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GREEN FINANCE AND FINANCIAL RISK: AN ANALYSIS BASED ON VALUE-AT-RISK AND PORTFOLIO OPTIMIZATION

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Abstract

Il ruolo della finanza verde nell'analisi del VaR e nell'ottimizzazione del portafoglio nell'ambito della gestione del rischio finanziario viene analizzato in questa tesi. Non si tratta infatti di un lavoro limitato alle misure di rischio tradizionali, ma ha un'affinità con metriche sofisticate, come il Conditional Value-at-Risk e i modelli GARCH per la volatilità. La ricerca si divide in due parti fondamentali. Una parte è teorica, in cui si considerano varie misure di rischio e si esaminano i modi in cui sono state utilizzate nella pratica per l'ottimizzazione del portafoglio. In questo contesto, vengono considerate in dettaglio le tecniche di simulazione storica, parametrica e Monte Carlo e vengono utilizzati i modelli GARCH per prevedere la volatilità. La parte empirica riguarda l'analisi di dati reali per verificare come l'inclusione di indici verdi possa influire sui portafogli di investimento. Ciò sarà particolarmente appropriato nel contesto odierno, in cui la finanza sostenibile sta ricevendo una crescente attenzione. In questo studio si cerca di capire come l'inclusione di indici verdi nei portafogli sia paragonabile a quella dei portafogli che ne sono privi e se questi possano migliorare il rapporto rischio-rendimento dei portafogli di investimento, soprattutto in tempi di turbolenza dei mercati. I risultati sono interessanti: sembra che l'integrazione di indici verdi possa davvero aumentare la resilienza dei portafogli riducendo i rischi di coda. La combinazione di queste due tecniche più avanzate di ottimizzazione, basate sul CVaR e su un modello GARCH, fornisce un modo robusto di gestire il rischio nei portafogli contemporanei. I risultati di questa ricerca sottolineano la crescente necessità di adottare le più recenti strategie di gestione del rischio nell'ambito della finanza sostenibile e gettano nuova luce su ulteriori ricerche e applicazioni reali.

The role of green finance in VaR analysis and portfolio optimization in the backdrop of financial risk management is investigated in this thesis. Indeed, this is not a work limited to traditional risk measures but has an affinity for sophisticated metrics, such as Conditional Value-at-Risk and GARCH models for volatility. The research is split into two key parts. One part is theoretical, where various risk measures are considered and ways in which they have been used in practice when carrying out portfolio optimization are examined. In this context, it considers historical, parametric, and Monte Carlo simulation techniques in some detail and uses GARCH models to predict volatility. The empirical part covers real data analysis with a view to testing how the inclusion of green indexes may affect investment portfolios. This will particularly be appropriate in today's context, whereby sustainable finance is meeting growing attention. In this study, an attempt is made to understand how including green indexes into portfolios compares to portfolios without them and whether these might improve the risk-return ratio of investment portfolios, more so in times of turbulence on markets. Results are interesting: it seems that integrating green indexes can really increase portfolio resilience by cutting down tail risks. The combination of these two most advanced techniques for optimization, based on CVaR and a GARCH model, provides a robust way of managing the risk in contemporary portfolios. The findings of this research

underline the growing need to adopt the latest state-of-the-art risk management strategies in the area of sustainable finance and shed new light on further research and real-world applications.

Contents

Introduction

The risk management department experienced rapid growth in investment and commercial banks during the 1990s, and again following the crash of 2008. In response to the volatile and complex risks they faced, banks established specialized risk management departments with functions encompassing the measurement and management of risk. Value at Risk (VaR), a quantitative measure of an institution's risk exposure, is among the primary tools employed by risk managers. Initially perceived as somewhat inaccessible and reserved for mathematicians and quantitative analysts, VaR is grounded in statistical techniques that might pose challenges for the layperson. While recognized as a measure based on strong assumptions and suited primarily for approximating market risk exposure (rather than measuring risk exposure in the banking book), VaR faces criticism for its association with scientific rigor. During the crash of 2007–08, bank losses surpassed their VaR values, leading to further criticism. The primary objective of the risk management function within a financial institution is to ensure a comprehensive understanding of the risks and exposures inherent in the firm's operations, thereby rendering monetary loss acceptable to the organization. If the bank's risk management function is effective, there will be no overreaction to any unexpected losses, which may increase eventual costs to many times the original loss amount. Quantitative risk management aims to provide precise definitions and mathematical connotations to risk concepts, describing the interdependence and concentration of risks related to extreme events, drawing from various related quantitative disciplines such as financial and actuarial mathematics, statistics, and econometrics.

In this thesis, we delve deeper into the realm of risk management by exploring not only the traditional measures such as VaR but also more advanced metrics like Conditional Value at Risk (CVaR) and sophisticated modeling techniques such as Generalized Autoregressive Conditional Heteroskedasticity (GARCH) models. CVaR addresses some of the limitations of VaR by providing a measure of the expected loss given that the loss has exceeded the VaR threshold, offering a more comprehensive view of tail risk. The GARCH models, on the other hand, allow for the modeling of time-varying volatility, capturing the clustering of volatility and providing better risk forecasts.

Furthermore, this thesis investigates portfolio optimization from a theoretical perspective, incorporating these risk measures into the optimization process to construct portfolios that are not only efficient in terms of returns but also robust against risks. Traditional mean-variance optimization is extended to mean-CVaR optimization, providing a framework for constructing portfolios that minimize potential extreme losses.

In the practical part of this study, we apply these theoretical concepts to the analysis of the integration of green indices into investment portfolios. The growing importance of sustainable finance and the increasing integration of Environmental, Social, and Governance (ESG) factors into investment decisions necessitate a thorough understanding of how these green indices behave in terms of risk and return. We analyze the performance of portfolios that include green indices, comparing them with traditional portfolios to assess whether the inclusion of green assets can enhance the risk-return profile of the portfolios, particularly in periods of market stress.

By bridging the gap between theoretical risk measures and practical portfolio management, this thesis aims to provide valuable insights for both academics and practitioners in the field of finance. The findings of this study contribute to the ongoing discourse on the effectiveness of advanced risk management tools and the role of sustainable finance in modern investment strategies. Through rigorous quantitative analysis and real-world application, this work underscores the importance of robust risk management practices and the potential benefits of incorporating green indices into investment portfolios.

1 Value-at-Risk and other risk measures

The measurement of market risk has evolved from simple naive indicators that distort the measurement of risk to sophisticated risk measures such as the latest Value-at-Risk (VaR) methodology for whole portfolios of securities. In this chapter, we'll explain the principles behind VaR and clarify the strengths and weaknesses of the approach.

1.1 Value At Risk

Value-at-Risk (VaR) represents the potential maximum loss that could occur from holding a security or portfolio within a specified time frame, at a given level of probability, known as the confidence level.

For instance, stating a daily VaR of \$10 million at the 99% confidence level implies that, on average, daily losses from the position will exceed \$10 million only once every 100 trading days (or approximately two to three days annually).

Calculating VaR involves two primary steps:

1. determining the distribution of the portfolio or portfolio returns at the selected horizon (e.g., one day)
2. identifying the appropriate percentile of this distribution to ascertain a specific loss value.

Before focusing on the statistical and mathematical notations, let's try to provide an initial theoretical explanation in words.

Let's suppose we want to compute VaR at 99% confidence level assuming that the distribution is a normal bell-shaped curve. The VaR of the position or portfolio is simply the maximum loss at this 99 percent confidence level, measured relative to the expected value of the portfolio at the target horizon. That is, VaR is the distance of the first percentile from the mean of the distribution:

Var = expected profit/loss - worst case loss at the 99% confidence level

Definition 1.1 (Value-at-Risk) *Given some confidence level* $\alpha \in (0, 1)$*, the VaR of our portfolio at the confidence level* α *is given by the smallest number* l *such that the probability that the loss* L *exceeds* l *is no larger than* $(1-\alpha)$.

$$\mathrm{VaR}_\alpha = inf\{l \in \mathbb{R} : P(L > l) \leq 1 - \alpha\} \tag{1}$$

In general, α takes values of 0.90, 0.95, or 0.99. If $\alpha = 0.95$, then with 95% confidence (i.e., with a probability of 5%), we can incur losses equal to or greater than $\mathrm{VaR}_{0.95}$. Alternatively, VaR_α indicates the maximum value that one can expect to lose with a probability of α.

$$\mathbb{P}(L < \mathrm{VaR}) \leq 1 - \alpha \tag{2}$$

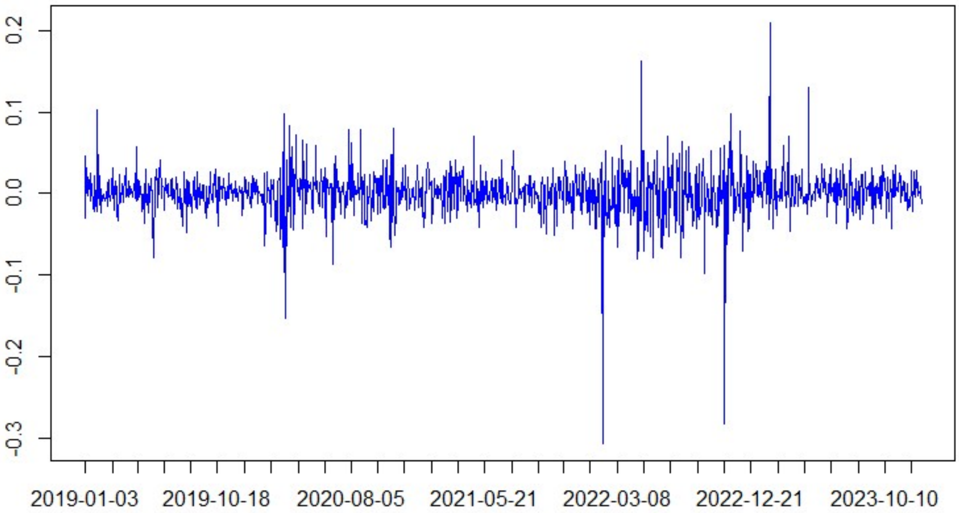


Figure 1: Log-returns of META of the last five years

1.2 Returns distribution

When computing VaR, we endeavor to model the behavior of financial assets, specifically focusing on capturing the dynamics of price fluctuations. The assumption of normality is the largest assumption made in many VAR models. In reality, financial price change histories deviate from perfect normal distributions. This assumption implies that successive price changes are independent and that the likelihood of an upward change in prices equals that of a downward one.[1] One commonly observed characteristic of financial price series is the presence of numerous 'outliers', which manifest as large outlying price changes often referred to as *fat tails*. Additionally, financial markets experience non-normality in the form of *skewed distributions*, arising from prolonged movements in the value of an asset in one direction.

