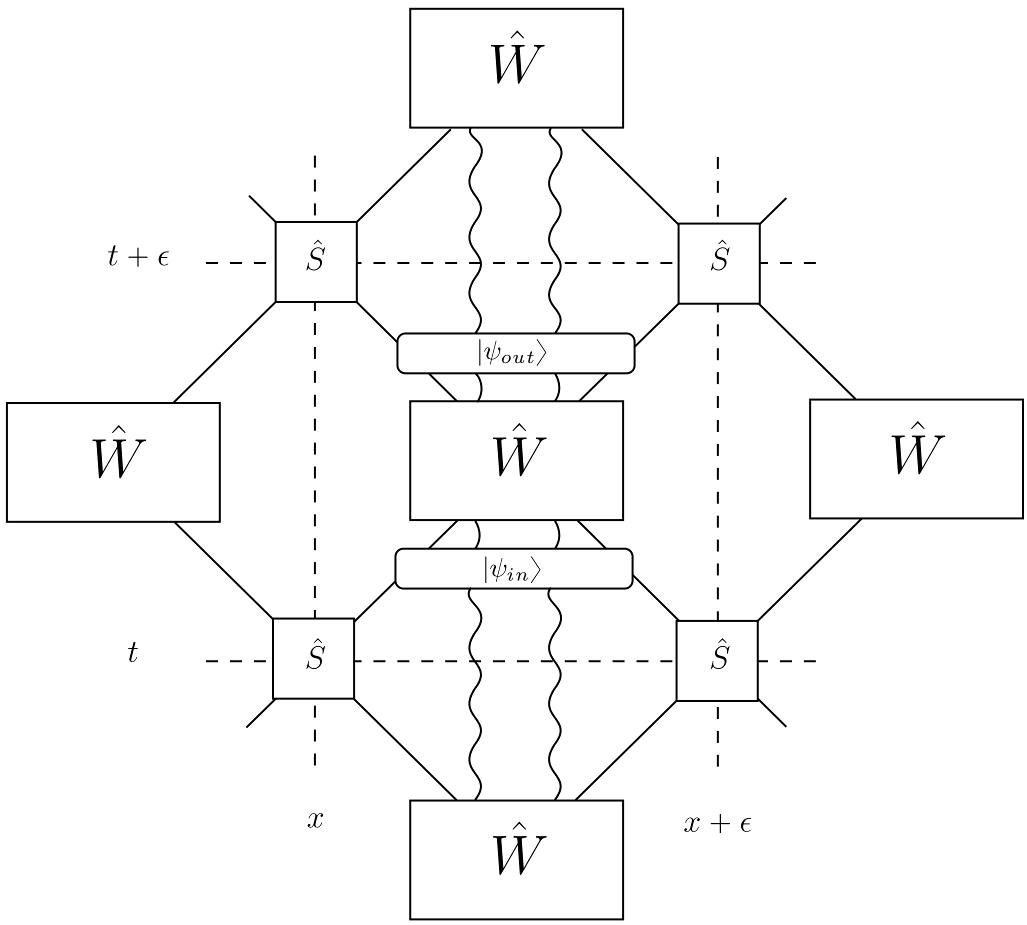

Figure 1.3: A diagramatic representation of the gauge field

We emphasize that, if G is a group of the Standard Model, the physical interpretation of the gauge field could be the G-charge of the bosons as in the standard *QFT*. However, for now, we will avoid speaking about their spatial degrees of freedom in order to see if we can recover G-gauge invariance without them.

Remark 1.17: We conclude this chapter with some necessary considerations about the gauge field. Interpreting it as bosonic modes is as intriguing as risky: allowing our theory to account for an unbounded number of particles in the same place is something we would like to avoid. This issue is beyond the scope of the present manuscript. We just briefly mention here a possible approach to the problem, based on the hypothesis that the bosonic behaviour might be recovered as an approximation of the dynamics of pairs of fermionic excitations. A development of this line of reasoning can be found in Ref.[20].

Finally, since we are not even sure that this is a problem, in this thesis we allow the states of the gauge field on each link to live in an infinite dimensional Hilbert space. However, the framework we are building, due to its simplicity, could be easily adapted to different interpretations in the future. Moreover, it would be interesting to present a general approach that works for every matrix Lie group G. However, since this is not the aim of the present project, we proceed by delving into the ideas we outlined for the case of our interest: $SU(2)$

Chapter 2

SU(2)-gauge invariant evolution

We want to build an *SU(2)-gauge invariant massless (1+1) QCA*. In *Sec.(2.1)* and *Sec.(2.2)*, we will define the $SU(2)$-charges and the gauge field through some preliminary results for the representation theory of $SU(2)$. The following *Secs.(2.3-2.6)* will be devoted to explaining an heuristic way of building an $SU(2)$-gauge invariant evolution for a massless right-mover[1]. The generalization for left-movers and the multi-particle sectors will be discussed later

2.1 $SU(2)$-charges

This branch of mathematics is widely known thanks to its link to the angular momentum in *NRQM* [2]. Therefore, we will adopt usual notations, results and conventions to introduce this topic. However, it is important to underline that we are dealing with *SU(2)-charges* and not angular momenta here.

Let us start by defining the irreducible representations of $SU(2)$:

Definition 2.1 (Irreducible representation of $SU(2)$): Given the matrix Lie group $SU(2)$ and $j \in \frac{1}{2}\mathbb{N}_0$, its *$j$-th finite-dimensional complex irreducible representation* is the group homomorphism

$$R_j : SU(2) \to GL(\mathbb{C}^{2j+1}) \tag{2.1}$$

We will adopt the notation $\mathcal{H}_j \equiv \mathbb{C}^{2j+1}$ according to *Def.(1.9)*.

[1]From now on, unless needed, we will just write "right/left-mover"

[2]We take [21] as reference for this part

Moreover, we will write the image of the representation as $\hat{R}_j(g)\ \ \forall g \in SU(2)$ since it is a linear action of $SU(2)$ on $\mathcal{H}_j$ (see *App.(A)*) and we want to align with the usual notation for operators in quantum theory.

Definition 2.2 (Linear action of $SU(2)$): Given the representation R_j of $SU(2)$ on $\mathcal{H}_j$, $\hat{R}_j(g) \in GL(\mathcal{H}_j)$ is a *linear action of SU(2)* on $\mathcal{H}_j$, where $g \in SU(2)$ identifies an axis $\hat{n}$ and an angle θ. It is defined as

$$\hat{R}_j(g) \equiv \hat{R}_j(\hat{n},\theta) = e^{-i\theta\hat{n}\cdot\vec{\hat{J}}^{(j)}} : \mathcal{H}_j \to \mathcal{H}_j \tag{2.2}$$

where $J^{(j)} : \mathfrak{su}(2) \to \mathfrak{gl}(\mathcal{H}_j)$ is the representation of $\mathfrak{su}(2)$ associated to R_j. We emphasize that $\hat{R}_j(g)$ is a unitary operator $\forall\ j \in \frac{1}{2}\mathbb{N}_0$ and $\forall\ g \in SU(2)$ [3].

Notation: fixed a direction in space, namely $\hat{z}$, we adopt the usual notation for the *eigenvectors* of $\hat{R}_j(\hat{z},\theta)$:

$$\hat{R}_j(\hat{z},\theta)\,|j,m\rangle = e^{-i\theta m}\,|j,m\rangle \tag{2.3}$$

$$\hat{J}^{(j)}_z\,|j,m\rangle = m\,|j,m\rangle \tag{2.4}$$

We notice that, according to the spectral theorem, $\{|j,m\rangle\}_{m\in\{-j,\ldots,j\}}$ form an orthonormal basis for $\mathcal{H}_j$. Therefore, we can write $\hat{R}_j(g)$ as follows:

$$\hat{R}_j(g) = \sum_{m,n} R^{m,n}_{(j)}(g)\,|j,m\rangle\langle j,n| \tag{2.5}$$

Where $R^{m,m}_{(j)} = e^{-i\theta m}$, see *Eq.(2.4)*

Moreover, we define two special elements of $\mathfrak{gl}(\mathcal{H}_j)$ whose utility will be clear in the next sections:

$$\hat{J}^{(j)}_{\pm}\,|j,m\rangle = \left(\hat{J}^{(j)}_x \pm i\hat{J}^{(j)}_y\right)|j,m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\,|j,m\pm 1\rangle \tag{2.6}$$

Remark 2.3: We underline that we pick $\mathcal{H}_{SU(2)} = \mathcal{H}_{\frac{1}{2}}$ for the fermions since we want to align with the standard *QFT*. We will explain how we describe the gauge field in the next section

[3] See *Prop.(3.24)* of Ref.[11]

2.2 Gauge field

We start the section by stating this result

Proposition 2.4 the *Free Dirac-SU(2) (1+1) QCA* is not $SU(2)$-gauge invariant.

Proof. It can be easily checked by direct calculations □

In the spirit of *Sec.(1.3)*, we introduce a gauge field. In order to recover the desired property, we need to split the link in two semi-links and use at least 3 parameters j,m and m'. This mathematical requirement could be aligned with the physical model of rishons [1] that is widely used in Hamiltonian quantum simulations, *e.g.* Refs.[26, 19]. There, the gauge field is mathematically allowed to assume all the possible values of j, m and m'. However, we must wonder whether it is necessary to put extra constraints on the physically admissible states. In standard *QFT*, the Euler-Lagrange equations encode that information, *e.g.* the Gauss law, but the theory of *QCA* does not provide something like that. How can we make it emerge naturally for this framework? We want to deal with this issue in future works, but, for the moment, we will consider all the mathematically admissible states

The segment state of the *SU(2)-gauge invariant massless (1+1) QW* lives in the Hilbert space [4]

$$\mathcal{H}_{LOC} \doteq \left(\mathbb{C} \oplus \mathcal{H}_{\frac{1}{2}}\right) \otimes \mathcal{H}_G^L \otimes \mathcal{H}_G^R \otimes \left(\mathbb{C} \oplus \mathcal{H}_{\frac{1}{2}}\right) \tag{2.7}$$

Where $\mathcal{H}_G^L \cong \mathcal{H}_G^R = \bigoplus_{j\in\frac{1}{2}\mathbb{N}_0} \mathcal{H}_j$ are the Hilbert spaces of the gauge field on the left semi-link that is attached on the site x (left) and on the right semi-link attached on the site $x+\epsilon$ (right) respectively. We emphasize that *Eq.(2.7)* allows the presence of one fermion at maximum on each cell (see *Re.(1.10)*). Accounting for the presence of two fermions on each cell will just be a matter of adding through direct sum a one-dimensional Hilbert space on them.

[4] Notice that we used *LOC* for this one and *loc* for the one without gauge field in order to clearly distinguish them

Therefore, since we want to start from a right-mover, a possible segment state reads [5]

$$|\psi_{in}\rangle = \left|\frac{1}{2}, \frac{1}{2}s\right\rangle \otimes |j, m\rangle \otimes |j, m'\rangle \otimes |vac\rangle \in \mathcal{H}_{LOC} \tag{2.8}$$

where $s \in \{-1, 1\}$, *vac* means that we have nothing in the second cell of the *QCA* and we avoided superpositions for the sake of simplicity. We will call *SU(2)-gauge invariant massless (1+1) right-QW* the related *QCA*.

In order to build an $SU(2)$-gauge invariant evolution for it, we have to spend some time speaking about Clebsch-Gordan theory: In *Sec.(2.3)* we will introduce the usual framework, *i.e.* for fixed j_1 and j_2, in *Sec.(2.4)* we will generalize it for variable j of the gauge field and in *Sec.(2.5)* we will show a more compact way of writing it for our particular case. We will delve into long and boring calculations, but the prize will be worth it!

2.3 Clebsch-Gordan Theory

The first tool we need is the theory of Clebsch-Gordan coefficients. It is a natural way of encoding information about the $SU(2)$-charge of the cell and the semi-link state in the updated semi-link state. They will be the main building block of the *SU(2)-gauge invariant massless (1+1) QCA* we want to build.

Given two irreducible representations j_1 and j_2 of $SU(2)$, the following Hilbert spaces are isomorphic (see *App.(B)*)

$$\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \cong \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \mathcal{H}_j \tag{2.9}$$

In fact,

$$dim\left(\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}\right) = dim\left(\bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \mathcal{H}_j\right) \tag{2.10}$$

[5]We dropped the indeces + and − to simplify the notation

We are interested in the Hilbert space isomorphisms [6] that are also intertwining maps (see *App.(A)*). In the formulas, let $R_{j_1} \otimes R_{j_2}$ be the representation of $SU(2)$ on $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}$ and $R_{j_1+j_2} \oplus \ldots \oplus R_{|j_1-j_2|}$ be the representation of $SU(2)$ on $\bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \mathcal{H}_j$. We want to define the *Clebsch-Gordan map* $\hat{U}_{j_1,j_2}$ *s.t.* it is an isomorphism and satisfies

$$\hat{U}_{j_1,j_2} \circ \left[\hat{R}_{j_1}(g) \otimes \hat{R}_{j_2}(g)\right] = \left[\hat{R}_{j_1+j_2}(g) \oplus \ldots \oplus \hat{R}_{|j_1-j_2|}(g)\right] \circ \hat{U}_{j_1,j_2} \ , \ \forall g \in SU(2) \tag{2.11}$$

This condition can be written in terms of the elements of the Lie algebra instead of the matrix Lie group's ones because the equations become easier. We are allowed to do so because $SU(2)$ is simply connected and, therefore (see *App.(A)*),

$$Hom_{\mathfrak{su}(2)}(V_1, V_2) = Hom_{SU(2)}(V_1, V_2) \tag{2.12}$$

In other words, a map is an intertwiner between representations of $SU(2)$ iff it is an intertwiner between representations of $\mathfrak{su}(2)$.

