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BV solutions for rate-independent evolutions in brittle fracture

Tesi di Laurea Magistrale in Matematica

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Introduction

Fracture mechanics was originally developed in 1921, when A.A. Griffith introduced in [6] a mechanical description of crack behaviour in brittle elastic materials. This framework is based on the so-called Griffith criterion, which can be derived through a variational approach, where crack evolution is described in terms of energies and their energy variations with respect to changes in the crack set. The central difficulty of this approach is the dependence of the energy functional on a varying domain: given Ω and an initial crack K_0, the elastic energy is defined on the cracked configuration $\Omega \setminus K_0$ as:

$$E_0 = \int_{\Omega \setminus K_0} \varepsilon(u_0) : \sigma(u_0)\, dx$$

where u_0 denotes the displacement field in $\Omega \setminus K_0$, while $\varepsilon(u_0)$ and $\sigma(u_0)$ are the associated strain and stress tensors, respectively. When the crack evolves to a new configuration K_s, the domain changes accordingly to $\Omega \setminus K_s$, and the energy becomes:

$$E_s = \int_{\Omega \setminus K_s} \varepsilon(u_s) : \sigma(u_s)\, dx.$$

The problem is that a fundamental object, i.e. the energy release rate, is defined as the derivative of E:

$$G = \lim_{s \to 0} \frac{E_s - E_0}{s}.$$

Since the parameter s affects the domain of integration, the computation of this derivative presents significant analytical difficulties. To overcome this issue, in this thesis we reformulate the problem by pulling back the energy onto the fixed reference configuration $\Omega \setminus K_0$. In particular, we focus on curved crack evolutions and assume that the crack evolution can be described through suitable diffeomorphisms of the initial configuration.
The main objective of the thesis is to show that, at least in dimension one, the crack length as a function of the evolution parameter is a function of bounded variation.
In the first chapter, we introduce the preliminary notions on the spaces BV and SBV. In particular, we present the definition of variation and its main properties, together with weak* and strict convergence results and their characterisations. Special attention is devoted to the one-dimensional case, where several properties of BV functions are specific to the setting and do not extend to higher dimensions. We introduce the notions of pointwise variation and essential variation, and their relationship with the variation of a function. We also discuss the concept of the good representative of a BV function and its properties. Furthermore, we present the decomposition of a BV function into its absolutely continuous, jump, and Cantor parts. Finally, we briefly introduce the space SBV (special functions of bounded variation), consisting of BV functions without Cantor part, and discuss basic compactness and extension properties.
In the second chapter, we first introduce the notions of strain and stress and define the elastic energy in terms of these quantities. We then begin by studying the following classical boundary value problem:

$$\begin{cases} -\underline{\operatorname{div}}(\sigma(u)) = f & \Omega \\ u = g & \partial\Omega \end{cases}$$

To prove the existence and uniqueness of the solution, we introduce the associated variational formulation in terms of a bilinear form $a(u, v)$. We then establish the standard properties of this form and conclude the analysis by applying the Lax–Milgram theorem. Subsequently, we move to the cracked domain and consider the following boundary value problem:

$$\begin{cases} -\operatorname{div}(\sigma(u_s)) = 0 & \Omega \setminus K_s \\ u_s = g & \partial_D\Omega \\ \sigma(u_s) \cdot \nu = 0 & \partial_N\Omega \\ \sigma(u_s) \cdot \nu_{K_s} = 0 & K_s^{\pm} \end{cases}$$

where $K_s^{\pm}$ denotes the two faces of the crack. To analyse this system, we derive a suitable variational formulation. As previously noted, it is not convenient to treat the variations in the domain, therefore, we pull back the formulation to the reference configuration $\Omega \setminus K_0$. This approach assumes that K_s can be described as a diffeomorphism of K_0. Consequently, we define the energy functionals on both $\Omega \setminus K_0$ and $\Omega \setminus K_s$ and prove a differentiation under the integral sign result. This proves to be quite convenient because, as shown in the proof, the derivative "enters" the integral as the derivative of a matrix, effectively reducing the shape sensitivity analysis to a more manageable algebraic calculation on the reference domain. Finally, we introduce the energy release rate as the derivative of the elastic energy with respect to the space parameter.

In the third chapter, we begin by stating the classical Griffith criterion for fractures in brittle materials, followed by the introduction of the evolution problem. We first establish the setting by introducing the critical value G_c, which depends on the toughness of the materials (i.e., the resistance of the material to the propagation of a pre-existing crack). The reference domain is a rectangular domain Ω with Dirichlet conditions on the upper and lower edges, and homogeneous Neumann conditions on the remaining sides. Furthermore, prescribe a time-dependent boundary condition $\hat{g} = \alpha(t)g$.

Starting with the classical case, we introduce the free energy functional $\mathscr{F}(t, l)$ and study the crack evolution governed by the Kuhn-Tucker conditions:

$$\begin{cases} \dot{\ell}(t) \geq 0, \\ G(t, \ell(t)) \leq G_c, \\ \big(G(t, \ell(t)) - G_c\big)\dot{\ell}(t) = 0. \end{cases}$$

which express the irreversibility condition of the growth of the crack, the activation condition of the propagation that occurs when the critical value is reached, and the equilibrium condition. Through subsequent computations, we obtain the following energy balance formula:

$$\mathcal{E}(T, \ell(T)) = \mathcal{E}(0, \ell(0)) + L_{\text{ext}}(0, T) - G_c\ell(T).$$

However, in this setting, the existence of such a crack path is not guaranteed. Consequently, the problem must be reformulated in a weak sense:

$$\begin{cases} G(t,\, \ell(t)) \leq G_c, & \forall t \in [0, T], \\ \big(G(t,\, \ell^-(t)) - G_c\big)\, d\ell(t) = 0, & \\ \int_A \big(G(t, \ell^-(t)) - G_c\big)\, d\,\ell(t) = 0 & \text{for every Borel set } A \subset [0, T] \end{cases}$$

Here, ℓ is treated as a Bounded Variation function, implying that $\dot{\ell}$ is a measure. Accordingly, the integral is understood in the sense of measures. The primary differences lie in the use of the left limit $G(t, \ell^-(t))$ (which is natural, as activation depends on the evolution up to time t) and the measure $d\ell$. We then observe that $\mathscr{F}(\cdot, \ell(\cdot))$ belongs to BV. Nevertheless, this formulation remains insufficient, as the evolution also depends on the behaviour of G at the jump points.

In the final part of the chapter, we discuss evolution through stationary points. We employ a time-discretization technique, setting Δt^k as a positive sequence of time increments such that $\Delta t^k \searrow 0$. For each k, we define a uniform discretization $t_n^k = n\Delta t^k$ and the sequence ℓ_n^k as:

$$\ell_0^k = l_0, \qquad \ell_n^k = \text{argmin}\{G_c(l - \ell_{n-1}^k) : l \in [\ell_{n-1}^k, L], \text{ with } G(t_n^k, l) \leq G_c\}.$$

The evolution is then presented as the limit of ℓ_n^k, as $\Delta t_k \searrow 0$. Under the assumption of strict monotonicity for α, we prove the following variational representation of ℓ_G:

$$\ell_G(t) = \min\{l \in [l_0, L] : G(t, l) \leq G_c\}.$$

Moreover, we get the properties of uniqueness of ℓ_G, the (left) continuity and the BV regularity. By introducing the "last time" the crack length measures l_0 and T as the "time of failure" (i.e. when $\ell_G = L$) as:

$$t_0 = \sup\{t \in [0, +\infty) : \ell_G(t) = \ell_0\}, \qquad T = \inf\{t \in [0, +\infty) : \ell_G(t) = L\}.$$

we obtain the following improved description of the evolution.:

$$\begin{cases} G(t, \ell_G(t)) \leq G_c & \text{in } [0, +\infty); \\ G(t, \ell_G(t)) & \text{is continuous in } [0, +\infty) \setminus \{T\}; \\ (G(t, \ell_G^-(t)) - G_c) d\,\ell_G(t) = 0 & \text{in the sense of measures in } [0, +\infty) \\ (G(t, l) - G_c) \geq 0 & \text{for every } l \in [\ell_G^-(t), \ell_G^+(t)] \setminus \{L\}. \end{cases}$$

The fourth condition in this formulation clarifies that, at jump points, the crack propagates in an "uncontrolled" (unstable) manner. We then present a rate-independent representation result for ℓ_G, confirming that the evolution depends only on the sequence of states rather than the velocity. The uniqueness of ℓ_g can be stated even in the case of convex energy.
Finally, we briefly examine the case where α is non-monotonic. In this context, a variational representation is less straightforward, and uniqueness is no longer guaranteed. By introducing the non-decreasing envelope $\overline{\alpha}$, we demonstrate that $\ell_G(t) = \lambda_G(\overline{\alpha}(t))$ constitutes a quasi-static evolution. The chapter concludes with an overview of snap-back behaviour, where the crack evolves even when the condition $G(t, l) \geq G_c$ is not satisfied at the current configuration $(t, l(t))$.

Chapter 1

Functions of Bounded Variation

In this chapter, we start with the classical theory, first for the space BV and then for the space SBV.

1.1 The space BV

Definition 1.1 (Function of Bounded Variation)**.** *Let $u \in L^1(\Omega)$. We say that u is a* function of bounded variation in Ω *if the distributional derivative $\mathcal{D}u$ of u is representable by a finite Radon measure $\mu \in \mathcal{M}_b(\Omega)$, that is:*

$$\int_\Omega uv'\,dx = -\int_\Omega v\,d\mu, \qquad \forall v \in C_c^1(\Omega).$$

The space of all functions of bounded variation in Ω is denoted by $BV(\Omega)$.

Remark. The Sobolev space $W^{1,1}(\Omega)$ is contained in $BV(\Omega)$. The inclusion is strict: an example is given by the Heaviside function, whose distributional derivative is the Dirac measure δ_0.

Proposition 1.1 (Properties of $\mathcal{D}u$)**.** *Let $u \in BV_{\text{loc}}(\Omega)$. Then the following hold:*

1. *For any locally Lipschitz function $v : \Omega \to \mathbb{R}$, we have $uv \in BV_{\text{loc}}(\Omega)$. Moreover:*

$$\langle \mathcal{D}(uv), \phi\rangle = \langle v\,\mathcal{D}u, \phi\rangle + \langle u\,v', \phi\rangle \qquad \forall \phi \in C_c^\infty(\Omega).$$

2. *Let ρ_ε be a convolution kernel, and define:*

$$\Omega_\varepsilon = \{x \in \Omega : d(x, \partial\Omega) > \varepsilon\}.$$

Then:

$$(u * \rho_\varepsilon)' = \mathcal{D}u * \rho_\varepsilon.$$

3. *If $\mathcal{D}u = 0$, then u is equivalent to a constant on each connected component of Ω.*

Proof. 1. Let $v_n \in C_c^\infty(\Omega)$ be such that $v_n \to v$ in $W^{1,1}_{\text{loc}}(\Omega)$. Then:

$$\langle \mathcal{D}(v_n u), \phi\rangle = -\langle v_n u, \phi'\rangle = -\int_\Omega v_n u \phi'\,dx \to -\int_\Omega vu\phi'\,dx = -\langle v\,u, \phi'\rangle = \langle \mathcal{D}(vu), \phi\rangle, \;\; \forall \phi \in C_c^\infty(\Omega).$$

On the other hand:

$$\langle \mathcal{D}(v_n u), \phi\rangle = \langle u\mathcal{D}v_n + v_n\mathcal{D}u, \phi\rangle = \langle u\mathcal{D}v_n, \phi\rangle + \langle v_n\mathcal{D}u, \phi\rangle \to \langle u\mathcal{D}v, \phi\rangle + \langle v\mathcal{D}u, \phi\rangle, \qquad \forall \phi \in C_c^\infty(\Omega).$$

2. This follows from standard properties of convolutions in the theory of distributions.

3. It follows from point (2).

□

Now, we introduce the definition of the *variation* of a function $u \in L^1_{\text{loc}}(\Omega)$.

Definition 1.2 (Variation)**.** *Let $u \in L^1_{\text{loc}}(\Omega)$. We define the* variation of u in Ω *as follows:*

$$V(u,\Omega) := \sup_{\substack{v\in C_c^\infty(\Omega)\\ \|v\|_\infty\le 1}} \left\{ \int_\Omega uv'\,dx \right\} = \sup_{\substack{v\in C_c^\infty(\Omega)\\ \|v\|_\infty\le 1}} \langle \mathcal{D}u, v\rangle_{((C_c^\infty)^*, C_c^\infty)} = \sup_{\substack{v\in C_c^\infty(\Omega)\\ \|v\|_\infty\le 1}} \langle \mathcal{D}u, v\rangle_{(\mathcal{M}_b, C_c)}.$$

Remark. The definition is still valid taking $v \in C_c^1(\Omega)$. Moreover, suppose u is regular enough. Then, using integration by parts, we obtain $V(u,\Omega) = \int_\Omega |\mathcal{D}u|\,dx$.

Proposition 1.2. *Let $u \in L^1(\Omega)$. Then the following hold:*

1. *if $u \in BV(\Omega)$, then $V(u,\Omega) = |\mathcal{D}u|(\Omega)$.*
2. *$u \in BV(\Omega)$ if and only if $V(u,\Omega) < +\infty$.*
3. *The mapping $u \mapsto |\mathcal{D}u|(\Omega)$ is lower semi-continuous in the $L^1_{\text{loc}}(\Omega)$ topology.*

Proof. 1.

$\boxed{\le}$ Since $u \in BV(\Omega)$:

$$\int_\Omega uv'\,dx = -\int_\Omega v\,d\mathcal{D}u \le \|v\|_\infty |\mathcal{D}u|(\Omega).$$

Therefore, $V(u,\Omega) \le |\mathcal{D}u|(\Omega)$.

$\boxed{\ge}$ It is sufficient to prove that:

$$\left| \int_\Omega uv'dx \right| \le V(u,\Omega)\|v\|_\infty \quad \forall v \in C_c^1(\Omega). \tag{1.1}$$

In fact, we know that $C_c^1(\Omega) \hookrightarrow C_0(\Omega)$, the inclusion being dense. Thus, we can define $L \in (C_0(\Omega))^*$ by $L(v) = \int_\Omega uv'\,dx$. Hence:

$$\|L\| = \sup_{\substack{v\in C_c^1\\ \|v\|_\infty\le 1}} \left| \int_\Omega uv'dx \right| \le V(u,\Omega).$$

By the Riesz representation theorem, there exists a unique $\mu \in \mathcal{M}_b(\Omega)$ such that $\|L\| = |\mu|(\Omega)$. Since $u \in BV(\Omega)$, we have $\mu = \mathcal{D}u$ and we can say that $|\mathcal{D}u|(\Omega) \le V(u,\Omega)$.
In order to prove (1.1), we consider a sequence of mollifiers ρ_n and $u_n = u * \rho_n$. Let $v \in C_c^1(\Omega)$. Since $u_n \in C^\infty(\Omega)$, then, for all n, we have:

$$\left| \int_\Omega u_n v'\,dx \right| = \left| \int_\Omega u_n' v\,dx \right| \le \|v\|_\infty \int_\Omega |u_n'|\,dx = \|v\|_\infty V(u_n,\Omega).$$

We know that $u_n \xrightarrow{L^1(\Omega)} u$. Therefore:

$$\|u_n v' - uv'\|_{L^1} \le \|u_n - u\|_{L^1}\|v'\|_{L^\infty} \longrightarrow 0 \Longrightarrow u_n v' \xrightarrow{L^1(\Omega)} uv'.$$

As a consequence, since the variation is lower semi-continuous in the $L^1_{\text{loc}}(\Omega)$ topology, we obtain that $V(u_n,\Omega) \to V(u,\Omega)$. Passing to the limit, we have (1.1).

2. Similar to 1.

3. The thesis follows from the lower semi-continuity of the variation.

□

Remark. From now on, we denote by $V(u,\Omega)$ the variation of a function u in Ω (when it is defined). We will use $|\mathcal{D}u|(\Omega)$ to specifically denote the variation of a BV function.

Remark. $BV(\Omega)$ is a Banach space with the following norm:

$$\|u\|_{BV} = \int_\Omega |u|\,dx + |\mathcal{D}u|(\Omega).$$

However, the norm-topology is too strong in many applications. For example, the space $C^1(\Omega)$ is not dense in $BV(\Omega)$, not even in the scalar case. To see this, consider $u \in BV(\Omega)$ such that $\mathcal{D}u$ is nonzero and $\mathcal{D}u \perp \mathscr{L}^1$. Let $v \in C^1(\Omega) \cap BV(\Omega)$ (thus $\mathcal{D}v = v'\,dx \ll \mathscr{L}^1$). Then:

$$|\mathcal{D}(u-v)|(\Omega) = |\mathcal{D}u|(\Omega) + |\mathcal{D}v|(\Omega) \geq |\mathcal{D}u|(\Omega) > 0.$$

This is in contrast with the fact that $BV(\Omega)$ functions can be approximated in the $L^1(\Omega)$ topology by "smooth functions" whose gradients are bounded in $L^1(\Omega)$. Assume $\Omega = \mathbb{R}$. Let ρ_ε be a convolution kernel and let $u_\varepsilon = u * \rho_\varepsilon$. Using Proposition 1.1(2) and some results about the convolution between a continuous function and a Radon measure, we have the following:

$$|\mathcal{D}u_\varepsilon|(\mathbb{R}) = \int_\mathbb{R} |u_\varepsilon'|\,dx = \int_\mathbb{R} |\mathcal{D}u * \rho_\varepsilon|\,dx \leq |\mathcal{D}u|(\mathbb{R}).$$

In particular, by the lower semi-continuity of the variation, $|\mathcal{D}u_\varepsilon|(\mathbb{R})$ converges to $|\mathcal{D}u|(\mathbb{R})$ as $\varepsilon \to 0$. We now give a local version of this result.

Proposition 1.3. *Let $u \in BV(\Omega)$ and let $U \Subset \Omega$ such that $|\mathcal{D}u|(\partial U) = 0$. Then:*

$$\lim_{\varepsilon\to 0} |\mathcal{D}u_\varepsilon|(U) = |\mathcal{D}u|(U).$$

In particular, $\lim_{\varepsilon\to 0} |\mathcal{D}u_\varepsilon|(B_\varrho(x)) = |\mathcal{D}u|(B_\varrho(x))$ for any ball $B_\varrho(x) \subset\subset \Omega$ such that $|\mathcal{D}u|(\partial B_\varrho(x)) = 0$.

Proof. By the lower semi-continuity of the variation, we have $\liminf_{\varepsilon\to 0} |\mathcal{D}u_\varepsilon|(U) \geq |\mathcal{D}u|(U)$. On the other hand, denoting by U_ε the open ε-neighbourhood of U, we infer (using standard results from measure theory):

$$\limsup_{\varepsilon\to 0} |\mathcal{D}u_\varepsilon|(U) \leq \limsup_{\varepsilon\to 0} |\mathcal{D}u|(U_\varepsilon) = |\mathcal{D}u|(\overline{U}) = |\mathcal{D}u|(U).$$

□

Now, we show that, for a general domain Ω, the approximation by smooth functions with bounded gradient in $L^1(\Omega)$ is a characteristic property of BV functions. We may consider this theorem as the analogue of the Meyers-Serrin theorem for Sobolev spaces.

Theorem 1.1 (Approximation by smooth functions)**.** *Let $u \in L^1(\Omega)$. Then $u \in BV(\Omega)$ if and only if there exists a sequence $(u_n) \subset C^\infty(\Omega)$ converging to u in $L^1(\Omega)$ such that:*

$$L = \lim_{n\to+\infty} \int_\Omega |u_n'|\,dx < \infty. \tag{1.2}$$

Furthermore, the least such constant L in (1.2) is $|\mathcal{D}u|(\Omega)$.

Proof. $\Leftarrow$) Let u be approximated in $L^1(\Omega)$ by smooth functions satisfying (1.2). Possibly extracting a subsequence, we can assume that the measures $u_n' \mathscr{L}^1$ converge weakly* in Ω to some measure μ in $\mathcal{M}_b(\Omega)$ such that $|\mu|(\Omega) \le L$ (by weak* compactness of finite Radon measures). Passing to the limit as $n \to +\infty$ in the classical integration by parts formula:

$$\int_\Omega u_n \phi' \, dx = -\int_\Omega \phi \, u_n' \, dx \qquad \forall \phi \in C_c^1(\Omega).$$

we obtain $u \in BV(\Omega)$ with $\mathcal{D}u = \mu$. In particular, $|\mathcal{D}u|(\Omega) = |\mu|(\Omega) \le L$.