1.2.1 Normal distribution

Assuming that returns are distributed normally as $X \sim N\left(\mu, \sigma^2\right)$, we have:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \quad , \quad -\infty \leq x \leq \infty \tag{3}$$

$$E[X] = \mu$$

$$var[X] = \sigma^2$$

Let's see an example. Consider the daily log-returns of META in Figure 1

To verify if returns are normally distributed, the QQ-Plot test can be conducted, comparing

[1] The distribution of price changes is symmetrical around the mean

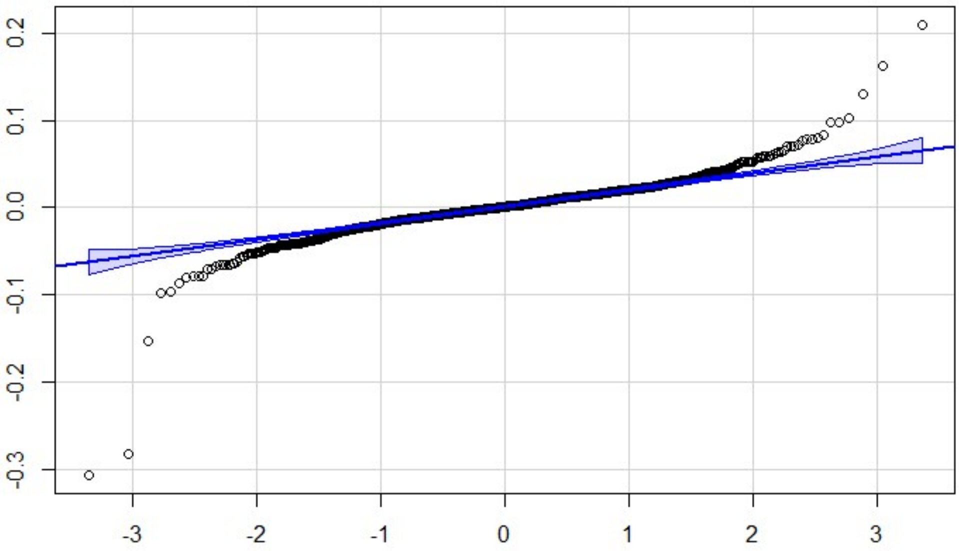


Figure 2: QQ plot displaying the empirical quantiles against the theoretical ones of the normal distribution. It is observed that the distribution does not adequately capture the extreme values. The blue lines denote the 95% confidence intervals.

standardized empirical quantiles with theoretical quantiles of the considered distribution.

The points should follow the regression line if the data are normally distributed. As we can see in figure 2, the points don't fall within the confidence interval (blue lines), therefore, we can reject the hypothesis of normality.

1.2.2 t-Student distribution

Let $Z \sim N(0,1)$, $W \sim \chi^2(\upsilon)$ such that Z and W are independent. Then

$$X = \frac{Z}{\sqrt{\frac{W}{\upsilon}}} \sim t_\upsilon \tag{4}$$

where t_ϵ denotes the t-distribution with υ degree of freedom.

$$E[X] = 0$$

$$var(X) = \frac{\upsilon}{\upsilon - 2}, \quad \upsilon > 2$$

As we can see in Figure 3, the t-distribution offers a viable alternative to the normal distribution when modeling VaR. It exhibits fatter tails compared to the normal distribution which suggests a higher likelihood of experiencing exceptionally large losses compared to what would be expected under the normal distribution. When the degree of freedom is unknown we can use a simple method to estimate them:

$$\upsilon = \frac{6}{kurt - 3} + 4$$

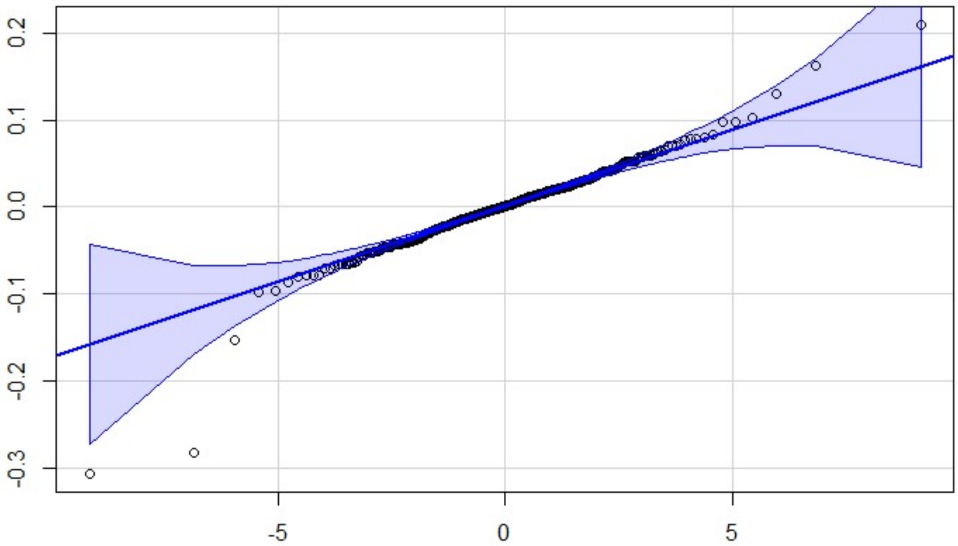


Figure 3: QQ-plot of the log-returns of META stock assuming a t-student distribution with 4 degrees of freedom (υ): graph of standardized empirical quantiles versus the theoretical quantiles of the t-student distribution. Note how extreme values now fall within the confidence interval.

1.3 Choice of VaR parameters

In the field of financial risk management, the selection of an appropriate time horizon T and a confidence level α for the Value at Risk (VaR) represents a crucial aspect, although there is no universally optimal solution for defining these parameters. Such choices should be guided by strategic considerations that reflect the specificities of the entity under consideration and the operational context in which it operates.

The risk horizon adopted should essentially correspond to the period during which the financial entity expects to hold its investment portfolio. This decision is influenced by a range of factors, including contractual and legal constraints, as well as considerations related to market liquidity. Consequently, the risk horizon can vary significantly depending on the reference market. In practice, for an effective implementation of enterprise-wide risk management, it is essential to select a horizon that faithfully reflects the characteristics of the entity's main market segment. For example, for insurance companies, the constraint to hold the policy portfolio for a year, without the possibility of making substantial changes or renegotiating premiums, makes this timeframe the most suitable horizon for assessing the market risk associated with their investments.

Furthermore, the choice of risk horizon must take into account the liquidity of the market for the assets of interest. In the presence of illiquid markets, where selling losing positions can prove problematic, opting for a relatively extended time horizon may be appropriate. Liquidity, varying greatly among different markets, requires careful evaluation to determine the risk management horizon that best suits the entity's needs, thus ensuring optimal management of its risk exposures.

Regarding the choice of the confidence level α, it is again difficult to provide a definitive recommendation, as different values of α are suitable for different purposes. Fortunately, once we have an estimate for the loss distribution, it is straightforward to compute quantiles at various confidence levels simultaneously. For capital adequacy purposes, a high confidence level is certainly required to ensure a sufficient safety margin. For example, the Basel Committee recommends using the VaR at the 99% level with T equal to 10 days for market risk. To set limits for traders, a bank would typically use 95% with T equal to one day.

This reflection underscores the importance of a considered and customized approach in choosing the fundamental elements for calculating the VaR, to effectively support financial risk management strategies.

1.4 Different approaches for VaR computation

There are three main methods for calculating VaR. As with all statistical models, they depend on certain assumptions:

- the variance/covariance method
- historical simulation
- Monte Carlo simulation

1.4.1 Historical simulation

The historical simulation approach calculates Value at Risk by analyzing historical data of asset returns, without making any assumptions about the distribution of returns. Instead, it is based on past data to estimate future risk.

Furthermore, the historical approach incorporates any extraordinary events absent in parametric models. However, it also possesses limitations. The computed VaR heavily relies on the selected historical timeframe and overlooks market changes or unforeseen events not captured in past data. Consequently, if a market crash or an unprecedented economic crisis unfolds in the future, the VaR derived from the historical approach may inadequately portray the actual risk.

1.4.2 Parametric approach

The parametric method, also known as the "variance-covariance" method, is the most widely used approach for calculating Value at Risk (VaR). This is because of its computational simplicity compared to other methods. This approach is based on the assumption that returns are distributed normally.

$$\mathrm{VaR}_\alpha = \mu + \sigma\Phi^{-1}(\alpha) \tag{5}$$

μ : is the mean

Φ : is the standard normal density function

$$\Phi^{-1}(\alpha) : \text{is the } \alpha\text{-quantile of} \quad \Phi$$

Example Suppose we hold an asset with an expected return (μ) of 5%, standard deviation of returns (σ) of 2% and $100,000 invested in that asset. We want to compute Var at 99% confidence interval.

$\mu = 0.05$

$\sigma = 0.2 \times 100000 = 2000$[2]

$\alpha = 0.99$

Now, using the (5) we can compute VaR.

$$\text{VaR}_{0.99} = 0.05 + 2000 \times \Phi^{-1}(0.99) = \$4660$$

So, the 99% confidence level VaR for this single asset, using this data, is $4,660. This means there is a 1% probability that losses exceed this value.

1.4.3 Monte Carlo simulation

The third method, Monte Carlo simulation, is more flexible than the previous two. Monte Carlo simulation allows the risk manager to use actual historical distributions for risk factor returns rather than having to assume normal returns. In practice, the model generates simulations using volatility and correlation estimates. Subsequently, each scenario is evaluated to calculate VaR. Furthermore, Monte Carlo allows the simulation of extreme scenarios that have not been observed in the past to evaluate the risk of these scenarios.

However, this method necessitates the utilization of intricate mathematical models to generate the random numbers, and the simulation process incurs significant computational costs and time resources.

Moreover, the accuracy of the outcome largely depends on the selection of the simulation model, which must be appropriately calibrated to mirror the market's characteristics.

Example To better understand what has been explained in the previous paragraphs, an example follows to calculate the Value-at-Risk.