Therefore, we can substitute *Eq.(2.11)* with

$$\hat{U}_{j_1,j_2} \circ \left[\vec{\hat{J}}^{(j_1)}(G) \otimes \hat{\mathbb{I}} + \hat{\mathbb{I}} \otimes \vec{\hat{J}}^{(j_2)}(G)\right] = \left[\vec{\hat{J}}^{(j_1+j_2)} \oplus \ldots \oplus \vec{\hat{J}}^{(|j_1-j_2|)}(G)\right] \circ \hat{U}_{j_1,j_2} \ , \ \forall G \in \mathfrak{su}(2) \tag{2.13}$$

The intertwining condition can be further rewritten as follows:

$$\hat{U}_{j_1,j_2} \circ \left[\hat{J}_a^{(j_1)} \otimes \hat{\mathbb{I}} + \hat{\mathbb{I}} \otimes \hat{J}_a^{(j_2)}\right] = \left[\hat{J}_a^{(j_1+j_2)} \oplus \ldots \oplus \hat{J}_a^{(|j_1-j_2|)}\right] \circ \hat{U}_{j_1,j_2} \ , \ \forall a \in \{z, +, -\} \tag{2.14}$$

That is because, given the j-th irreducible representation of $SU(2)$, $\{\hat{J}_z^{(j)}, \hat{J}_+^{(j)}, \hat{J}_-^{(j)}\}$ is a basis for the Lie algebra $\mathfrak{gl}(\mathcal{H}_j)$

We provide a practical criterion to check whether an isomorphism is an intertwining map

[6] From now on, unless needed, we will simply write isomorphism

Proposition 2.5: The isomorphism $\hat{U}_{j_1,j_2}$ is an intertwining map, *i.e.* satisfies *Eq.(2.14)*, if its coefficients $C^{j',m}_{j_1,j_2,m_1,m_2}$ satisfy the following relations:

$$m = m_1 + m_2 \tag{2.15}$$

$$\begin{aligned}&\sqrt{(j \mp m)(j \pm m + 1)}C^{j_1,j_2,m_1,m_2}_{j,m\pm 1} = \\ &= \sqrt{(j_1 \pm m_1)(j_1 \mp m_1 + 1)}C^{j_1,j_2,m_1\mp 1,m_2}_{j,m} + \\ &\sqrt{(j_2 \pm m_2)(j_2 \mp m_2 + 1)}C^{j_1,j_2,m_1,m_2\mp 1}_{j,m}\end{aligned} \tag{2.16}$$

In this last equation, the condition *Eq.(2.15)* has been shifted to $m_1 + m_2 = m \pm 1$ according to the case we are considering.

Proof. We divide the proof in *steps*

z) We start by applying both sides to $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$:

$$\begin{aligned}LHS &= \hat{U}_{j_1,j_2} \circ \left[\hat{J}^{(j_1)}_z \otimes \mathbb{I} + \mathbb{I} \otimes \hat{J}^{(j_2)}_z\right] (|j_1, m_1\rangle \otimes |j_2, m_2\rangle) = \\ &= (m_1 + m_2) \sum_j C^{j,m}_{j_1,j_2,m_1,m_2} |j_1, j_2; j, m\rangle\end{aligned} \tag{2.17}$$

$$\begin{aligned}RHS &= \left[\hat{J}^{(j_1+j_2)}_z \oplus \ldots \oplus \hat{J}^{(|j_1-j_2|)}_z\right] \circ \hat{U}_{j_1,j_2} (|j_1, m_1\rangle \otimes |j_2, m_2\rangle) = \\ &= m \sum_j C^{j,m}_{j_1,j_2,m_1,m_2} |j_1, j_2; j, m\rangle\end{aligned}$$

$$\tag{2.18}$$

Therefore, in order to have $RHS = LHS$ for the $z)$ equation, the following equality must hold:

$$m = m_1 + m_2 \tag{2.19}$$

+) Again, we apply both sides to $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$:

$$\begin{aligned}
LHS &= \hat{U}_{j_1,j_2} \circ \left[\hat{J}_+^{(j_1)} \otimes \mathbb{I} + \mathbb{I} \otimes \hat{J}_+^{(j_2)}\right] (|j_1, m_1\rangle \otimes |j_2, m_2\rangle) = \\
&= \hat{U}_{j_1,j_2} \Big(\sqrt{(j_1 - m_1)(j_1 + m_1 + 1)}\, |j_1, m_1 + 1\rangle \otimes |j_2, m_2\rangle + \\
&\sqrt{(j_2 - m_2)(j_2 + m_2 + 1)}\, |j_1, m_1\rangle \otimes |j_2, m_2 + 1\rangle \Big) = \\
&= \sqrt{(j_1 - m_1)(j_1 + m_1 + 1)} \sum_{j'} C^{j',n}_{j_1,j_2,m_1+1,m_2} |j_1, j_2; j', n\rangle + \\
&\sqrt{(j_2 - m_2)(j_2 + m_2 + 1)} \sum_{j'} C^{j',k}_{j_1,j_2,m_1,m_2+1} |j_1, j_2; j', k\rangle
\end{aligned} \tag{2.20}$$

$$\begin{aligned}
RHS &= \left[\hat{J}_+^{(j_1+j_2)} \oplus \ldots \oplus \hat{J}_+^{(|j_1-j_2|)}\right] \circ \hat{U}_{j_1,j_2} (|j_1, m_1\rangle \otimes |j_2, m_2\rangle) = \\
&\left[\hat{J}_+^{(j_1+j_2)} \oplus \ldots \oplus \hat{J}_+^{(|j_1-j_2|)}\right] \left(\sum_{j'} C^{j',m}_{j_1,j_2,m_1,m_2} |j_1, j_2; j', m\rangle \right) \\
&= \sum_{j'} \sqrt{(j' - m)(j' + m + 1)} C^{j',m}_{j_1,j_2,m_1,m_2} |j_1, j_2; j', m + 1\rangle
\end{aligned} \tag{2.21}$$

If we adjust the labels

$$n, k \to m+1 \quad and \quad j' \to j \tag{2.22}$$

We clearly see that $RHS = LHS$ for the +) equation can be rewritten as:

$$\begin{aligned}
&\sqrt{(j-m)(j+m+1)}C^{j,m}_{j_1,j_2,m_1,m_2} = \\
&= \sqrt{(j_1-m_1)(j_1+m_1+1)}C^{j,m+1}_{j_1,j_2,m_1+1,m_2} + \\
&\sqrt{(j_2-m_2)(j_2+m_2+1)}C^{j,m+1}_{j_1,j_2,m_1,m_2+1}
\end{aligned} \tag{2.23}$$

For all $j \in \frac{1}{2}\mathbb{N}_0$. If we further adopt the change of labels $m = M-1$ and we recall that in our convention $C^{ab}_{cdef} = C^{cdef}_{ab}$, we end up with:

$$\begin{aligned}
&\sqrt{(j+M)(j-M+1)}C^{j_1,j_2,m_1,m_2}_{j,M-1} = \\
&= \sqrt{(j_1-m_1)(j_1+m_1+1)}C^{j_1,j_2,m_1+1,m_2}_{j,M} + \\
&\sqrt{(j_2-m_2)(j_2+m_2+1)}C^{j_1,j_2,m_1,m_2+1}_{j,M}
\end{aligned} \tag{2.24}$$

In conclusion, *Eq.(2.24)* is exactly one of *Eq.(2.16)* (we only have to rename again the label $M = m$).

−) The line of reasoning carried out for $a = +$ works also for $a = -$. We will obtain the other condition in *Eq.(2.16)*

□

Remark 2.6: We emphasize that *Eq.(2.15)* and *Eq.(2.16)* don't uniquely define the Clebsch-Gordan map and its coefficients. In order to do so we need to adopt the *Condon-Shortley phase convention*. It fixes the phases of the Clebsch-Gordan coefficients by requiring that all $C^{j',m}_{j_1,j_2,m_1,m_2}$ are real and $C^{j,j}_{j_1,j_2,j_1,j-j_2} > 0$. Under the Condon-Shortley phase convention, the numerical values of $C^{j,m}_{j_1,j_2,m_1,m_2}$ and $C^{j_1,j_2,m_1,m_2}_{j,m}$ are equal. That's why they're usually not distinguished in the physics literature.

Finally, we can provide the formal definition of the intertwining isomorphism $\hat{U}_{j_1,j_2}$:

Definition 2.7 (Clebsch-Gordan map): Given j_1 and j_2 two fixed irreducible representation of $SU(2)$, we call *Clebsch-Gordan map* the intertwining isomorphism that statisfies the Condon-Shortley phase convention and is defined as follows:

$$\begin{aligned} \hat{U}_{j_1,j_2} : \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \to \bigoplus_{j'=|j_1-j_2|}^{j_1+j_2} \mathcal{H}_{j'} \\ |j_1, m_1\rangle \otimes |j_2, m_2\rangle \mapsto \sum_{j'=|j_1-j_2|}^{j_1+j_2} C^{j',m}_{j_1,j_2,m_1,m_2} |j_1, j_2; j', m\rangle \end{aligned} \tag{2.25}$$

and the inverse is

$$\begin{aligned} \hat{U}^{-1}_{j_1,j_2} = \hat{U}^{\dagger}_{j_1,j_2} : \bigoplus_{j'=|j_1-j_2|}^{j_1+j_2} \mathcal{H}_{j'} \to \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \\ |j_1, j_2; j, m\rangle \mapsto \sum_{m'_1,m'_2 : m'_1+m'_2=m} C^{j_1,j_2,m'_1,m'_2}_{j,m} |j_1, m'_1\rangle \otimes |j_2, m'_2\rangle \end{aligned} \tag{2.26}$$

From now on, unless needed, we will simply write $\sum_{j'}$ and $\sum_{m'_1,m'_2}$.

Notation: In the physics literature, a notation such as $\langle j_1, m_1; j_2, m_2 | j_1, j_2; j, m\rangle$ is also normally used for the Clebsch-Gordan coefficients. Moreover, we prefer to put a $'$ on the labels that are varying under the sum in order to make things clearer

2.3.1 Explicit coefficients of $\hat{U}_{j_1,j_2}$

Solving the linear system composed by *Eq.(2.15)* and *Eq.(2.16)* while imposing the Condon-Shortley phase convention would give us the explicit formula of every Clebsch-Gordan coefficient. However, since there is no need to make life harder, let's think to the case of our interest: We want to describe the $SU(2)$-charge of the system composed by the fermion on the cell and the gauge field on the semi-link. The former is $\frac{1}{2}$, but what about the second j? For the moment, let's imagine that it's fixed in order to provide an explicit form of the Clebsch-Gordan coefficients. The derivation of this coefficients and the fact that they form a unitary map can be found in Ref.[21]. We think it's interesting to check here the intertwining condition.

Given $j \in \frac{1}{2}\mathbb{N}_0$ and $s \in \{-1, 1\}$, $\hat{U}_{\frac{1}{2},j}$ is defined as follows:

$$\begin{gathered}\hat{U}_{\frac{1}{2},j} : \mathcal{H}_{\frac{1}{2}} \otimes \mathcal{H}_j \to \bigoplus_{j'=|j-\frac{1}{2}|}^{j+\frac{1}{2}} \mathcal{H}_{j'} \\ \left|\frac{1}{2}, \frac{1}{2}s\right\rangle \otimes |j,m\rangle \mapsto \sum_{J'} C^{J',M}_{\frac{1}{2},j,\frac{1}{2}s,m} \left|\frac{1}{2}, j; J', M\right\rangle\end{gathered} \tag{2.27}$$

and the inverse is

$$\begin{gathered}\hat{U}^{-1}_{\frac{1}{2},j} = \hat{U}^{\dagger}_{\frac{1}{2},j} : \bigoplus_{J'=|j_1-j_2|}^{j_1+j_2} \mathcal{H}_{J'} \to \mathcal{H}_{\frac{1}{2}} \otimes \mathcal{H}_j \\ \left|\frac{1}{2}, j; J, M\right\rangle \mapsto \sum_{\frac{1}{2}s',m'} C^{\frac{1}{2},j,\frac{1}{2}s',m'}_{J,M} \left|\frac{1}{2}, \frac{1}{2}s'\right\rangle \otimes |j,m'\rangle\end{gathered} \tag{2.28}$$

Where

$$C^{j+\frac{1}{2},m+\frac{1}{2}s}_{\frac{1}{2},j,\frac{1}{2}s,m} = \sqrt{\frac{j+sm+1}{2j+1}} \tag{2.29}$$

$$C^{j-\frac{1}{2},m+\frac{1}{2}s}_{\frac{1}{2},j,\frac{1}{2}s,m} = -s\sqrt{\frac{j-sm}{2j+1}} \tag{2.30}$$

and, according to *Re.(2.6)*

$$C^{\frac{1}{2},j,\frac{1}{2}s',m'}_{J,M} = C^{J,M}_{\frac{1}{2},j,\frac{1}{2}s',m'} \tag{2.31}$$

Moreover, these coefficients satisfy *Eq.(2.15)*, *i.e.* $C^{J',N}_{\frac{1}{2},j,\frac{1}{2}s',m} = 0$ if $N \neq \frac{1}{2}s + m$.