$\Rightarrow$) Let $u \in BV(\Omega)$. For any $\delta > 0$, we construct $v_\delta \in C^\infty(\Omega)$ such that:

$$\int_\Omega |u - v_\delta| \, dx < \delta; \qquad \int_\Omega |v_\delta'| \, dx \le |\mathcal{D}u|(\Omega) + \delta. \tag{1.3}$$

To this end, we notice that Ω can be written as the union of a countable family of sets $\{\Omega_n\}_{n\ge1}$ with compact closure in Ω and such that any point of Ω belongs to at most four sets Ω_n. For instance, this family can be obtained by setting:

$$\Omega_{k,1} = \left\{ x \in \Omega \cap B_{k+1} \setminus \overline{B}_{k-1} : d(x, \partial\Omega) > \frac{1}{2} \right\}$$

and:

$$\Omega_{k,p} = \left\{ x \in \Omega \cap B_{k+1} \setminus \overline{B}_{k-1} : \frac{1}{p-1} > d(x, \partial\Omega) > \frac{1}{p+1} \right\}$$

for integers $k \ge 1$, $p > 1$, with $B_0 = \emptyset$. Let $\{\varphi_n\}$ a partition of unity relative to the covering $\{\Omega_n\}$, we can find $\varepsilon_n > 0$ such that $\operatorname{supp}\big((u\varphi_n) * \rho_{\varepsilon_n}\big) \subset \Omega_n$ and:

$$\int_\Omega \Big[|u\varphi_n * \rho_{\varepsilon_n} - u\varphi_n| + |(u\varphi_n') * \rho_{\varepsilon_n} - u\varphi_n'| \Big] \, dx < 2^{-n}\delta. \tag{1.4}$$

The function $v_\delta = \sum_n (u\varphi_n) * \rho_{\varepsilon_n}$ is smooth in Ω because the sum is locally finite. Moreover, our choice of ε_n gives:

$$\int_\Omega |v_\delta - u| \, dx \le \sum_{n=1}^{+\infty} \int_\Omega |(u\varphi_n) * \rho_{\varepsilon_n} - u\varphi_n| \, dx < \delta.$$

Using Proposition 1.1(1), we obtain:

$$\begin{aligned} v_\delta' &= \sum_{n=1}^{+\infty} \big((u\varphi_n) * \rho_{\varepsilon_n}\big)' = \sum_{n=1}^{+\infty} \big(\mathcal{D}(u\varphi_n)\big) * \rho_{\varepsilon_n} = \sum_{n=1}^{+\infty} (\varphi_n \mathcal{D}u) * \rho_{\varepsilon_n} + \sum_{n=1}^{+\infty} (u\varphi_n') * \rho_{\varepsilon_n} = \\ &= \sum_{n=1}^{+\infty} (\varphi_n \mathcal{D}u) * \rho_{\varepsilon_n} + \sum_{n=1}^{+\infty} [(u\varphi_n') * \rho_{\varepsilon_n} - u\varphi_n']. \end{aligned}$$

Thanks to (1.4) and general results on convolutions, we obtain:

$$|\mathcal{D}v_\delta|(\Omega) = \int_\Omega |v_\delta'| dx < \delta + \sum_{n=1}^{+\infty} \int_\Omega \varphi_n \, d|\mathcal{D}u| = \delta + |\mathcal{D}u|(\Omega).$$

This proves the existence of v_δ. Choosing $\delta_n = 2^{-n}$ and setting $u_n = v_{\delta_n}$, from (1.3) we find:

$$\lim_{n\to+\infty} \int_\Omega |u_n - u| \, dx = 0, \qquad \limsup_{n\to+\infty} \int_\Omega |u_n'| \, dx \le |\mathcal{D}u|(\Omega).$$

The lower semi-continuity of the variation implies that $|\mathcal{D}u_n|(\Omega)$ converges to $|\mathcal{D}u|(\Omega)$, hence $|\mathcal{D}u|(\Omega)$ is the least constant in (1.2).

□

Definition 1.3 (Weak* convergence)**.** *Let u, $(u_n) \in BV(\Omega)$. We say that (u_n)* weakly* converges *in $BV(\Omega)$ to u if:*

1. *$u_n \to u$ in $L^1(\Omega)$;*
2. *$\mathcal{D}u_n \overset{*}{\rightharpoonup} \mathcal{D}u$ in Ω, meaning that:*

$$\lim_{n\to+\infty} \int_\Omega \phi \, d\mathcal{D}u_n = \int_\Omega \phi \, d\mathcal{D}u \qquad \forall \phi \in C_0(\Omega).$$

Remark (*BV as a dual space*). In general, weak convergence in $BV(\Omega)$ as a Banach space is not used, because this convergence is hard to characterise. In fact, little is known about the dual of BV. On the other hand, it can be proved that $BV(\Omega)$ is the dual of a separable space, and that the convergence in Definition 1.3 corresponds to the usual weak* convergence.

Proposition 1.4 (Criterion for weak* convergence)**.** *Let $(u_n) \subset BV(\Omega)$. Then (u_n) weakly* converges to u if and only if (u_n) is bounded in $BV(\Omega)$ and converges to u in $L^1(\Omega)$.*

Proof. $\Leftarrow$) Let (u_n) be bounded in BV and $u_n \to u$ in $L^1(\Omega)$. Then, it is sufficient to prove that $(\mathcal{D}u_n)$ weakly* converges to $\mathcal{D}u$ in Ω. From the hypothesis, we know that $u_n \in BV(\Omega)$ for all n. Then $|\mathcal{D}u_n|(\Omega) < +\infty$ for all n. By standard results in measure theory, any sequence of finite Radon measures admits a weakly* convergent subsequence. Thus, we only need to prove that the limit $\mu = \lim_{k\to\infty} \mathcal{D}u_{n_k}$ is equal to $\mathcal{D}u$:

$$\lim_{k\to+\infty} \int_\Omega u_{n_k} \phi' \, dx = -\int_\Omega \phi \, d\mathcal{D}u_{n_k} \qquad \forall \phi \in C_c(\Omega),$$

$$\lim_{k\to+\infty} \int_\Omega u_{n_k} \phi' \, dx = \int_\Omega u\phi' \, dx = -\int_\Omega \phi \, d\mu \qquad \forall \phi \in C_c(\Omega).$$

Hence, $\mu = \mathcal{D}u$.

$\Rightarrow$) It follows from the Banach-Steinhaus theorem, because weak* convergence of finite Radon measures in Ω corresponds to weak* convergence in $(C_0(\Omega))'$.

□

The following is a direct consequence:

Corollary 1.1. *Let $(u_n) \subset BV(\Omega)$. Then, the following hold:*

1. *$\sup_{n\in\mathbb{N}} \|u_n\|_{BV} < +\infty$;*
2. *If $u_n \to u$ in $L^1(\Omega)$, then $u \in BV(\Omega)$;*
3. *$\mathcal{D}u_n \overset{*}{\rightharpoonup} \mathcal{D}u$.*

Definition 1.4 (Strict convergence)**.** *Let u and (u_n) be $\in BV(\Omega)$. We say that (u_n)* strictly converges *to u in $BV(\Omega)$ if:*

1. *$u_n \to u$ in $L^1(\Omega)$;*
2. *$|\mathcal{D}u_n|(\Omega) \to |\mathcal{D}u|(\Omega)$ as $n \to +\infty$.*

Remark. It can be easily checked that:

$$d(u,v) = \int_\Omega |u-v|\,dx + \Big| |\mathcal{D}u|(\Omega) - |\mathcal{D}v|(\Omega) \Big|$$

defines a metric on $BV(\Omega)$ and induces strict convergence. From Proposition 1.4, strict convergence implies weak* convergence. The converse is not true: for instance, the sequence $\frac{1}{n}\sin(nx)$ weakly* converges in $BV((0,2\pi))$ to 0, but the convergence is not strict because $|\mathcal{D}u_n|((0,2\pi)) = 4$ for any $n \geq 1$.

Proposition 1.5. *Let u and $(u_n) \in BV(\Omega)$ and suppose that (u_n) strictly converges to u. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(\lambda x) = \lambda f(x)$ for all $\lambda \geq 0$. Then, for any bounded continuous function $\phi : \Omega \to \mathbb{R}$, we have:*

$$\lim_{n\to+\infty} \int_\Omega \phi\, f\left(\frac{\mathcal{D}u_n}{|\mathcal{D}u_n|}\right) d|\mathcal{D}u_n| = \int_\Omega \phi\, f\left(\frac{\mathcal{D}u}{|\mathcal{D}u|}\right) d|\mathcal{D}u|$$

Consequently:

$$f\left(\frac{\mathcal{D}u_n}{|\mathcal{D}u_n|}\right)|\mathcal{D}u_n| \overset{*}{\rightharpoonup} f\left(\frac{\mathcal{D}u}{|\mathcal{D}u|}\right)|\mathcal{D}u|$$

in Ω. In particular, $|\mathcal{D}u_n| \to |\mathcal{D}u|$ weakly in Ω.*

We now define two useful quantities.

Definition 1.5 (Proper Lipschitz function / push-forward)**.** *Let $\Omega, \Omega' \subset \mathbb{R}$ be open sets.*

1. *A map $\varphi : \Omega \to \Omega'$ is called a* proper Lipschitz function *if:*

   i. *φ is Lipschitz;*

   ii. *for any compact set $K \subset \Omega'$, the preimage $\varphi^{-1}(K)$ is compact in Ω.*

2. *For $u \in L^1(\Omega)$ and a proper Lipschitz function $\varphi : \Omega \to \Omega'$, the* push forward of u by φ *is defined as:*

$$\varphi_\# u(y) := \sum_{x\in\varphi^{-1}(y)} u(x)\,\sigma(x),$$

   where $\sigma(x) := \mathrm{sign}(\varphi'(x))$.

Theorem 1.2. *Let Ω, $\Omega' \subset \mathbb{R}$ be open sets, and let $\varphi : \Omega \to \Omega'$ be a proper Lipschitz function. If $u \in BV(\Omega)$, then, $\varphi_\# u \in BV(\Omega')$ and:*

$$|\mathcal{D}(\varphi_\# u)| \leq \varphi_\# |\mathcal{D}u|,$$

where L is the Lipschitz constant of φ.

Corollary 1.2. *Let Ω be a bounded open interval, and let $\varphi : \Omega \to \Omega'$ be a proper, Lipschitz and invertible map. Assume that $\psi = \varphi^{-1}$ maps $\mathscr{L}^1$-negligible sets to $\mathscr{L}^1$-negligible sets. Then, the map $u \mapsto u \circ \psi$ maps $W^{1,1}(\Omega)$ into $W^{1,1}(\Omega')$, and:*

$$(u\circ\psi)'(y) = u'(\psi(y))\, [\varphi'(\psi(y))]^{-1} \qquad \text{for } \mathscr{L}^1\text{-a.e. } y \in \Omega'.$$

1.2 BV functions in one-dimensional domains

In this section, we examine the pointwise behaviour of BV functions of one variable. The results in this section are fundamental, since the one-dimensional case gives an insight into the structure of BV functions of many variables.

Definition 1.6 (Pointwise variation)**.** *Let $a, b \in \overline{\mathbb{R}}$, with $a < b$ and $I = (a, b)$. Let δ be a subdivision of I, that is $\delta = \{a < t_1 < \cdots < t_n < b\}$. Then, for any $u : I \to \mathbb{R}$, we define the* pointwise variation of u in I *as follows:*

$$pV(u, I) := \sup_{\delta} \left\{ \sum_{i=1}^{n-1} \big|u(t_{i+1}) - u(t_i)\big| \right\}.$$

Remark. Consider $u : \Omega \to \mathbb{R}$ and Ω an open subset of $\mathbb{R}$. Let $\mathcal{C}$ be the set of the connected components of Ω. Then $pV(u, \Omega)$ is defined by $\displaystyle\sum_{I \in \mathcal{C}} pV(u, I)$

Proposition 1.6 (Properties for pV)**.** *Let u and I be as in the previous definition. Then the following hold:*

1. *$u \mapsto pV(u, I)$ is lower semi-continuous with respect to pointwise convergence in I;*
2. *If u is such that $pV(u, I) < +\infty$, then u is bounded;*
3. *If u is a bounded monotone function, then $pV(u, I) = |u(b_-) - u(a_+)| < +\infty$;*
4. *If u is such that $pV(u, I) < +\infty$, then there exist bounded monotone functions g, h such that $u = g - h$.*

Proof. 1. Let δ be a partition of I. Consider $u_n \to u$ pointwise on I. Then:

$$\sum_{i=1}^{n-1} |u(t_{i+1}) - u(t_i)| = \liminf_{n \to +\infty} \left(\sum_{i=1}^{n-1} |u(t_{i+1}) - u(t_i)| \right) \leq \liminf_{n \to +\infty} pV(u_n, I).$$

The claim follows by taking the supremum over all the partitions of I.

2. Let $\overline{x} \in I$ be a point such that $u(x) < +\infty$. Consider $x > \overline{x}$. Since $pV(u, I) < +\infty$, $|u(x) - u(\overline{x})| \leq pV(u, I) < +\infty$, thus $|u(x)| \leq |u(\overline{x})| + |u(x) - u(\overline{x})| \leq c$. The argument is similar for $x \leq \overline{x}$.

3. Assuming that u is non-decreasing, we have $u(a_+) \leq u(x) \leq u(b_-)$. Therefore:

$$\sum_{i=1}^{n-1} \big|u(t_{i+1}) - u(t_i)\big| = u(t_n) - u(t_1) \Longrightarrow pV(u, I) = |u(b_-) - u(a_+)| = u(b_-) - u(a_+).$$

The proof is analogous if u is non-increasing.

4. Let $g(t) = pV(u, (a, t])$, and $h(t) = g(t) - u(t)$ with $t \in I$. Then:

   i. $g(t)$ is monotone non-decreasing by definition;

   ii. $h(t)$ is monotone non-decreasing. In fact, let $t_1 \leq t_2$ in I. Then:

$$\begin{aligned} g(t_1) - u(t_1) \leq g(t_2) - u(t_2) &\Longleftrightarrow pV(u, (a, t_1]) - u(t_1) \leq pV(u, (a, t_2]) - u(t_2) \\ &\Longleftrightarrow pV(u, (a, t_1]) + u(t_2) - u(t_1) \leq pV(u, (a, t_2]). \end{aligned}$$

   On the other hand:

$$pV(u, (a, t_1]) + u(t_2) - u(t_1) \leq pV(u, (a, t_1]) + |u(t_2) - u(t_1)| \leq pV(u, (a, t_2]).$$

   Therefore, the inequality holds.

□

Remark. Let u and I be as above, with $pV(u, I) < +\infty$. Let $h \geq 1$ be fixed and consider a (possibly infinite) collection $\{x_i^h\}_{1 \leq i \leq n_h}$ of points such that:

1. $x_1^h = a$, $x_{n_h}^h = b$;

2. $0 < x_{i+1}^h - x_i^h \leq \frac{1}{h} \quad \forall i \in \{1, \dots, n_h - 1\}$.

Consider the following step function:

$$u_h(t) := \sum_{i=1}^{n_h-1} u(y_i^h)\chi_{(x_i^h, x_{i+1}^h]}(t). \tag{1.5}$$

Then, for any choice of $y_i^h \in (x_i^h, x_{i+1}^h)$, $u_h \xrightarrow[h\to+\infty]{L^1} u$. Since $pV(u, I) < +\infty$, from Proposition 1.6, we can find bounded monotone functions g, h such that $u = g - h$. But if $f : [a, b] \to \mathbb{R}$ is monotone, then f is Riemann-integrable. Thus, u is Riemann integrable.

Clearly, $pV(u, I)$ is very sensitive to the value of u at each point. This suggests the following definition.

Definition 1.7 (Essential variation)**.** *Let u, I be the same as before. Then, we define the* essential variation of u in I *as:*

$$eV(u, I) := \inf\Big\{pV(v, I) : v = u \quad \mathscr{L}^1\text{-a.e. in } I\Big\}.$$

Theorem 1.3. *For any $u \in L^1_{\mathrm{loc}}(\Omega)$ with $eV(u, \Omega) < +\infty$, the infimum is achieved in the definition of essential variation. Moreover, $V(u, \Omega) = eV(u, \Omega)$.*

Proof. Assume $\Omega = I = (a, b)$. It is sufficient to consider the case $V(u, I) < +\infty$.

$\boxed{\leq}$ By definition, we prove that $V(u, I) \leq pV(v, I)$ for all v such that $v = u$ a.e. For $h \geq 1$, we consider v_h the step functions defined in eq. (1.5). Using the definition of variation, we obtain:

$$V(v_h, I) = \sum_{i=1}^{n_h-1} \left|v(y_{i+1}^h) - v(y_i^h)\right| \leq pV(v, I).$$

Now we can pass to the limit for $h \to +\infty$. Using the lower semi-continuity of the variation, we get:

$$V(u, I) = V(v, I) \leq \liminf_{h\to+\infty} V(v_h, I) \leq pV(v, I).$$

$\boxed{\geq}$ Assume $V(u, I) < +\infty$. Then $u \in BV_{\mathrm{loc}}(I)$ and $|\mathcal{D}u|(J) = V(u, J)$ for all $J \Subset I$. Since $|\mathcal{D}u| \in \mathcal{M}_b(I)$ and:

$$\sup_{J\Subset I} |\mathcal{D}u|(J) = \sup_{J\Subset I} V(u, J) = V(u, I) < +\infty,$$

from measure theory we know that $\mathcal{D}u$ is a finite Radon measure on I such that $|\mathcal{D}u|(I) = V(u, I)$. Let $\mu = \mathcal{D}u$ and define $w(t) := \mu((a, t))$. Let $\phi \in C_c^\infty(I)$. Then, using Fubini's theorem, we have:

$$\begin{aligned}\langle \mathcal{D}w, \phi\rangle = -\langle w, \phi'\rangle = -\int_a^b w(t)\phi'(t)\,dt = -\int_a^b \phi'(t)\mu((a, t))dt = -\int_a^b \int_a^t \phi'(t)\,d\mu(s)\,dt = \\ = -\int_a^b \int_s^b \phi'(t)\,dt\,d\mu(s) = \int_a^b \phi(s)\,d\mu(s).\end{aligned}$$

Therefore, $\mathcal{D}w = \mu$. As a consequence, by Proposition 1.1(3) there exists $c \in \mathbb{R}$ such that $u(t) - w(t) = c$ for $\mathscr{L}^1$-a.e. $t \in I \Rightarrow w(t) + c = u(t)$ a.e. Moreover, for any collection of points $t_1 < \dots < t_n$ in I, the following holds:

$$\sum_{i=1}^{n-1} |w(t_{i+1}) - w(t_i)| = \sum_{i=1}^{n-1} |\mu([t_i, t_{i+1}))| \leq \sum_{i=1}^{n-1} |\mu|([t_i, t_{i+1})) \leq |\mu|(I).$$

In particular:

$$eV(u,I) \le pV(w+c,I) = pV(w,I) \le |\mu|(I) = V(u,I).$$

As a direct consequence, we see that $w+c$ is a minimizer in the definition of $eV(u,I)$.
Now, consider $\Omega \subset \mathbb{R}$ a generic open set. Then $eV(u,\Omega) = \sum_{I\in\mathcal{C}} eV(u,I)$, where $\mathcal{C}$ is the set of connected components of Ω (this is possible because Ω has at most countably many connected components). Hence, by additivity, $V(u,\Omega) = eV(u,\Omega)$. □

Thanks to the previous theorem, we can give the notion of good representative of a BV function (in one-dimensional domain) and the characterization theorem.

Definition 1.8 (Good representative)**.** *Let $u \in BV(\Omega)$. We say that $\overline{u}$ is a* good representative of u *if:*

1. $\overline{u} = u$ *a.e.;*
2. $pV(\overline{u},\Omega) = eV(u,\Omega) = V(u,\Omega)$.

Remark. We can always find at least one good representative. In fact, we have seen that if $u \in BV(\Omega)$, then $V(u,\Omega) = |\mathcal{D}u|(\Omega) < +\infty$. Moreover, we know that $V(u,\Omega) = eV(u,\Omega)$. Then, we choose $\overline{u}$ to be an element such that $pV(\overline{u},\Omega) = eV(u,\Omega)$.

Theorem 1.4. *Let $I = (a,b) \subset \mathbb{R}$ be an interval and $u \in BV(I)$. Let $A = \{t \in I : \mathcal{D}u(\{t\}) \neq 0\}$. Then, the following statements hold:*

1. *there exists a unique $c \in \mathbb{R}$ such that:*

$$u^l(t) := c + \mathcal{D}u((a,t)) \quad \text{and} \quad u^r(t) := c + \mathcal{D}u((a,t])$$

are good representatives of u;

2. *a function $\overline{u} : I \to \mathbb{R}$ is a good representative if and only if $\overline{u}$ is a convex combination of u^l and u^r, that is:*

$$\overline{u}(t) = \theta u^l(t) + (1-\theta)u^r(t) \qquad \text{for some } \theta \in [0,1]; \tag{1.6}$$

3. *any good representative $\overline{u}$ is continuous in $I \setminus A$ and has a jump discontinuity at any point in A. In particular:*

$$\overline{u}(t_-) = u^l(t) = u^r(t_-), \qquad \overline{u}(t_+) = u^l(t_+) = u^r(t) \quad \text{for all } t \in A;$$

4. *any good representative $\overline{u}$ is differentiable at $\mathscr{L}^1$ a.e.-point of I and the derivative $\overline{u}'$ is the density of $\mathcal{D}u$ with respect to the $\mathscr{L}^1$ Lebesgue measure.*

Proof. 1. We have already seen in the proof of Theorem 1.3 that u^l is a good representative of u, with a suitable $c \in \mathbb{R}$. The proof is similar for u^r.