Let's consider the price series of the NASDAQ index of the last five years in figure (4). Each point represents the daily closing price. The analysis focuses on NASDAQ returns due to its representativeness of the technology sector and its importance in the global economy.
Computing the logarithmic returns and fixing the confidence level $\alpha = 0.95$ we can estimate the $\text{VaR}_{0.95}$ using the Historical simulation and the Parametric approach. After the computations in R program, the results obtained are:

[2]standard deviation of returns in monetary terms

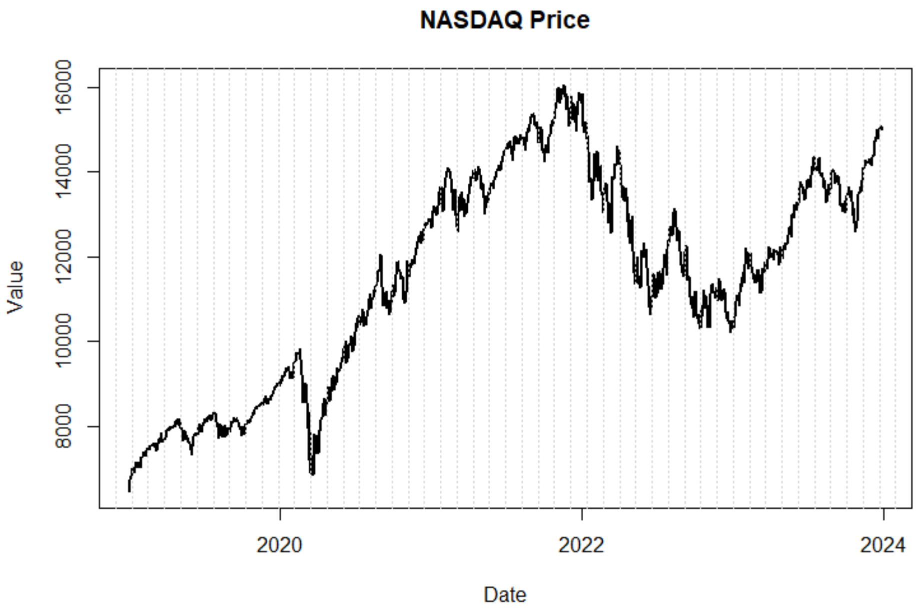


Figure 4: NASDAQ price series from January 1, 2019 to December 31, 2023

Historical $\text{VaR}_{0.95} = -0,0253$

Parametric $\text{VaR}_{0.95} = -0,0255$

Monte Carlo $\text{VaR}_0.95 = -0.0256$

Given the estimated results we can say that the maximum potential loss is 2.53%, based on historical data, over the considered time horizon with a 95% probability. In the same way, based on the parametric approach, losses exceeding 2.55% of the asset's value are expected not to occur in 95% of the cases. Below in figure (5),(6) and (7) are represented the results.

The above figure (8) illustrates the daily returns of the NASDAQ index, along with both historical and parametric Value at Risk (VaR), calculated using a 250-day rolling window. Through this graph, we aim to provide a clear view of how risk, measured through VaR, evolves over time in response to market dynamics, using an approach that balances reactivity and stability in risk estimation.

1.5 VaR as capital regulation

In this paragraph, we will dedicate ourselves to an in-depth examination of a key formula in the regulation of bank capital, a crucial element in the field of financial risk management. The formula in question models the calculation of the Initial Margin (IM) based on Market Risk (MR), incorporating the Value at Risk (VaR) and a specific risk component (CSR), essential elements for the evaluation and hedging of risk markets that banks face daily.

In the context of growing demands for financial stability and regulatory transparency, financial institutions must adjust their capital requirements to accurately reflect the risks they assume.

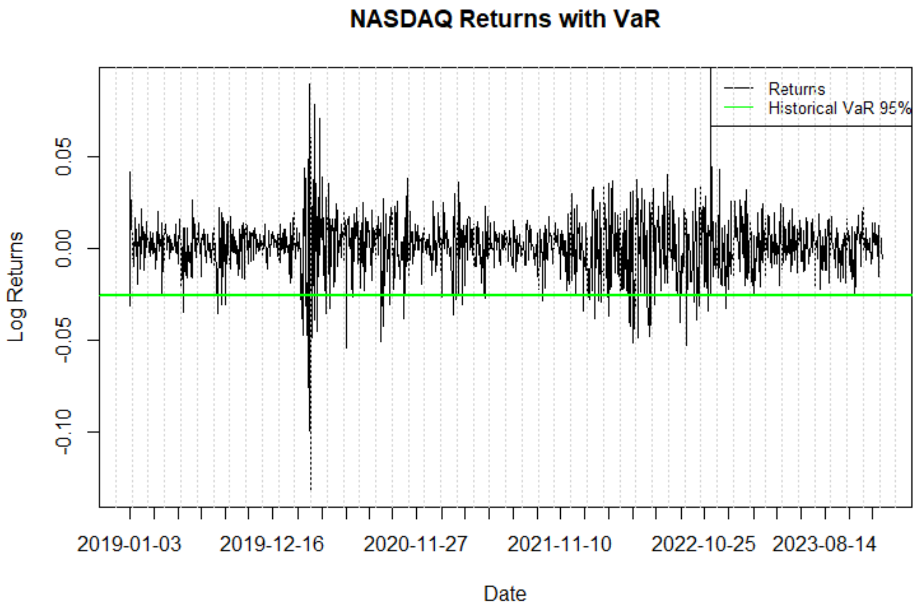


Figure 5: Representation of logarithmic returns of NASDAQ with historical $\mathrm{VaR}_{0.95}$

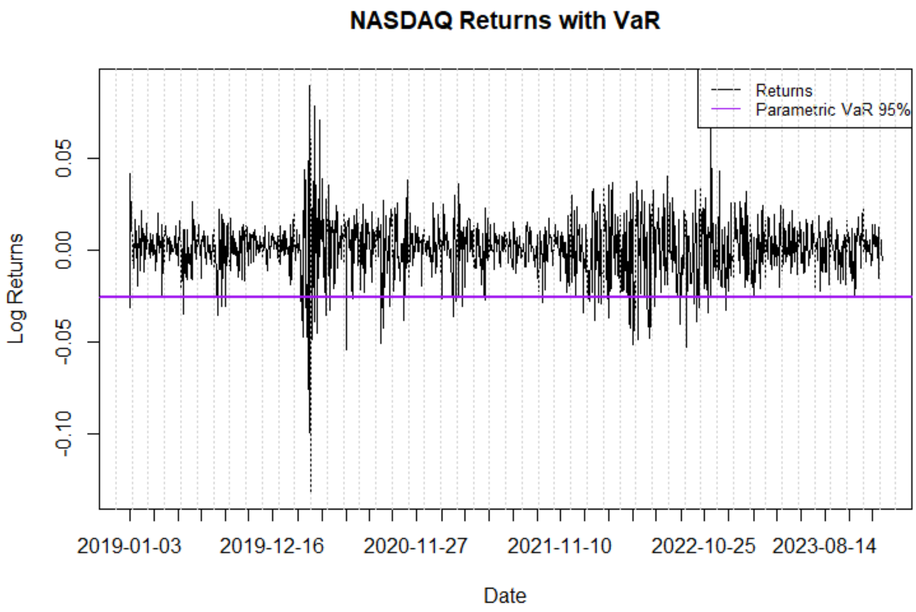


Figure 6: Representation of logarithmic returns of NASDAQ with parametric $\mathrm{VaR}_{0.95}$

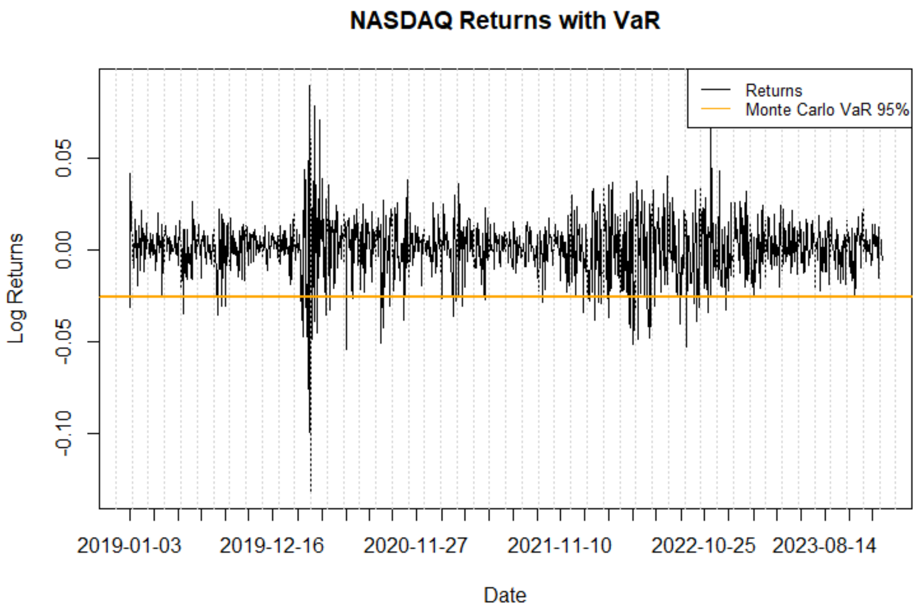


Figure 7: Monte Carlo simulation using the historical mean and historical standard deviation of NASDAQ's returns

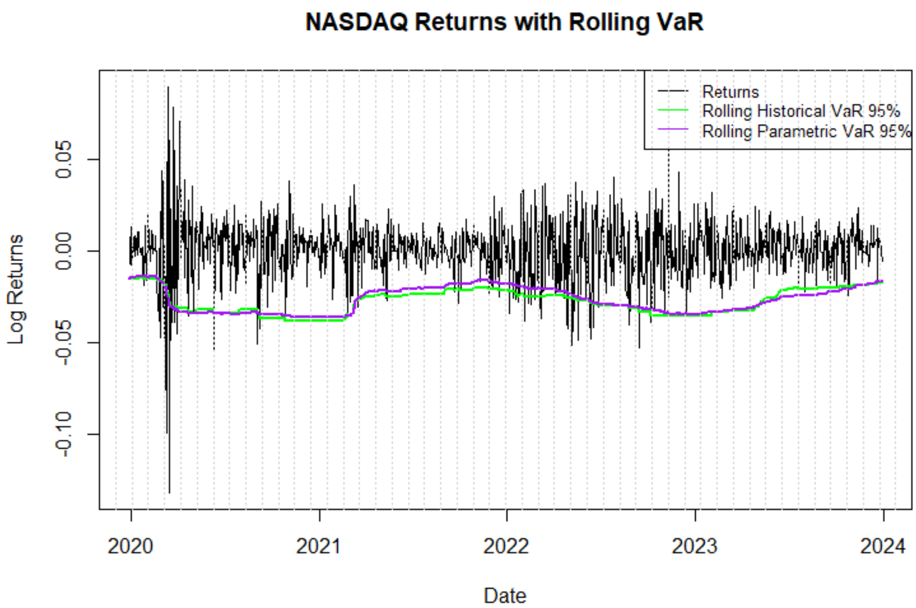


Figure 8: Hisorical and parametric VaR using 250-days rolling window

Backtesting zones	Number of violations	Stress factor
Green	0	1.50
	1	1.50
	2	1.50
	3	1.50
	4	1.50
Amber	5	1.70
	6	1.76
	7	1.83
	8	1.88
	9	1.92
Red	10 or more	2.00

Table 1: Stress factor according to Basel III based on backtesting of bank's internal model

The formula examined offers a methodological approach to quantify the capital needed to cover potential extreme losses, highlighting the importance of prudent risk management based on solid principles. The formula of the required capital based on VaR is:

$$\mathrm{RC}_{\mathrm{IM}}^{t}(\mathrm{MR}) = \max\left\{\mathrm{VaR}_{0.99}^{t,10}, \frac{k}{60}\sum_{i=1}^{60}\mathrm{VaR}_{0.99}^{t-i+1,10}\right\} + \mathrm{C}_{\mathrm{SR}} \tag{6}$$

$\max\{\cdot\}$: this part of the formula calculates the greater of the 10-day VaR at the 99% confidence level calculated for the current day (t) and the average of the 10-day VaR calculated over the previous 60 days, multiplied by a stress factor (k). The factor $k/60$ serves to normalize the sum of the VaR of the previous 60 days, making it comparable with the single VaR of the current day. Where:

- $\mathrm{VaR}_{0.99}^{t,10}$: indicates the 10-day VaR calculated at the 99% confidence level for the current day.
- $\sum_{i=1}^{60}\mathrm{VaR}_{0.99}^{t-i+1,10}$: Represents the sum of the 10-day VaR calculated at the 99% confidence level for the previous 60 days.
- $k/60$: $1.5 \leq k \leq 2$ is a stress factor that is determined based on the overall quality of the bank's internal model. This factor modulates the impact of the average of the previous VaRs on the calculation of the capital requirement.