Let's check the other condition. In this notation, *Eq.(2.16)* reads:

$$\sqrt{(J \mp M)(J \pm M + 1)}C^{\frac{1}{2},j,\frac{1}{2}s,m}_{J,M\pm 1} =$$

$$= \sqrt{\left(\frac{1}{2} \pm \frac{1}{2}s\right)\left(\frac{1}{2} \mp \frac{1}{2}s + 1\right)}C^{\frac{1}{2},j,\frac{1}{2}s\mp 1,m}_{J,M} + \tag{2.32}$$

$$\sqrt{(j \pm m)(j \mp m + 1)}C^{\frac{1}{2},j,\frac{1}{2}s,m\mp 1}_{J,M}$$

For the sake of clarity and completeness, we will analyze every possible case separately. We will vary $\pm$, J and s. Everytime, $RHS = LHS$ must hold.

- $+$, $J = j + \frac{1}{2}$ and $s = +1$

$$RHS = \sqrt{(j-m+1)(j+m+1)}\sqrt{\frac{j+m+1}{2j+1}} = \sqrt{\frac{j-m+1}{2j+1}}(j+m+1)$$

$$LHS = \sqrt{\frac{j-m+1}{2j+1}} + \sqrt{(j+m)(j-m+1)}\sqrt{\frac{j+m}{2j+1}} = \sqrt{\frac{j-m+1}{2j+1}}(j+m+1) \tag{2.33}$$

Therefore, $RHS = LHS$.

- $+$, $J = j + \frac{1}{2}$ and $s = -1$

$$
\begin{aligned}
RHS &= \sqrt{(j-m+2)(j+m)}\sqrt{\frac{j-m+1}{2j+1}} \\
LHS &= \sqrt{(j+m)(j-m+1)}\sqrt{\frac{j-m+2}{2j+1}}
\end{aligned}
\tag{2.34}
$$

Therefore, $RHS = LHS$.

- $+$, $J = j - \frac{1}{2}$ and $s = +1$

$$
\begin{aligned}
RHS &= \sqrt{(j-m)(j+m)}\left(-\sqrt{\frac{j-m}{2j+1}}\right) = -\sqrt{\frac{j+m}{2j+1}}(j-m) \\
LHS &= \sqrt{\frac{j+m}{2j+1}} + \sqrt{(j+m)(j-m+1)}\left(-\sqrt{\frac{j-m+1}{2j+1}}\right) = -\sqrt{\frac{j+m}{2j+1}}(j-m)
\end{aligned}
\tag{2.35}
$$

Therefore, $RHS = LHS$.

- $+$, $J = j - \frac{1}{2}$ and $s = -1$

$$
\begin{aligned}
RHS &= \sqrt{(j-m+1)(j+m-1)}\sqrt{\frac{j+m}{2j+1}} \\
LHS &= \sqrt{(j+m)(j-m+1)}\sqrt{\frac{j+m-1}{2j+1}}
\end{aligned}
\tag{2.36}
$$

Therefore, $RHS = LHS$.

- $-$, $J = j + \frac{1}{2}$ and $s = +1$

$$RHS = \sqrt{(j+m+2)(j-m)}\sqrt{\frac{j+m+1}{2j+1}}$$
$$LHS = \sqrt{(j-m)(j+m+1)}\sqrt{\frac{j+m+2}{2j+1}} \tag{2.37}$$

Therefore, $RHS = LHS$.

- $-$, $J = j + \frac{1}{2}$ and $s = -1$

$$RHS = \sqrt{(j+m+1)(j-m+1)}\sqrt{\frac{j-m+1}{2j+1}} = \sqrt{\frac{j+m+1}{2j+1}}(j-m+1)$$
$$LHS = \sqrt{\frac{j+m+1}{2j+1}} + \sqrt{(j-m)(j+m+1)}\sqrt{\frac{j-m}{2j+1}} = \sqrt{\frac{j+m+1}{2j+1}}(j-m+1) \tag{2.38}$$

Therefore, $RHS = LHS$.

- $-$, $J = j - \frac{1}{2}$ and $s = +1$

$$RHS = \sqrt{(j+m+1)(j-m-1)}\left(-\sqrt{\frac{j-m}{2j+1}}\right)$$
$$LHS = \sqrt{(j-m)(j+m+1)}\left(-\sqrt{\frac{j-m-1}{2j+1}}\right) \tag{2.39}$$

Therefore, $RHS = LHS$.

- $-$, $J = j - \frac{1}{2}$ and $s = -1$

$$RHS = \sqrt{(j+m)(j-m)}\sqrt{\frac{j+m}{2j+1}} = \sqrt{\frac{j-m}{2j+1}}(j+m)$$

$$LHS = -\sqrt{\frac{j-m}{2j+1}} + \sqrt{(j-m)(j+m+1)}\sqrt{\frac{j+m+1}{2j+1}} = \sqrt{\frac{j-m}{2j+1}}(j+m) \tag{2.40}$$

Therefore, $RHS = LHS$.

Finally, we can conclude that the $\hat{U}_{\frac{1}{2},j}$ provided by *Eq.(2.29)* and *Eq.(2.30)* is an intertwining map $\forall j \in \frac{1}{2}\mathbb{N}_0$.

However, in the spirit of *Sec.(1.3)*, we should wonder what would happen if we want to deal with a bigger space? What if we want to consider all the possible j and their superpositions? How should $\hat{U}_{j_1,j_2}$ be updated? We want to *lift* the Clebsch-Gordan theory for composite systems of fixed j_1 and j_2 $SU(2)$-charges to composite system of $\frac{1}{2}$ and variable j $SU(2)$-charges. This is the aim of the next section

2.4 Lifting the Clebsch-Gordan theory

We start by defining the new Hilbert spaces we will work with. We consider a bipartite Hilbert space

$$\mathcal{H}_{\frac{1}{2}} \otimes \mathcal{H}_G^L \tag{2.41}$$

It is clear that we are aligning with *Eq.(2.7)*. In fact, we consider now the more complex Hilbert space

$$\bigoplus_{j\in\frac{1}{2}\mathbb{N}_0} \left(\mathcal{H}_{j-\frac{1}{2}} \oplus \mathcal{H}_{j+\frac{1}{2}}\right) \tag{2.42}$$

Notation: We will refer to *Eq.(2.41)* and *Eq.(2.42)* as $\mathcal{H}_{fact}$ and $\mathcal{H}_{comp}$, respectively

We wish to define a unitary operator $\hat{U}_{CG} : \mathcal{H}_{fact} \to \mathcal{H}_{comp}$ that generalizes the Clebsch-Gordan couplings between the left and right registers. If we think carefully about what we wrote in the previous lines, we will understand that this aim is quite easy to accomplish: the new operator $\hat{U}_{CG}$ will be the direct sum (see *App.(B)*) over $j \in \frac{1}{2}\mathbb{N}_0$ of all the operators $\hat{U}_{\frac{1}{2},j}$. If we rewrite $\mathcal{H}_{fact}$ as

$$\mathcal{H}_{fact} = \mathcal{H}_{\frac{1}{2}} \otimes \left(\bigoplus_{j\in\frac{1}{2}\mathbb{N}_0} \mathcal{H}_j \right) \cong \bigoplus_{j\in\frac{1}{2}\mathbb{N}_0} \left(\mathcal{H}_{\frac{1}{2}} \otimes \mathcal{H}_j \right) \tag{2.43}$$

the idea should become clearer. We emphasize that this result holds since the distributivity property for $\otimes$ and $\oplus$ is well-defined for Hilbert spaces.

Let j be the variable label for the representations of $SU(2)$, the following two separable Hilbert spaces are isomorphic:

$$\mathcal{H}_{fact} = \bigoplus_{j\in\frac{1}{2}\mathbb{N}_0} \left(\mathcal{H}_{\frac{1}{2}} \otimes \mathcal{H}_j \right) \cong \bigoplus_{j\in\frac{1}{2}\mathbb{N}_0} \left(\mathcal{H}_{j-\frac{1}{2}} \oplus \mathcal{H}_{j+\frac{1}{2}} \right) = \mathcal{H}_{comp} \tag{2.44}$$

Remark 2.8: We emphasize that the labels j_1 and j_2 in the elements of $\bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \mathcal{H}_j$ were redundant when we were considering fixed representations. They become crucial now since we have to deal with different representations. In fact, the orthogonality relations for the elements of $\mathcal{H}_{fact}$ and $\mathcal{H}_{comp}$ are

$$\begin{gathered} \left(\left\langle \frac{1}{2}, \frac{1}{2}s \right| \otimes \langle j, m| \right) \left(\left| \frac{1}{2}, \frac{1}{2}s' \right\rangle \otimes |j', m'\rangle \right) = \delta_s^{s'} \delta_j^{j'} \delta_m^{m'} \\ \langle j_1, j_2; j, m | j_1', j_2'; j', m' | j_1, j_2; j, m | j_1', j_2'; j', m' \rangle = \delta_j^{j'} \delta_m^{m'} \delta_{\{j_1,j_2\}}^{\{j_1',j_2'\}} \end{gathered} \tag{2.45}$$

where $\delta_{\{j_1,j_2\}}^{\{j_1',j_2'\}} = \delta_{j_1}^{j_1'}\delta_{j_2}^{j_2'} + \delta_{j_1}^{j_2'}\delta_{j_2}^{j_1'}$. Moreover, from now on, unless needed, we will simply write $\left|\frac{1}{2}s\right\rangle$ instead of $\left|\frac{1}{2}, \frac{1}{2}s\right\rangle$

We can now provide a definition of the extended Clebsch-Gordan map:

Definition 2.9 (Lifted Clebsch-Gordan map): Let j be the variable label for the representations of $SU(2)$, we call *lifted Clebsch-Gordan map* the intertwining isomorphism that satisfies the Condon-Shortley phase convention and is defined as follows:

$$\begin{aligned}&\hat{U}_{CG} := \bigoplus_{j\in\frac{1}{2}\mathbb{N}_0} \hat{U}_{\frac{1}{2},j} : \mathcal{H}_{fact} \rightarrow \mathcal{H}_{comp}\\ &\qquad\qquad \left|\frac{1}{2}s\right\rangle \otimes |j,m\rangle \mapsto \sum_{J'} C^{J',M}_{\frac{1}{2},j,\frac{1}{2}s,m} \left|\frac{1}{2},j;J',M\right\rangle\end{aligned} \tag{2.46}$$

and the inverse is

$$\begin{aligned}&\hat{U}^{-1}_{CG} = \hat{U}^{\dagger}_{CG} : \mathcal{H}_{comp} \rightarrow \mathcal{H}_{fact}\\ &\qquad \left|\frac{1}{2},j;J,M\right\rangle \mapsto \sum_{s',m'} C^{\frac{1}{2},j,\frac{1}{2}s',m'}_{J,M} \left|\frac{1}{2}s'\right\rangle \otimes |j,m'\rangle\end{aligned} \tag{2.47}$$

We provide an example of the action of this map related to *Re.(2.8)*:

Example 2.10: According to the previous definition and to the explicit form of the Clebsch-Gordan coefficients, we know that:

$$\hat{U}_{CG}\left(\left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2},\frac{1}{2}\right\rangle\right) = \left|\frac{1}{2},\frac{1}{2};1,1\right\rangle \tag{2.48}$$

$$\hat{U}_{CG}\left(\left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes \left|\frac{3}{2},\frac{3}{2}\right\rangle\right) = \frac{1}{2}\left|\frac{1}{2},\frac{3}{2};2,1\right\rangle + \frac{\sqrt{3}}{2}\left|\frac{1}{2},\frac{3}{2};1,1\right\rangle \tag{2.49}$$

According to *Eq.(2.45)*, the starting vector are orthogonal and, since $\hat{U}_{CG}$ is a unitary map, it must preserve orthogonality.