2 $\Leftarrow$) It follows from the following estimate:

$$\sum_{i=1}^{n-1} |\overline{u}(t_{i+1}) - \overline{u}(t_i)| \le \sum_{i=1}^{n-1} |u^l(t_{i+1}) - u^l(t_i)| + \sum_{i=1}^{n-1} |u^l(t_i) - u^r(t_i)| + \sum_{i=1}^{n-1} |u^l(t_{i+1}) - u^r(t_{i+1})|.$$

Noticing that $u^l = u^r$ in $I \setminus A$, we obtain:

$$pV(\overline{u},I) \le |\mathcal{D}u|(I) + 2|\mathcal{D}u|(A) < +\infty,$$

hence $\overline{u}$ has left and right limits in I. Moreover:

$$\lim_{s\to t^+} \overline{u}(s) = \lim_{s\to t^+} u^l(s) = u^l(t) \qquad \lim_{s\to t^-} \overline{u}(s) = \lim_{s\to t^-} u^r(s) = u^r(t)$$

for any $t \in I$.

2 $\Rightarrow$) We consider $pV(\overline{u}, \cdot)$ as a function of the interval $J \subset I$. From this point of view, $pV(\overline{u}, \cdot)$ has the following properties:

(a) inner regularity:

$$pV(\overline{u}, J) = \sup_{J' \Subset J} pV(\overline{u}, J');$$

(b) super-additivity: let $\{J_i\}_{i \leq p} \subset J$ with:

$$\bigcup_{i=1}^{p} J_i \subset J, \qquad J_i \cap J_k = \emptyset \text{ if } i \neq k.$$

Then:

$$pV(\overline{u}, J) \geq \sum_{i=1}^{p} pV(\overline{u}, J_i).$$

Using these properties of $pV(\overline{u}, \cdot)$, we can show that $\overline{u}$ is a good representative on any interval $J = (c, d) \subset I$. Indeed, since A is at most countable and $pV(\overline{u}, \cdot)$ is inner regular, we can assume, in order to prove the inequality $pV(\overline{u}, J) \leq |\mathcal{D}u|(J)$, that neither c nor d belong to A. Under this assumption, we obtain:

$$\begin{aligned} pV(\overline{u}, J) &\leq -pV(\overline{u}, (a, c)) - pV(\overline{u}, (d, b)) + pV(\overline{u}, I) \leq \\ &\leq -|\mathcal{D}u|((a, c)) - |\mathcal{D}u|((d, b)) + |\mathcal{D}u|((a, b)) = |\mathcal{D}u|((c, d)). \end{aligned}$$

Since $pV(\overline{u}, I) < +\infty$, $\overline{u}$ can be written as a difference of monotone functions. Then, the right and left limits of $\overline{u}$ exist at any point in I. Since $\overline{u}$ is a representative of u, the following hold:

$$\begin{aligned} \overline{u}(t_-) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t-\varepsilon}^{t} \overline{u}(\tau)\, d\tau = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t-\varepsilon}^{t} u^l(\tau)\, d\tau = u^l(t) \quad \text{for all } t \in I; \\ \overline{u}(t_+) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t}^{t+\varepsilon} \overline{u}(\tau)\, d\tau = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t}^{t+\varepsilon} u^r(\tau)\, d\tau = u^r(t) \quad \text{for all } t \in I. \end{aligned}$$

Hence, passing to the limit as $\varepsilon \to 0$:

$$|\overline{u}(t-\varepsilon) - \overline{u}(t)| + |\overline{u}(t) - \overline{u}(t+\varepsilon)| \leq pV(\overline{u}, B_{2\varepsilon}(t)) = |\mathcal{D}u|(B_{2\varepsilon}(t))$$

and consequently:

$$|u^l(t) - \overline{u}(t)| + |\overline{u}(t) - u^r(t)| \leq |u^l(t) - u^r(t)|$$

which implies eq. (1.6).

3. By definition, u^l and u^r are continuous and coincide at any point of $I \setminus A$. By eq. (1.6), any other good representative enjoys the same properties. In particular, its left and right limits coincide with those of u^l and u^r.

4. Let v be the density of $\mathcal{D}u$ with respect to $\mathscr{L}^1$ and let $\mathcal{D}^s u = \mathcal{D}u - v\mathscr{L}^1$ be the singular part of $\mathcal{D}u$. We prove that any good representative $\overline{u}$ is differentiable at any Lebesgue point t of v such that $|\mathcal{D}^s u|(B_\rho(t)) = o(\rho)$. It is known that this statement holds true for $\mathscr{L}^1$-a.e. $t \in I$. Using the definition

of u^l, we have:

$$(u^l)'_+(t) = \lim_{\rho\to 0^+} \frac{u^l(t+\rho) - u^l(t)}{\rho} = \lim_{\rho\to 0^+} \frac{\mathcal{D}u([t,t+\rho))}{\rho} =$$
$$= \lim_{\rho\to 0^+} \frac{1}{\rho}\int_t^{t+\rho} v(\tau)\,d\tau + \lim_{\rho\to 0^+} \frac{\mathcal{D}^s u([t,t+\rho))}{\rho} = v(t).$$

A similar argument yields $(u^l)'_-(t) = v(t)$. The differentiability of any other good representative follows from that of u^l and the following inequality:

$$|\overline{u}(t) - u^l(t)| \le |\mathcal{D}u(\{t\})| = |\mathcal{D}^s u|(\{t\}).$$

□

It is straightforward to show that any monotone function is a good representative in its equivalence class. Therefore, applying Theorem 1.4 to monotone functions, we obtain the following result.

Corollary 1.3. *Let $u : (a,b) \to \mathbb{R}$ be a monotone function. Then u is differentiable at $\mathscr{L}^1$-a.e. $t \in (a,b)$. Moreover:*

$$|u(b_-) - u(a_+)| \ge \int_a^b |u'(t)|\,dt + \sum_{t\in\Gamma_u} |u(t_+) - u(t_-)|$$

where Γ_u is the discontinuity set of u. Note that, since u is monotone, it has at most countably many discontinuity points.

Theorem 1.5. *Let $(a,b) \subset \mathbb{R}$ be a bounded interval. Then, the linear map $(c,\mu) \mapsto u$, where $u(t) := c + \mu((a,t))$ is an isomorphism between the Banach space $\mathbb{R} \times \mathcal{M}_b(a,b)$ and $BV(a,b)$.*

Proof. Let $T(c,\mu)(t) := c + \mu((a,t))$.

1. $T(c,\mu) \in BV(a,b)$. This follows from the proof of Theorem 1.3.
2. T is linear.
3. T is continuous. In fact, we have:

$$\|T(c,\mu)\|_{BV} = \|T(c,\mu)\|_{L^1} + |\mu|((a,b)) \le |c|(b-a) + (b-a)|\mu|((a,b)) + |\mu|((a,b)) =$$
$$= |c|(b-a) + (b-a+1)|\mu|((a,b)).$$

4. $\ker(T) = \{(0,0)\}$. In fact, if $T(c,\mu) = 0$, then $\mu = \mathcal{D}T(c,\mu) = 0$ and by definition of T, $c = 0$.
5. T is surjective: this follows from Theorem 1.4.

Then, the closed graph theorem implies that T is an isomorphism. □

Remark. This result is still valid with vector-valued functions.

The equations defining u^l and u^r can be written without involving c:

$$u^l(s) - u^l(t) = \mathcal{D}u([t,s)), \quad u^r(s) - u^r(t) = \mathcal{D}u((t,s]); \qquad \text{with } a < t < s < b$$

and can be regarded as the fundamental theorem of calculus in BV. However, we have seen in the definitions of u^l and u^r that good representatives are not uniquely determined by the distributional derivative, because the measure may have an atomic part. This difficulty is not present when dealing with absolutely continuous functions.

Definition 1.9 (Absolutely continuous functions)**.** *Let $\Omega \subset \mathbb{R}$ be an open set and let $u \in L^1(\Omega)$. We say that u is* absolutely continuous in Ω *if $u \in BV(\Omega)$ and $\mathcal{D}u$ is absolutely continuous with respect to $\mathscr{L}^1$.*

The space of absolutely continuous functions in Ω coincides with the Sobolev space $W^{1,1}(\Omega)$. From Theorem 1.4, we deduce that for any absolutely continuous function, there exists a unique continuous representative $\overline{u}$ differentiable $\mathscr{L}^1$-a.e. in Ω and which satisfies the fundamental theorem of calculus $\overline{u}(t)-\overline{u}(s) = \int_s^t \overline{u}'(\tau)\,d\tau$ for every interval $[s,t]\subset\Omega$, with $s<t$.
In general, any measure μ on an open set $\Omega\subset\mathbb{R}$ can be split into three parts: the *absolutely continuous part* (with respect to $\mathscr{L}^1$), denoted by μ^a; the *purely atomic part*, denoted by μ^j; and the *diffuse part* (i.e., without atoms), also called the *singular part*, denoted by μ^c. To obtain this decomposition, we first decompose μ into its absolutely continuous part μ^a and its singular part μ^s (using the Radon-Nikodým theorem). Then, we define $\mu^j=\mu^s|_A$ and $\mu^c=\mu^s|_{\Omega\setminus A}$ with $A=\{t\in\Omega : \mu(\{t\})\neq 0\}$ the set of atoms of μ (notice that A is at most countable). In this way, we have obtained the following decomposition:

$$\mu=\mu^a+\mu^s=\mu^a+\mu^j+\mu^c.$$

The decomposition is unique and, since μ^a, μ^j, μ^c are mutually singular, we have also $|\mu|=|\mu^a|+|\mu^j|+|\mu^c|$. From a result from the theory of measures, we know that μ^s can also be represented by the restriction of μ to the following $\mathscr{L}^1$-negligible set:

$$S:=\left\{t\in\Omega : \lim_{\rho\to 0}\frac{|\mu|(B_\rho(t))}{\rho}=+\infty\right\}$$

containing A. Hence, we can describe these three measures in a more constructive way:

$$\mu^a=\mu|_{(\Omega\setminus S)} \qquad \mu^j=\mu|_A \qquad \mu^c=\mu|_{(S\setminus A)}.$$

This decomposition suggests the following definition.

Definition 1.10 (Jump function, Cantor function)**.** *Let $u\in BV(\Omega)$. We say that u is a* jump function *if $\mathcal{D}u=\mathcal{D}^j u$, i.e. $\mathcal{D}u$ is a purely atomic measure. We say that u is a* Cantor function *if $\mathcal{D}u=\mathcal{D}^c u$, i.e. $\mathcal{D}u$ is a singular measure without atoms.*

Using the previous definition and Theorem 1.5, we can give the following useful representation of BV functions in intervals.

Corollary 1.4 (Decomposition of BV functions)**.** *Let $\Omega=(a,b)\subset\mathbb{R}$ be a bounded interval. Then, any $u\in BV(\Omega)$ can be represented by $u^a+u^j+u^c$, where $u^a\in W^{1,1}(\Omega)$, u^j is a jump function and u^c is a Cantor function. The three functions are uniquely determined up to additive constants. Moreover:*

$$|\mathcal{D}u|(\Omega)=|\mathcal{D}u^a|(\Omega)+|\mathcal{D}u^j|(\Omega)+|\mathcal{D}u^c|(\Omega)=\int_a^b|\overline{u}'(t)|\,dt+\sum_{t\in A}|\overline{u}(t_+)-\overline{u}(t_-)|+|\mathcal{D}u^c|(\Omega)$$

where $\overline{u}$ is any good representative of u.

This decomposition of BV functions is typical of dimension 1.
Example (jump function). Let $(a,b)=(0,1)$. Let $(d_n)_{n\in\mathbb{N}}$ be a sequence in $(0,1)$. We define:

$$u(t):=\sum_{n:\,d_n<t}2^{-n}.$$

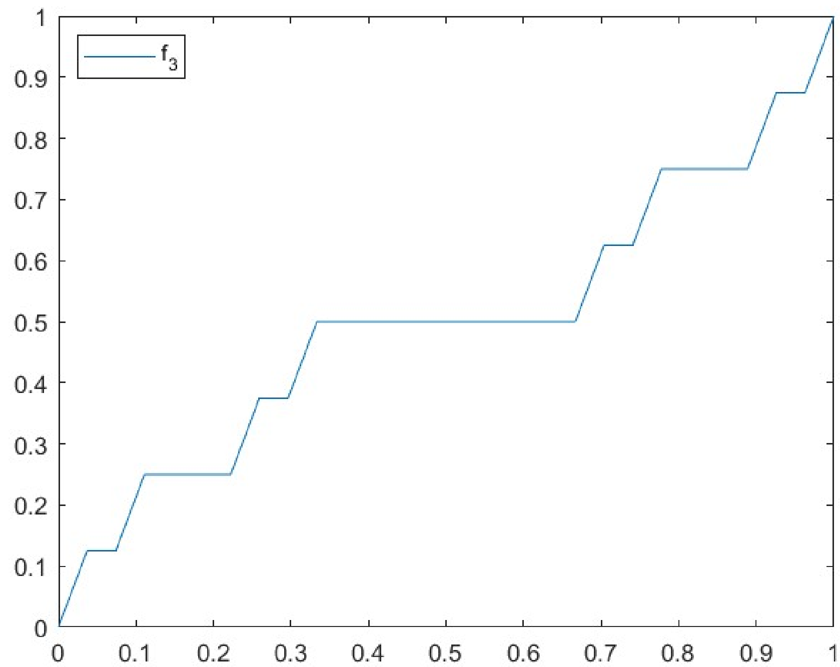
The distributional derivative of u is the finite measure $\mu=\sum_n 2^{-n}\delta_{d_n}$. Indeed, $u(t)=\mu((0,t))$ for any $t\in(0,1)$. In general, the distributional derivative of a jump function can be recovered from the left and right limits of a good representative, as shown in Theorem 1.4.

In general, it is more complicated to give an example of a Cantor function: given a good representative, these functions are continuous in their domain and differentiable with derivative equal to 0 almost everywhere.

This means that, unlike absolutely continuous functions and jump functions, the derivative of a Cantor function can be seen only as a measure in the distributional sense.

Example (Cantor function). Let $(a, b) = (0, 1)$. Let $\psi_1(t) = \frac{t}{3}$ and $\psi_2(t) = \frac{t+2}{3}$ be scaling functions. We define by induction the following sequence of increasing functions $f_n : (0, 1) \to (0, 1)$:

$$f_0(t) = t;$$
$$f_{n+1}(t) = \begin{cases} \frac{1}{2} f_n \circ \psi_1^{-1}(t) & t \in [0, \frac{1}{3}], \\ \frac{1}{2} & t \in [\frac{1}{3}, \frac{2}{3}], \\ \frac{1}{2}\big(f_n \circ \psi_2^{-1}(t)\big) & t \in [\frac{2}{3}, 1], \end{cases} \quad \forall n > 0$$



Graph of f_3 on $[0, 1]$

The functions f_n have the following properties:

(a) f_n is constant in any connected component of $\mathbb{R} \setminus C_n$, where C_n is the n-th Cantor middle third;

(b) $\|f_{n+1} - f_n\|_\infty \leq \frac{1}{2^{n+1}} \frac{1}{3}$.

By (b), f_n is a Cauchy sequence in $C([0, 1])$, hence uniformly convergent in $[0, 1]$ to some continuous function f. The limit f is still increasing and maps $[0, 1]$ to $[0, 1]$. In particular, $f \in BV([0, 1])$ and $\mathcal{D}f$ is a probability measure in $(0, 1)$. On the other hand, we infer that f is constant in any connected component of $(0, 1) \setminus C$, where C is the Cantor set. These properties allow us to conclude that f is a Cantor function, because $\mathcal{D}f$ has no atoms (since f is continuous) and $\mathcal{D}^a f = 0$ (since $f' = 0$ on $(0, 1) \setminus C$, which has full measure in $(0, 1)$).

1.3 The space SBV

Definition 1.11 (SBV function)**.** *Let $u \in BV(I)$. We say that u is a* special function of bounded variation*, and we write $u \in SBV(I)$, if the Cantor part of its derivative $\mathcal{D}^c u$ is zero, that is:*

$$\mathcal{D}u = \mathcal{D}^a u + \mathcal{D}^j u = \int_I u'(t)\,dt + \sum_{t \in A} |\overline{u}(t_+) - \overline{u}(t_-)|.$$

Remark. The inclusion of $W^{1,1}(I)$ in $SBV(I)$ is strict. Let $J = (a,b) \subset I$. Then, $u = \chi_J \in SBV(I)$, since $\mathcal{D}\chi_J = \delta_b - \delta_a$. Clearly, $u \notin W^{1,1}(I)$.

Remark. By definition, it is clear that SBV is a subspace of BV. Moreover, this inclusion is strict. Let $I = (0,1)$ and f be the Cantor function introduced above. Then, we know that $f \in BV(I)$ and $\mathcal{D}f = \mathcal{D}^c f$; thus $f \notin SBV(I)$.

Proposition 1.7. *$SBV(I)$ is a closed subspace of $BV(I)$.*

Proof. As observed, $SBV(I)$ is a proper subspace of $BV(I)$. Let $u_n \in SBV(I)$ for any n such that u_n converges to $u \in BV(I)$ in the BV norm. Let v_n be as follows:

$$v_1 = u_1, \quad v_k = u_k - u_{k-1} \qquad \forall k > 1.$$

Notice that v_n is an SBV function for any n. Then:

$$u_N = \sum_{k=1}^{N} v_k, \quad \mathcal{D}u = \sum_{k=1}^{\infty} \mathcal{D}v_k.$$

and $u_N \to u$ in BV.In particular, $\sum_{k=1}^{\infty} \mathcal{D}^a v_k$ is absolutely continuous while $\sum_{k=1}^{\infty} \mathcal{D}^s v_k$ is a singular measure (both with respect to $\mathscr{L}^1$). Consequently:

$$\mathcal{D}^a u = \sum_{k=1}^{\infty} \mathcal{D}^a v_k, \qquad \mathcal{D}^s u = \sum_{k=1}^{\infty} \mathcal{D}^s v_k.$$

Since $\mathcal{D}^s u$ is concentrated on $\bigcup_k S_k$, we deduce that $u \in SBV(I)$, where S_k denotes the set S defined in the previous chapter, for each u_k. □

Remark. The space $SBV(I)$ is not closed in $BV(I)$ with respect to L^1 convergence. Let $(u_n) \subset W^{1,\infty}(I)$ be the sequence introduced in the Cantor function example. Then $u_n \xrightarrow{L^1} u$, where u is the Cantor function.

Proposition 1.8. *Any $u \in BV(I)$ belongs to $SBV(I)$ if and only if $\mathcal{D}^s u$ is concentrated on a countable set.*

Proof. $\Rightarrow$) If $u \in SBV(I)$, then $\mathcal{D}^s u = \mathcal{D}^j u$ is concentrated, by definition, on A, which is the set of the atoms of the measure. Since $\mathcal{D}u$ is finite (is the distributional derivative of a BV function), then A is at most countable.

$\Leftarrow$) Since $\mathcal{D}^c u$ is diffuse and has no atoms (by Definition 1.10, and $\mathcal{D}^s u$ is concentrated on a countable set, it follows that $\mathcal{D}^c u = 0$.

□

Remark. When S is countable, it coincides with A.

Theorem 1.6. *For any $w \in L^1(\mathbb{R})$, there exists $u \in SBV_{\mathrm{loc}}(\mathbb{R})$ such that $u' = w$ $\mathscr{L}^1$-a.e. in $\mathbb{R}$ and* $|\mathcal{D}u|(\mathbb{R}) \le C\|w\|_{L^1}$.

We now state the closure and compactness of SBV, in dimension 1. The proofs are omitted.