In summary, this formula calculates the capital requirement for the initial margin taking into account the worst of the current VaR and the weighted average of the previous 60 days VaR, adjusted for a stressor, and adds a component to cover the specific risk. This approach aims to ensure that the bank maintains an adequate level of capital to cover potential extreme market losses, taking into account both recent market fluctuations and the quality of its risk model.

According to Basel III, the stress factor k is based on backtesting of the bank's internal model. The regulator has fixed the coefficient to be adopted based on the number of violations in the backtesting.

Through the analysis of this formula, we highlight not only the intrinsic complexity in the evaluation of market risks but also the sophistication of the strategies adopted by banks to comply with current regulations. In particular, the role of the stress factor and the CSR component in the calculation of the initial margin highlights the attention to specific risks not fully captured by standard VaR estimates. The in-depth analysis of these issues reveals how crucial it is for banks to maintain a balance between taking risks and preserving financial stability, in a context in which market fluctuations can rapidly alter risk conditions. Furthermore, examining this formula allows us to reflect on the evolution of capital regulation practices and their impact on the strategic management of banks globally

1.6 Coherent risk measures

A risk measure wich satisfy the four axioms of translation invariance, subadditivity, positive homogeneity, and monotonicity is called *coherent*.[3]

1. *Translation invariance*

$$\rho(X+k) = \rho(X) - k \quad \forall X, k \in \mathbb{R}$$

   In other words, adding cash k to a portfolio should reduce its risk by k. This reduces the lowest portfolio value.

2. *Subadditivity*

$$\rho(X+Y) \leq \rho(X) + \rho(Y) \quad \forall X, Y$$

   It means that the risk of a portfolio must be less than the sum of separate risks. Merging portfolios cannot increase risk. This is linked to the concept of diversification: the more diversified a portfolio is, the less risky it is.

3. *Positive homogeneity*

$$\rho(bX) = b\rho(X) \quad \forall b, X \geq 0$$

   Increasing the size of a portfolio by a factor b should scale its risk measure by the same factor b.

4. *Monotonicity*

$$\rho(X) \geq \rho(Y) \quad \forall X, Y \quad with \quad X \leq Y$$

   If a portfolio has systematically lower values than another (in each state of the world), it must have greater risk. In other words, the relationship between the value of an investment and its risk is monotonically increasing: if the investment is riskier, it tends to have lower values, and vice versa.

[3] Artzner P., Delbaen F., Eber J. e Heath D. - *Coherent Measures of Risk*, Mathematical Finance (1999) The authors have defined properties that characterize a broad class of risk measures, referred to as coherent.

The Value at Risk (VaR) has faced significant criticism as a measure of risk, primarily due to its inadequate aggregation properties. The root of this critique can be traced back to the seminal work of Artzner et al. (1999), which demonstrated that VaR does not qualify as a coherent risk measure because it fails to adhere to the principle of subadditivity.

1.7 Expected Shortfall

The Expected Shortfall (ES), also known as Conditional Value at Risk (CVaR), is a financial risk measure that was introduced to overcome some of the limitations of the Value at Risk (VaR). While the VaR provides an estimate of the maximum loss that can be expected with a certain level of confidence over a given time horizon, it does not provide information on the size of losses that can occur beyond this threshold. As a result, the VaR can underestimate risk in the presence of return distributions with heavy tails, typical of many financial instruments.

VaR is sometimes criticized for several different reasons. Most important is the fact that it completely ignores the severity of losses in the far tail of the distribution and its lack of the subadditivity property. This is where the Expected Shortfall comes into play.

Definition 1.2 (Expected Shortfall) *For a loss L with $E(|L|) < \infty$ and density function F_L the expected shortfall at confidence level $\alpha \in (0,1)$ is defined as*

$$\mathrm{ES}_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 q_u(F_L)\mathrm{d}u \tag{7}$$

where $q_u(F_L)$ is the quantile function of F_L.

The expected shortfall is thus related to VaR by

$$\mathrm{ES}_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u(L)\mathrm{d}u$$

The Expected Shortfall (ES) emerges as a critical advancement in risk measurement, addressing the shortcomings of traditional metrics by offering a more nuanced understanding of extreme risk scenarios:

- Response to Heavy Tails: financial instruments and markets often exhibit return distributions with "heavy tails," where the probability of extreme loss events is greater than predicted by normal distributions. The ES, by considering the average of losses in the distribution's tail beyond the VaR, provides a more accurate estimate of risk in these extreme scenarios.

- Improvement in Risk Management: the ES offers financial operators and regulators a more reliable tool for assessing risk and determining the necessary capital to cover potential extreme losses, promoting more robust risk management and greater financial stability.

- Regulatory Demands: in response to the financial crisis of 2007-2008, regulators sought more effective risk assessment methods that could better capture tail risk. Basel III and other

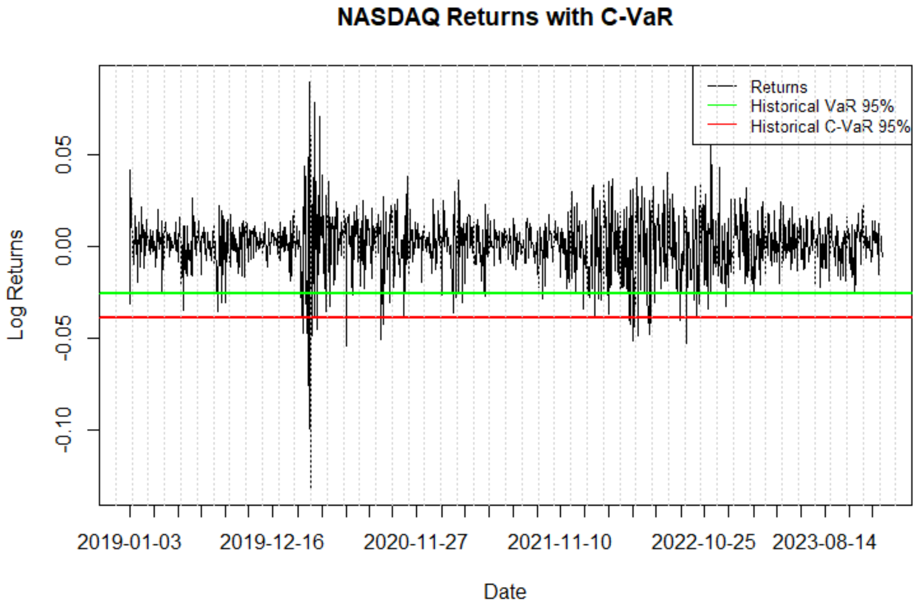


Figure 9: Historical simulation of the Expected Shortfall at 95% confidence level

regulatory frameworks have begun to prefer the ES over the VaR for certain applications, recognizing its ability to provide a more comprehensive view of risk.

As with VaR, expected shortfall can be calculated in the three methods described above (historical, parametric[4] and Monte Carlo).

Considering the NASDAQ example (1.4.3), after computations in R software, the results obtained are:

Historical $\mathrm{ES}_{0.95} = -0,0385$

Parametric $\mathrm{ES}_{0.95} = -0,0322$

Monte Carlo $\mathrm{ES}_0.95 = -0.0321$

The results obtained from calculating the Expected Shortfall (ES) at 95% for the NASDAQ index using three different methods - historical, parametric, and Monte Carlo - reveal estimates of the expected loss beyond the worst 5% of losses. based on historical data, in the worst 5% of cases, an average loss of at least 3.85% of the portfolio's value is expected. The parametric approach, which assumes a specific statistical distribution of returns (typically normal), provides an estimate of an average loss of 3.22% in the worst 5% of cases. In figures (9),(10) and (11) below are shown the results compared with VaR.

As before, we compute the CVaR using a 250-day rolling window to make the representation of its evolution in time with a Historical and parametric approach.

[4]Parametric $\mathrm{ES}_\alpha = \mu + \sigma \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}$ where ϕ is the density of the standard normal distribution.

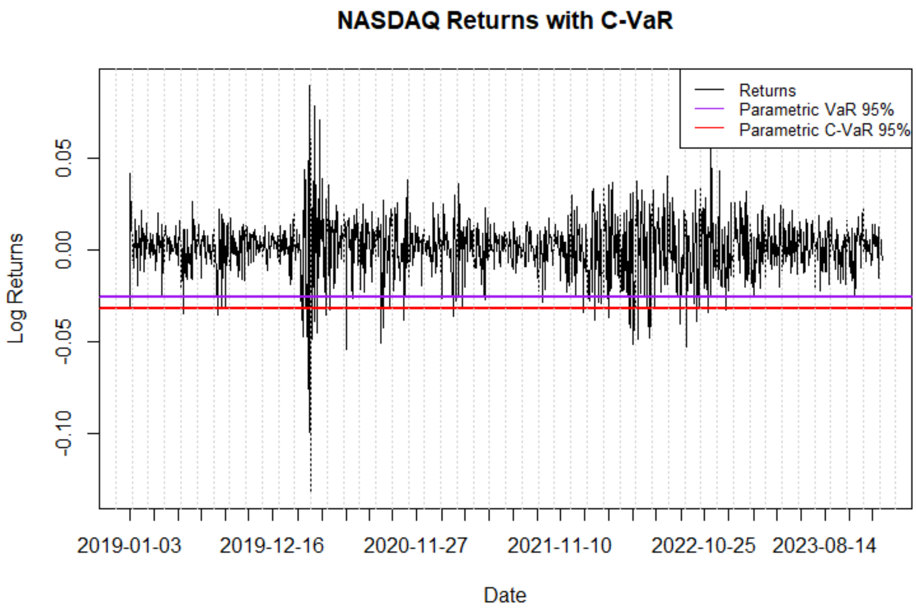


Figure 10: Parametric Expected Shortfall compared with VaR at 95% confidence level