Moreover, we underline that the linear action of $SU(2)$ on $\mathcal{H}_{comp}$ doesn't care about some labels. An example will probably make everything clearer:

Example 2.11: $\left|\frac{1}{2},\frac{1}{2};1,1\right\rangle$ and $\left|\frac{1}{2},\frac{3}{2};1,1\right\rangle$ are both elements of $\mathcal{H}_{comp}$ and, according to *Eq.(2.45)*, they are orthogonal. However,

$$\bigoplus_{j\in\frac{1}{2}\mathbb{N}_0}\left(\hat{R}_{j-\frac{1}{2}}(\hat{z},\theta)\oplus\hat{R}_{j+\frac{1}{2}}(\hat{z},\theta)\right)\left|\frac{1}{2},\frac{1}{2};1,1\right\rangle = e^{-i\theta 1}\left|\frac{1}{2},\frac{1}{2};1,1\right\rangle \tag{2.50}$$

$$\bigoplus_{j\in\frac{1}{2}\mathbb{N}_0}\left(\hat{R}_{j-\frac{1}{2}}(\hat{z},\theta)\oplus\hat{R}_{j+\frac{1}{2}}(\hat{z},\theta)\right)\left|\frac{1}{2},\frac{3}{2};1,1\right\rangle = e^{-i\theta 1}\left|\frac{1}{2},\frac{3}{2};1,1\right\rangle \tag{2.51}$$

This feature holds $\forall g \in SU(2)$. The linear action doesn't care about the labels j_1 and j_2.

Finally, it's useful to show that, passing from $\hat{U}_{\frac{1}{2},j}$ to $\hat{U}_{CG}$, also the property of intertwinig is preserved. The importance of this result will be clear in the next sections

Proposition 2.12: Let $\bigoplus_{j\in\frac{1}{2}\mathbb{N}_0}\left(R_{\frac{1}{2}}\otimes R_j\right)$ be the representation of $SU(2)$ on $\mathcal{H}_{fact}$ and $\bigoplus_{j\in\frac{1}{2}\mathbb{N}_0}\left(R_{j-\frac{1}{2}}\oplus R_{j+\frac{1}{2}}\right)$ be the representation of $SU(2)$ on $\mathcal{H}_{comp}$. The map $\hat{U}_{CG}:\mathcal{H}_{fact}\to\mathcal{H}_{comp}$ is an intertwining map, *i.e.* satisfies

$$\hat{U}_{CG}\circ\left[\bigoplus_{j\in\frac{1}{2}\mathbb{N}_0}\left(\hat{R}_{\frac{1}{2}}(g)\otimes\hat{R}_j(g)\right)\right] = \left[\bigoplus_{j\in\frac{1}{2}\mathbb{N}_0}\left(\hat{R}_{j-\frac{1}{2}}(g)\oplus\hat{R}_{j+\frac{1}{2}}(g)\right)\right]\circ\hat{U}_{CG} \tag{2.52}$$

Proof. In order to prove this result, let's write *Eq.(2.54)* in the following form:

$$\left(\bigoplus_{j\in\frac{1}{2}\mathbb{N}_0}\hat{U}_{\frac{1}{2},j}\right)\circ\left[\bigoplus_{j\in\frac{1}{2}\mathbb{N}_0}\left(\hat{R}_{\frac{1}{2}}(g)\otimes\hat{R}_j(g)\right)\right] = \tag{2.53}$$

$$= \left[\bigoplus_{j\in\frac{1}{2}\mathbb{N}_0}\left(\hat{R}_{j-\frac{1}{2}}(g)\oplus\hat{R}_{j+\frac{1}{2}}(g)\right)\right]\circ\left(\bigoplus_{j\in\frac{1}{2}\mathbb{N}_0}\hat{U}_{\frac{1}{2},j}\right) \tag{2.54}$$

Since the composition of two maps in direct sum acts block-diagonally, leaving each sector invariant, we can further rewrite the equation

$$\bigoplus_{j\in\frac{1}{2}\mathbb{N}_0}\left[\hat{U}_{\frac{1}{2},j}\circ\left(\hat{R}_{\frac{1}{2}}(g)\otimes\hat{R}_j(g)\right)\right]=\bigoplus_{j\in\frac{1}{2}\mathbb{N}_0}\left[\left(\hat{R}_{j-\frac{1}{2}}(g)\oplus\hat{R}_{j+\frac{1}{2}}(g)\right)\circ\hat{U}_{\frac{1}{2},j}\right] \tag{2.55}$$

Which we know is true $\forall j\in\frac{1}{2}\mathbb{N}_0$ according to *Def.(2.7)* □

2.5 The auxiliary channel

If we look carefully at the structure of $\mathcal{H}_{comp}$, we notice that the only case in which the labels $\frac{1}{2}, j$ of the kets are crucial is the following:

Let's j' and j'' be two irreducible representations of $SU(2)$ *s.t.* $j'=j''+1$. In this case, the labels j' and j'' allow to preserve the orthogonality relations in the states

$$\left|\frac{1}{2},j';j'-\frac{1}{2},m\right\rangle\perp\left|\frac{1}{2},j'';j''+\frac{1}{2},m\right\rangle \tag{2.56}$$

In fact, $j'-\frac{1}{2}=j''+\frac{1}{2}$.

Therefore, instead of using $\frac{1}{2},j'$ and $\frac{1}{2},j''$, it's enough to use $|\uparrow\rangle$ and $|\downarrow\rangle$. This saves us from the only source of misunderstanding in a minimal way and leads us to the following chain of isomorphisms:

$$\mathcal{H}_{comp}=\bigoplus_{j\in\frac{1}{2}\mathbb{N}_0}\left(\mathcal{H}_{j-\frac{1}{2}}\oplus\mathcal{H}_{j+\frac{1}{2}}\right)\cong\bigoplus_{j\in\frac{1}{2}\mathbb{N}_0}\left(\mathcal{H}_j\otimes\mathcal{H}_{ch}\right)\cong\left(\bigoplus_{j\in\frac{1}{2}\mathbb{N}_0}\mathcal{H}_j\right)\otimes\mathcal{H}_{ch}=$$

$$=\mathcal{H}_G^L\otimes\mathcal{H}_{ch}\cong\mathbb{C}\otimes\mathcal{H}_G^L\otimes\mathcal{H}_{ch} \tag{2.57}$$

Where $H_{ch}=span\{|\uparrow\rangle,|\downarrow\rangle\}$ and $\mathbb{C}$ has been added to align with *Eq.(2.7)*.

Definition 2.13 (Extended Clebsch-Gordan map): We call *extended Clebsch-Gordan map*

$$\hat{U}_{CG}^{(PA)} : \mathcal{H}_{fact} \to \mathbb{C} \otimes \left(\bigoplus_{j \in \frac{1}{2}\mathbb{N}_0} \mathcal{H}_j \right) \otimes \mathcal{H}_{ch} \tag{2.58}$$

but keep referring to it as $\hat{U}_{CG}$. Notice that also the properties of unitarity and intertwining are preserved (See *App.(B)*). Therefore, from now on, unless needed, we will simply write

$$\hat{U}_{CG} : \mathcal{H}_{fact} \to \mathcal{H}_{comp} \tag{2.59}$$

with $\mathcal{H}_{comp} = \mathbb{C} \otimes \left(\bigoplus_{j \in \frac{1}{2}\mathbb{N}_0} \mathcal{H}_j \right) \otimes \mathcal{H}_{ch}$

We now describe the action of $\hat{U}_{CG}$ explicitly to give the brain a bit of a workout

Example 2.14: Let us take $e_1^{(j,m)}, e_2^{(j,m)} \in \mathcal{H}_{fact}$ and $f_{\uparrow,+}^{(j,m)}, f_{\downarrow,+}^{(j,m)}, f_{\uparrow,-}^{(j,m)}, f_{\downarrow,-}^{(j,m)} \in \mathcal{H}_{comp}$, where

$$e_1^{(j,m)} = \left| +\frac{1}{2} \right\rangle \otimes |j,m\rangle \quad , \quad e_2^{(j,m)} = \left| -\frac{1}{2} \right\rangle \otimes |j,m\rangle$$

$$f_{\uparrow,+}^{(j,m)} = |vac\rangle \otimes \left| j+\frac{1}{2}, m+\frac{1}{2} \right\rangle \otimes |\uparrow\rangle \quad , \quad f_{\downarrow,+}^{(j,m)} = |vac\rangle \otimes \left| j-\frac{1}{2}, m+\frac{1}{2} \right\rangle \otimes |\downarrow\rangle \tag{2.60}$$

$$f_{\uparrow,-}^{(j,m)} = |vac\rangle \otimes \left| j+\frac{1}{2}, m-\frac{1}{2} \right\rangle \otimes |\uparrow\rangle \quad , \quad f_{\downarrow,-}^{(j,m)} = |vac\rangle \otimes \left| j-\frac{1}{2}, m-\frac{1}{2} \right\rangle \otimes |\downarrow\rangle$$

We now give the action of $\hat{U}_{CG}$ on $e_1^{(j,m)}$ and $e_2^{(j,m)}$,

$$\begin{aligned} \hat{U}_{CG}(e_1^{(j,m)}) &= \frac{1}{\sqrt{2j+1}} \left(\sqrt{j+m+1} f_{\uparrow,+}^{(j,m)} - \sqrt{j-m} f_{\downarrow,+}^{(j,m)} \right) \\ \hat{U}_{CG}(e_2^{(j,m)}) &= \frac{1}{\sqrt{2j+1}} \left(\sqrt{j-m+1} f_{\uparrow,-}^{(j,m)} + \sqrt{j+m} f_{\downarrow,-}^{(j,m)} \right) \end{aligned} \tag{2.61}$$

The coefficients, in fact, are exactly the same as *Eq.(2.29)* and *Eq.(2.30)*

2.6 A proposal for an SU(2)-gauge invariant evolution for a right mover

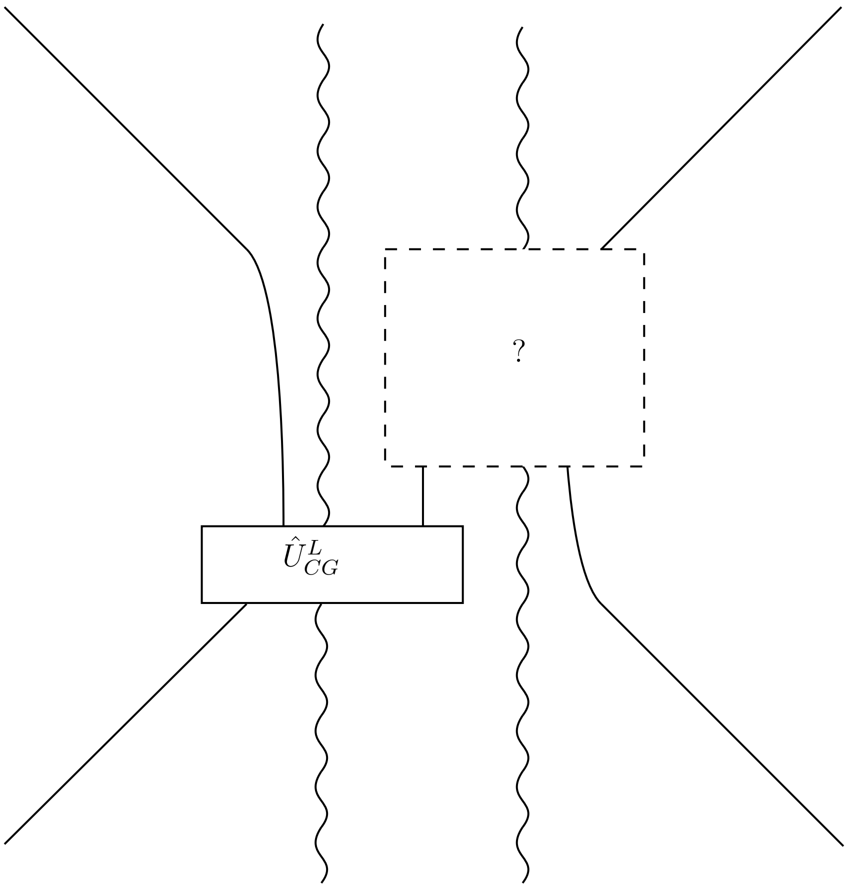

Figure 2.1: A diagramatic representation of the first step of the $SU(2)$-gauge invariant evolution of a right-mover

The lifted Clebsch-Gordan map can be clearly interpreted as the first step of the $SU(2)$-gauge invariant evolution of a right-mover. In order to complete the transport to the other cell, we must act on the right part of the segment state, *i.e.* the one that lives in $\mathcal{H}_{ch} \otimes \mathcal{H}^R_G \otimes \mathbb{C}$ (see *Fig.(2.1)*). However, the tensor product of Hilbert spaces is generally not symmetric and $\hat{U}_{CG}$ is not symmetric in its arguments. Therefore, we will adopt the following notation to distinguish what happens on the left and right parts of the segment state:

$$\begin{aligned}\mathcal{H}^L_{fact} &= \mathcal{H}_{\frac{1}{2}} \otimes \mathcal{H}^L_G \neq \mathcal{H}^R_G \otimes \mathcal{H}_{\frac{1}{2}} = \mathcal{H}^R_{fact}\\ \mathcal{H}^L_{comp} &= \mathbb{C} \otimes \mathcal{H}^L_G \otimes \mathcal{H}_{ch} \neq \mathcal{H}_{ch} \otimes \mathcal{H}^R_G \otimes \mathbb{C} = \mathcal{H}^R_{comp}\end{aligned} \tag{2.62}$$

and provide

Definition 2.15 (Left and right Clebsch-Gordan map): We call *left and right Clebsch-Gordan map*, respectively,

$$\begin{aligned}\hat{U}^L_{CG} &: \mathcal{H}^L_{fact} \to \mathcal{H}^L_{comp}\\ \hat{U}^R_{CG} &: \mathcal{H}^R_{fact} \to \mathcal{H}^R_{comp}\end{aligned} \tag{2.63}$$

Where $\hat{U}^L_{CG}$ is the map defined in *Def.(2.12)* and $\hat{U}^R_{CG}$ can be deduced keeping in mind that

$$C^{J,M}_{\frac{1}{2},j,\frac{1}{2}s,m} = (-1)^{J-\frac{1}{2}-j} C^{J,M}_{j,\frac{1}{2},m,\frac{1}{2}s} \tag{2.64}$$

Unfortunately, we can't use $\hat{U}^{R\,\dagger}_{CG} : \mathcal{H}^R_{comp} \to \mathcal{H}^R_{fact}$ to complete the evolution. It would give us a final segment state that doesn't respect the constraints introduced in *Sec.(2.2)*. We can check this fact by looking at the explicit form of $\hat{U}^{R\,\dagger}_{CG}$.