Theorem 1.7 (Closure of SBV)**.** *Let $\varphi : [0,\infty) \to [0,\infty]$, $\theta : (0,\infty) \to (0,\infty]$ be lower semi-continuous increasing function and assume that:*

$$\lim_{t\to+\infty} \frac{\varphi(t)}{t} = \infty, \qquad \lim_{t\to 0} \frac{\theta(t)}{t} = \infty.$$

Let $I \subset \mathbb{R}$ be an open interval and let $(u_n) \subset SBV$ such that:

$$\sup_n \left\{ \int_I \varphi(|\mathcal{D}^a u_n|)dx + \sum_{t\in A} \theta\big(|\overline{u}_n(t_+) - \overline{u}_n(t_-)|\big) \right\} < +\infty. \tag{1.7}$$

with $\overline{u}_n$ a good representative of u_n. Suppose u_n weakly converges in $BV(I)$ to u. Then:*

1. *$u \in SBV$;*
2. *$\mathcal{D}^a u_n$ weakly converge to $\mathcal{D}^a u$;*
3. *$\mathcal{D}^c u_n$ weakly converge to $\mathcal{D}^c u$;*
4. *$\mathcal{D}^j u_n$ weakly* converge to $\mathcal{D}^j u$.*

Moreover:

$$\begin{aligned} &\int_I \varphi(|\mathcal{D}^a u|)\, dx \le \liminf_{n\to+\infty} \int_I \varphi(|\mathcal{D} u_n|)\, dx, && \textit{if } \varphi \textit{ is convex;} \\ &\sum_{t\in A} \theta\big(|u(t_+) - u(t_-)|\big) \le \liminf_{n\to+\infty} \sum_{t\in A} \theta\big(|\overline{u}_n(t_+) - \overline{u}_n(t_-)|\big) && \textit{if } \theta \textit{ is concave.} \end{aligned}$$

Theorem 1.8 (Compactness of SBV)**.** *Let φ, θ, I as in Theorem 1.7. Let $(u_n) \subset SBV(I)$ be satisfying* (1.7) *and assume that $\|u_n\|_\infty$ is uniformly bounded in n. Then, there exists a subsequence (u_{n_k}) weakly* converging in $BV(I)$ to u in $SBV(I)$.*

Appendix

Extension domains and compactness in BV

In many situations, it is useful to extend a function $u \in BV(\Omega)$ to a function $\overline{u} \in BV(\mathbb{R})$. For example, this extension is used several times to obtain the trace theorem in BV.

Definition 1.12 (Extension domains)**.** *We say that an open set $\Omega \subset \mathbb{R}^N$ is an* extension domain *if $\partial\Omega$ is bounded and for any open set A with $\overline{\Omega} \subset A$ there exists a linear and continuous extension operator $T : BV(\Omega) \to BV(\mathbb{R}^N)$ satisfying:*

1. *$Tu = 0$ a.e. in $\mathbb{R}^N \setminus A$ for any $u \in BV(\Omega)$;*
2. *$|\mathcal{D}Tu|(\partial\Omega) = 0$ for any $u \in BV(\Omega)$;*
3. *for any $p \in [1, +\infty]$, the restriction of T to $W^{1,p}(\Omega)$ induces a linear continuous map from $W^{1,p}(\Omega)$ into $W^{1,p}(\mathbb{R}^N)$.*

Proposition 1.9. *Any open set Ω with compact Lipschitz boundary is an extension domain.*

Proof. Since $\partial\Omega$ is bounded, we can assume without loss of generality that Ω is bounded. Since $\overline{\Omega}$ is compact, we can find a finite collection $\{R_i\}_{i\in I}$ of open rectangles whose union B contains $\overline{\Omega}$. Then B is contained in A and has the following property: for any $i \in I$, either $R_i \subset \Omega$ or $\partial\Omega \cap R_i$ is the graph of a Lipschitz function defined on one face L_i of R_i. Possibly reducing the rectangles, we can also assume that the closure of $\partial\Omega \cap R_i$ intersects neither $\overline{L_i}$ nor the closure of the face opposite to L_i. Let $\{\eta_i\}_{i\in I}$ be a partition of unity relative to $\{R_i\}_{i\in I}$. We define $\Omega_i = \Omega \cap R_i$ and

$$Tu := \sum_{i\in I} T_i(u\,\eta_i)$$

where $T_i : BV(\Omega_i) \to BV(R_i)$ are suitable linear and continuous extension operator satisfying the following:

(a) $|\mathcal{D}T_i\,u|(R_i \cap \partial\Omega) = 0$ for any $u \in BV(\Omega)$;

(b) for any $p \in [1, +\infty]$ the restriction of T to $W^{1,p}$ induces a linear continuous map between this space and $W^{1,p}(R_i)$.

Using Proposition 1.1(1), it is easy to see that T satisfies the conditions of Definition 1.12. Hence, we only need to show the existence of T_i. Let $i \in I$ be fixed. We exclude the trivial case $\Omega_i = R_i$. then, up to rotation, translation and a homothety, in the construction of T_i we can assume with no loss of generality that:

$$R_i = L \times (-1, 1), \qquad \Omega_i = \{x = (y, z) \in L \times (-1, 1) : z > \phi(y)\}$$

for some $L \subset \mathbb{R}^{N-1}$ and ϕ Lipschitz with $\inf \phi > -1$ and $\sup \phi < 1$. We can transform Ω_i in $R_i^+ = L \times (0, 1)$

using the deformation $\varphi : R_i \to R_i$ defined by:

$$\varphi(y,z) = \begin{cases} \left(y, \frac{z-\phi(y)}{1-\phi(y)}\right) & \text{if } z \geq \phi(y) \\ \left(y, \frac{z-\phi(y)}{1+\phi(y)}\right) & \text{if } z \leq \phi(y) \end{cases}$$

Since both φ and its inverse are Lipschitz orientation-preserving maps, we can construct a linear continuous extension operator $T_i' : BV(R_i^+) \to BV(R_i)$ satisfying (a), (b) with R_i^+ instead of Ω_i. Then, we can recover T_i as $T_i(u) = T_i'(u \circ \varphi^{-1}) \circ \varphi$ and using Theorem 1.2, T_i inherits property (a) from T_i'. The verification of (b) is straightforward, using Corollary 1.2.
In conclusion, let us define $T_i' : BV(R_i^+) \to BV(R_i)$ by reflection, setting $T_i[u](y,z) = u(y,|z|)$. We can prove that:

$$|\mathcal{D}T_i'[u]|(L \times (-r,r)) \leq 2|\mathcal{D}u|(L \times (0,r)) \qquad \forall r \in (0,1]$$

in three steps:

i) $u \in C^1(\overline{R}_i^+)$ using the Gauss-Green theorem;

ii) $u \in C^1(R_i^+)$ using the approximation with the functions $u_n(y,z) = u(y, z+t_n)$, with $t_n \to 0$;

iii) the general case, using Theorem 1.1 in $L \times (0,r)$.

Letting $r \to 0$ in the previous inequality we obtain that T_i' verifies (a). The verification of (b) is similar. □

Remark. This result can be extended to $[BV(\Omega)]^m$ with a componentwise argument.

Remark. Not all bounded open sets are extension domains. In $\mathbb{R}$, an example is given by a generic open bounded set Ω with countably many connected component I_n such that $\sum_n |I_n|^{\frac{1}{2}} < +\infty$, and we consider the function $u = \sum_n |I_n|^{-\frac{1}{2}} \chi_{I_n}$. It is easy to verify that $u \in BV(\Omega) \setminus L^\infty(\Omega)$, thus u cannot be extended to a BV function in $\mathbb{R}$.

We now state a compactness theorem for BV. Since the Sobolev space $W^{1,1}$ has no comparable compactness properties, this provides another justification for the introduction of the BV space.

Theorem 1.9 (Compactness)**.** *Let $(u_n) \subset BV_{\text{loc}}(\Omega)$ such that:*

$$\sup_{n \in \mathbf{N}} \left\{ \int_A |u_n|\, dx + |\mathcal{D}u_n|(A) \right\} < +\infty \qquad \forall A \subset \Omega \textit{ open }.$$

Then u_n admits a subsequence u_{n_k} converging in $L^1_{\text{loc}}(\Omega)$ to some u in $BV_{\text{loc}}(\Omega)$. Moreover, if Ω is a bounded extension domain and the sequence is bounded in $BV(\Omega)$, then $u \in BV(\Omega)$ and the subsequence weakly converges to u.*

In order to prove Theorem 1.9, we need the following:

Proposition 1.10. *Let $u \in BV(\Omega)$ and $K \subset \Omega$ be a compact set. Then:*

$$\int_K |u * \rho_\varepsilon - u|\, dx \leq \varepsilon |\mathcal{D}u|(\Omega) \qquad \forall \varepsilon \in (0, \operatorname{dist}(K; \partial\Omega)).$$

Proof. (*Proposition 1.10*) By Theorem 1.1 we can assume without loss of generality that $u \in C^1(\Omega)$. Let ε be as in the statement. We start from the following identity:

$$u(x - \varepsilon y) - u(x) = -\varepsilon \int_0^1 \langle \nabla u(x - \varepsilon t y), y \rangle\, dt \qquad x \in K \text{ and } y \in B_1.$$

(it can be seen using the fundamental theorem of calculus). Then, for $x \in K$ and $\varepsilon < \mathrm{dist}(K, \partial\Omega)$:

$$\int_K |u(x-\varepsilon y) - u(x)|\,dx \le \varepsilon \int_0^1 \int_K |\langle \nabla u(x - \varepsilon t y), y\rangle|\,dx\,dt \le \varepsilon |\mathcal{D}u|(\Omega).$$

Multiplying both sides by $\rho(y)$ and integrating, we obtain:

$$\int_K \left(\int_{\mathbb{R}^N} |u(x-\varepsilon y) - u(x)|\rho(y)\,dy \right) dx \le \varepsilon |\mathcal{D}u|(\Omega).$$

Since:

$$u * \rho_\varepsilon - u(x) = \int_{\mathbb{R}^N} |u(x-\varepsilon y) - u(x)|\rho(y)\,dy$$

the statement follows. □

Proof. (*Theorem 1.9*) Let $\Omega' \Subset \Omega$ be an open set. By a diagonal argument, we only need to show the existence of a subsequence (u_{n_k}) converging in $L^1(\Omega)$ to some $u \in BV(\Omega)$ (we can conclude that $u \in BV(\Omega)$ thanks to Corollary 1.1). Let:

1. $\delta = \mathrm{dist}(\Omega', \partial\Omega) > 0$;
2. $U \subset \Omega$ the open $\frac{\delta}{2}$-neighbourhood of Ω';
3. $u_{n,\varepsilon} = u_n * \rho_\varepsilon$.

If $\varepsilon \in (0, \frac{\delta}{2})$, then the functions $u_{n,\varepsilon}$ are $C^\infty(\overline{\Omega'})$ and satisfy:

$$\|u_{n,\varepsilon}\|_{C(\overline{\Omega'})} \le \|u_n\|_{L^1(U)} \|\rho_\varepsilon\|_\infty, \qquad \|\nabla u_{n,\varepsilon}\|_{C(\overline{\Omega'})} \le \|u_n\|_{L^1(U)} \|\nabla \rho_\varepsilon\|_\infty.$$

By our assumption on (u_n), the sequence $(u_{n,\varepsilon})$ is uniformly bounded and equicontinuous for ε fixed. Then, we can extract a convergent subsequence of $(u_{n,\varepsilon})$ in $C(\overline{\Omega'})$. By a diagonal argument, we can extract a subsequence (n_k) such that $(u_{n_k,\varepsilon})$ converges in $C(\overline{\Omega'})$ for any $\varepsilon = \frac{1}{p}$, with $p > \frac{2}{\delta}$. Applying Proposition 1.10 and the triangle inequality, we find:

$$\limsup_{k,k' \to +\infty} \int_{\Omega'} |u_{n_k} - u_{n_{k'}}|\,dx \le \limsup_{k,k' \to +\infty} \int_{\Omega'} |u_{n_k,\frac{1}{p}} - u_{n_{k'},\frac{1}{p}}|\,dx +$$
$$+ \limsup_{k,k' \to +\infty} \int_{\Omega'} \left[|u_{n_k} - u_{n_{k,\frac{1}{p}}}| + |u_{n_{k',\frac{1}{p}}} - u_{n_{k'}}| \right] dx \le \frac{2}{p} \sup_{n \in \mathbf{N}} |\mathcal{D}u_n|(U).$$

Since we can choose an arbitrarily large p and $L^1(\Omega)$ is a Banach space, this proves that (u_{n_k}) is a Cauchy sequence in $L^1(\Omega)$. The second part of the thesis follows applying the first part of the statement to the extensions Tu_n. The weak* convergence follows from Corollary 1.1. □

Chapter 2

Elasticity

2.1 Basic elements.

In this part, we introduce the classical elements often used in elasticity problems. We consider $\mathbb{R}^{N\times N}$ as the space of square matrices, with the standard scalar product:

$$A : B = \sum_{i,j=1}^{N} a_{ij} b_{ij}$$

and the associated norm :

$$|A| = \left(\sum_{i,\, j=1}^{N} a_{ij}^2 \right)^{1/2}.$$

Then, we have the following two subspaces:

1. $\mathbb{R}^{N\times N}_{\text{sym}} := \{A = A^T\}$ the space of symmetric matrices;
2. $\mathbb{R}^{N\times N}_{\text{skew}} := \{A = -A^T\}$ the space of skew-symmetric (or antisymmetric) matrices.

Remark.

1. $\mathbb{R}^{N\times N}_{\text{sym}} \perp \mathbb{R}^{N\times N}_{\text{skew}}$. In fact, let $A \in \mathbb{R}^{N\times N}_{\text{sym}}$, $B \in \mathbb{R}^{N\times N}_{\text{skew}}$. Then, using the definition of the scalar product:
$$A : B = \text{tr}(A^T B) = \text{tr}(AB^T) = \text{tr}(A(-B)) = -(A : B).$$
Thus, $A : B = 0$.
2. $\mathbb{R}^{N\times N} = \mathbb{R}^{N\times N}_{\text{sym}} \oplus \mathbb{R}^{N\times N}_{\text{skew}}$. We recall that the two subspaces are orthogonal. Now, given $A \in \mathbb{R}^{N\times N}$, we can take $B = \frac{1}{2}(A + A^T)$ and $C = \frac{1}{2}(A - A^T)$. Clearly, $B \in \mathbb{R}^{N\times N}_{\text{sym}}$, $C \in \mathbb{R}^{N\times N}_{\text{skew}}$ and $A = B + C$;
3. Given $A \in \mathbb{R}^{N\times N}_{\text{sym}}$, $C \in \mathbb{R}^{N\times N}$, from 1. we have $A : C = A : C_{\text{sym}}$. Indeed:
$$A : C = A : (C_{\text{sym}} + C_{\text{skew}}) = A : C_{\text{sym}}.$$

In this section, we consider $\Omega \subset \mathbb{R}^2$ a bounded open Lipschitz set; $K \subset \Omega$ a crack (its regularity will be specified later), and $u \in H^1(\Omega \setminus K; \mathbb{R}^2)$. We consider the linear elasticity regime, which is valid for small deformations. In this context, we have the following:

Definition 2.1 (Strain, stress). *Given Ω, K, u as above, we define:*

1. $\varepsilon(u) = \mathcal{D}u_{\text{sym}}$;
2. $\sigma(u) = 2\mu\varepsilon(u) + \lambda\,\text{tr}(\varepsilon(u))\,\mathrm{I}$.

In particular, $\varepsilon(u) \in L^2(\Omega \setminus K; \mathbb{R}^{2\times 2}_{\text{sym}})$ *and* $\sigma(u) \in L^2(\Omega \setminus K; \mathbb{R}^{2\times 2}_{\text{sym}})$. I *is the identity matrix, while* μ *and* λ *are the so-called* Lamé constants*.*

Proposition 2.1 (Properties of ε and σ)**.** *The following hold:*

1. $|\varepsilon(u)|^2 \leq |\mathcal{D}u|^2$;
2. $|\sigma(u)|^2 \leq C|\varepsilon(u)|^2$.

Proof. The first inequality follows directly from the definition of $\varepsilon(u)$. For the other one, it is sufficient to give an estimate of $\lambda \operatorname{tr}(\varepsilon(u))\,\mathrm{I}$. we notice that $\lambda \operatorname{tr}(\varepsilon(u))$ is a scalar, thus $\lambda \operatorname{tr}(\varepsilon(u))\,\mathrm{I}$ is diagonal. Moreover, the map $A \mapsto \operatorname{tr}(A)$ is linear from $\mathbb{R}^{N\times N}$ to $\mathbb{R}$ and then $A \mapsto \operatorname{tr}(A)\mathrm{M}$ is linear in the same spaces, with $M \in \mathbb{R}^{N\times N}$ fixed. □

Now, we introduce the notion of energy.

Definition 2.2 (Linear elastic energy)**.** *Given* Ω, u, σ, ε *as above, the* linear elastic energy *is:*

$$E(u) := \frac{1}{2}\int_\Omega \sigma(u) : \varepsilon(u)\,dx.$$

Remark. We notice that $\sigma(u) : \varepsilon(v) = \sigma(v) : \varepsilon(u)$. In fact:

$$\sigma(u) : \varepsilon(v) = 2\mu\,\varepsilon(u) : \varepsilon(v) + \lambda \operatorname{tr}(\varepsilon(u)) \operatorname{tr}(\varepsilon(v)) = \sigma(v) : \varepsilon(u).$$

Thanks to Proposition 2.1, we can define a symmetric bilinear and continuous form a:

$$a : H^1(\Omega; \mathbb{R}^2) \times H^1(\Omega; \mathbb{R}^2) \to \mathbb{R}$$

such that:

$$a(u, v) = \int_\Omega \sigma(u) : \varepsilon(v)\,dx.$$

Using Definition 2.2, we have $a(u, u) = 2E(u)$.

Proposition 2.2. $a(u, v) = dE(u)[v]$.

Proof.

$$\begin{aligned} dE(u)[v] &= \lim_{h\to 0^+}\left(\frac{E(u+hv) - E(u)}{h}\right) = \lim_{h\to 0}\frac{1}{2h}\big(a(u+hv, u+hv) - a(u,u)\big) = \\ &= \lim_{h\to 0}\frac{1}{2h}\Big(a(u,u) + h^2 a(v,v) + 2h a(u,v) - a(u,u)\Big) = a(u,v) + \lim_{h\to 0}\frac{h}{2}a(v,v) = a(u,v). \end{aligned}$$

□

The coercivity of the bilinear form a introduced before is obtained by the following result.

Proposition 2.3 (Korn inequality)**.** *Let* $\Omega \subset \mathbb{R}^2$ *be an open Lipschitz set. Then:*

$$\int_\Omega |\varepsilon(u)|^2\,dx \geq \frac{1}{2}\int_\Omega |\mathcal{D}u|^2\,dx \quad \forall u \in H^1_0(\Omega; \mathbb{R}^2).$$

Proof. We consider $u \in C^\infty_c(\Omega; \mathbb{R}^2)$.

$$\begin{aligned} |\varepsilon(u)|^2 &= \left|\begin{pmatrix} u_{1,1} & \frac{1}{2}(u_{1,2} + u_{2,1}) \\ \frac{1}{2}(u_{1,2} + u_{2,1}) & u_{2,2} \end{pmatrix}\right|^2 = u_{1,1}^2 + u_{2,2}^2 + \frac{1}{2}(u_{1,2} + u_{2,1})^2 = \\ &= \frac{1}{2}(u_{1,1}^2 + u_{2,2}^2 + u_{1,2}^2 + u_{2,1}^2) + \frac{1}{2}u_{1,1}^2 + \frac{1}{2}u_{2,2}^2 + u_{1,2}u_{2,1} = \frac{1}{2}|\mathcal{D}u|^2 + \frac{1}{2}u_{1,1}^2 + \frac{1}{2}u_{2,2}^2 + u_{1,2}u_{2,1}. \end{aligned}$$

Then:

$$\int_\Omega |\varepsilon(u)|^2\,dx = \frac{1}{2}\int_\Omega |\mathcal{D}u|^2\,dx + \frac{1}{2}\int_\Omega u_{1,1}^2 + u_{2,2}^2 + 2u_{1,2}u_{2,1}\,dx.$$

We notice that:

$$\int_\Omega u_{1,2}u_{2,1}\,dx = \langle u_{1,2}, u_{2,1}\rangle = -\langle u_{1,21}, u_2\rangle = -\int_\Omega u_{1,21}u_2\,dx.$$

Therefore:

$$\int_\Omega u_{1,2}u_{2,1}\,dx = -\int_\Omega u_{1,21}u_2\,dx = \int_\Omega u_{1,1}u_{2,2}\,dx.$$

This implies:

$$\int_\Omega |\varepsilon(u)|^2\,dx = \frac{1}{2}\int_\Omega |\mathcal{D}u|^2\,dx + \frac{1}{2}\int_\Omega \operatorname{tr}(\varepsilon(u))^2\,dx \geq \frac{1}{2}\int_\Omega |\mathcal{D}u|^2\,dx.$$

By density of $C_c^\infty(\Omega;\mathbb{R}^2)$ in $H_0^1(\Omega;\mathbb{R}^2)$, we conclude. □

Proposition 2.4. *Let $u \in H^1(\Omega;\mathbb{R}^2)$. Then, $a(u,v) = 0$ for every $v \in H_0^1(\Omega;\mathbb{R}^2)$ if and only if*

$$-\underline{\operatorname{div}}(\sigma(u)) = 0 \quad \textit{in } H^{-1}(\Omega;\mathbb{R}^2).$$

Proof. If $a(u,v) = 0$, then:

$$\int_\Omega \sigma(u) : \varepsilon(v)\,dx = 0.$$

On the other hand:

$$0 = \int_\Omega \sigma(u) : \varepsilon(v)\,dx = \int_\Omega \sigma(u) : \mathcal{D}v\,dx = \sum_{i=1}^2 \int_\Omega \sigma_i(u)\nabla v_i\,dx = \sum_{i=1}^2 \langle -\operatorname{div}(\sigma_i(u)), v_i\rangle.$$

□

Remark. 1. Let u be a minimum for E. Then, since $a(u,v) = dE(u)[v]$, we obtain $a(u,v) = 0$.