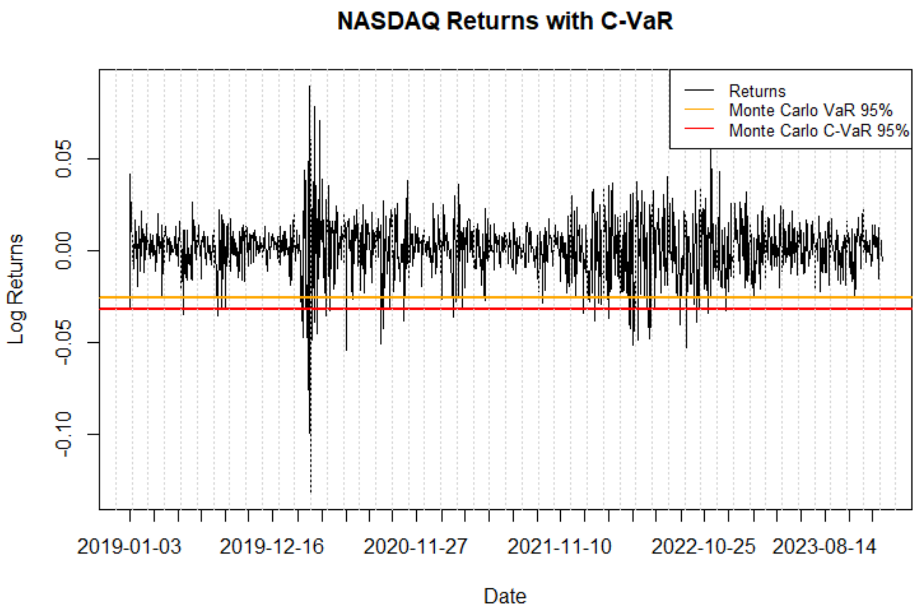


Figure 11: Monte Carlo C-VaR simulation using the historical mean and historical standard deviation of NASDAQ's returns

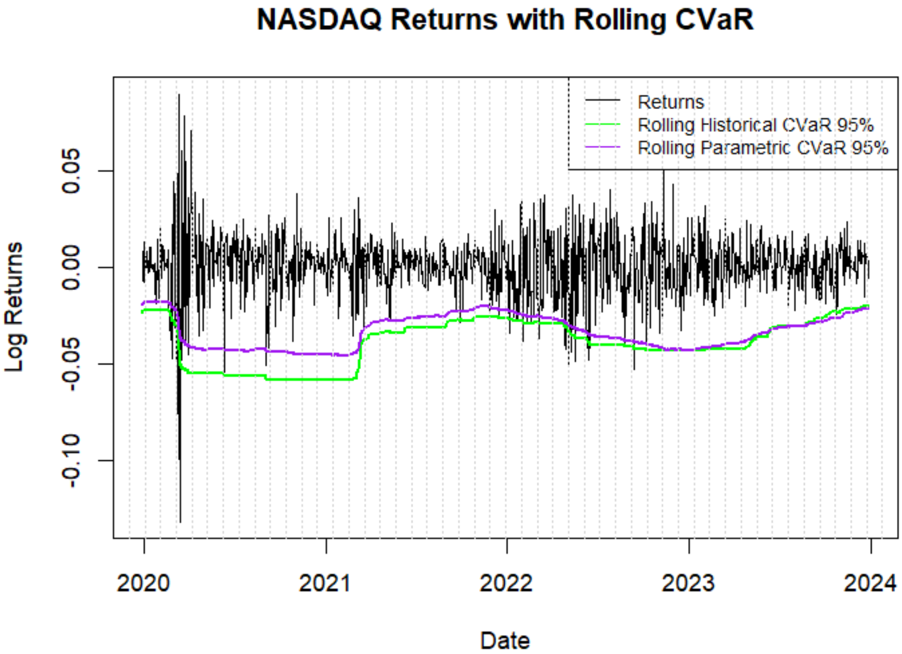


Figure 12: Hisorical and parametric VaR using 250-days rolling window

2 Volatility models

In a financial environment marked by constant uncertainties and unpredictable movements, interpreting market volatility represents a crucial challenge for investors, fund managers, and analysts. The ability to decipher, model, and anticipate market fluctuations is fundamental not only for effective risk management but also for evaluating assets, and derivative instruments and developing solid investment strategies. At the heart of these advanced analyses, we find volatility models, sophisticated tools designed to capture the complex dynamics of financial markets. Among these, GARCH (Generalized Autoregressive Conditional Heteroskedasticity) models emerge as distinctive tools, capable of describing how volatility evolves, offering a robust framework for the analysis of fluctuations in financial returns.

This chapter delves into volatility models, paying special attention to the different variations of GARCH models. Initially introduced by Robert Engle in 1982 and then extended by Tim Bollerslev in 1986, GARCH models have radically transformed the way we interpret volatility, highlighting its conditional behavior and tendency to form clusters. Starting from the basic GARCH model, numerous versions have been developed to respond to specific needs related to the analysis of financial data, such as adaptation to financial leverage phenomena, reaction to asymmetric shocks, and modeling of volatility trends on an extended storm.

2.1 ARMA process

Before dealing with the GARCH model, it is necessary to briefly focus on the ARMA process to better understand what we are going to do. First, we define the components of the ARMA process.

A *qth-order moving average process*, denoted MA(q), is characterized by:

$$Y_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q}$$

which can be written as:

$$Y_t = \mu + \sum_{j=0}^{q} \theta_j \varepsilon_{t-j} \tag{8}$$

A *pth-order autoregression*, denoted AR(p), satisfies

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + \varepsilon_t \tag{9}$$

An ARMA(p,q) *process* include both autoregressive and moving average terms:

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q}$$

or, in lag operator form,

$$(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p) Y_t = c + (1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q)\varepsilon_t$$

Now, dividing both sides by $(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p)$, we obtain

$$Y_t = \mu + \psi(L)\varepsilon_t \tag{10}$$

where

$$\psi(L) = \frac{(1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q)}{(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p)}$$
$$\mu = c/(1 - \phi_1 - \phi_2 - \cdots - \phi_p)$$

Thus, the stationarity of an ARMA process depends entirely on the autoregressive parameters $(\phi_1, \phi_2, \ldots, \phi_p)$ and not on the moving average parameters $(\theta_1, \theta_2, \ldots, \theta_q)$.

Specifying an ARMA model before applying a GARCH ensures that autocorrelation in the data is adequately captured and removed before modeling volatility. This step is crucial to ensure that the volatility estimated by GARCH truly reflects the intrinsic variability of returns and is not distorted by the presence of unmanaged time dependencies. This leads to more precise volatility estimates and improves the reliability of future forecasts, as we are going to see later.

2.2 The ARCH(q) model

The ARCH(q) model introduced by Engle (1982) is a linear function of past squared disturbances:

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 \tag{11}$$

In this model, to assure a positive conditional variance, the parameters have to satisfy the following constraints: $\omega > 0$ and $\alpha_1, \alpha_2, \ldots, \alpha_q \geq 0$.
To accurately describe how volatility is maintained over time across ARCH models, it is often necessary to resort to a rather high order of these models. This approach involves estimating a considerable number of parameters, potentially leading to a complication known as over-parameterization, which can compromise the effectiveness and efficiency of the estimates obtained.

In terms of computational complexity, this need to estimate a large set of parameters makes ARCH models particularly burdensome from the point of view of the necessary calculations. This becomes particularly critical when working with large datasets, significantly slowing down the estimation process.

As for the ability of these models to reflect the persistence of volatility over the long term, ARCHs may sometimes not fully capture this intrinsic characteristic of financial returns. Persistence means that periods of high volatility tend to be followed by periods of equally high volatility, and vice versa, a behavior that long-form ARCH models can struggle to represent effectively.

2.3 The GARCH(p,q) model

To model the conditional heteroscedasticity and overcome the problems described above, Bollerslev(1986) proposed the GARCH(p,q) model

$$\sigma_t^2 = \omega + \alpha(L)\varepsilon_t^2 + \beta(L)\sigma_t^2 \tag{12}$$

where $\alpha(L) = \alpha_1 L + \cdots + \alpha_q L^q$, $\beta(L) = \beta_1 L + \cdots + \beta_p L^p$. GARCH(1,1) is the most popular model:

$$\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \tag{13}$$

The model can be written as an ARCH(∞):

$$\begin{aligned} \sigma_t^2 &= \left(1 - \sum_{i=1}^{p} \beta_i L_i\right)^{-1} \left[\omega + \sum_{j=1}^{q} \alpha_j \varepsilon_{t-j}^2\right] \\ &= \omega^* + \sum_{k=0}^{\infty} \phi_k \varepsilon_{t-k-1}^2 \end{aligned} \tag{14}$$

$\sigma_t^2 \geq 0$ if $\omega^* \geq 0$ and all $\phi_k \geq 0$. The non-negativity of ω^* and ϕ_k is also a necessary condition for the non-negativity of σ_t^2. In order to make ω^* and $\{\phi_k\}_{k=0}^{\infty}$ well defined, let's assume that:

1. The roots of the polynomial $\beta(x) = 1$ lie outside the unit circle and $\omega \geq 0$. This is a condition for ω^* to be positive and finite.

2. $\alpha(x)$ and $1 - \beta(x)$ have no common roots.

Considering the GARCH(1,1), the positivity of σ_t^2 requires (Nelson and Cao, 1992), along with conditions (1) and (2), that

$$\omega \geq 0, \quad \beta_1 \geq 0, \quad \alpha_1 \geq 0.$$

2.3.1 Forecasting volatility

We can use GARCH(p,q) models to forecast volatility (Engle and Bollerslev 1986):

$$\sigma_{t+k}^2 = \omega + \sum_{i=1}^{q} \alpha_j \varepsilon_{t+k-i}^2 + \sum_{i=1}^{p} \beta_i \sigma_{t+k-i}^2$$

Now, we can write the process in two parts, before and after time t:

$$\sigma_{t+k}^2 = \omega + \sum_{i=1}^{n} \left[\alpha_i \varepsilon_{t+k-i}^2 + \beta_i \sigma_{t+k-i}^2\right] + \sum_{i=k}^{m} \left[\alpha_i \varepsilon_{t+k-i}^2 + \beta_i \sigma_{t+k-i}^2\right]$$

where $n = \min\{m, k-1\}$ and by definition summation from 1 to 0 and from $k > m$ to m both are equal to zero. Thus:

$$E_t[\sigma^2_{t+k}] = \omega + \sum_{i=1}^{n}\left[(\alpha_i + \beta_i)E_t(\sigma^2_{t+k-i})\right] + \sum_{i=k}^{m}\left[\alpha_i \varepsilon^2_{t+k-i} + \beta_i \sigma^2_{t+k-i}\right]$$

In particular for a GARCH(1,1) and $k > 2$:

$$\begin{aligned} E_t[\sigma^2_{t+k}] = & \sum_{i=0}^{k-2}(\alpha_1+\beta_1)^i\omega + (\alpha_1+\beta_1)^{k-1}\sigma^2_{t+1} \\ = & \ \omega\frac{\left[1-(\alpha_1+\beta_1)^{k-1}\right]}{\left[1-(\alpha_1+\beta_1)\right]} + (\alpha_1+\beta_1)^{k-1}\sigma^2_{t+1} \\ = & \ \sigma^2\left[1-(\alpha_1+\beta_1)^{k-1}\right] + (\alpha_1+\beta_1)^{k-1}\sigma^2_{t+1} \\ = & \ \sigma^2 + (\alpha_1+\beta_1)^{k-1}\left[\sigma^2_{t+1} - \sigma^2\right] \end{aligned}$$

2.3.2 Fitting a GARCH model

To understand what is explained above, we consider an example showing the necessary steps to fit a GARCH model in R software.