Example 2.16: Let us start from the definition of the following states $d^{(j,m)}_1, d^{(j,m)}_2 \in \mathcal{H}^R_{comp}$ and $g^{(j,m)}_{\uparrow,+}, g^{(j,m)}_{\downarrow,+}, g^{(j,m)}_{\uparrow,-}, g^{(j,m)}_{\downarrow,-} \in \mathcal{H}^R_{fact}$

$$\begin{aligned}
&d^{(j,m)}_{\uparrow} = |\uparrow\rangle \otimes |j, m'\rangle \otimes |vac\rangle \quad , \quad \tilde{d}^{(j,m)}_{\downarrow} = |\downarrow\rangle \otimes |j, m'\rangle \otimes |vac\rangle\\
&g^{(j,m)}_{1,-} = \left|j - \frac{1}{2}, m' - \frac{1}{2}\right\rangle \otimes \left|+\frac{1}{2}\right\rangle \quad , \quad g^{(j,m)}_{2,-} = \left|j - \frac{1}{2}, m' + \frac{1}{2}\right\rangle \otimes \left|-\frac{1}{2}\right\rangle\\
&g^{(j,m)}_{1,+} = \left|j + \frac{1}{2}, m' - \frac{1}{2}\right\rangle \otimes \left|+\frac{1}{2}\right\rangle \quad , \quad g^{(j,m)}_{2,+} = \left|j + \frac{1}{2}, m' + \frac{1}{2}\right\rangle \otimes \left|-\frac{1}{2}\right\rangle
\end{aligned} \tag{2.65}$$

The action of $\hat{U}^{R\,\dagger}_{CG}$ on $d_1^{(j,m)}$ and $d_2^{(j,m)}$, implementing *Eq.(2.64)*, is

$$\begin{aligned}\hat{U}^{R\,\dagger}_{CG}(d_\uparrow^{(j,m)}) &= \frac{1}{\sqrt{2j}}\left(\sqrt{j+m'}g_{1,-}^{(j,m)} + \sqrt{j-m'}g_{2,-}^{(j,m)}\right)\\ \hat{U}^{R\,\dagger}_{CG}(d_\downarrow^{(j,m)}) &= \frac{1}{\sqrt{2j+2}}\left(\sqrt{j-m'+1}g_{1,+}^{(j,m)} + \sqrt{j+m'+1}g_{2,+}^{(j,m)}\right)\end{aligned} \tag{2.66}$$

Therefore, if we combine the step given by *Eq.(2.31)* and the one given by *Eq.(2.66)*, we end-up with a state for the gauge field that needs to be described by 4 parameters j,j',m and m'.

In order to solve the puzzle, we define a new unitary operator:

$$\hat{A}^R_{CG} : \mathcal{H}^R_{comp} \rightarrow \mathcal{H}^R_{fact} \tag{2.67}$$

that acts on $d_1^{(j,m)}$ and $d_2^{(j,m)}$ as follows:

$$\begin{aligned}\hat{A}^R_{CG}(d_\uparrow^{(j,m)}) &= \frac{1}{\sqrt{2j+2}}\left(+\sqrt{j-m'+1}g_{1,+}^{(j,m)} + \sqrt{j+m'+1}g_{2,+}^{(j,m)}\right)\\ \hat{A}^R_{CG}(d_\downarrow^{(j,m)}) &= \frac{1}{\sqrt{2j}}\left(\sqrt{j+m'}g_{1,-}^{(j,m)} + \sqrt{j-m'}g_{2,-}^{(j,m)}\right)\end{aligned} \tag{2.68}$$

We may be wondering if $\hat{A}^R_{CG}$ is unitary. The answer is yes, and the reason is that we've just exchanged the columns of $\hat{U}^{R\,\dagger}_{CG}$ in order to have a total unitary transformation that preserves the Gauss law. In fact, according to linear algebra, exchanging the columns of a unitary matrix gives us a new matrix that is still unitary.

We also notice that we can decompose $\hat{A}^R_{CG}$ as $\hat{A}^R_{CG} = \hat{U}^{R\,\dagger}_{CG} \circ \left(\hat{T} \otimes \hat{\mathbb{I}} \otimes \hat{\mathbb{I}}\right)$. Looking carefully at *Eq.(2.66)* and *Eq.(2.68)*, it's easy to see that $\hat{T}$ is nothing but a flip of the auxiliary channel.

Finally, we have built a proposal for the $SU(2)$-gauge invariant transport of a right-mover. In the spirit of *Def.(1.14)*, we want to test whether $\hat{W}$ commutes with every local linear action of $SU(2)$. However, before doing so, we need to update the definitions given in *Sec.(1.3)* and fix the notation according to *Sec.(2.2)*.

Chapter 3

SU(2)-gauge invariant massless (1+1) QW

In this last chapter, we will exploit the results obtained in the previous one in order to define the *SU(2)-gauge invariant massless (1+1) right/left-QW* and prove that they are actually $SU(2)$-gauge invariant. This will be the topic of *Sec.(3.1,3.2)*. Moreover, we will present some observations and conjectures regarding the two-particles sector in *Sec.(3.3)*. A complete treatment of each particle sector, including the resolution of several related open issues, goes beyond the scope of the present work and is left to future projects.

3.1 *SU(2)-gauge invariant massless (1+1) right-QW*

We emphasize that, from now on, we will consider the extension of the maps defined in *Sec.(2.6)* in order to take into account also the vacuum sector. Since they act as the identity on the vacuum, we will use the same notation as before.

Definition 3.1 (*SU(2)-gauge invariant massless (1+1) right-QW*): The *SU(2)-gauge invariant massless (1+1) right-QW* is an *SU(2) (1+1) QCA* whose gate $\hat{W}_{rm} : \mathcal{H}_{LOC} \to \mathcal{H}_{LOC}$ reads

$$\hat{W}_{rm} = \left(\hat{\mathbb{I}} \otimes \hat{\mathbb{I}} \otimes \hat{U}_{CG}^{R\,\dagger}\right) \circ \left(\hat{\mathbb{I}} \otimes \hat{\mathbb{I}} \otimes \hat{T} \otimes \hat{\mathbb{I}} \otimes \hat{\mathbb{I}}\right) \circ \left(\hat{U}_{CG}^{L} \otimes \hat{\mathbb{I}} \otimes \hat{\mathbb{I}}\right) \tag{3.1}$$

We emphasize we don't need to antysimmetrize by hand here since we are considering only the one particle sector. Moreover, see *Fig.(3.1)* for a diagrammatic representation of $\hat{W}_{rm}$

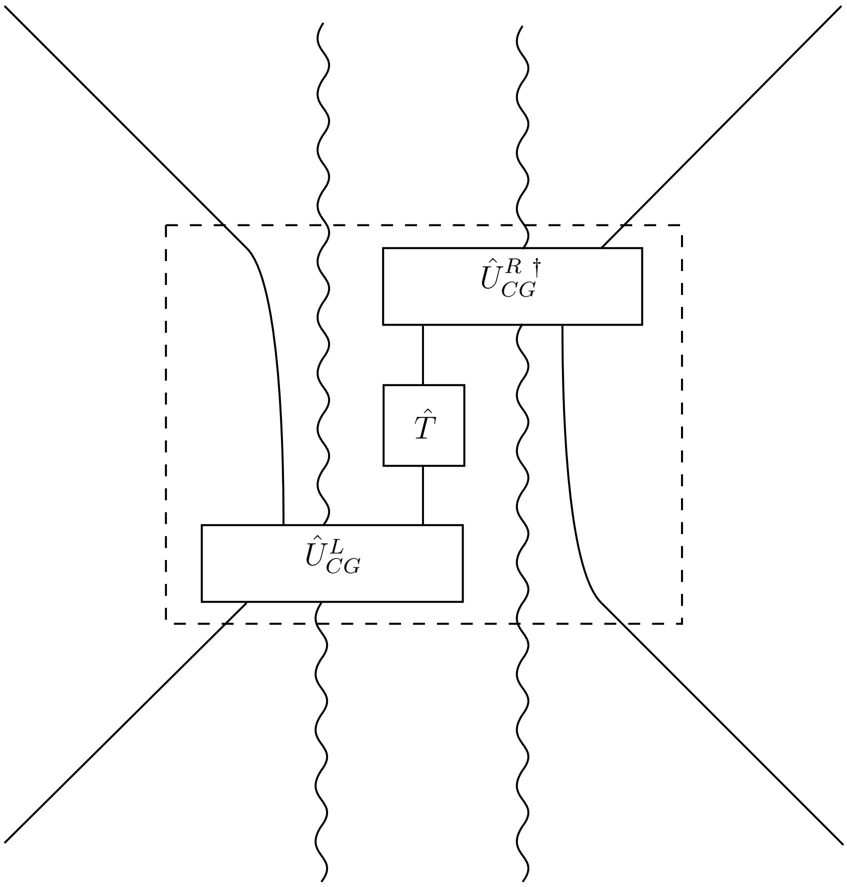

Figure 3.1: A diagramatic representation of the $\hat{W}_{rm}$

Moreover, a *local linear action of SU(2)* on a segment of the *SU(2)-gauge invariant massless (1+1) right-QW* is:

Definition 3.2 (Local linear action of $SU(2)$ for the *right-QW*): Given the *SU(2)-gauge invariant massless (1+1) right-QW*, let $\bigoplus_{j\in\frac{1}{2}\mathbb{N}_0}\left(R_{\frac{1}{2}}\otimes R_j\right)$ be the representation of $SU(2)$ on $\mathcal{H}^L_{fact}$ and $\bigoplus_{j\in\frac{1}{2}\mathbb{N}_0}\left(R_j\otimes R_{\frac{1}{2}}\right)$ be the representation of $SU(2)$ on $\mathcal{H}^R_{fact}$. $\hat{R}(g,g')$ is a *local linear action of* $SU(2)$ on $\mathcal{H}_{LOC}$, with $g,g'\in SU(2)$. It is defined as

$$\hat{R}(g,g') = \left\{ \bigoplus_{j\in\frac{1}{2}\mathbb{N}_0} \left[\left(\hat{\mathbb{I}} \oplus \hat{R}_{\frac{1}{2}}(g) \right) \otimes \hat{R}_j(g) \right] \right\} \otimes \left\{ \bigoplus_{j\in\frac{1}{2}\mathbb{N}_0} \left[\hat{R}_j(g') \otimes \left(\hat{\mathbb{I}} \oplus \hat{R}_{\frac{1}{2}}(g') \right) \right] \right\} \tag{3.2}$$

From now on, unless needed, we will just write:

$$\hat{R}(g,g') = \hat{R}^L(g) \otimes \hat{R}^R(g') \tag{3.3}$$

Finally, we are ready to prove the main result of this chapter

Proposition 3.3: The *(1+1) right-QW* defined in *Def.(3.1)* is *SU(2)-gauge invariant*

Proof. According to *Def.(1.14)*, we must prove that $\hat{W}_{rm}$ satisfies the following equation

$$\hat{W}_{rm} \circ \hat{R}(g,g') = \hat{R}(g,g') \circ \hat{W}_{rm} \quad \forall g, g' \in SU(2) \tag{3.4}$$

where $\hat{R}(g,g')$ is the local linear action of $SU(2)$ as in *Def.(2.17)*.

We emphasize that we must use the right representations in each step, but we already have what we need to show the aimed result. In fact, $\hat{W}$ is a composition of intertwining maps: $\hat{U}^L_{CG}$, and so $\hat{U}^{R\,\dagger}_{CG}$, has been proven to be an intertwining map in *Prop.(2.12)*, while $\hat{\mathbb{I}} \otimes \hat{\mathbb{I}} \otimes \hat{T}$ and $\hat{T} \otimes \hat{\mathbb{I}} \otimes \hat{\mathbb{I}}$ can be trivially proven to be an intertwining map due to *Ex.(2.11)*.