2. The above result is valid taking $a(u,v) = f$ for a suitable f.

Now we study the following problem:

$$\begin{cases} -\underline{\operatorname{div}}(\sigma(u)) = f & \Omega \\ u = g & \partial\Omega \end{cases}$$

with $f \in L^2(\Omega,\mathbb{R}^2)$. Let $\tilde{g}$ be a lifting of g and define $w := u - \tilde{g}$. Since σ and $\underline{\operatorname{div}}$ are linear operators, we have:

$$\begin{cases} -\underline{\operatorname{div}}(\sigma(w)) = f + \underline{\operatorname{div}}(\sigma(\tilde{g})) & \Omega \\ w = 0 & \partial\Omega \end{cases}$$

Using the previous results, we obtain the following equivalent formulation:

$$P = \begin{cases} a(w,v) = l(v) & \forall v \in H_0^1(\Omega;\mathbb{R}^2) \\ w \in H_0^1(\Omega;\mathbb{R}^2) \end{cases}$$

where:

$$a(w,v) = \int_\Omega \sigma(w) : \varepsilon(v)\,dx; \qquad l(v) = \int_\Omega fv\,dx - \int_\Omega \sigma(\tilde{g}) : \varepsilon(v)\,dx$$

Proposition 2.5. *There exists a unique solution w for the problem P.*

Proof. It is enough to verify the hypothesis of the Lax-Milgram theorem. $H^1_0(\Omega;\mathbb{R}^2)$ is a Hilbert space, while the operator l is linear in v, and continuous. We need to control the continuity and coercivity of a.

- *Continuity.*

$$a(w,v)=\int_\Omega \sigma(w):\varepsilon(v)\,dx \le \|\sigma(w)\|_{L^2(\Omega;\mathbb{R}^2)}\|\varepsilon(v)\|_{L^2(\Omega;\mathbb{R}^2)} \le c\|w\|_{H^1_0(\Omega;\mathbb{R}^2)}\|v\|_{H^1_0(\Omega;\mathbb{R}^2)}.$$

- *Coercivity.* Using the Korn inequality:

$$\begin{aligned}a(w,w)&=\int_\Omega \sigma(w):\varepsilon(w)\,dx=\int_\Omega\Big(2\mu\varepsilon(w)+\lambda\,\mathrm{tr}(\varepsilon(w))\,\mathrm{I}\Big):\varepsilon(w)\,dx=\\&=\int_\Omega 2\mu|\varepsilon(w)|^2dx+\int_\Omega\lambda\,\mathrm{tr}^2(\varepsilon(w))\,dx\ge\int_\Omega 2\mu|\varepsilon(w)|^2\,dx\ge c\int_\Omega|\mathcal{D}w|^2\,dx=c\|w\|^2_{H^1_0(\Omega;\mathbb{R}^2)}.\end{aligned}$$

□

We now introduce the problem we want to resolve, in which we aim to study the behaviour of a crack inside a domain Ω. Let $\Omega\subset\mathbb{R}^2$ be a bounded, connected, open set with Lipschitz boundary $\partial\Omega$, and let $s\in[0,L]$. Assume that the initial crack $K_0\subset\Omega$ is closed and of class $W^{1,\infty}$. Then, the problem is:

$$P_g=\begin{cases}-\operatorname{div}(\sigma(u_s))=0 & \Omega\setminus K_s\\ u_s=g & \partial_D\Omega\\ \sigma(u_s)\cdot\nu=0 & \partial_N\Omega\\ \sigma(u_s)\cdot\nu_{K_s}=0 & K_s^\pm\end{cases}\tag{2.1}$$

with ν_{K_s} the normal to the crack K_s. The variational formulation is straightforward, using the same reasoning seen before. For each s let:

$$\mathcal{V}_s=\{v\in H^1(\Omega\setminus K_s): v=0\quad \partial_D\Omega\}.$$

We look for $w_s\in\mathcal{V}_s$ such that:

$$\int_{\Omega\setminus K_s}\sigma(w_s):\varepsilon(v_s)\,dx=-\int_{\Omega\setminus K_s}\sigma(\tilde g):\varepsilon(v_s)\,dx\qquad\forall v_s\in\mathcal{V}_s.$$

Although it may seem reasonable, it is not convenient to work directly with variations of the crack. For example, let $g=0$, $s>0$ be fixed and consider u_0, u_s be the equilibrium configuration for the two states 0 and s. Then, the variational formulations are:

$$\begin{aligned}&\int_{\Omega\setminus K_s}\sigma(u_s):\varepsilon(v_s)dx=0\qquad\forall v_s\in\mathcal{V}_s,\\&\int_{\Omega\setminus K_0}\sigma(u_0):\varepsilon(v_0)dx=0\qquad\forall v_0\in\mathcal{V}_0.\end{aligned}$$

Using the same notation seen in the previous section, we equivalently have:

$$\begin{aligned}a(u_s,v_s)=0&\qquad\forall v_s\in\mathcal{V}_s;\\a(u_0,v_0)=0&\qquad\forall v_0\in\mathcal{V}_0.\end{aligned}\tag{2.2}$$

Moreover:

- a is a bilinear, symmetric and coercive form;

- by a simple computation, $a(u_0 + u_s, u_0 - u_s) = a(u_s, u_s) - a(u_0, u_0)$;
- since the crack is increasing, $K_0 \subset K_s$, which implies that $\Omega \setminus K_s \subset \Omega \setminus K_0$ and thus $\mathcal{V}_0 \subset \mathcal{V}_s$. Then, using (2.2), we have $a(u_s, u_0 - u_s) = 0$.

Then, the variation of the energy gives the following estimate:

$$a(u_0, u_0) - a(u_s, u_s) = a(u_0 + u_s, u_0 - u_s) = a(u_0, u_0 - u_s) = a(u_0 - u_s, u_0 - u_s) \geq C \left\|u_0 - u_s\right\|^2,$$

where $\|\cdot\|$ denotes the norm $H^1_0(\Omega \setminus K)$. We see that, if the variation of the energy is $\mathcal{O}(h)$, then $u_0 - u_s = \mathcal{O}(h^{1/2})$. In particular, u_0' is not well defined. This is a problem, because when "deriving" the energy, we expect the following quantity:

$$\frac{dE}{dh} = \int_{\Omega \setminus K_0} \sigma(u_0) : \varepsilon(u_0')\, dx$$

For this reason, it seems not convenient, even if natural, to frame the energy release in terms of linear outer variations.

2.2 Assumptions on the variations of the crack.

Let s be fixed. Assume the variations K_s of K_0 to be of the form:

$$K_s = \Psi_s(K_0)$$

for a suitable family of bi-Lipschitz diffeomorphisms Ψ_s. In particular, we take $\Psi_s(x) = x + h\Phi_s(x)$, where:

1. Φ_s is uniformly bounded in $W^{1,\infty}(\Omega; \mathbb{R}^2)$;
2. $\Phi_s(x) = 0$ on $\partial\Omega$;
3. $\mathcal{D}\Phi_s \to \mathcal{D}\Phi$ a.e. in Ω.

Note that configurational variations of this type are not necessarily “incremental”, i.e. the family of cracks K_s is not necessarily increasing.

2.3 Two dimensional setting.

We start by considering the problem with the geometrical state $s \in [0, L]$ fixed: we have $E(u_s, s)$ and we want to find a minimizer u_s. Assume that the initial crack K and its incremental variations K_0 are both represented by the graph of a function k of class $W^{2,\infty}$ that is:

$$K_s = \{(x_1, k(x_1)) : x_1 \in [-l, s]\}.$$

Assume also, for simplicity, that $k(0) = k'(0) = 0$. Now, we want to represent the variations K_s in terms of configurational variations of the initial crack K. To this end, let $\phi \in W^{1,\infty}(\Omega; [0, 1])$ be a cut off function of the origin, i.e., $\phi = 1$ in a neighbourhood of 0 and $\mathrm{supp}(\phi) \Subset \Omega$. Then, we define:

$$\Psi_s(x_1, x_2) = \Big(x_1 + s\phi(x_1, x_2),\, x_2 + k(x_1 + s\phi(x_1, x_2)) - k(x_1)\Big).$$

Choosing:

$$\Phi(x_1, x_2) = \big(\phi(x_1, x_2),\, \phi(x_1, x_2)k'(x_1)\big); \quad \Phi_s(x_1, x_2) = \left(\phi(x_1, x_2), \frac{k(x_1 + s\phi(x_1, x_2)) - k(x_1)}{s}\right)$$

we can write:

$$\Psi_s(\underline{x}) = \underline{x} + s\Phi_s(\underline{x}).$$

We just need to verify the convergence property for Φ_s.

Proposition 2.6. *$\Phi_s \to \Phi$ strongly in $L^\infty(\Omega, \mathbb{R}^2)$. Moreover $\mathcal{D}\Phi_s \to \mathcal{D}\Phi$ a.e. in Ω and $\mathcal{D}\Phi_s$ is bounded in $L^\infty(\Omega, \mathbb{R}^{2\times 2})$.*

Proof. Since by hypothesis, k is of class, $W^{2,\infty}$ and ϕ is of class $W^{1,\infty}$, it is easy to verify that $\Phi_s \to \Phi$. A simple calculation shows that:

$$\mathcal{D}\Phi_s = \begin{pmatrix} \partial_{x_1}\phi & \partial_{x_2}\phi \\ \frac{k'(x_1+s\phi)-k'(x_1)}{s} + k'(x_1+s\phi)\partial_{x_1}\phi & k'(x_1+s\phi)\partial x_2\phi \end{pmatrix};$$

$$\mathcal{D}\Phi = \begin{pmatrix} \partial_{x_1}\phi & \partial_{x_2}\phi \\ \partial_{x_1}\phi\, k'(x_1) + \phi k''(x_1) & k'(x_1)\,\partial_{x_2}\phi \end{pmatrix}$$

Then, by continuity and a.e. differentiability of the Lipschitz function k', it follows that $\mathcal{D}\Phi_s \to \mathcal{D}\Phi$ a.e. in Ω. Thanks to the $W^{2,\infty}$ regularity of k, we have that $\mathcal{D}\Phi_s$ is bounded in $L^\infty(\Omega; \mathbb{R}^{2\times 2})$. □

Remark. Within these variations it is not possible to include kinked cracks, since in that case only $s\mathcal{D}\Phi_s$ is bounded in $L^\infty(\Omega, \mathbb{R}^{2\times 2})$.

2.4 Energy.

We use the following notation:

$$W(\mathcal{D}u) = \frac{1}{2}\mathbf{C}(\mathcal{D}u) : \mathcal{D}u$$

with $\mathbf{C}(\mathcal{D}u) = \sigma(u)$ and $\mathcal{D}u = \varepsilon(u)$. Using this notation, the energy takes the form:

$$E(u_s, s) = \int_{\Omega\setminus K_s} W(\mathcal{D}u_s)\, dx \tag{2.3}$$

with $u_s \in \mathcal{U}_s$, where:

$$\mathcal{U}_s = \{v \in H^1(\Omega \setminus K_s) : v = g \quad \partial_D\Omega.\}$$

As $K_s = \Psi_s(K_0)$, we compute a change of variable in eq. (2.3). Using the chain-rule:

$$\mathcal{D}(u_s \circ \Psi_s) = (\mathcal{D}u_s \circ \Psi_s)\mathcal{D}\Psi_s \Rightarrow \mathcal{D}u_s \circ \Psi_s = \mathcal{D}(u_s \circ \Psi_s)\mathcal{D}\Psi_s^{-1}.$$

Then:

$$\begin{aligned}\frac{1}{2}\int_{\Omega\setminus K_s} \mathcal{D}(u_s) : \mathbf{C}[\mathcal{D}(u_s)]\, dx &= \frac{1}{2}\int_{\Omega\setminus K_0} [\mathcal{D}(u_s)\circ\Psi_s] : \mathbf{C}[\mathcal{D}(u_s)\circ\Psi_s] \det(\mathcal{D}\Psi_s)\, dx = \\ &= \frac{1}{2}\int_{\Omega\setminus K_0} [\mathcal{D}(u_s\circ\Psi_s)\mathcal{D}\Psi_s^{-1}] : \mathbf{C}[\mathcal{D}(u_s\circ\Psi_s)\mathcal{D}\Psi_s^{-1}] \det(\mathcal{D}\Psi_s)\, dx. \end{aligned} \tag{2.4}$$

At this point, we use the equivalence $AB : C = A : CB^T$, with $A = \mathcal{D}(u_s \circ \Psi_s)$, $B = \mathcal{D}\Psi_s^{-1}$ and $C = \mathbf{C}[\mathcal{D}(u_s \circ \Psi_s)\mathcal{D}\Psi_s^{-1}]$. Finally, we obtain:

$$E(u_s, s) = \int_{\Omega\setminus K_0} \mathcal{D}(u_s \circ \Psi_s) : \mathbf{C}_s[\mathcal{D}(u_s \circ \Psi_s)]\, dx$$

with $\mathbf{C}_s[F] = \mathbf{C}[F\mathcal{D}\Psi_s^{-1}]\mathcal{D}\Psi_s^{-T}\det(\mathcal{D}\Psi_s)$. Since Ψ_s induces a one-to-one correspondence between $H_0^1(\Omega\setminus K_0)$ and $H_0^1(\Omega\setminus K_s)$ we will equivalently consider the energy $E_s(u_s, s)$ in $H_0^1(\Omega\setminus K_0)$. Now, we give some useful results.

Proposition 2.7. *$\Psi_s \to \mathrm{I}$ in $W^{1,\infty}(\Omega, \mathbb{R}^2)$. Moreover, for $s \ll 1$ we have the following expansions:*

$$\mathcal{D}\Psi_s^{-1} = \sum_{n=0}^{+\infty}(-s\mathcal{D}\Phi_s)^n = \mathrm{I} - s\mathcal{D}\Phi_s + o(s),$$
$$\det\mathcal{D}\Psi_s = 1 + s\,\mathrm{tr}(\mathcal{D}\Phi_s) + o(s).$$

which hold in $L^\infty(\Omega, \mathbb{R}^{n\times n})$.

Proof. Since $\mathcal{D}\Psi_s(x) = \mathrm{I} + s\mathcal{D}\Phi_s(x)$, with $\mathcal{D}\Phi_s$ uniformly bounded, we have $\Psi_s \to \mathrm{I}$ in $W^{1,\infty}(\Omega;\mathbb{R}^2)$. Moreover, for $s \ll 1$, with a simple calculation the two expansions are straightforward in $L^\infty(\Omega;\mathbb{R}^{2\times 2})$. □

Proposition 2.8. *Let $w \in H^1(\Omega\setminus K_0;\mathbb{R}^2)$. Then, $\mathbf{C}_s(\mathcal{D}w) \to \mathbf{C}(\mathcal{D}w)$ in $L^2(\Omega\setminus K_0;\mathbb{R}^{2\times 2})$ and:*

$$\frac{\mathbf{C}_s(\mathcal{D}w) - \mathbf{C}(\mathcal{D}w)}{s} \to \mathbf{C}'(\mathcal{D}w) = -\mathbf{C}(\mathcal{D}w\mathcal{D}\Phi_0) - \mathbf{C}(\mathcal{D}w)\mathcal{D}\Phi_0^T + \mathbf{C}(\mathcal{D}w)\,\mathrm{tr}\,\mathcal{D}\Phi_0$$

in $L^2(\Omega\setminus K_0;\mathbb{R}^{2\times 2})$.

Proof. We know that:

$$\mathbf{C}_s(\mathcal{D}w) = \mathbf{C}(\mathcal{D}w\mathcal{D}\Psi_s^{-1})\mathcal{D}\Psi_s^{-T}\det\mathcal{D}\Psi_s.$$

Since $\mathcal{D}\Psi_s^{-1} \to \mathrm{I}$ in $L^\infty(\Omega\setminus K_0;\mathbb{R}^{2\times 2})$, it follows that:

$$\mathbf{C}_s(\mathcal{D}w) \to \mathbf{C}(\mathcal{D}w)$$

in $L^2(\Omega\setminus K_0, \mathbb{R}^{2\times 2})$. From the expansions of Proposition 2.7, the following is valid a.e. in $\Omega\setminus K_0$:

$$\begin{aligned}\mathbf{C}_s(\mathcal{D}w) &= \mathbf{C}(\mathcal{D}w\mathcal{D}\Psi_s^{-1})\mathcal{D}\Psi_s^{-T}\det\mathcal{D}\Psi_s = \\ &= \mathbf{C}\Big(\mathcal{D}w\big(\mathrm{I} - s\mathcal{D}\Phi_s + o(s)\big)\Big)(\mathrm{I} - s\mathcal{D}\Phi_s + o(s))^T(1 + s\,\mathrm{tr}\,\mathcal{D}\Phi_s + o(s)) = \\ &= \mathbf{C}(\mathcal{D}w) + s\Big(-\mathbf{C}(\mathcal{D}w\mathcal{D}\Phi_s) - \mathbf{C}(\mathcal{D}w)\mathcal{D}\Phi_s^T + \mathbf{C}(\mathcal{D}w)\,\mathrm{tr}\,\mathcal{D}\Phi_s\Big) + o(s) = \\ &= \mathbf{C}(\mathcal{D}w) + s\mathbf{C}_s'(\mathcal{D}w) + o(s)\end{aligned}$$

where

$$\mathbf{C}_s'(\mathcal{D}w) = -\mathbf{C}(\mathcal{D}w\mathcal{D}\Phi_s) - \mathbf{C}(\mathcal{D}w)\mathcal{D}\Phi_s^T + \mathbf{C}(\mathcal{D}w)\,\mathrm{tr}\,\mathcal{D}\Phi_s.$$

Since $\mathcal{D}\Phi_s$ is uniformly bounded in $L^2(\Omega\setminus K_0;\mathbb{R}^{2\times 2})$ and converges a.e. to $\mathcal{D}\Phi$, by the theorem of dominated convergence, it follows that $\mathbf{C}_s'(\mathcal{D}w) \to \mathbf{C}'(\mathcal{D}w)$ in $L^2(\Omega\setminus K_0;\mathbb{R}^{2\times 2})$. □

2.5 Integral representation.

Thanks to the theorem of Lax-Milgram, we know that, for each s, there is a unique minimizer u_s for the quantities $E(\cdot, s)$. More precisely, let:

$$v_s \in \mathrm{argmin}\{E(w_s, s) : w_s \in \mathcal{U}_s\}, \qquad u_s \in \mathrm{argmin}\{E_s(w_s, s) : u \in \mathcal{U}_0\}$$

be respectively the displacement field in $\Omega \setminus K_s$ and its pull-back in $\Omega \setminus K_0$. We will denote by $G_\Phi(u_0)$ the variation of energy, i.e. :

$$G_\Phi(u_0) = \lim_{s\to 0} \frac{E(u_s, s) - E(u_0, 0)}{s} = \lim_{s\to 0} \frac{E_s(u_s, s) - E(u_0, 0)}{s}.$$

Theorem 2.1.

$$G_\Phi(u_0) = \frac{1}{2} \int_{\Omega\setminus K_0} \mathcal{D}u_0 : \mathbf{C}'(\mathcal{D}u_0)\, dx \tag{2.5}$$

where $\mathbf{C}'$ *has been defined in Proposition 2.8*

Proof. We define the following bilinear forms:

$$A_s(u_s, v) = \int_{\Omega\setminus K_0} \mathcal{D}v : \mathbf{C}_s(\mathcal{D}u_s)\, dx, \qquad A(u, v) = \int_{\Omega\setminus K_0} \mathcal{D}v : \mathbf{C}(\mathcal{D}u)\, dx,$$

so that the Euler–Lagrange equation for u_s reads $A_s(u_s, v) = 0$ for all $v \in \mathcal{V}_0$. First, we verify that A_s is elliptic and continuous uniformly with respect to s. We will need the next estimates, which follow from the hypothesis of section 2.2, together with the expansions seen Proposition 2.7:

$$\|\Psi_s\|_{L^\infty(\Omega\setminus K_0;\mathbb{R}^2)} \le c_1; \quad \|\Psi_s^{-1}\|_{L^\infty(\Omega\setminus K_s;\mathbb{R}^2)} \le c_2;$$
$$\|\mathcal{D}\Psi_s^{-1}\|_{L^\infty(\Omega\setminus K_s;\mathbb{R}^{2\times 2})} \le c_3; \quad \|\det \mathcal{D}\Psi_s\|_{L^\infty(\Omega\setminus K_0)} \le c_4.$$

We now verify the two properties for A_s:

1. *Continuity.* We recall that:

$$\mathbf{C}_s(\mathcal{D}w) = \mathbf{C}(\mathcal{D}w\mathcal{D}\Psi_s^{-1})\mathcal{D}\Psi_s^{-T} \det(\mathcal{D}\Psi_s).$$