Let's consider the daily price series of Chevron Corporation (CVX) of the last ten years, from 1 January 2014 to 31 December 2023, and compute the logarithmic returns in figure (13). CVX is one of the largest energy companies in the world, primarily engaged in the exploration, production, refining, and sale of oil and natural gas, as well as the production of chemicals. Chevron operates globally and is known for being one of the "supermajors" in the oil and gas industry.

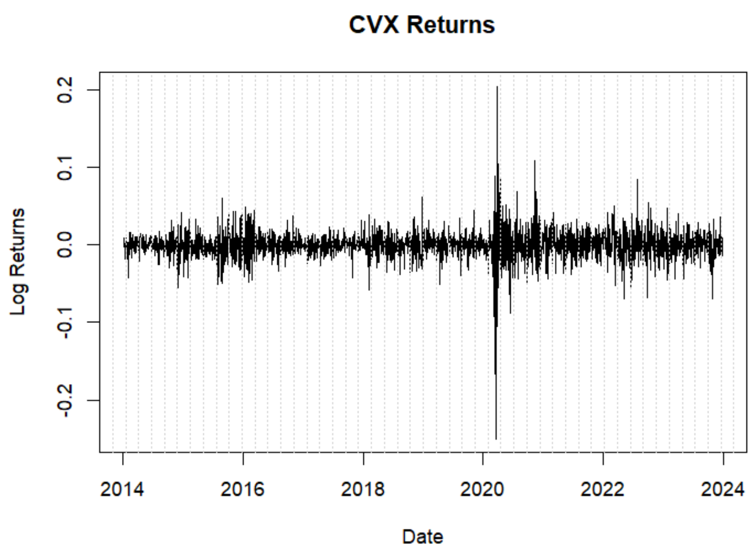


Figure 13: Chevron Corporation's log-returns of the last ten years. Volatility clusters can be observed, periods with high fluctuations in returns that alternate with periods with lower fluctuations.

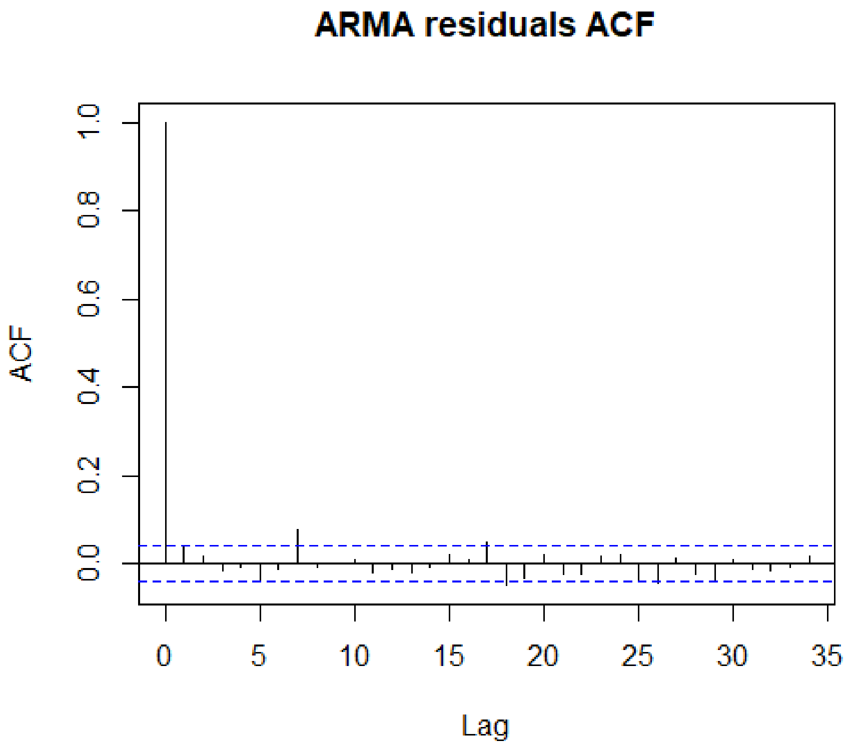


Figure 14: The Autocorrelation Function (ACF) shows the autocorrelation of the residuals of an ARMA model at different lags.

Now we start with the choice of the ARMA model in R software. Use *auto.arima* to automatically select the best ARMA model based on the Akaike information criterion (AIC)[5]. This model attempts to capture any autocorrelation in returns. Based on the results, we select the ARMA(2,2) process. Then, after fitting the ARMA model, extract the model residuals, which represent fluctuations in returns not explained by the ARMA model.

Looking at the figure (14) it seems that most of the bars, which represent the value of the autocorrelation at the different lags, are within the confidence bands (the blue dotted lines). These bands are typically set to plus or minus two standard deviations from the null hypothesis of no autocorrelation. Given that all bars are within the bands, this suggests that there is no significant autocorrelation in the residuals at any of the lags. This is a sign that the ARMA model may have adequately captured temporal relationships in the data.

Now we are ready to model the returns time series. First, *ugarchspec* in R, defines the specifications of the model to be optimized. It is also possible to choose the conditional distribution to use for the error terms (we use the normal distributions but many options are available). Finally, we estimate the model using the *ugarchfit* function.

To evaluate the model, the standardized residuals are used to verify the model because, if this

[5] See section 2.5.1 for further information about Akaike criterion

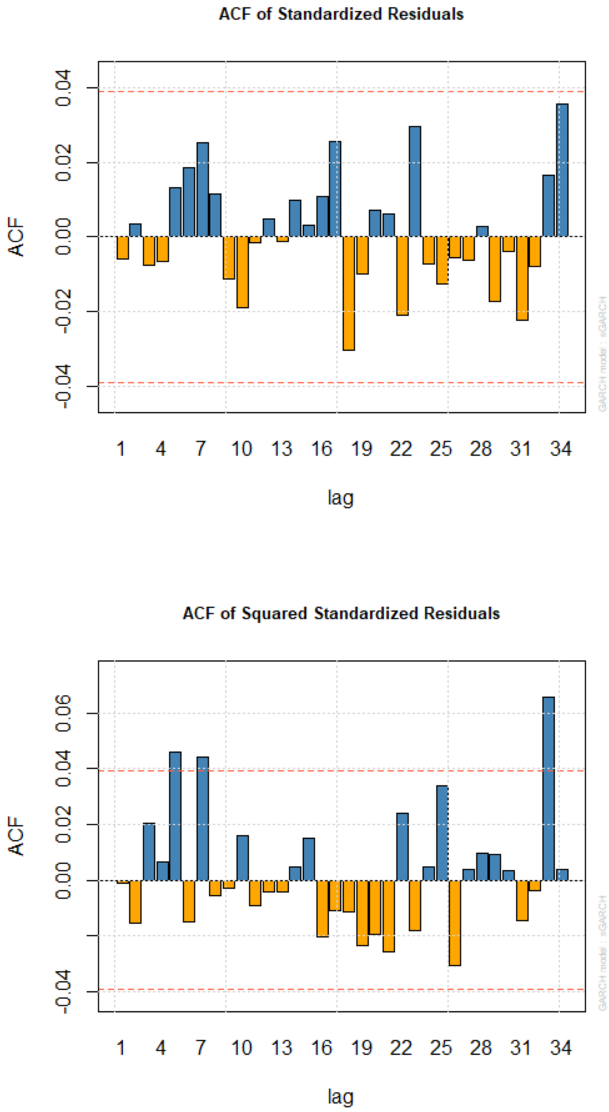


Figure 15: Autocorrelation function of standardized residual and squared standardized residuals

is good, neither the residuals themselves nor their squares must show autocorrelation. As we can see in figure (15) the upper and lower bands of the correlograms up to the thirty-fifth lag, equal to 0.04 and -0.04, are reached overall only on three occasions.

When we specified the *ugarchspec* object, the normal distribution was chosen as the conditional distribution to use for the error terms. To verify the validity of this choice The second tool is the graphical analysis of two important graphs: the first measures the differences between the empirical and theoretical distribution of the standardized residuals while the second is a Q-Q Plot of the latter which has been explained in section (1.2).
The graphs confirm that the standardized residuals do not follow the normal distribution.

We therefore repeat our analysis of the ARMA(2,2)-GARCH(1,1) model, this time specifying that the residuals follow the Student's t-distribution. As we can see in figure (17), Student's t

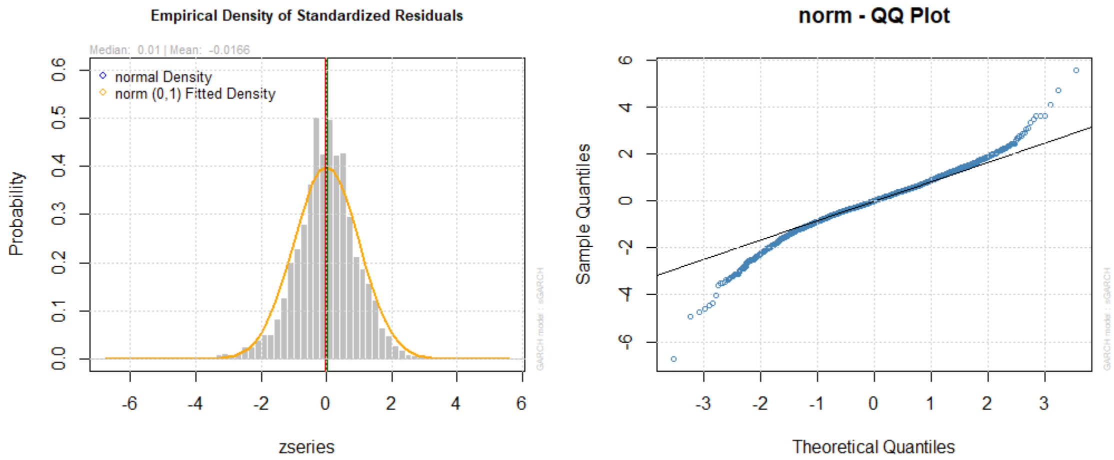


Figure 16: Empirical and theoretical density and Q-Q Plot of standardized residuals (normal)

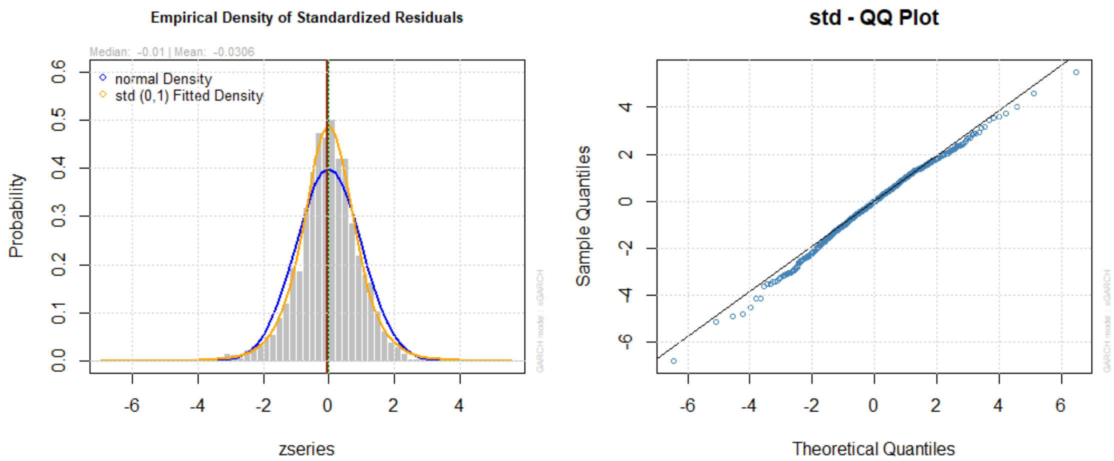


Figure 17: Empirical and theoretical density and Q-Q Plot of standardized residuals (t-Student)

is not the definitive answer but the result is more than satisfactory. In particular, the Q-Q Plot follows a straighter line than that followed by the normal distribution, and, all things considered, the use of the Student's t-distribution instead of the normal one seems justified.