Therefore, each "block" $\hat{R}^X(x)$ can pass through $\hat{W}_{rm}$ without changing the final outcome, *i.e.* *Eq.(3.4)* holds. See *Fig.(3.2)* for a diagrammatic representation

□

3.1.1 Explicit coefficients of $\hat{W}_{rm}$

Given a segment state

$$|\psi_{in}(s,j,m,m')\rangle = \left|\frac{1}{2}s\right\rangle \otimes |j,m\rangle \otimes |vac\rangle \otimes |j,m'\rangle \otimes |vac\rangle \in \mathcal{H}_{LOC} \tag{3.5}$$

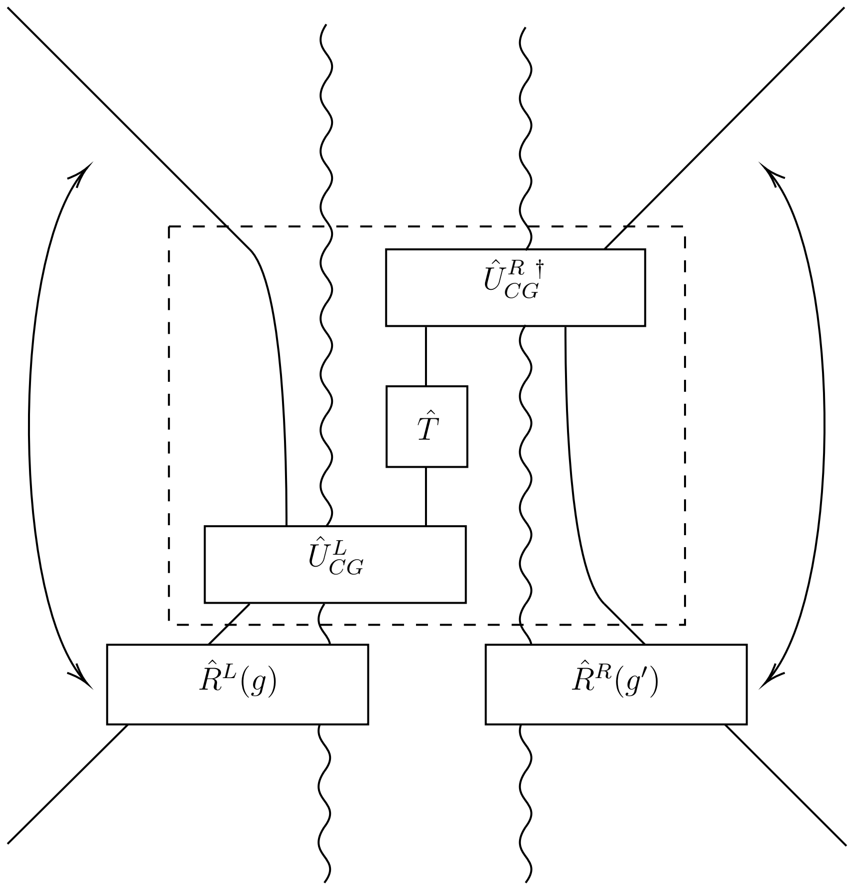

Figure 3.2: A diagramatic representation of *Prop.(2.19)*

we are able to write its unitary and $SU(2)$-gauge invariant evolution as follows:

$$\hat{W}_{rm}\left(|\psi_{in}(s,j,m,m')\rangle\right) =$$

$$= \sqrt{\frac{j+sm+1}{2j+1}}\sqrt{\frac{j-m'+1}{2j+2}}\,|vac\rangle \otimes \left|j+\frac{1}{2}, m+\frac{1}{2}s\right\rangle \otimes \left|j+\frac{1}{2}, m'-\frac{1}{2}\right\rangle \otimes \left|+\frac{1}{2}\right\rangle +$$

$$\sqrt{\frac{j+sm+1}{2j+1}}\sqrt{\frac{j+m'+1}{2j+2}}\,|vac\rangle \otimes \left|j+\frac{1}{2}, m+\frac{1}{2}s\right\rangle \otimes \left|j+\frac{1}{2}, m'+\frac{1}{2}\right\rangle \otimes \left|-\frac{1}{2}\right\rangle -$$

$$s\sqrt{\frac{j-sm}{2j+1}}\sqrt{\frac{j+m'}{2j}}\,|vac\rangle \otimes \left|j-\frac{1}{2}, m+\frac{1}{2}s\right\rangle \otimes \left|j-\frac{1}{2}, m'-\frac{1}{2}\right\rangle \otimes \left|+\frac{1}{2}\right\rangle -$$

$$s\sqrt{\frac{j-sm}{2j+1}}\sqrt{\frac{j-m'}{2j}}\,|vac\rangle \otimes \left|j-\frac{1}{2}, m+\frac{1}{2}s\right\rangle \otimes \left|j-\frac{1}{2}, m'+\frac{1}{2}\right\rangle \otimes \left|-\frac{1}{2}\right\rangle =$$

$$= |\psi_{out}(s,j,m,m')\rangle \tag{3.6}$$

We end this section with a remark that is significant also for the next sections:

Remark 3.4: Looking at *Eq.(3.6)*, we immediately notice that superposition of different j sectors appears in the outcome. Whether this is physically admissible depends critically on whether j is subject to a superselection rule in the relevant physical setting. At the purely kinematical level treated here, the superposition is formally well-defined. As we pointed out in *Sec.(2.2)*, the imposition of physical constraints on the initial state, which may select or forbid such superpositions, is left to future work.

3.2 *SU(2)-gauge invariant massless (1+1) left-QW*

The generalization for left-movers is straightforward.

Definition 3.5 (*SU(2)-gauge invariant massless (1+1) left-QW*): The *SU(2)-gauge invariant massless (1+1) left-QW* is an *SU(2) (1+1) QCA* whose gate $\hat{W}_{lm} : \mathcal{H}_{LOC} \to \mathcal{H}_{LOC}$ reads

$$\hat{W}_{lm} = \left(\hat{U}^{L\,\dagger}_{CG} \otimes \hat{\mathbb{I}} \otimes \hat{\mathbb{I}}\right) \circ \left(\hat{\mathbb{I}} \otimes \hat{\mathbb{I}} \otimes \hat{T} \otimes \hat{\mathbb{I}} \otimes \hat{\mathbb{I}}\right) \circ \left(\hat{\mathbb{I}} \otimes \hat{\mathbb{I}} \otimes \hat{U}^{R}_{CG}\right) \tag{3.7}$$

We emphasize we don't need to antisymmetrize by hand here since we are considering only the one particle sector

Finally, *Def.(3.2)* and *Prop.(3.3)* can be straightforwardly generalized to the case of left-movers paying the necessary attention. We will not delve into it here

3.2.1 Explicit coefficients of $\hat{W}_{lm}$

Given a segment state

$$|\psi_{in}(s,j,m,m')\rangle = |vac\rangle \otimes |j,m\rangle \otimes |vac\rangle \otimes |j,m'\rangle \otimes \left|\frac{1}{2}s\right\rangle \in \mathcal{H}_{LOC} \tag{3.8}$$

we are able to write its unitary and $SU(2)$-gauge invariant evolution as follows:

$$\hat{W}_{lm}\left(|\psi_{in}(s,j,m,m')\rangle\right) =$$

$$= \sqrt{\frac{j+sm'+1}{2j+1}}\left(-\sqrt{\frac{j-m'+1}{2j+2}}\right)\left|+\frac{1}{2}\right\rangle \otimes \left|j+\frac{1}{2}, m-\frac{1}{2}\right\rangle \otimes \left|j+\frac{1}{2}, m'+\frac{1}{2}s\right\rangle \otimes |vac\rangle +$$

$$\sqrt{\frac{j+sm'+1}{2j+1}}\left(-\sqrt{\frac{j+m'+1}{2j+2}}\right)\left|-\frac{1}{2}\right\rangle \otimes \left|j+\frac{1}{2}, m+\frac{1}{2}\right\rangle \otimes \left|j+\frac{1}{2}, m'+\frac{1}{2}s\right\rangle \otimes |vac\rangle +$$

$$s\sqrt{\frac{j-sm'}{2j+1}}\sqrt{\frac{j+m'}{2j}}\left|+\frac{1}{2}\right\rangle \otimes \left|j-\frac{1}{2}, m-\frac{1}{2}\right\rangle \otimes \left|j-\frac{1}{2}, m'+\frac{1}{2}s\right\rangle \otimes |vac\rangle +$$

$$s\sqrt{\frac{j-sm'}{2j+1}}\sqrt{\frac{j-m'}{2j}}\left|-\frac{1}{2}\right\rangle \otimes \left|j-\frac{1}{2}, m+\frac{1}{2}\right\rangle \otimes \left|j-\frac{1}{2}, m'+\frac{1}{2}s\right\rangle \otimes |vac\rangle =$$

$$= |\psi_{out}(s,j,m,m')\rangle \tag{3.9}$$

3.3 Preliminary observations on the two-particle sector

Exploiting the results obtained for the one particle sector, we should be able to deduce the structure of the *SU(2)-gauge invariant massless (1+1) QCA*. This means allowing the presence of more fermions on each cells. In the following, we provide a glimpse into the two-particles sector.

We highlight that each new segment state we will deal with lives in the Hilbert space

$$\mathcal{H}_{LOC} \doteq \left(\mathbb{C} \oplus \mathcal{H}_{\frac{1}{2}} \oplus \mathbb{C}\right) \otimes \mathcal{H}_G^L \otimes \mathcal{H}_G^R \otimes \left(\mathbb{C} \oplus \mathcal{H}_{\frac{1}{2}} \oplus \mathbb{C}\right) \tag{3.10}$$

where the new $\mathbb{C}$ accounts for the possible presence of two fermions with opposite $SU(2)$-charge in accordance with the *CAR*.

When considering two particles, we must deal with two main cases: the two massless fermions can have the same direction or opposite direction. The framework changes considerably depending on this detail. That is why we will deal with each case separately.

3.3.1 Two-particle sector: opposite direction

When we deal with two massless fermions with opposite direction, the generalization should be straightforward. In fact, given a segment state

$$|\psi_{in}(s, s', j, m, m')\rangle = \left|\frac{1}{2}s\right\rangle \otimes |j, m\rangle \otimes |j, m'\rangle \otimes \left|\frac{1}{2}s'\right\rangle \in \mathcal{H}_{LOC} \tag{3.11}$$

we hope that a quantum circuit arranged as in *Fig.(3.3)* satisfies the required properties.

Remark 3.6: We highlight that, according to *Fig.(3.3)* and if this is the right path, the massless fermions would interact with the gauge field, while no direct fermion-fermion interaction would arise, consistently with standard massless *QFT* in $(1+1)$ dimensions. The fermions do not occupy the same cell simultaneously due to their opposite directions of propagation. The only subtlety is a minus sign arising from the *CAR* when the two fermions are exchanged through $\hat{S}$, which must be tracked carefully.

3.3.2 Two-particle sector: same direction

This case requires further efforts. In fact, we never defined a way to make two massless fermions with the same direction evolve. As foreshadowed in *Re.(1.6)*, it is the moment to move to the first-quantization basis since the combination of 3 angular momenta needs the particles to be labeled.

We will provide an outlook of our first idea for the case of two right-movers. The quantum circuit can be easily generalized to the case of two left-movers. Given a segment state

$$|\psi_{in}(s, s', j, m, m')\rangle = \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}\right\rangle\left|-\frac{1}{2}\right\rangle - \left|-\frac{1}{2}\right\rangle\left|\frac{1}{2}\right\rangle\right) \otimes |j, m\rangle \otimes |j, m'\rangle \otimes |vac\rangle \tag{3.12}$$

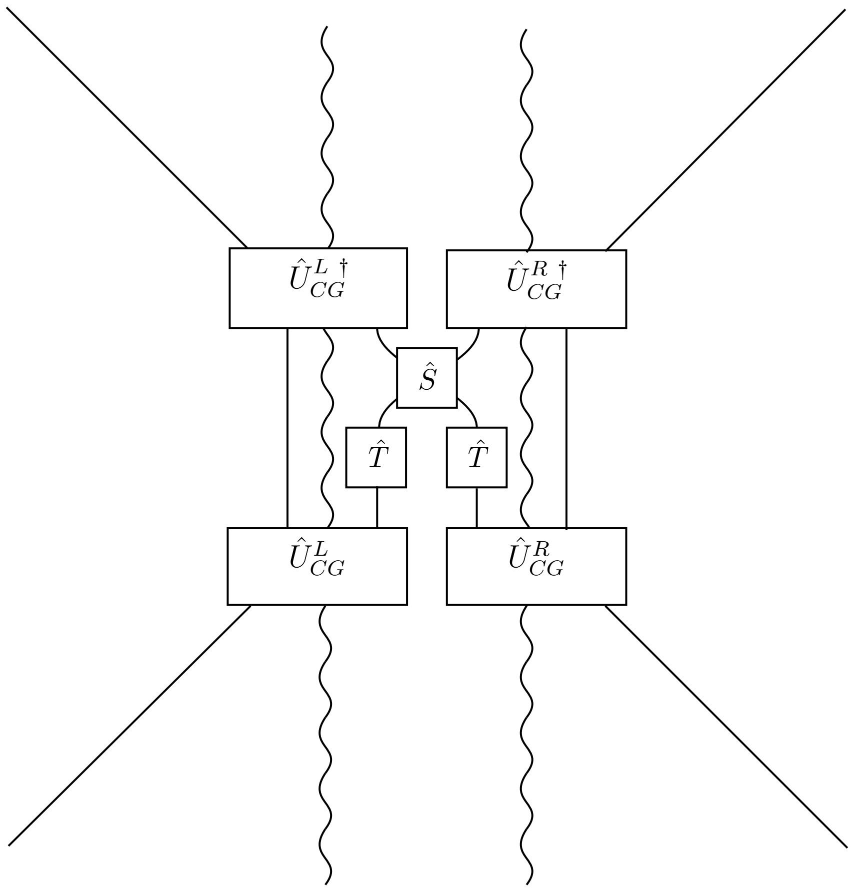

Figure 3.3: A diagramatic representation of our proposal for the two particles with opposite direction sector

Note that the antisymmetric combination in Eq.(3.12) is the unique state (up to a global phase) in the 2-particle sector, consistently with the $\mathbb{C}$ factor in $\mathcal{H}_{LOC}$, which is isomorphic to the antisymmetric subspace $\mathcal{H}_{1/2} \wedge \mathcal{H}_{1/2}$. we hope that a quantum circuit arranged as in *Fig.(3.4)* satisfies the required properties.