   Then:

$$\|\mathbf{C}_s(\mathcal{D}w)\|_{L^2(\Omega\setminus K_0)} \le \|\mathbf{C}(\mathcal{D}w\mathcal{D}\Psi_s^{-1})\|_{L^2(\Omega\setminus K_0)} \|\mathcal{D}\Psi_s^{-T}\|_{L^\infty(\Omega\setminus K_0)} \|\det(\mathcal{D}\Psi_s)\|_{L^\infty(\Omega\setminus K_0)} \le$$
$$\le \hat{c}_1 \|\mathcal{D}w\mathcal{D}\Psi_s^{-1}\|_{L^2(\Omega\setminus K_0)} \le \hat{c}_2 \|\mathcal{D}w\|_{L^2(\Omega\setminus K_0)}.$$

   In conclusion:

$$|A_s(w, v)| \le \int_{\Omega\setminus K_0} |\mathcal{D}w| : |\mathbf{C}_s(\mathcal{D}v)|\ dx \le \|\mathcal{D}w\|_{L^2(\Omega\setminus K_0;\mathbb{R}^{2\times 2})} \|\mathbf{C}_s(\mathcal{D}v)\|_{L^2(\Omega\setminus K_0;\mathbb{R}^{2\times 2})} \le$$
$$\le C\|\mathcal{D}w\|_{L^2(\Omega\setminus K_0;\mathbb{R}^{2\times 2})} \|\mathcal{D}v\|_{L^2(\Omega\setminus K_0;\mathbb{R}^{2\times 2})} \le C\|w\|_{H^1(\Omega\setminus K_0;\mathbb{R}^2)} \|v\|_{H^1(\Omega\setminus K_0;\mathbb{R}^2)}$$

2. *Coercivity.* Thanks to the theorem of change of variables between Sobolev spaces, we have:

$$\|\mathcal{D}w\|^2_{L^2(\Omega\setminus K_0)} = \int_{\Omega\setminus K_0} |\mathcal{D}w|^2\, dx = \int_{\Omega\setminus K_0} \left|\mathcal{D}\left[(w \circ \Psi_s) \circ \Psi_s^{-1}\right]\right|^2\, dx =$$
$$= \int_{\Omega\setminus K_0} \left|\mathcal{D}(w \circ \Psi_s)(\Psi_s^{-1}(x))\, \mathcal{D}\Psi_s^{-1}\right|^2\, dx =$$
$$= \int_{\Omega\setminus K_s} \left|\mathcal{D}(w \circ \Psi_s)(y)\mathcal{D}\Psi_s^{-1}\right|^2 |\det \mathcal{D}\Psi_s|\ dy \le c_2 \|\mathcal{D}(w \circ \Psi_s)\|^2_{L^2(\Omega\setminus K_s;\mathbb{R}^{2\times 2})}.$$

Now, using the Korn inequality, we obtain:

$$\begin{aligned}A_s(w,w) &= \int_{\Omega\setminus K_0} \mathcal{D}w : \mathbf{C}_s[\mathcal{D}w]\,dx = \int_{\Omega\setminus K_s} \mathcal{D}(w\circ\Psi_s) : \mathbf{C}[\mathcal{D}(w\circ\Psi_s)]\,dx \geq \\ &\geq c_1\|\mathcal{D}(w\circ\Psi_s)\|^2_{L^2(\Omega\setminus K_s;\mathbb{R}^{2\times 2})} \geq c_2\|\mathcal{D}w\|^2_{L^2(\Omega\setminus K_0;\mathbb{R}^{2\times 2})} \geq c\|w\|^2_{H^1(\Omega\setminus K_0;\mathbb{R}^2)}.\end{aligned}$$

which proves the coercivity of A_s.

Now, we prove the weak convergence of u_s to u_0 in $H^1(\Omega\setminus K_0;\mathbb{R}^2)$. From the previous computations, it follows that u_s is bounded in $H^1(\Omega\setminus K_0;\mathbb{R}^2)$ and thus (up to subsequences) that $u_s \rightharpoonup w$ in $H^1(\Omega\setminus K_0;\mathbb{R}^2)$. Passing to the limit in the variational formulation $A_s(u_s,v)=0$, we obtain $A(w,v)=0$ for all $v\in\mathcal{V}_0$. Then, $w=u_0$ and $u_s\rightharpoonup u_0$ in $H^1(\Omega\setminus K_0;\mathbb{R}^2)$. By the variational formulations, we have:

$$A_s(u_s,u_s) = A_s(u_s-u_0,u_s) + A_s(u_0,u_s) = A_s(u_0,u_s).$$

Hence:

$$E_s(u_s,s) = \frac{1}{2}A_s(u_s,u_s) = \frac{1}{2}A_s(u_0,u_s), \qquad E(u_0,0) = \frac{1}{2}A(u_0,u_0).$$

Writing explicitly, we get:

$$\frac{1}{2}A_s(u_0,u_s) - \frac{1}{2}A(u_0,u_0) = \frac{1}{2}\int_{\Omega\setminus K_0} \mathcal{D}u_s : \big(\mathbf{C}_s(\mathcal{D}u_0) - \mathbf{C}(\mathcal{D}u_0)\big)\,dx$$

Hence, by Proposition 2.8:

$$\frac{E_s(u_s,s) - E(u_0,0)}{s} \to \frac{1}{2}\int_{\Omega\setminus K_0} \mathcal{D}u_0 : \mathbf{C}'(\mathcal{D}u_0)\,dx$$

which concludes the proof. □

Note that Theorem 2.1 employs only the weak convergence of u_s. In particular, it does not rely on quantitative estimates of $u_s - u_0$. Finally, introducing $E'(u) = \frac{1}{2}A'(u,u)$, with:

$$A'(u,w) = \int_{\Omega\setminus K_0} \mathcal{D}u : \mathbf{C}'(\mathcal{D}w)\,dx,$$

we can say, together with eq. (2.5) that $G_\Phi(u_0) = E'(u_0)$.

Remark. In the case of "incremental variations", G_Φ provides the energy release up to a multiplicative constant. Strictly speaking, the classical definition of elastic energy release (for incremental cracks) would be:

$$\mathcal{G}(u_0) = -\lim_{s\to 0}\frac{E_s(u_s) - E(u_0)}{\mathscr{H}^{n-1}(K_s\setminus K_0)},$$

i.e. the negative derivative with respect to variations of crack surface area, and not with respect the parametrization parameter s. Thus, apart from the minus sign, $G_\Phi(u_0)$ differs from $\mathcal{G}(u_0)$ by a multiplicative constant, which measures the variation of surface area with respect to s. Finally, note that $G_\Phi(u_0)$ linearly depends on Φ through $\mathbf{C}'$. As a matter of fact, the energy release depends only on the variations of the crack and it is independent of the choice of the configurational map.

Appendix

Three dimensional setting

In this part, we give some information for the three dimensions. Assume again that the initial crack $K \subset \Omega$ is represented by a graph, $K = h(K^b)$, where:

- $h(x_1, x_2) = \big(x_1,\, x_2,\, k(x_1, x_2)\big)$ for a suitable function k of class $W^{2,\infty}$;
- $K^b \subset \mathbb{R}^2$ a compact, connected set (the "base").

Let $\phi \in W^{1,\infty}(\Omega; \mathbb{R}^3)$ with $|\phi| \leq 1$ and $\text{supp}(\phi) \Subset \Omega$. For $s > 0$, we define:

$$\Psi_s(x) = x + \Big(s\phi_1(x),\, s\phi_2(x),\, k\big(x_1 + s\phi_1(x),\, x_2 + s\phi_2(x)\big) - k(x_1, x_2)\Big)$$

and the corresponding variations of the crack $K_s = \Psi_s(K)$. Denote also $h_s(x_1, x_2) = \Psi_s \circ h(x_1, x_2)$ so that $K_s = h_s(K^b)$. Introducing $\phi^b(x_1, x_2) = \phi(x_1, x_2, k(x_1, x_2))$, the map h_s can be written as:

$$h_s(x_1, x_2) = \Big(x_1 + s\phi_1^b(x),\, x_2 + s\phi_2^b(x),\, k(x_1 + s\phi_1^b(x),\, x_2 + s\phi_2^b(x))\Big) = h(x_1^s, x_2^s)$$

where $(x_1^s, x_2^s) = (x_1, x_2) + s\phi^b(x_1, x_2)$ represents the corresponding variation of the domain K^b.

Chapter 3

Evolution of the crack.

3.1 Introduction

In order to study the evolution of the crack in an elastic body, we review the model proposed by A. A. Griffith.

Theorem 3.1 (Minimum energy theorem)**.** *The equilibrium state of an elastic solid body, subjected to prescribed surface forces, is such that the potential energy of the whole system is at its minimum.*

Since the total potential energy is not convex, we have, in general, local minima. This non-convexity motivates the following rupture criterion:

Theorem 3.2 (Rupture criterion)**.** *The equilibrium position must be one in which rupture of the solid has occurred, if the system can pass from the unbroken to the broken condition by a process involving a continuous decrease of the total potential energy.*

However, in order to apply this extended theorem to the problem of finding the breaking loads of real solids, it is necessary to take into account the increase in potential energy. When a rupture occurs, new surfaces are formed in the interior of the solid, and it is known that work must be done against the cohesive forces of the molecules on either side of the crack. This work appears as potential surface energy, which per unit area is a material constant. On the other hand, the rupture of the material also releases some elastic potential energy. Thus, a balance must be considered between the released elastic potential energy, which decreases as the crack opens, and the surface potential energy, which increases with the formation of new surfaces. From the physical point of view, the validity of this model is quite narrow in the range of phenomena related to cracks, as it is confined to the case of quasi-static propagation of a brittle, pre-existing crack. However, it provided the basis on which fracture mechanics developed.

3.2 Notation.

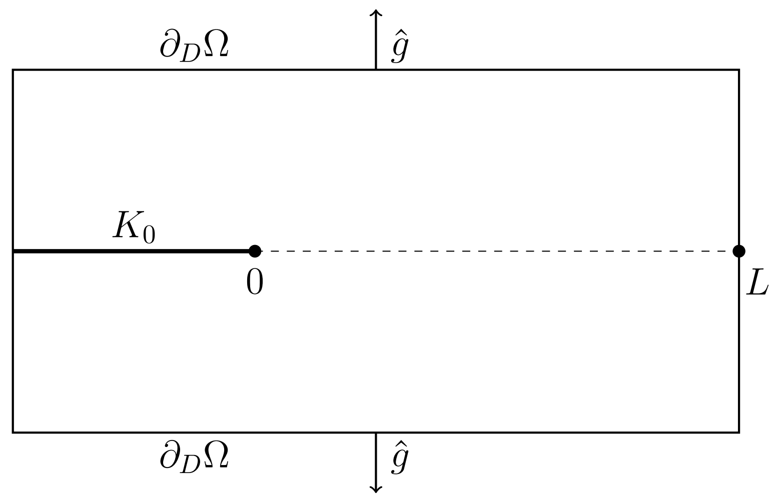
In this chapter, we consider:

1. $\Omega \subset \mathbb{R}^2$;
2. K_0 the initial crack.

For simplicity, we can consider Ω to be a rectangle and the admissible cracks to be restricted to the family of closed line segments K_l of the form $[0,l] \times \{0\}$, with $l \in [0, L]$. In the theory of Griffith, the cohesive forces across the crack are neglected, hence the potential energy of the fracture is just of the form $G_c\, l$, where $G_c > 0$ is a material parameter, often written as $G_c = 2\gamma$, where γ is the toughness of the material. As before, we have the elastic deformation u_l on $\Omega \setminus K_l$. Moreover, we set a boundary displacement $\hat{g} = \alpha(t) g$, with:

- $g \in H^{1/2}(\partial\Omega; \mathbb{R}^2)$;
- $\alpha \in C^1([0,T])$;
- α is strictly monotonically increasing;
- $\alpha(0) = 0$;

imposed on $\partial_D\Omega$. In our example, we consider $\partial_D\Omega$ the upper and lower edges of Ω. The remaining part of the boundary $\partial_N\Omega = \partial\Omega \setminus \partial_D\Omega$ is traction free. In this context, we have the previously seen elastic energy. Moreover, since the material is brittle, the faces of the crack K_l are traction free. A schematic representation of the initial configuration is shown below:



To summarize, we have two parameters: the spatial parameter $\ell \in [0, L]$ and the time parameter $t \in [0, T]$. The model we are considering is quasi-static, which means that, for any time $t \in [0, T]$, the system is in equilibrium.
Let us consider now the initial configuration $\Omega \setminus K_0$ and let $t = 0$. Then, we have $u_0 = 0$ on $\partial_D\Omega$. From the previous section, we know that there exists a unique $\hat{u}_0$ such that:

$$\hat{u}_0 = \operatorname{argmin} \left\{ E(v) \,:\, v \in H^1(\Omega \setminus K_0; \mathbb{R}^2), \quad v = g \quad \partial_D\Omega \right\}.$$

with $E(v)$ the elastic energy seen in the previous section. Now, we consider the evolution in time, with the spatial parameter $\ell = 0$ fixed (i.e. the crack is not propagating) and $t > 0$. Then, we define:

$$u_0(t) = \operatorname{argmin} \left\{ E(v) \,:\, v \in H^1(\Omega \setminus K_0; \mathbb{R}^2), \quad v = \hat{g} \quad \partial_D\Omega \right\}. \tag{3.1}$$

The eq. (3.1) has a unique solution of the form $u(t) = \alpha(t)\hat{u}_0$. Consider the operator

$$\begin{aligned} \mathcal{S} : H^{1/2}(\partial\Omega; \mathbb{R}^2) &\longrightarrow H^1(\Omega \setminus K_0; \mathbb{R}^2), \\ h &\longmapsto u_h, \end{aligned} \tag{3.2}$$

where u_h denotes the unique solution of 3.1 with boundary condition $u = h$. Since the associated bilinear form is continuous and the problem admits a unique solution, the operator $\mathcal{S}$ is well-defined, linear, and continuous. Then:

$$\alpha(t)\hat{u}_0 = \alpha(t)\mathcal{S}(g) = \mathcal{S}(\alpha(t)g) = u_{\alpha(t)g}.$$

Thus, we write $u(t) := u_{\alpha(t)\hat{u}_0}$.

Remark. The result holds for any $l \in [0, L]$.

We denote:

$$\widehat{\mathcal{E}}(l) := \min\left\{E(u)\ :\ u \in H^1(\Omega \setminus K_l;\mathbb{R}^2), \quad u = g \quad \partial_D\Omega\right\} \tag{3.3}$$

Definition 3.1. *The free energy functional* $\widehat{\mathscr{F}} : [0, L] \to [0, +\infty)$ *is given by:*

$$\widehat{\mathscr{F}}(l) = \widehat{\mathcal{E}}(l) + G_c\, l$$

This form of the free energy is not optimal, since the value of the energy is not explicit with respect to its argument. In many applications, including Griffith's work, the exact value of the energy is replaced by suitable estimates in special cases, but, for our purpose, this is not possible. At this point, let us introduce a fundamental quantity, the energy release rate, which measures the decrease in elastic energy with respect to crack increments. In the open interval $[0, L]$, the energy release rate is defined as:

$$\widehat{G}(l) := -\frac{\partial}{\partial \ell}\widehat{\mathcal{E}}(l). \tag{3.4}$$

Note that the quantity $\widehat{G}$ is analogous to G_Φ introduced in section 2.5.

3.3 Classic framework

Thanks to the previous definitions, the Griffith's criterion stated before can be formulated in:

Proposition 3.1. *A crack* K_l *is in equilibrium when* $\widehat{G}(l) \le G_c$. *It may propagate in quasi-static regime when* $\widehat{G}(l) = G_c$. *The case* $\widehat{G}(l) \ge G_c$ *is related to unstable propagation.*

The unstable propagation should be out of our quasi-static framework, but this is not always possible. The Griffith's criterion can be written conveniently in terms of the free energy $\widehat{\mathscr{F}}$. In fact:

$$\frac{\partial}{\partial l}\widehat{\mathscr{F}}(l) = \frac{\partial}{\partial l}\widehat{\mathcal{E}}(l) + G_c = -\widehat{G}(l) + G_c. \tag{3.5}$$

It follows that the crack is in equilibrium when the free energy is non-decreasing, i.e. when $-\widehat{G}(l) + G_c \ge 0$. Note that for $l = L$, the inequality $-\widehat{G}(L) + G_c \ge 0$ is trivially satisfied, since $\widehat{G}(L) = 0$ and $G_c > 0$. This means that, independently of any material parameter, the fracture $K_L = [0, L]$ is stationary.
Now, consider that in the time frame $[0, +\infty)$, the evolution of the state variables l and u is driven by the Dirichlet boundary condition seen before: $u(y) = \hat{g}(t, y) = \alpha(t)g(y)$ for $y \in \partial_D(\Omega)$, with the previous hypothesis for α. Then, we define $\mathcal{E}(t, l)$ and $G(t, l)$ as in eq. (3.4) - eq. (3.5) and, thanks to eq. (3.2):

$$\mathcal{E}(t, l) = \alpha^2(t)\widehat{\mathcal{E}}(l), \qquad G(t, l) = \alpha^2(t)\widehat{G}(l).$$

In analogy, we have:

$$\mathscr{F}(t, l) = \mathcal{E}(t, l) + G_c\, l = \alpha^2(t)\widehat{\mathcal{E}}(l) + G_c\, l. \tag{3.6}$$

We can present a set of equations that conveniently describes the quasi-static propagation of the crack K_ℓ. Let $\ell(t)$ denote the position of the crack tip in the quasi-static propagation. At time t, the fracture set is $K_{\ell(t)} = [0, \ell(t)]$. Griffith's criterion can be written conveniently in terms of the Kuhn–Tucker conditions: for every time $t \in [0, +\infty)$:

$$\begin{cases} \dot{\ell}(t) \ge 0, \\ G(t, \ell(t)) \le G_c, \\ \big(G(t, \ell(t)) - G_c\big)\dot{\ell}(t) = 0. \end{cases} \tag{3.7}$$

The first equation gives the stationarity of the crack, while the third gives the propagation condition: if the crack advances ($\dot{\ell}(t) > 0$), then $G(t, \ell(t)) = G_c$ (the release rate G has reached the critical value G_c). The second equation represents an equilibrium condition. Indeed, from eq. (3.6), we have that $\partial_\ell \mathscr{F} \geq 0$, which means that, as the crack grows, more energy is released. Therefore, taking $\ell(t)$ as the reference point, we obtain that $\mathscr{F}(t, \ell(t))$ is a local minimum. Moreover, by the chain rule, we have:

$$\frac{d}{dt}\mathscr{F}(t, \ell(t)) = \partial_t \mathscr{F}(t, \ell(t)) + \partial_\ell \mathscr{F}(t, \ell(t))\dot{\ell}(t).$$

Recalling eq. (3.6) and the Kuhn-Tucker conditions, it follows that:

$$\partial_\ell \mathscr{F}(t, \ell(t))\dot{\ell}(t) = 0.$$

By the power balance, we conclude:

$$\frac{d}{dt}\mathscr{F}(t, \ell(t)) = \partial_t \mathscr{F}(t, \ell(t)) = \mathcal{P}_{\text{ext}}(t),$$

where $\mathcal{P}_{\text{ext}}(t)$ is the power of the external forces. Then:

$$\int_0^T \frac{d}{dt}\mathscr{F}(t, \ell(t))\, dt = \int_0^T \mathcal{P}_{\text{ext}}(t)\, dt = L_{\text{ext}}(0, T); \tag{3.8}$$

where L_{ext} is the work done by the external forces between 0 and T. On the other hand:

$$\int_0^T \frac{d}{dt}\mathscr{F}(t, \ell(t))\, dt = \mathscr{F}(T, \ell(T)) - \mathscr{F}(0, \ell(0)) = \mathcal{E}(T, \ell(T)) + G_c \ell(T) - \mathcal{E}(0, \ell(0)). \tag{3.9}$$

From eq. (3.8) - eq. (3.9), we obtain:

$$\mathcal{E}(T, \ell(T)) = \mathcal{E}(0, \ell(0)) + L_{\text{ext}}(0, T) - G_c \ell(T).$$

The evolution problem formulated above in terms of classical derivatives is, in general, not well-given, as the existence of a solution is not guaranteed.
We will see in the following that the function ℓ is, in general, not differentiable in the classical sense and may not even be continuous. In particular, ℓ may have jump discontinuities. This behaviour may seem inconsistent with the physical setting. However, the quasi-static evolution should be interpreted as a limit process, obtained by time rescaling. From this perspective, jump discontinuities correspond to fast transitions occurring on a very small time intervals. This is the case for the evolution by stationary points. In general, the admissible evolutions ℓ are only of bounded variation; hence, their time derivative $\dot{\ell}$ is a measure. As a consequence, eq. (3.7) must be reformulated in a suitable weak sense. Finally, we remark that the jump discontinuities are related to the non-convexity of the energy functional. The natural generalization of the Kuhn-Tucker conditions is the following:

$$G(t,\, \ell(t)) \leq G_c, \quad \forall t \in [0, T], \tag{3.10}$$

$$\big(G(t,\, \ell^-(t)) - G_c\big)\, d\ell(t) = 0, \tag{3.11}$$

where eq. (3.10) holds for every $t \in [0, T]$, while eq. (3.11)is to be interpreted in the sense of measures:

$$\int_A \big(G(t, \ell^-(t)) - G_c\big)\, d\,\ell(t) = 0 \quad \text{for every Borel set } A \subset [0, T]$$

There are two important differences in the activation condition compared to the classical formulation. First, $G(t, \ell(t)$ is replaced by the left limit $G(t, \ell^-(t)) = \big(G(t, \ell(t))\big)^-$. Using left limits is natural because the activation depends only on the evolution up to time t. Second, the measure $d\ell$ is used instead of the derivative $\dot{\ell}$. Note that $d\ell$ is a space variable, being a measure, and not a velocity as $\dot{\ell}$. Moreover, let us write $d\ell = d\ell^a + d\ell^c + d\ell^j$,where $d\ell^a$ is the absolutely continuous part, $d\ell^c$ is the Cantor part, and $d\ell^j$ is concentrated on the set of jumps $S(\ell)$. Then:

$$d\ell^a = \dot{\ell}(t)dt \qquad d\ell^j = \sum_{t\in S(\ell)} (\ell^+(t) - \ell^-(t))\delta_t = \sum_{t\in S(\ell)} [\![\ell(t)]\!]\delta_t$$

where $\dot{\ell}(t)$ is the density with respect to the Lebesgue measure dt, and δ_t is the Dirac delta concentrated in t. Being $d\ell^k$, for $k = a, c, j$ mutually singular, the activation condition holds separately for each measure:

$$\big(G(t, \ell^-(t)) - G_c\big) d\ell^k(t) = 0 \quad \text{ for } k = a, c, j$$

in the sense of measures in $[0, T]$. In particular, for the velocity $\dot{\ell}(t)$ the Kuhn–Tucker condition becomes:

$$\big(G(t, \ell(t)) - G_c\big)\dot{\ell}(t) = 0 \quad \text{for a.e. } t \in [0, T],$$

while for the jumps it reads

$$\big(G(t, \ell^-(t)) - G_c\big)[\![\ell(t)]\!] = 0 \Leftrightarrow G(t, \ell^-(t)) = G_c.$$

We remark that it is important to require that eq. (3.10)-eq. (3.11) holds true in the sense of measure and not only for a.e. $t \in [0, T]$. Our requirement is slightly stronger, but the difference is fundamental since characteristic features of the evolution, such as jump points, are concentrated on sets that are a.e. negligible. In the sequel, it will be clear that eq. (3.10)-eq. (3.11) is not enough to characterize the evolution in the presence of jump discontinuities; the characterization of the evolution at jumps requires knowledge of the behaviour of $G(t, \cdot)$ on the interval $[\ell^-, \ell^+]$.
The evolutions presented in the next sections are defined in terms of the energies.