Finally, using R software, we can compute the conditional volatility estimated by the model in figure (18 and the Value-at-Risk. Previously, we talked about the ability of GARCH to forecast future volatility. Through R it is possible to see the behavior of future volatility and future returns estimated by the model.

2.4 Asymmetric model

The traditional GARCH model assumes that the market's response to a shock is determined by the magnitude of the shock rather than its positive or negative nature. However, it is widely recognized

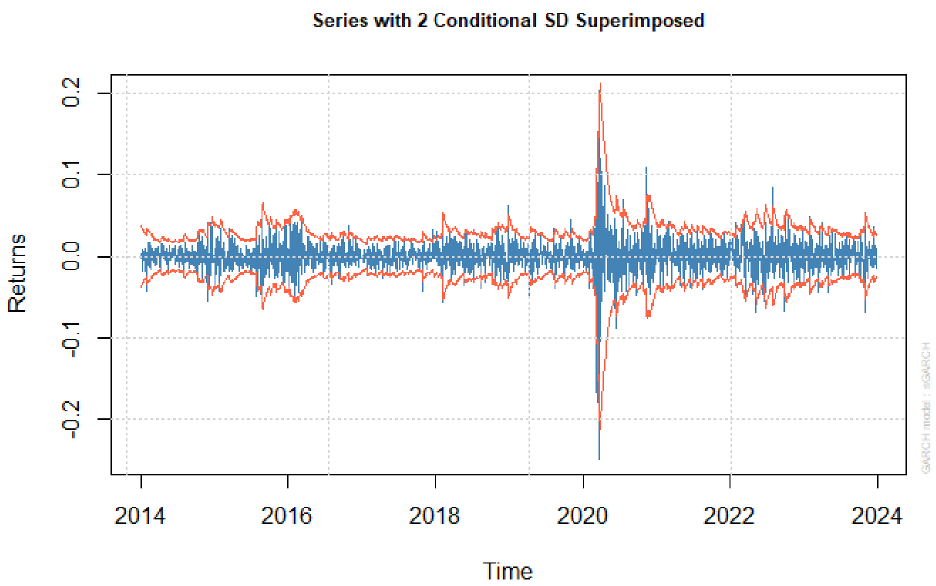


Figure 18: Series with 2 Conditional SD Superimposed of Chevron Corporation (CVX) from first January 2014 to 31 December 2023. The red lines represent the expected variability of returns based on the conditional volatility estimated by the GARCH model. They indicate how much we expect returns to fluctuate around the mean, given the estimated levels of volatility.

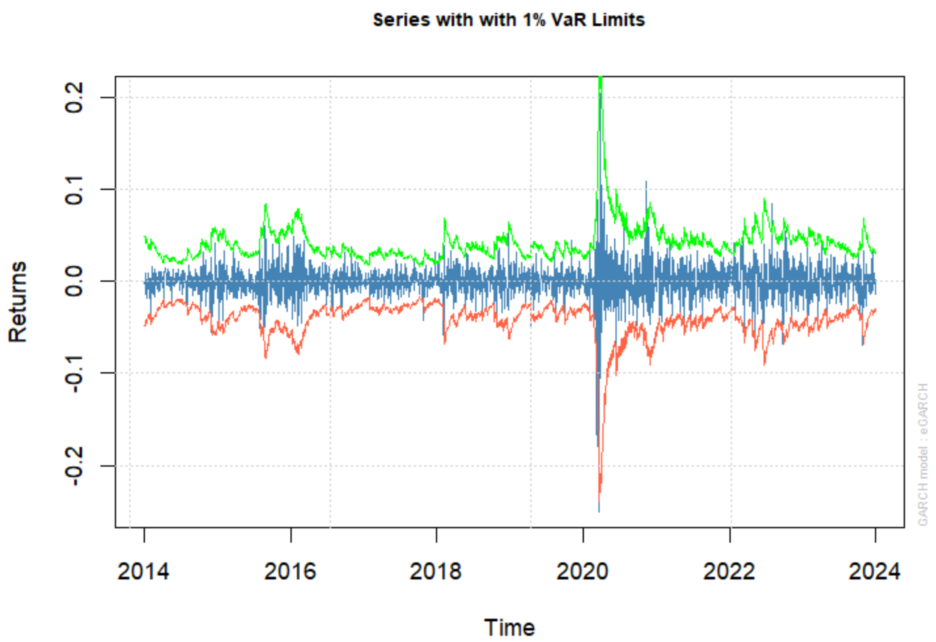


Figure 19: The green and red lines represent the 1% VaR of CVX, indicating the loss level that should not be exceeded with 99% probability. In this context, the red line is usually more relevant, as it represents the maximum expected losses.

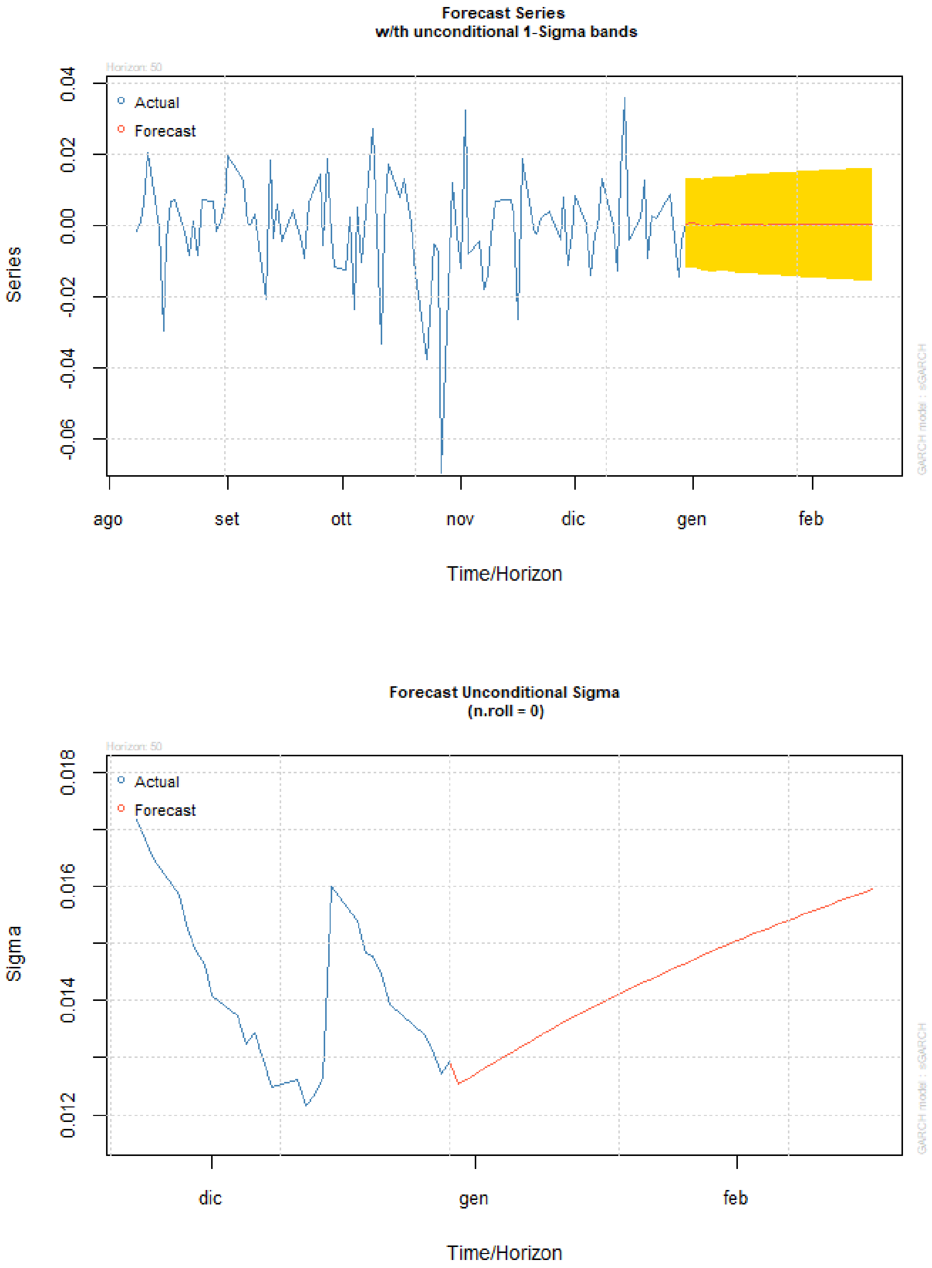


Figure 20: Graph of the estimate of conditional volatility and 50-days returns following the observation period.

that in the financial context, negative shocks influence volatility more than positive shocks, with volatility typically intensifying in periods of market decline compared to rising ones.

Black (1976) linked this phenomenon to the observation that negative news typically leads to a decrease in stock prices, thereby raising the stock's leverage (that is, the debt-to-equity ratio), which in turn results in increased volatility of the asset.

2.4.1 The EGARCH(p,q) model

The structure of (12) imposes some limitations on GARCH models:

- The negative correlation between stock returns and changes in returns volatility: the volatility tends to rise in response to 'bad news' and to fall in response to 'good news'. GARCH models, however, assume that only the magnitude and not the positivity or negativity unanticipated excess returns determine feature σ_t^2.
- The GARCH models are not able to explain the observed covariance between ε_t^2 and ε_{t-j}.
- GARCH models essentially specify the behavior of the square of the data. In this case, a few large observations can dominate the sample.