Remark 3.7: We highlight that the order in which we combine two or more angular momenta through $\hat{U}_{CG}^{R/L}$ is relevant and changes the result. The two kets tell two different orders of combining the $SU(2)$-charges. We must deal with this subtlety carefully if we want to translate this framework in the occupation mode basis. However, we will delve into this in the conclusions.

A physically motivated requirement for this sector is that a pair of mass-

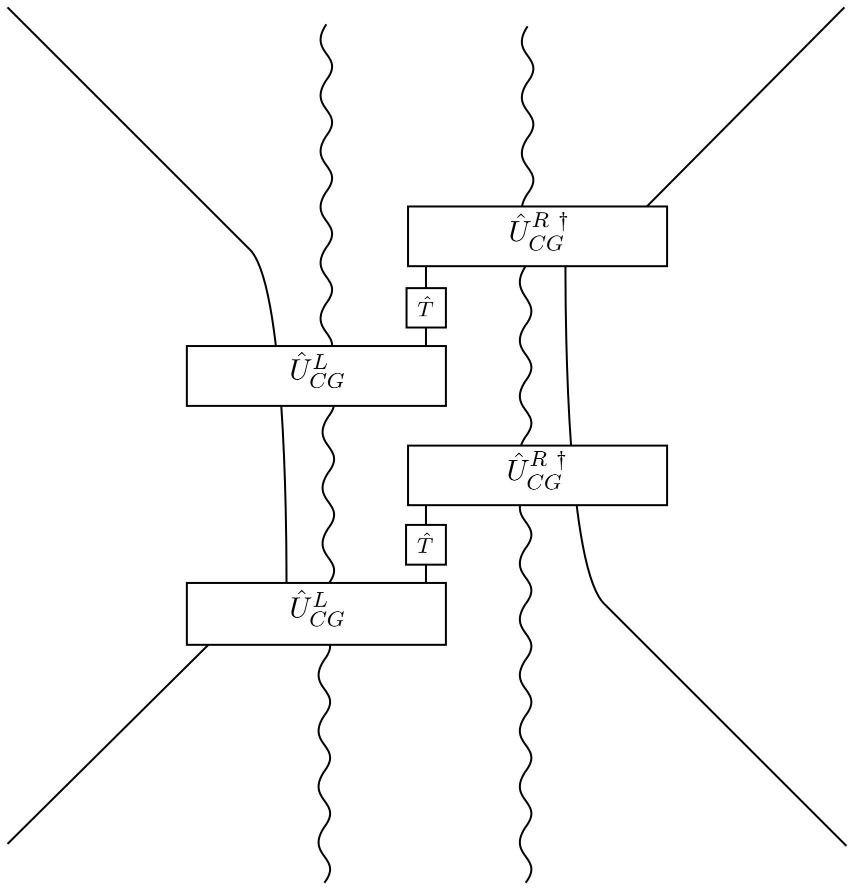

Figure 3.4: A diagramatic representation of our proposal for the two particles with the same direction sector

less fermions propagates through the link without altering the gauge field, consistently with the absence of mass and the neutrality of the singlet state under $SU(2)$.

In conclusion, this could be just one of the many ways of proceeding. In particular, it would be interesting to test whether the same aimed result can be obtained after each bloch that composes the unitary evolution. This would require the introduction of another auxiliary channel. However, we leave these considerations and analysis for future works.

Chapter 4

Conclusions

As we affirmed in the introduction, a rigorous definition of *SU(N)-gauge invariant QCA* is of great interest for three reasons: testing whether a fundamentally discrete quantum theory is tenable, the quantum simulation of fundamental particles and the mathematical classification of structures that are as simple as they are fruitful.

In the previous chapters, we translated the physical idea of $SU(2)$-gauge invariance into the mathematical language of *(1+1) QCA* for massless single fermions. We identified a minimal implementation of this property, highlighted all possible physical interpretations of this framework and gave some hints on how we intend to proceed for the multi-particle sector.

These results are just the beginning, but we expect they can pave the way for a rigorous definition of *SU(N)-gauge invariant (3+1) QCA for massive fermions*. Keeping in mind this aim, we can list which are likely to be the future challenges of this project:

1. Dealing with the physical constraints on the initial states. In *Sec.(3.1,3.2)* we noticed that explicit calculations can become really long if we consider all the mathematically allowed states. However, delving into the physical constraints of the initial state, *e.g.* non-abelian Gauss Law, could help us simplify the algorithm

2. Completing this framework for the multi-particle sector. As we pointed out in *Sec.(3.3)* there are many open issues and we are not sure about how and if it will be possible to deduce rigorously the multi-particle sector from the one-particle sector

3. Translating this framework into the occupation mode basis. Working

in the first quantization basis and encoding the CAR by hand is surely more complicated and less elegant. However, we emphasize again (see *Re.(1.6)*) that we thought it would have been clever, at least for the beginning, to deal with the Clebsch-Gordan theory in its natural basis. We will translate everything into the occupation mode basis and Heisenberg picture. We will probably adapt this framework to the definition of *fermionic cellular automata* (FCA) [28]

4. Understanding whether, and under which conditions, the mathematical structure uniquely determines the physical content. In fact, we never stated any result regarding the uniqueness of this approach. Wondering in how many different ways we can build an *SU(2)-gauge invariant massless (1+1) QCA* would be extremely useful.

5. Generalizing everything for massive fermions. This step introduces additional technical complications, primarily related to the mass terms in the *Free Dirac (1+1) QCA*.

6. Extending the dimensions to $(3 + 1)$. This extension could be guided by Ref.[25]

7. Formulating a theory for general $SU(N)$. This challenge will probably be one of the most difficult due to the fact that the complexity arises significantly with N.

8. Moving to the global picture. Iterating the fundamental quantum circuit and considering a global initial state will allow us to carry out explicit calculations for the framework we developed. As mentioned in the Introduction, we will have to deal with the problem of fermion doubling if we want to do the numerics. Luckily, this problem has already been solved in Ref.[27]. Therefore, this suggests that existing techniques could be adapted to the present framework.

9. Implementing the technology of QCA renormalisation to test the scientific value of these results. As already pointed out, the aim of this project is to understand whether QCA are fundamental structures or whether they can at least simulate QFT. In order to do so, we need a framework capable of reconciling a hypothetical Planck scale of QCA with the usual Fermi scale used in high-energy physics. The theory of QCA renormalisation developed in Ref.[29] may provide the appropriate framework to address this issue.

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Appendix A

This Appendix is dedicated to some useful results of the theory of Matrix Lie groups, associated Lie algebras and their representations[1]

If V is a finite-dimensional real or complex vector space, let $GL(V)$ denote the group of invertible linear transformations of V. If we choose a basis for V, we can identify $GL(V)$ with $GL(n;\mathbb{R})$ or $GL(n;\mathbb{C})$

Definition A.1 (Finite-dimensional complex representation of a group): Let G be a group. A *finite-dimensional complex representation* of G is a group homomorphism

$$\Pi : G \to GL(V) \tag{4.1}$$

where V is a finite-dimensional complex vector space (with $dim(V) \geqslant 1$). As a shorthand, we can say that Π is a representation of G on V

The groups we will deal with are *matrix Lie groups*.

Definition A.2 (Matrix Lie group): A *matrix Lie group* G is a subgroup and a closed submanifold of the general linear group $GL(n;\mathbb{F})$, with $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Equivalently, G is a Lie group whose elements are matrices and whose group structure is given by the usual matrix multiplication. Common examples of matrix Lie groups are: $GL(n;\mathbb{F})$, $SL(n;\mathbb{F})$, $O(n)$, $SO(n)$, $U(n)$ and $SU(n)$.

We let $\mathfrak{gl}(V) = End(V)$ denote the space of all linear operators from V to itself, which forms a *Lie algebra* under the bracket $[X, Y] = XY - YX$.

[1]We draw inspiration from Ref.[11] for this part

Definition A.3 (Finite-dimensional complex representation of a Lie algebra): Let $\mathfrak{g}$ be a complex Lie algebra. A a *finite-dimensional complex representation* of $\mathfrak{g}$ is a Lie algebra homomorphism

$$\pi : \mathfrak{g} \to \mathfrak{gl}(V) \tag{4.2}$$

where V is a finite-dimensional complex vector space (with $dim(V) \geqslant 1$). As a shorthand, we can say that π is a representation of $\mathfrak{g}$ on V

Notation: In the dedicated literature and in the following pages, the term representation is used both as defined above and for V and its elements.

Definition A.4 (Irreducible representation): Let Π be a finite-dimensional real or complex representation of a matrix Lie group G or Lie algebra $\mathfrak{g}$ on V. A subspace W of V is called *invariant* if $\Pi(A)\,(w) \in W \quad \forall w \in W$ and $\forall A \in G$. An invariant subspace W is called *nontrivial* if $W \neq \{0\}$ and $W \neq V$. A representation with no nontrivial invariant subspaces is called *irreducible*.

Moreover, we underline that images of representations are *linear actions* of the represented matrix Lie group or Lie algebra on the chosen vector space.

Definition A.5 (Linear action of G): Given a representation Π of G on V, $\Pi(A) \in GL(V)$ is a *linear action of* G on V, with $A \in G$

We can associate to each matrix Lie group G a Lie algebra $\mathfrak{g}$. Many questions involving a group can be studied by transferring them to the Lie algebra, where we can use tools of linear algebra.

Definition A.6 (Lie algebra associated to a matrix Lie group): Let G be a matrix Lie group. The *Lie algebra associated to* G, denoted $\mathfrak{g}$, is the set of all matrices a such that $e^{ta} \in G \;\; \forall t \in \mathbb{R}$. However, in physics conventions, it is common to work with e^{-ita}. We adopt this convention throughout this thesis.

Now, we can define a useful set of maps:

Definition A.7 (Intertwining map): Let Π be a representation of a matrix Lie group G on V and Σ a representation of the same group on W. A linear map $\phi : V \to W$ is called an *intertwining map* of representations if

$$\phi \circ \Pi(A)\,(v) = \Sigma(A) \circ \phi\,(v) \quad \forall A \in G \;\; and \;\; \forall v \in V \tag{4.3}$$

The analogous property defines intertwining maps of representations of a Lie algebra.