Proposition 3.2. *Let $\ell \in BV_{\text{loc}}(0, +\infty)$. Then, $\mathscr{F}(\cdot, \ell(\cdot)) \in BV_{\text{loc}}(0, +\infty)$. The distributional derivative $d\mathscr{F}$ is a measure and can be written as:*

$$d\mathscr{F} = \mathcal{P}_{\text{ext}}(t)dt - (G(t, \ell(t)) - G_c)(d^a\ell + d^c\ell) + d^j\mathscr{F}, \tag{3.12}$$

where $\mathcal{P}_{\text{ext}}$ is the same as before, defined as:

$$\mathcal{P}_{\text{ext}}(t) = \int_{\partial_D\Omega} \dot{g}(t, y)\,\big(\sigma(t, y)\,\nu(y)\big)\,dy.$$

Remark. In terms of the decompositions, we have:

$$\begin{aligned} d\mathscr{F} &= d^a\mathscr{F} + d^c\mathscr{F} + d^j\mathscr{F} \\ d\ell &= \dot{\ell}\,dt + d^c\ell + d^j\ell. \end{aligned}$$

Combining with eq. (3.12):

$$\begin{aligned} d^a\mathscr{F} &= (\mathcal{P}_{\text{ext}}(t) - (G(t, \ell(t)) - G_c)\dot{\ell}(t)\,dt, \\ d^c\mathscr{F} &= -(G(t, \ell(t)) - G_c)d^c\ell, \\ d^j\mathscr{F} &= \sum_{t\in S(\ell)} [\![\mathscr{F}(t, \ell(t))]\!]\delta_t, \end{aligned}$$

where

$$[\![\mathscr{F}(t, \ell(t))]\!] = \mathscr{F}(t, \ell^+(t)) - \mathscr{F}(t, \ell^-(t)) = -\int_{\ell^-(t)}^{\ell^+(t)} (G(t, s) - G_c)\,ds.$$

Proof. Since $\mathscr{F}(t, l)$ is of class C^1 with respect to l by the chain rule in BV it follows that $\mathscr{F}(t, \ell(t)) \in BV_{\text{loc}}(0, +\infty)$. Hence, its distributional derivative $d\mathscr{F}$ is a measure of the form $d\mathscr{F} = d^a\mathscr{F} + d^c\mathscr{F} + d^j\mathscr{F}$.

More precisely:

$$\begin{aligned} d^a \mathscr{F} + d^c \mathscr{F} &= \langle \nabla \mathscr{F}, (dt, d^a\ell + d^c\ell)\rangle = \\ &= \frac{\partial}{\partial t}\mathscr{F}(t,\ell(t))\,dt + \frac{\partial}{\partial l}\mathscr{F}(t,\ell(t))(d^a\ell + d^c\ell) = \\ &= \frac{\partial}{\partial t}\mathcal{E}(t,\ell(t))\,dt - (G(t,\ell(t)) - G_c)(d^a\ell + d^c\ell). \end{aligned}$$

Since $\mathcal{E}(t,\ell) = \alpha^2(t)\widehat{\mathcal{E}}(\ell)$ and recalling eq. (2.1):

$$\frac{\partial}{\partial t}\mathcal{E}(t,\ell(t)) = 2\alpha(t)\dot{\alpha}(t)\widehat{\mathcal{E}}(\ell) = \alpha(t)\dot{\alpha}(t)\int_{\partial_D\Omega}(\hat{\sigma}(t,y)\,\nu(y))\hat{g}(y)\,dy.$$

Remembering that $\dot{g}(t,y) = \dot{\alpha}(t)\hat{g}(y)$ and $u_{\ell(t)}(x) = \alpha(t)\hat{u}_{\ell(t)}(x)$:

$$\frac{\partial}{\partial t}\mathcal{E}(t,\ell(t)) = \int_{\partial_D\Omega}(\sigma(t,y)\,\nu(y))\,\dot{g}(t,y)\,dy = \mathcal{P}_{\text{ext}}(t).$$

Note that $\mathcal{P}_{\text{ext}} \in L^1$ since $\dot{\alpha} \in L^1$ and $\widehat{\mathcal{E}} \in L^\infty$. Moreover:

$$d^j\mathscr{F} = \sum_{t\in S(\ell)}\big(\mathscr{F}(t,\ell^+(t)) - \mathscr{F}(t,\ell^-(t))\big)\delta_t = -\sum_{t\in S(\ell)}\left(\int_{\ell^-(t)}^{\ell^+(t)}(G(t,s) - G_c)\,ds\right)\delta_t,$$

which concludes the proof. □

Remark. For all the evolutions considered here, $d^j\mathscr{F}$ is a negative measure and $(G(t,\ell(t)) - G_c)(d^a\ell + d^c\ell) = 0$. Hence, the energy $\mathscr{F}(t,\ell(t))$ belongs to $SBV_{\text{loc}}(0,T)$ and satisfies the inequality $d\mathscr{F} \le \mathcal{P}_{\text{ext}}\,dt$ (in the sense of measures).

3.4 Evolution by stationary points

In this section, we present the evolution using a time-discretization technique, following a constructive approach. Let Δt^k be a positive sequence of time increments such that $\Delta t^k \searrow 0$. For every index k, consider a uniform time discretization $t_n^k = n\Delta t^k$, for $n \in \mathbb{N}$. Then, we define the sequence ℓ_n^k by induction with respect to $n \in \mathbb{N}$ as:

$$\ell_0^k = l_0, \qquad \ell_n^k = \min\{l \in [\ell_{n-1}^k, L] : G(t_n^k, l) \le G_c\}. \tag{3.13}$$

Remark. If $\ell_n^k > \ell_{n-1}^k$, then $G(t_n^k, l) > G_c$ for $l \in [\ell_{n-1}^k, \ell_n^k)$. By continuity of G, we obtain $G(t_n^k, \ell_n^k) = G_c$. Furthermore, for $l \in [\ell_{n-1}^k, \ell_n^k)$:

$$\frac{\partial}{\partial l}\mathscr{F}(t_n^k, l) = -G(t_n^k, l) + G_c < 0,$$

which implies that $\mathscr{F}(t_n^k, \cdot)$ is strictly decreasing in $[\ell_{n-1}^k, \ell_n^k)$, in agreement with the Griffith criterion.

In this way, the crack advances only when the stationarity condition is violated, and stops as soon as it is satisfied. Indeed, ℓ_n^k is the stationary point of the free energy $\mathscr{F}(t_n^k, \cdot)$, closest to ℓ_{n-1}^k. Equivalently, from a physical point of view, eq. (3.13) can be written as:

$$\ell_0^k = l_0, \qquad \ell_n^k = \operatorname{argmin}\{G_c(l - \ell_{n-1}^k) : l \in [\ell_{n-1}^k, L],\ \text{with } G(t_n^k, l) \le G_c\}$$

Let $\ell^k : [0,+\infty) \to [l_0, L]$ be the piecewise constant function given by:

$$\ell^k(t) = \ell_n^k \qquad \text{for } t_n^k \le t < t_{n+1}^k.$$

We now define the evolution ℓ_G as the limit of ℓ_n^k defined above, as $\Delta t^k \searrow 0$.

Definition 3.2. *A function* $\ell_G : [0,+\infty) \to [l_0, L]$ *is* an evolution of the quasi-static crack propagation problem in the sense of Griffith *if there exists a subsequence of* $\{\ell^k\}$ *converging pointwise to* ℓ_G*.*

Remark. The existence of a limit function in the sense of the preceding definition is guaranteed by the following general result:

Theorem 3.3 (Helly's Theorem). *Let* $\ell_n : \mathbb{R} \to \mathbb{R}$ *be a sequence of uniformly bounded and non-decreasing functions. Then, there exists a subsequence that converges pointwise to a bounded and non-decreasing function* ℓ*.*

Proposition 3.3. *Let* α *be a strictly monotone control. Then, for every time* t_n^k*:*

$$\ell_n^k = \min\{l \in [l_0, L] : G(t_n^k, l) \leq G_c\}. \tag{3.14}$$

In particular, if $l_0 < \ell_n^k < L$*, then* $G(t_n^k, \ell_n^k) = G_c$*.*

Proof. We argue by induction.

$n = 1$. It follows directly from eq. (3.13).

$n > 1$. Assume eq. (3.14) holds true for time t_n^k. By definition:

$$\ell_{n+1}^k = \min\{l \in [\ell_n^k, L] : G(t_{n+1}^k, l) - G_c \leq 0\}.$$

If $\ell_n^k = l_0$, then eq. (3.14) is satisfied for ℓ_{n+1}^k. Otherwise, by induction, $G(t_n^k, l) - G_c > 0$ for $l \in [l_0, \ell_n^k)$. Denoting $\alpha_n^2 := (\alpha(t_n^k))^2$ and using the monotonicity of α:

$$G(t_{n+1}^k, l) - G_c = \alpha_{n+1}^2 \widehat{G}(l) - G_c \geq \alpha_n^2 \widehat{G}(l) - G_c = G(t_n^k, l) - G_c > 0 \quad \text{for } l \in [l_0, \ell_n^k).$$

Hence, eq. (3.14) holds true in t_{n+1}^k. Finally, if $l_0 < \ell_n^k < L$, then $G(t_n^k, l) - G_c > 0$ for $l \in [l_0, \ell_n^k)$ and $G(t_n^k, \ell_n^k) - G_c) \leq 0$. By the continuity of G, we get $G(t_n^k, \ell_n^k) - G_c = 0$.

□

The following proposition provides a variational characterization of ℓ_G.

Proposition 3.4. *Let* α *be a strictly monotone control. Then,* ℓ_G *satisfies the initial condition* $\ell_G(t_0) = l_0$ *and is non-decreasing. Moreover, for every* $t \in [0, +\infty)$*:*

$$\ell_G(t) = \min\{l \in [l_0, L] : G(t, l) \leq G_c\}. \tag{3.15}$$

Remark. An evolution satisfying the above properties is unique.

Proof. The property of monotonicity and the verification of the initial condition are a direct consequence of Helly's Theorem. Given $t \in [0, +\infty)$, and $k \in \mathbb{N}$, let $m \in \mathbb{N}$ depending on t and k (the dependence will be omitted) be defined by $m = \lfloor \frac{t}{\Delta t^k} \rfloor$ (where $\lfloor \cdot \rfloor$ denotes the integer part). Then, t_m and ℓ_m denote, respectively:

$$t_m = t_m^k = m\Delta t^k, \quad \ell_m = \ell_m^k = \ell^k(t_m) = \ell_k(t).$$

By the definition of ℓ_m we have $G(t_m, \ell_m) \leq G_c$. Passing to the limit with respect to m, the continuity of G gives $G(t, \ell_G(t)) \leq G_c$. Furthermore, let $l < \ell_G(t)$. Then $\ell_m > l$ for $k \gg 1$. By Proposition 3.3 we know that $G(t_m, l) > G_c$, then by the monotonicity of α:

$$G_c < G(t_m, l) = \alpha^2(t_m)\widehat{G}(l) \leq \alpha^2(t)\widehat{G}(l) = G(t, l).$$

Thus $G(t, l) > G_c$ for $l < \ell_G(t)$. □

Corollary 3.1. ℓ_G *is left-continuous in* $(0, +\infty)$*.*

Proof. It is enough to consider the case $l_0 < \ell_G^-(t)$. Let us check that $G(t, \ell_G^-(t)) \le G_c$ and that

$$G(t,l) > G_c \qquad \text{if } l_0 \le l < \ell_G^-(t).$$

Let $t' < t$, then $G(t', \ell_G(t')) \le G_c$. As $t' \nearrow t$, the equilibrium inequality becomes $G(t, \ell_G^-(t)) \le G_c$. Still assuming $t' < t$, let $l < \ell_G(t')$. Then, by monotonicity of α:

$$G_c < G(t', l) = \alpha^2(t')\widehat{G}(l) \le \alpha^2(t)\widehat{G}(l) = G(t,l).$$

Since t' is arbitrary, the above inequality follows for every $l < \ell_G^-(t)$.
In conclusion, $\ell_G^-(t)$ enjoys all the properties of Proposition 3.4, by uniqueness it must coincide with $\ell_G(t)$. ☐

Now, let us see the detailed properties of the solution ℓ_G. We define t_0 and T for ℓ_G as:

$$t_0 = \sup\{t \in [0, +\infty) : \ell_G(t) = \ell_0\}, \qquad T = \inf\{t \in [0, +\infty) : \ell_G(t) = L\}.$$

Theorem 3.4. *ℓ_G belongs to $BV_{\mathrm{loc}}(0, +\infty)$. Moreover:*

$$G(t, \ell_G(t)) \le G_c \text{ in } [0, +\infty); \tag{3.16}$$

$$G(t, \ell_G(t)) \text{ is continuous in } [0, +\infty) \setminus \{T\}; \tag{3.17}$$

$$(G(t, \ell_G^-(t)) - G_c) d\,\ell_G(t) = 0 \text{ in the sense of measures in } [0, +\infty). \tag{3.18}$$

Furthermore, for $t \in S(\ell_G)$ we have:

$$(G(t,l) - G_c) \ge 0 \quad \text{for every } l \in [\ell_G^-(t), \ell_G^+(t)] \setminus \{L\}. \tag{3.19}$$

Therefore:

$$\int_{\ell_G^-(t)}^{\ell_G^+(t)} (G(t,l) - G_c)\, dl \ge 0.$$

Proof. The proof is based on the variational property stated in Proposition 3.4 and on Corollary 3.1. For simplicity, we consider only the case $0 < t_0 < T < +\infty$.

1. eq. (3.16) comes directly from Proposition 3.4.

2. For eq. (3.17), we know that $G(t, \ell_G(t)) = \alpha^2(t)\widehat{G}(l_0)$ is continuous in $[0, t_0)$. Thanks to Corollary 3.1, ℓ_G is left-continuous in $(0, +\infty)$ and thus, since G is continuous, $G(t, \ell_G(t))$ is continuous in $[0, t_0]$. If $t_0 < t < T$, then $l_0 < \ell_G(t) < L$. Hence, by Proposition 3.4, $G(t,l) > G_c$ for $l_0 \le l < \ell_G(t)$. Passing to the limit as $l \nearrow \ell_G(t)$, it gives $G(t, \ell_G(t)) \ge G_c$. Since $G(t, \ell_G(t)) \le G_c$, it follows that $G(t, \ell_G(t)) = G_c$ in (t_0, T).

3. We now prove eq. (3.18). We just need to verify $G(t_0, \ell(t_0)) \ge G_c$, as the opposite inequality is true by Proposition 3.3. By monotonicity, $\ell_G(t) \ge \ell_G(t_0)$ for $t > t_0$. Then:

$$\begin{aligned} G(t, \ell_G(t_0)) &> G_c \qquad \text{if } \ell_G(t) > \ell_G(t_0), \\ G(t, \ell_G(t_0)) &= G_c \qquad \text{if } \ell_G(t) = \ell_G(t_0). \end{aligned}$$

   Therefore, $G(t, \ell_G(t_0)) \ge G_c$ for every $t > t_0$. By continuity, $G(t_0, \ell_G(t_0)) \ge G_c$. Finally, we can say that $G(t, \ell_G(t))$ is continuous in $[0, +\infty) \setminus \{T\}$ and it is explicitly given by:

$$G(t, \ell_G(t)) = \begin{cases} \alpha^2(t)\widehat{G}(l_0) & \text{for } t \le t_0, \\ G_c & \text{for } t_0 \le t < T, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the initiation time t_0 is characterized by $\alpha^2(t_0)\widehat{G}(l_0) = G_c$. From this representation of $G(t, \ell_G(t))$ and from Corollary 3.1, eq. (3.18) is straightforward.

4. It remains to consider eq. (3.19). Let $t \in S(\ell_G)$ and $t' > t$. Then, $\ell_G(t') \geq \ell_G^+(t)$ and by Proposition 3.4, $G(t', l) > G_c$ for every $l < \ell_G(t')$. Hence, $G(t', l) > G_c$ for every $l < \ell_G^+(t)$. As t' tends to t, we get $G(t, l) \geq G_c$ for every $l < \ell_G^+(t)$ and in particular for $l \in [\ell_G^-(t), \ell_G^+(t))$.

□

Remark. We see that the activation condition (eq. (3.16)) holds true for every time in $[0, +\infty)$. In particular, at $t = t_0$ the crack starts to propagate only when the critical value of the energy release rate is attained. As for jumps, eq. (3.19) says that for $t \in S(\ell_G)$ the crack "moves" instantaneously from ℓ_G^- to ℓ_G^+ through some unstable states $l \in (\ell_G^-, \ell_G^+)$, with $G(t, l) \geq G_c$. Strictly speaking, this behaviour is not compatible with the quasi-static framework; however, it should be considered as the limit behaviour of a fast dynamic propagation.

In the case of strictly monotone controls the Kuhn–Tucker conditions are equivalent to Definition 3.2.

Proposition 3.5. *Let α be a strictly monotone control. If ℓ is non-decreasing, left-continuous and satisfies eq.* (3.19)*-eq.* (3.19)*, then $\ell = \ell_G$ everywhere in* $[0, +\infty)$.

Proof. Clearly $G(t, \ell(t)) \leq G_c$. It remains to show that $G(t, l) > G_c$ for $l < \ell(t)$. Let $\tau < t$ such that $\ell(\tau) < l < \ell^+(\tau)$. If $\ell(\tau) = \ell^+(\tau)$, then $G(\tau, l) = G_c$. Otherwise, by eq. (3.19) we have $G(\tau, l) \geq G_c$. In both cases:

$$\alpha^2(t)\widehat{G}(l) > \alpha^2(\tau)\widehat{G}(l) = G(\tau, l) \geq G_c,$$

which concludes the proof. □

In general, if α is not strictly monotone, the above equivalence between the Kuhn–Tucker equations and Definition 3.2 is not true, and it is possible to find counter examples.

Proposition 3.6. *The energy $\mathscr{F}(t, \ell_G(t)) \in SBV_{\mathrm{loc}}(0, +\infty)$. Its derivative is a measure that can be represented as:*

$$d\mathscr{F} = \mathcal{P}_{\mathrm{ext}}(t)\, dt + d^j\mathscr{F}.$$

In particular, for every time interval (t_1, t_2) in $[0, +\infty)$, the following energy balance holds:

$$\mathscr{F}(t_2, \ell_G^-(t_2)) - \mathscr{F}(t_1, \ell_G^+(t_1)) = \int_{t_1}^{t_2} \mathcal{P}_{ext}(t)\, dt - \sum_{t \in S(\ell_G) \cap (t_1, t_2)} \int_{\ell_G^-(t)}^{\ell_G^+(t)} \big(G(t, s) - G_c\big)\, ds.$$

A similar formula holds true in $[t_1, t_2]$ taking into account possible jumps in the end points.

By eq. (3.19) and Proposition 3.2, it follows immediately that $d^j\mathscr{F}$ is a non-negative measure. Physically, it represents the amount of energy dissipated during an instantaneous propagation of the crack. Using the characterization of Theorem 3.4 it is possible to prove the following representation result, which gives the rate independence of ℓ_G.

Proposition 3.7. *Denote by $\lambda_G(\tau)$ the evolution for the boundary condition $\hat{g}(\tau, \cdot) = \tau g(\cdot)$ (i.e. we take $\alpha(\tau) = \tau$). Let ℓ_G be the solution for $\hat{g}(t, \cdot) = \alpha(t) g(\cdot)$. Then:*

$$\ell_G(t) = \lambda_G \circ \alpha(t)$$

Theorem 3.5. *If the energy $\widehat{\mathcal{E}}(\ell)$ is strictly convex (concave), then there exists a unique evolution, which is a steady (unsteady) state.*

Proof. We have seen that, under a strictly monotone control α, there is a unique evolution of the crack length satisfying Griffith's criterion, which has the form:

$$\ell_G(t) = \min\{l \in [\ell_0, \ell_T] : G(t, l) \leq G_c\} \quad \text{for every } t \in [0, T].$$

We recall that $\widehat{\mathcal{E}}(\ell)$ is $C^1(0,\ell_T)$, decreasing and in general non-convex. Now, $\widehat{\mathcal{E}}(\ell)$ is convex if and only if its derivative is monotone non-decreasing. By Griffith's criterion, a crack K of length ℓ can only propagate when $G(t,\ell) = G_c$. Then, recalling the definitions of t_0 and T, for $t < t_0$, $\ell_G(t) = \ell_0$, since $G(t,\ell) < G_c$. On the other hand, for $t \in [t_0, T]$, $\widehat{G}(\ell_G(t)) = G_c/\alpha^2(t)$. Furthermore, we know that $\widehat{G}$ is strictly decreasing. Hence, it is invertible. giving a uniquely determined value of $\ell_G(t)$. Moreover, $\widehat{\mathcal{E}}$ is strictly convex if and only if $\widehat{G}$ is strictly decreasing, and:

$$G(t, \ell_G(t)) = \alpha^2(t)\widehat{G}(\ell_G(t)) = G_c.$$

Then:

$$\widehat{G}(\ell_G(t)) = G_c\, \alpha^- 2(t)$$

and, thanks to the continuity of $\widehat{G}$:

$$\ell_G(t) = \widehat{G}^{-1}(G_c\, \alpha^- 2(t)).$$

Since:

1. $\widehat{G}$ is continuous and strictly decreasing;
2. α is continuous and strictly increasing;

then $\widehat{G}^{-1}$ is continuous and strictly increasing. If instead $\widehat{\mathcal{E}}(\ell)$ is concave, then $\widehat{G}(\ell)$ is strictly increasing. Therefore, once the threshold value G_c is reached by $\alpha^2(t_0)\widehat{G}(\ell_0)$, the crack starts propagating and, by the monotonicity of $\widehat{G}$, $\widehat{G}(\ell) > G_c$ for every $\ell > \ell_0$. Therefore, the system is in an unsteady state, and the crack propagates instantaneously ($t_0 = T$). ☐

If α is not monotonically increasing, it is still possible to define, in a simple way, a quasi-static evolution satisfying the Kuhn Tucker conditions; however, it is not clear how to employ a variational characterization. We remark that in this case the evolutions are not unique. Assume that $\alpha \in W^{1,\infty}_{\mathrm{loc}}(0,+\infty)$ and that $\alpha(0) = 0$. First of all, let us define the non-decreasing envelope $\overline{\alpha}$ by:

$$\overline{\alpha}(t) = \int_0^t [\alpha'(s)]^+\, ds,$$

where $[\cdot]^+$ is the positive part. By standard results on Sobolev spaces $\overline{\alpha}$ belongs to $W^{1,\infty}_{\mathrm{loc}}$, it is non-decreasing and non-negative. Note that $\overline{\alpha}$ could be defined equivalently as:

$$\overline{\alpha}(t) = \sup_{s\in(0,t)} \alpha(s).$$

Let $\lambda_G(\tau)$ be the (unique) quasi-static evolution obtained with control $\beta(\tau) = \tau$. Then, $\ell_G(t) = \lambda_G(\overline{\alpha}(t))$ will be a quasi-static evolution with control α. Its properties are listed in the next theorem.

Theorem 3.6. *ℓ_G is non-decreasing and satisfies the initial condition $\ell_G(0) = l_0$. It belongs to $BV_{\mathrm{loc}}(0,+\infty)$ and enjoys the Kuhn–Tucker conditions:*

$$G(t, \ell_G(t)) \le G_c \text{ in } [0,+\infty), \tag{3.20}$$

$$G(t, \ell_G(t)) \text{ is continuous in } [0,+\infty) \setminus \{T\}, \tag{3.21}$$

$$(G(t, \ell_G^-(t)) - G_c)\, d\ell_G(t) = 0 \text{ in the sense of measures in } [0,+\infty). \tag{3.22}$$

Furthermore, for $t \in S(\ell_G)$, we have:

$$(G(t, \ell_G(t)) - G_c) \ge 0 \quad \text{for every } l \in [\ell_G^-(t), \ell_G^+(t)] \setminus \{L\}. \tag{3.23}$$

Proof. It is immediate to check that ℓ_G is non-decreasing and that it satisfies the initial condition.
Since $\overline{\alpha}$ is Lipschitz, we can apply the chain for compositions of BV functions, obtaining $\ell_G^{\pm}(t) = \lambda_G^{\pm}(\overline{\alpha}(t))$

for every $t \in S(\ell_G)$. Let us see that, for every $t \in [0;+\infty)$ we have $\ell_G^-(t) = \lambda_G^-(\overline{\alpha}(t))$. On the contrary, the analogous property for ℓ_G^+ holds true only in the discontinuity points. In fact, if $\overline{\alpha}$ is constant in a neighbourhood of the jump, the property is false. We prove that ℓ_G is left-continuous, from which follows the above assertion. By monotonicity the left-continuity is equivalent to:

$$\ell_G(t) = \sup_{t'<t} \ell_G(t')$$

and then to:

$$\lambda_G(\overline{\alpha}(t)) = \sup_{t'<t} \lambda_G(\overline{\alpha}(t')).$$

As $\overline{\alpha}$ is monotone, the previous equation can be written as:

$$\lambda_G(\overline{\alpha}(t)) = \sup_{\alpha<\overline{\alpha}(t)} \lambda_G(\alpha)$$

which holds true thanks to the left-continuity of λ_G. For convenience, let us recall the Kuhn–Tucker relations for λ_G:

$$\begin{aligned}
&G(\tau, \lambda_G(\tau)) \leq G_c \text{ in } [0, +\infty),\\
&G(\tau, \lambda_G(\tau)) \text{ is continuous in } [0, +\infty),\\
&(G(\tau, \lambda_G^-(\tau)) - G_c)\, d\lambda_G = 0 \text{ in the sense of measures in } [0, +\infty).
\end{aligned}$$

Furthermore, for $\tau \in S(\lambda_G)$, we have:

$$(G(\tau, l) - G_c) \geq 0 \quad \text{for every } l \in [\lambda_G^-(t), \lambda_G^+(t)] \setminus \{L\}.$$

Then:

1. eq. (3.20) is true by a change of variables;
2. eq. (3.21) is true by the continuity of $\tau = \overline{\alpha}(t)$ and $G(\tau, \lambda_G^-(\tau)$;
3. eq. (3.22) is true by the chain rule using the fact that $\ell_G^-(t) = \lambda_G^-(\overline{\alpha}(t))$;
4. eq. (3.23) is true by a change of variables.

□

We now see an example of continuous evolution with snap back. Consider the following boundary condition:

$$g(x_1, x_2) = \frac{1}{2} x_2 \big(e^{-x_1^2} + \frac{3}{4} x_1^2 + 1\big) \tag{3.24}$$

with control $\alpha(t) = t$. Under this choice, the example provides a clear picture of the various evolutions and their properties. In general, there are no explicit formulas for $\widehat{\mathcal{E}}$ or $\widehat{G}$: the plots have been obtained by a polynomial interpolation of the numerical data (see [5]). From these plots, it is evident that $\widehat{\mathcal{E}}$ is non-increasing and non-convex, and $\widehat{G}$ is non-negative and non-monotone.

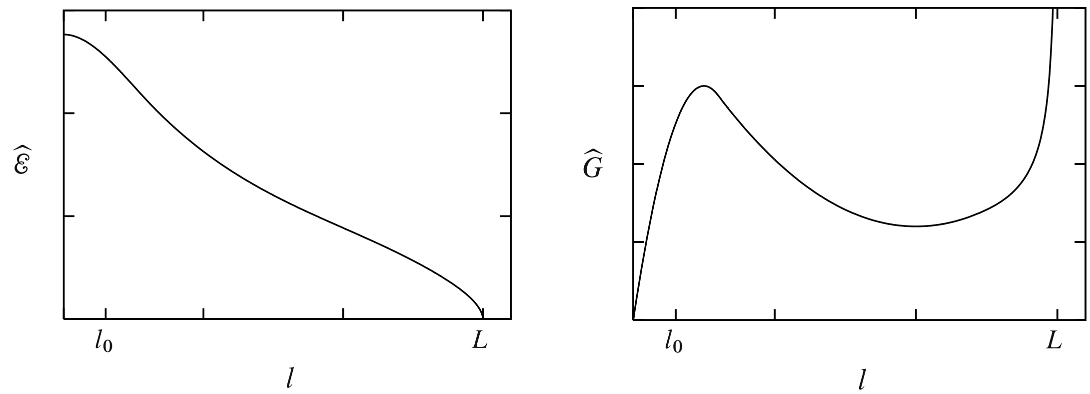


Figure 3.1: Plot of the elastic energy $\widehat{\mathcal{E}}$ and of the energy release rate $\widehat{G}$, as functions of l, for the boundary data eq. (3.24)

Our goal is: given $0 < t_0 < T < +\infty$, find a suitable control $\alpha(t)$ in such a way that:

$$\ell(t) = \begin{cases} l_0 & t \leq t_0, \\ m(t - t_0) + l_0 & t_0 \leq t \leq T, \quad \text{for } m = (L - l_0)/(T - t_0) \\ L & t \geq T \end{cases}$$

is a Lipschitz evolution satisfying the properties of Theorem 3.6. Consider $\widehat{G}(l) > 0$ in $[l_0, L]$ and that $\lim_{s \to L} -\widehat{G}(s) = \overline{G} < +\infty$. Let α_i for $i = 1, 2$ be defined respectively as :

$$\alpha_1 = \left(\frac{G_c}{\widehat{G}(l_0)} \right)^{\frac{1}{2}}, \qquad \alpha_2 = \left(\frac{G_c}{\widehat{G}(L)} \right)^{\frac{1}{2}}.$$

Let us define:

$$\alpha(t) = \begin{cases} (\alpha_1/t_0)t, & t \leq t_0, \\ \left(G_c/\widehat{G}(\ell(t)\right)^{1/2} & t_0 \leq t \leq T, \\ \alpha_2 & t \geq T. \end{cases}$$

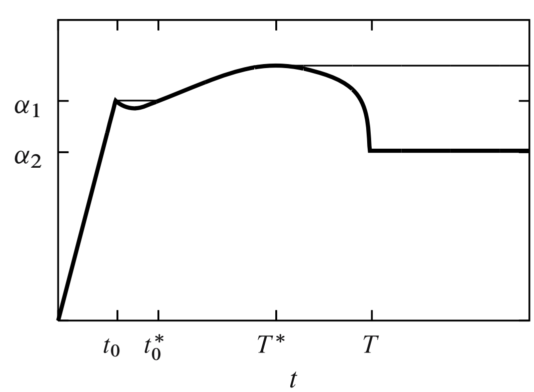


Figure 3.2: Graph of the controls α (bold) and $\overline{\alpha}$.

Note that $\alpha \in W^{1,\infty}_{\mathrm{loc}}(0,+\infty)$, since $\widehat{G}$ is locally Lipschitz and $\widehat{G} > 0$, but it is not monotonically increasing. The effect is clear in the plot of the conventional strain and stress. With this choice of α, the function ℓ defined above satisfies the properties of Theorem 3.6. In particular:

$$\dot{\ell} = \ell'(t)dt \quad \text{where } \ell'(t) = m\chi_{(t_0,T)}(t).$$

This evolution presents a typical snap-back behaviour: the crack propagates at time t_0 even if the external load (i.e. α) is decreasing. Here, considering that $\hat{g}$ is not constant and that $u(x,y)$ is anti-symmetric with respect to y, it is convenient to employ the conventional variables $(\overline{u}, \overline{s})$ defined as:

$$\overline{u} = \frac{1}{|\partial_D\Omega^+|}\int_{\partial_D\Omega^+} u\,dy = \frac{1}{|\partial_D\Omega^+|}\alpha\int_{\partial_D\Omega^+} g\,dy; \tag{3.25}$$

$$\overline{s} = \Big(\int_{\partial_D\Omega^+}(\hat{\sigma}\nu)\cdot\hat{g}\,dy\Big)\Big/\Big(\int_{\partial_D\Omega^+}\hat{g}\,dy\Big) = \alpha\Big(\int_{\partial_D\Omega^+}(\sigma\nu)\cdot g\,dy\Big)\Big/\Big(\int_{\partial_D\Omega^+} g\,dy\Big), \tag{3.26}$$

where $\partial_D\Omega^+$ denotes the upper face of Ω.

Remark. If g is constant, then $\overline{s}$ is simply:

$$\frac{1}{|\partial_D\Omega^+|}\int_{\partial_D\Omega^+}\sigma\nu\,dy$$

We denote:

$$\overline{g} = \frac{1}{|\partial_D\Omega^+|}\int_{\partial_D\Omega^+} g\,dy.$$

Then, $\overline{u}(t) = \alpha(t)\overline{g}$.

Proposition 3.8. *In the previous setting:*

$$\overline{s}(t) = \frac{\alpha(t)\widehat{\mathcal{E}}(\ell(t))}{L\,\overline{g}}.$$

Proof. We have $\Omega \subset \mathbb{R}^2$ and $u : \Omega \to \mathbb{R}^2$. By symmetry:

$$u_1(x,y) = u_1(x,-y) \qquad u_2(x,y) = -u_2(x,-y).$$

Then:

$$\frac{\partial}{\partial x}u_1(x,y) = \frac{\partial}{\partial x}u_1(x,-y), \qquad \frac{\partial}{\partial y}u_2(x,y) = \frac{\partial}{\partial y}u_2(x,-y),$$

Moreover, we can denote (from its definition):

$$g(x,y) = \frac{1}{2}xh(y).$$

Then:

$$\int_{\partial_D\Omega^+}(\sigma\nu)\cdot g\,dy = \int_{\partial_D\Omega^-}(\sigma\nu)\cdot g\,dy$$

since $\nu_+ = \hat{e}_2$ and $\nu_- = -\hat{e}_2$. Using:

1. integration by parts;
2. $\mathrm{div}(\sigma\,\nu) = 0$ on $\partial_N\Omega$ and $K^{\pm}_{\ell}$;
3. $\mathrm{div}\,\sigma = 0$;

we obtain:

$$\int_{\partial_D\Omega^+}(\sigma\nu)\cdot g\,dy = \frac{1}{2}\int_{\partial(\Omega\setminus K)}(\sigma\nu)\cdot\hat{u}\,dy = \frac{1}{2}\int_{\Omega\setminus K_\ell}\sigma : \mathcal{D}u\,dy = \frac{1}{2}\int_{\Omega\setminus K_\ell}\sigma : \varepsilon\,dy = \mathcal{E}(\ell)$$

Applying the calculations to the definition of $\overline{s}$, we obtain the claim. □

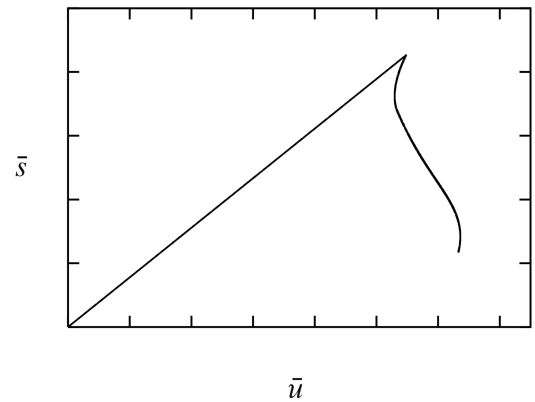


Figure 3.3: Curve $\overline{u}, \overline{s})$ of the conventional variables

For the evolution defined above, the plot of the curve $(\overline{u}, \overline{s})$ is contained in fig. 3.3. Note that during the snap back, the evolution follows an unstable branch of the energy (represented by the dotted branch of the graph in l_0 in fig. 3.4): $G(t, \ell(t)) = G_c$ but $G(t, l) > G_c$ ahead of the tip, i.e. for $l > \ell(t)$, and $G(t, l) < G_c$ for $l < \ell(t)$. Therefore, during the snap-back, $\ell(t)$ is a local maximum of the energy $\mathscr{F}$. Note also that this solution is not unique and in particular it is different from the evolution $\ell_G(t) = \lambda(\overline{\alpha}(t))$ defined previously, see fig. 3.5

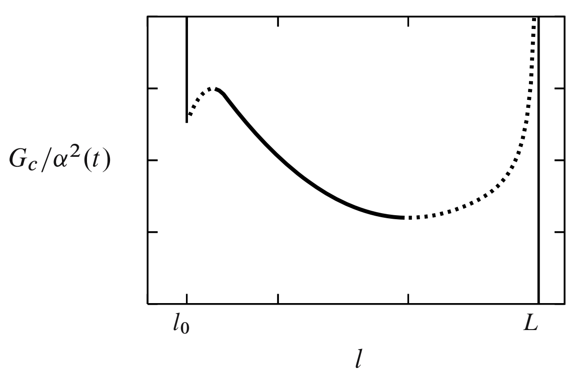


Figure 3.4: Locus of stationary points in the reference example

Here, it is given by:

$$\ell_G(t) = \begin{cases} l_0, & t \le t_0^*, \\ m(t-t_0)+l_0, & t_0^* \le t \le T^* \\ m(T^*-t_0)+l_0, & t \ge T^*, \end{cases} \quad \text{for } m = (L-l_0)/(T-t_0),$$

where:

$$\begin{aligned} t_0^* &= \max\{t : \alpha(t) \le \alpha_1\}, \\ T^* &= \min\{t : \overline{\alpha} < \alpha \text{ in } [t, +\infty)\}. \end{aligned}$$

Finally, we remark that for this example there are infinitely many bifurcations, indeed every function of the form:

$$\ell^{\circledast}(t) = \begin{cases} l_0, & t \le t_0^{\circledast}, \\ m(t-t_0)+l_0, & t_0^{\circledast} \le t \le T^{\circledast}, \\ m(T^{\circledast}-t_0)+l_0, & t \ge T^{\circledast}, \end{cases} \quad \text{for } m = (L-l_0)/(T-t_0),$$

defines an evolution which satisfies all the properties in Theorem 3.6 for every choice of $t_0^{\circledast} \in [t_0, t_0^*]$ and $T^{\circledast} \in [T^*, T]$.

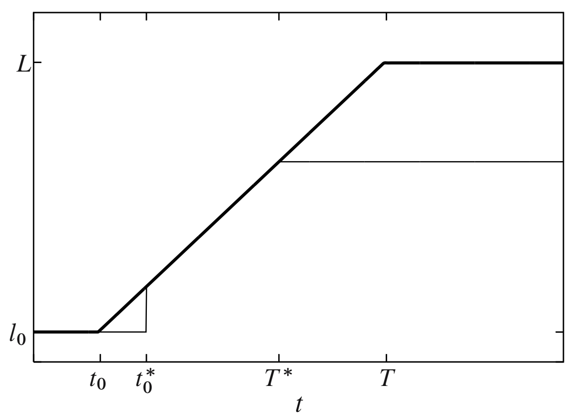


Figure 3.5: The evolutions ℓ (bold) and ℓ_G

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