Notation: Let Π_1 and Π_2 be two representations of a matrix Lie group G on V_1 and V_2, respectively. We denote the vector space of the intertwining maps between them as follows:

$$Hom_G(V_1, V_2) := \{\phi \in Hom(V_1, V_2) \mid \phi \circ \Pi_1(A) = \Pi_2(A) \circ \phi \;\; \forall A \in G\} \tag{4.4}$$

Similarly, let π_1 and π_2 be two representations of a matrix Lie algebra $\mathfrak{g}$ on V_1 and V_2, respectively. We denote the vector space of the intertwining maps between them as follows:

$$Hom_{\mathfrak{g}}(V_1, V_2) := \{\phi \in Hom(V_1, V_2) \mid \pi_1(a) \circ \phi = \phi \circ \pi_2(a) \;\; \forall a \in \mathfrak{g}\} \tag{4.5}$$

Moreover, if ϕ is an intertwining map of representations and, in addition, ϕ is invertible, then ϕ is said to be an isomorphism of representations. If there exists an isomorphism between V and W, then the representations are said to be *isomorphic*

Given the definition of interwining map, the following result holds:

Proposition A.8: Let ρ_V, ρ_W and ρ_X be representations of a matrix Lie group G on V, W and X, respectively. Moreover, let

$$f : V \to W, \qquad h : W \to X$$

be intertwining maps, *i.e.* linear maps satisfying

$$f \circ \rho_V(g) = \rho_W(g) \circ f \qquad \text{and} \qquad h \circ \rho_W(g) = \rho_X(g) \circ h \qquad \forall\, g \in G.$$

Then the composition $h \circ f : V \to X$ is again an intertwining map, i.e.

$$(h \circ f) \circ \rho_V(g) = \rho_X(g) \circ (h \circ f) \qquad \forall\, g \in G.$$

Proof. Let $g \in G$ be arbitrary. We compute the left-hand side:

$$\begin{aligned}
(h \circ f) \circ \rho_V(g) &= h \circ \big(f \circ \rho_V(g)\big) \\
&= h \circ \big(\rho_W(g) \circ f\big) && (f \text{ is intertwining}) \\
&= \big(h \circ \rho_W(g)\big) \circ f \\
&= \big(\rho_X(g) \circ h\big) \circ f && (h \text{ is intertwining}) \\
&= \rho_X(g) \circ (h \circ f).
\end{aligned}$$

Hence $h \circ f$ satisfies the intertwining condition. □

For the sake of completeness, we provide the following result:

Theorem A.9: Let G and H be matrix Lie groups, with associated Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Let $\Psi : G \to H$ be a Lie group homomorphism. Then there exists a unique real-linear map $\psi : \mathfrak{g} \to \mathfrak{h}$ *s.t.*

$$\Psi(e^a) = e^{\psi(a)} \tag{4.6}$$

For all $a \in \mathfrak{g}$ and $e^a \in G$. The map ψ has following additional properties:

1. $\psi(AaA^{-1}) = \Psi(A)\psi(a)\Psi(A)^{-1} \quad \forall a \in \mathfrak{g},\ A \in G$
2. $\psi([a, b]) = [\psi(a), \psi(b)] \quad \forall a, b \in \mathfrak{g}$
3. $\psi(a) = \frac{d}{dt}\Psi(e^{ta})\big|_{t=0} \quad \forall a \in \mathfrak{g}$

Proof. See *Th.(3.28)* in Ref.[11] □

and one of its corollary:

Corollary A.10: Let Π_1, Π_2 be two representations of a matrix Lie group G on V_1 and V_2, respectively, and π_1, π_2 be two representations of the associated Lie algebra $\mathfrak{g}$ on V_1 and V_2, respectively, it holds that

$$Hom_G(V_1, V_2) \subset Hom_{\mathfrak{g}}(V_1, V_2) \tag{4.7}$$

Proof. Remember that a representation is a matrix Lie group (resp. Lie algebra) homomorphism into $GL(V)$ (resp. $\mathfrak{gl}(V)$). Therefore, given a generic element ϕ of $Hom_G(V_1, V_2)$, *Th.(A.9)* tells us that the following chain of equalities holds:

$$\phi \circ e^{\pi_1(a)} = \phi \circ \Pi_1(e^a) = \Pi_2(e^a) \circ \phi = e^{\pi_2(a)} \circ \phi \tag{4.8}$$

For all $a \in \mathfrak{g}$ and $e^a \in G$. Moreover, differentiating in $t = 0$, we get

$$\phi \circ \pi_1(a) = \pi_2(a) \circ \phi \tag{4.9}$$

For all $a \in \mathfrak{g}$.

In conclusion, *Eq.(4.9)* is exactly the definition of intertwining map for Lie algebras. This completes the proof □

We can strengten this result for simply connected matrix Lie groups. We begin by stating the following theorem:

Theorem A.11: Let G and H be matrix Lie groups with associated Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Let $\psi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. If G is simply connected, there exists a unique Lie group homomorphisms $\Psi : G \to H$ *s.t.*

$$\Psi(e^a) = e^{\psi(a)} \tag{4.10}$$

For all $a \in \mathfrak{g}$ and $e^a \in G$

Proof. The proof of this theorem can be found in Ref.[11] □

Now, we can prove the following corollary:

Corollary A.12: Let Π_1, Π_2 be two representations of a matrix Lie group G on V_1 and V_2, respectively, and π_1, π_2 be two representations of the associated Lie algebra $\mathfrak{g}$on V_1 and V_2, respectively. If G is simply connected, it holds that

$$Hom_{\mathfrak{g}}(V_1, V_2) \subset Hom_G(V_1, V_2) \tag{4.11}$$

Proof. Remember that a representation is a matrix Lie group (resp. Lie algebra) homomorphism into $GL(V)$ (resp. $\mathfrak{gl}(V)$). A generic element ϕ of $Hom_{\mathfrak{g}}(V_1, V_2)$ is *s.t.*

$$\phi \circ \pi_1(a) = \pi_2(a) \circ \phi \tag{4.12}$$

for all $a \in g$. The definition of exponential of a matrix guarantees that also the following equation holds

$$\phi \circ e^{\pi_1(a)} = e^{\pi_2(a)} \circ \phi \tag{4.13}$$

For all $a \in g$. Moreover, *Th.(A.11)* tells us that the following chain of equalities holds:

$$\phi \circ \Pi_1(e^a) = \phi \circ e^{\pi_1(a)} = e^{\pi_2(a)} \circ \phi = \Pi_2(e^a) \circ \phi \tag{4.14}$$

For all $a \in g$ and $e^a \in G$.

In conclusion, *Eq.(4.14)* is exactly the definition of intertwinig map for matrix Lie groups. This completes the proof □

Combining *Cor.(A.10)* and *Cor.(A.12)*, we get

Corollary A.13: Let Π_1, Π_2 be two representations of a matrix Lie group G on V_1 and V_2, respectively, and π_1, π_2 be two representations of the associated Lie algebra $\mathfrak{g}$ on V_1 and V_2, respectively. If G is simply connected, it holds that

$$Hom_{\mathfrak{g}}(V_1, V_2) = Hom_G(V_1, V_2) \tag{4.15}$$

Proof. It's a consequence of *Cor.(A.10)* and *Cor.(A.12)* □

Finally, one way of generating representations is to take some representations one knows and combine them in some fashion. We will now consider two standard methods of obtaining new representations from old, namely direct sums and tensor products.

Definition A.14 (Direct sum of representations): Let G be a matrix Lie group and let $\Pi_1, \Pi_2, \ldots, \Pi_m$ be representations of G on $V_1, V_2, \ldots, V_m$. Then, the *direct sum* $\Pi_1 \oplus \Pi_2 \oplus \ldots \oplus \Pi_m$ is a representation of G on $V_1 \oplus V_2 \oplus \ldots \oplus V_m$ and defined by

$$[\Pi_1 \oplus \Pi_2 \oplus \ldots \oplus \Pi_m(A)]\,(v_1, v_2, \ldots, v_m) = (\Pi_1(A)v_1, \Pi_2(A)v_2, \ldots, \Pi_m(A)v_m) \tag{4.16}$$

For all $A \in G$. It is straightforward to check that this is really a representation of G.

The definition can be adapted to the Lie algebra case as follows:

$$[\pi_1 \oplus \ldots \oplus \pi_m(a)]\,(v_1, \ldots, v_m) = (\pi_1(a)v_1, \ldots, \pi_m(a)v_m) \tag{4.17}$$

Definition A.15 (Tensor product of representations): Let G be a matrix Lie group and let Π_1 and Π_2 be representations of G on V_1 and V_2. Then, the *tensor product* $\Pi_1 \otimes \Pi_2$ is a representation of G on $V_1 \otimes V_2$ and defined by

$$[\Pi_1 \otimes \Pi_2(A)]\,(v_1 \otimes v_2) = \Pi_1(A)v_1 \otimes \Pi_2(A)v_2 \tag{4.18}$$

For all $A \in G$. It is straightforward to check that this is really a representation of G.

The definition can be adapted to the Lie algebra case as follows:

$$[\pi_1 \otimes \pi_2(a)]\,(v_1 \otimes v_2) = \pi_1(a)v_1 \otimes v_2 + v_1 \otimes \pi_2(a)v_2 \tag{4.19}$$

Remark A.16: We emphasize that, if the original representation are irreducible, the direct sum and tensor product representation will typically not be irreducible when viewed as representations of G

Appendix B

This Appendix is dedicated to some useful results of the Hilbert space theory

Definition B.1 (Hilbert space): Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A *Hilbert space* $\mathcal{H}$ is a vector space over $\mathbb{K}$ equipped with an inner product

$$\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{K} \tag{4.20}$$

satisfying the following properties:

- *Sesquilinearity:*

$$\langle \alpha u + \beta v, w \rangle = \alpha \langle u, w \rangle + \beta \langle v, w \rangle, \qquad \langle u, \alpha v + \beta w \rangle = \bar{\alpha} \langle u, v \rangle + \bar{\beta} \langle u, w \rangle; \tag{4.21}$$

- *Hermitian symmetry:*

$$\langle u, v \rangle = \overline{\langle v, u \rangle}; \tag{4.22}$$

- *Positive definiteness:*

$$\langle u, u \rangle \geqslant 0, \qquad \langle u, u \rangle = 0 \iff u = 0 \tag{4.23}$$

The inner product induces a norm $\|u\| = \sqrt{\langle u, u \rangle}$ and a metric $d(u, v) = \|u - v\|$. The space $\mathcal{H}$ is furthermore required to be:

- *Complete:* every Cauchy sequence in $\mathcal{H}$ converges to an element of $\mathcal{H}$.

We now collect some results and definitions on Hilbert spaces that are needed throughout the thesis

Definition B.2 (Hilbert space isomorphism): Two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are *isomorphic* if there exists at least one *Hilbert space isomorphism*, *i.e.* a linear unitary map, between them.

Theorem B.3: Every separable Hilbert space $\mathcal{H}$ is isomorphic to l^2 (if $dim(\mathcal{H}) = \infty$) or to $\mathbb{C}^N$ (if $dim(\mathcal{H}) = N < \infty$).

Proof. For the proof, see *e.g.* Ref.[2]. □

Corollary B.4: All the separable Hilbert spaces of equal dimension are isomorphic.

Proof. Using *Th.(B.3)* and the transitive property. □

Definition B.5 (Direct sum of linear maps between Hilbert spaces): Given four Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ and $\mathcal{H}_4$ and two linear maps between them $\hat{A} : \mathcal{H}_1 \to \mathcal{H}_2$ and $\hat{B} : \mathcal{H}_3 \to \mathcal{H}_4$, the direct sum of $\hat{A}$ and $\hat{B}$ is a linear map defined as follows:

$$\begin{gathered} \hat{A} \oplus \hat{B} : \mathcal{H}_1 \oplus \mathcal{H}_3 \to \mathcal{H}_2 \oplus \mathcal{H}_4 \\ (x, y) \mapsto (\hat{A}x, \hat{B}y) \end{gathered} \tag{4.24}$$

Moreover, if $\hat{A}$ and $\hat{B}$ are bounded and/or unitary, $\hat{A} \oplus \hat{B}$ is bounded and/or unitary.

This definition can be generalized for the direct sum of a countable family of bounded linear operators as follows:

Definition B.6 (Direct sum of a countable family of linear maps between Hilbert spaces): Given two countable families of Hilbert spaces $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$ and $\{\mathcal{H}'_n\}_{n\in\mathbb{N}}$ and a countable family of bounded linear maps between them $\left\{\hat{A}_n\right\}_{n\in\mathbb{N}}$ *s.t.* $\hat{A}_{\bar{n}} : \mathcal{H}_{\bar{n}} \to \mathcal{H}'_{\bar{n}}$ and $\sup_{n\in\mathbb{N}} \|\hat{A}_n\| < \infty$, the direct sum of the $\hat{A}_n$ is a bounded linear map defined as follows:

$$\begin{gathered} \bigoplus_n \hat{A}_n : \bigoplus_n \mathcal{H}_n \to \bigoplus_n \mathcal{H}'_n \\ (x_n)_n \mapsto (\hat{A}_n x_n)_n \end{gathered} \tag{4.25}$$

Moreover, if all the $\hat{A}_n$ are unitary, $\bigoplus_n \hat{A}_n$ is unitary.

Remark B.7: In *Def.(B.5)* the hypothesis of boundedness is fundamental. Otherwise the direct sum could be badly defined

Remark B.8: Given a countable family of irreducible representations $\{R_j\}_{j\in\frac{1}{2}\mathbb{N}_0}$ of $SU(2)$, the condition $\sup_{j\in\frac{1}{2}\mathbb{N}_0}\|R_j(g)\| < \infty$ is automatically satisfied $\forall g \in SU(2)$ since $\hat{R}_j(g)$ is a unitary operator $\forall\ j \in \frac{1}{2}\mathbb{N}_0$ and $\forall\ g \in SU(2)$ (see *Def.(2.2)*). Therefore, $\bigoplus_j R_j$ as defined above is a well-defined representation of $SU(2)$.

Proposition B.9: Let $\phi : \mathcal{H} \to \mathcal{K}_1$ and $\chi : \mathcal{K}_1 \to \mathcal{K}_2$ be two Hilbert space isomorphism. Then, $\psi = f\circ\phi$ is a Hilbert space isomorphism. Moreover, if ϕ is an intertwining map between the representations ρ and ρ_1 of a matrix Lie group G, $\psi = f \circ \phi$ is an intertwining map between ρ and $\rho_2 := f \circ \rho_1 \circ f^{-1}$

Proof. The first point is trivial. Let us check the intertwining condition:

$$\psi \circ \rho(g) = f \circ \phi \circ \rho(g) = f \circ \rho_1(g) \circ \phi = f \circ \rho_1(g) \circ f^{-1} \circ f \circ \phi = \rho_2(g) \circ \psi \tag{4.26}$$

This completes the proof □