The *exponential* GARCH(p,q) model by Nelson (1991) provides a parametrization where σ_t^2 depends on both size and sign of lagged residual:

$$\ln(\sigma_t^2) = \omega + \sum_{i=1}^{q} \alpha_i[\phi z_{t-i} + \psi(|z_{t-i}| - E\,|z_{t-i}|)] + \sum_{i=1}^{p} \beta_i \ln(\sigma_{t-i}^2) \tag{15}$$

with $\alpha_1 \equiv 1$, $E\,|z_t| = (2/\pi)^{1/2}$ given that $z_t \sim$ i.i.d$N(0,1)$[6]. If $\psi = 0$ and $\phi < 0$, the innovation in $\ln(\sigma_{t-i}^2)$ is now positive (negative) when returns innovations are negative (positive). A negative shock to the returns which would increase the debt-to-equity ratio and therefore increase uncertainty of future returns could be accounted for when $\alpha_i > 0$ and $\phi < 0$. As in the standard GARCH case, the first-order model is the most popular EGARCH in practice.

2.4.2 The news impact curve

The news has an asymmetric effect on volatility. In the asymmetric volatility models, good news and bad news have different predictability for future volatility. The News Impact Curve (NIC), introduced by Pagan and Schwert (1990), is used to visualize how news, or more generally market shocks, influence the volatility predicted by a financial model, such as GARCH. NIC is very useful as investors can use it to estimate potential future volatility based on specific news scenarios, helping them build portfolios that best suit their risk profile. So, the NIC relates past return shocks (news) to current volatility by measuring how new information is incorporated into volatility estimates.

[6]Nelson (1991) have found that standardized residuals from estimated GARCH models are leptokurtic relative to the normal. Nelson assume that z_t has a GED distribution

For the GARCH model, the NIC is centered on $\varepsilon_{t-1} = 0$ while in the case of the EGARCH model, the curve has its minimum at ε_{t-1} and is exponentially increasing in both directions but with different parameters. Let's see in detail the NIC of the models explained before.

GARCH(1,1):

$$\sigma_t^2 = \omega + \alpha\varepsilon_t^2 + \beta\sigma_{t-1}^2$$

Given that $\sigma_{t-1}^2 = \sigma^2$, the NIC has the following expression:

$$\begin{aligned} \sigma_t^2 = & \quad A + \alpha\varepsilon_{t-1}^2 \\ A \equiv & \quad \omega + \beta\sigma^2 \end{aligned}$$

EGARCH(1,1):

$$\ln(\sigma_t^2) = \omega + \beta\ln(\sigma_{t-1}^2) + \phi z_{t-1} + \psi(|z_{t-1}| - E\,|z_{t-1}|)$$

where $z_t = \varepsilon/\sigma_t$. The NIC is:

$$\sigma_t^2 = \begin{cases} A\exp\left[\frac{\phi+\psi}{\sigma}\varepsilon_{t-1}\right], & \varepsilon_{t-1} > 0 \\ A & \varepsilon_{t-1} = 0 \\ A\exp\left[\frac{\phi-\psi}{\sigma}\varepsilon_{t-1}\right], & \varepsilon_{t-1} < 0 \end{cases}$$

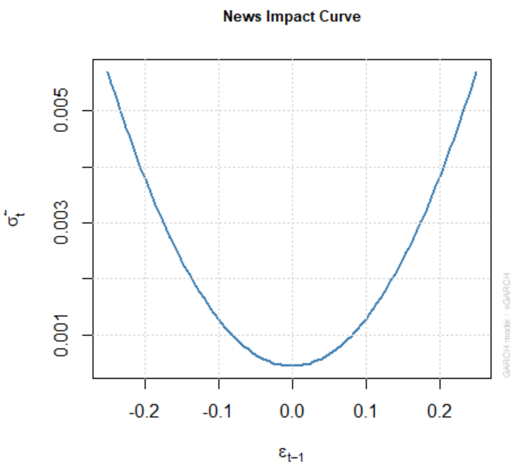
$$A \equiv \quad \sigma^{2\beta}\exp\left[\omega - \psi\sqrt{2/\pi}\right]$$

It is evident from the NIC that the EGARCH allows good and bad news to have a different impact on the volatility, while the standard GARCH does not. Moreover, it allows big news to have a greater impact on the volatility than in the GARCH model.

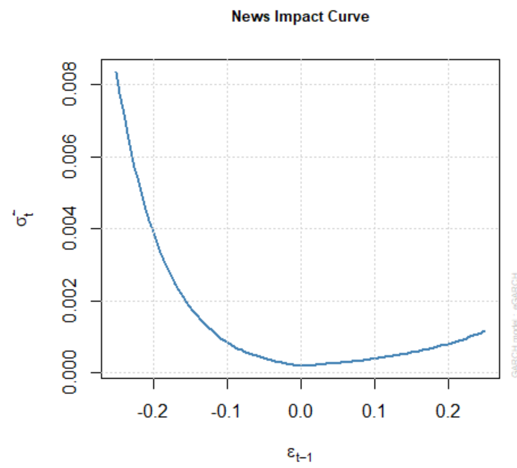
To better understand the curve, let's take the CVX example above by first representing the curve of GARCH(1,1) and that of EGARCH(1,1)[7]. As we might expect, the GARCH(1,1) curve shows a typical U-shape. This implies that both good news (positive shocks) and bad news (negative shocks) increase future volatility, but they do so in a symmetrical. The magnitude of the shock is directly proportional to the increase in volatility, regardless of the sign. In practical terms, this means that for Chevron, according to the GARCH(1,1) model, news, both positive and negative, has a uniform impact on the volatility forecast.

The EGARCH(1,1) curve, on the other hand, shows an asymmetric characteristic. It is flatter for positive shocks and more pronounced for negative shocks, indicating that bad news has a greater impact on volatility than good news. This is in line with what would be expected in the presence of leverage, where markets tend to react more to bad news.

[7] We repeat the steps used in (2.3.2) to model the EGARCH(1,1)



(a) GARCH NIC



(b) EGARCH NIC

Figure 21

2.5 GARCH selection Criteria

Model selection refers to the challenge of choosing the best model from a list of competing alternatives using the available data. Essentially, this process involves applying a model selection criterion to identify the model that best fits the data (Wasserman, 2000). The use of information criteria for model selection has been developed to summarize the evidence in favor of one model over others. Specifically, these techniques focus on minimizing the amount of information needed to describe the data and the model. Consequently, this leads to the selection of models that efficiently represent the data.

2.5.1 Akaike's Information Criteria (AIC)

One of the most used information criteria is AIC (Akaike, 1973). The the idea of AIC is to select the model that minimizes the negative likelihood penalized by the number of parameters as specified in the following equation

$$AIC = -2\log p(L) + 2p \tag{16}$$

where L is the likelihood under the fitted model and p is the number of parameters in the model. In particular, AIC is aimed at finding the best approximating model for the unknown true data-generating process.

2.5.2 Bayesian information criteria (BIC)

Unlike Akaike Information Criteria, BIC is derived within a Bayesian framework as an estimate of the Bayes factor for two competing models (Schwarz, 1978; Kass and Rafftery, 1995). BIC is defined as:

$$BIC = -\log p(L) + p\log(n) \tag{17}$$

Now, the second term depends on sample size n. BIC is designed to find the most probable model given the data from a Bayesian perspective. For practical purposes, both the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC) share the common goal of identifying good models, despite differing in their exact definitions of what constitutes a "good model." Therefore, comparing them is justified, at least to evaluate how each criterion performs in terms of identifying the correct model or how they behave when they are expected to prefer the same model.

3 Portfolio Optimization and Value-at-Risk

Portfolio optimization has always been a challenging proposition in the financial market and management to facilitate the selection of portfolios in a volatile market situation. One of the most important problems faced by every investor is asset allocation.

An investor during making investment decisions has to search for an equilibrium between risk and returns that are uncertain parameters in the suggested portfolio optimization models and should be estimated to solve the problem. Portfolio optimization aims to choose the weights of a given set of assets so that the selected portfolio is the best one according to specific criteria. This chapter delves into the intricacies of portfolio optimization, with a particular focus on the Mean-CVaR (Conditional Value at Risk) approach.

During the chapter, a practical example will accompany the theoretical explanation to better understand the topics that will be covered regarding portfolio optimization.

3.1 Mean Variance (MV) Portfolio

Modern portfolio theory explains how rational investors can use diversification to optimize their portfolios of risky assets. The foundational concepts of this theory trace back to Harry Markowitz's 1952 work on diversification and the efficient frontier of portfolios. Markowitz's model treats asset returns as random variables and represents a portfolio as a weighted combination of these assets. His model considers asset returns as a random variable and models a portfolio as a weighted combination of assets. Being a random variable, a portfolio's returns have an expected mean and variance. The assumption of Markowitz's mean-variance analysis is based on the following:

- Investors are risk averse so they prefer less risk to more for the same level of expected return.
- All investors have information concerning expected returns, variances, and covariances of assets and they need only know this information to determine optimal portfolios.
- No taxes and no transaction costs.

The expected return for a particular portfolio p of N assets is:

$$E(r_p) = \sum_{i=1}^{N} x_i r_i \tag{18}$$

where x_i denotes the weight percentage of ith asset in portfolio and r_i is the expected return of ith asset. The variance for the generic portfolio p can be expressed as:

$$\sigma_p^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \sigma_{ij} \tag{19}$$

where $sigma_{ij}^2$ is the covariance between the return on the ith and jth asset which measures how the returns on two assets move together.

The basic mean-variance optimization model can be stated as the minimization of the portfolio's variance:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j \sigma_{ij} \\
\text{s.t.} \quad & \sum_{i=1}^{N} x_i r_i = \overline{r} \\
& \sum_{i=1}^{N} x_i = 1 \\
& x_i \geq 0, \forall x_i \in [i = 1, 2, \ldots, N]
\end{aligned}
\tag{20}
$$

These constraints are expected return of portfolio must be equal to the target return; the sum of the proportions of financial assets that are in the portfolio must be equal to 1 and finally non-negativity condition for assets' proportions.

Minimun risk MV portfolio

Based on Markowitz (1952) we can write the portfolio selection problem as:

$$
\begin{aligned}
\min_{w} \quad & w^T \hat{\Sigma} w \\
\text{s.t.} \quad & w^T \hat{\mu} = \overline{r} \\
& w^T 1 = 1
\end{aligned}
\tag{21}
$$

The problem expresses that we want to minimize the variance-covariance risk, where the matrix $\hat{\Sigma}$ is the estimate of the covariance of assets. The vector w is the individual investments subject to the second condition that is the available capital is fully invested while $\overline{r}$ represents the expected/target return. The solution to the portfolio problem is unique:

$$
w^* = \hat{\mu} w_0^* + w_1^* \tag{22}
$$

where

$$
\begin{aligned}
w_0^* &= \frac{1}{\Delta}\left(B\hat{\Sigma}^{-1}\hat{\mu} - C\hat{\Sigma}^{-1}1\right) \\
w_1^* &= \frac{1}{\Delta}\left(C\hat{\Sigma}^{-1}\hat{\mu} - A\hat{\Sigma}^{-1}1\right) \\
\Delta &= AB - C^2
\end{aligned}
$$

with

$$
\begin{aligned}
A &= \hat{\mu}^T\hat{\Sigma}^{-1}\hat{\mu} \\
B &= 1^T\hat{\Sigma}^{-1}1 \\
C &= 1^T\hat{\Sigma}^{-1}\hat{\mu}
\end{aligned}
$$

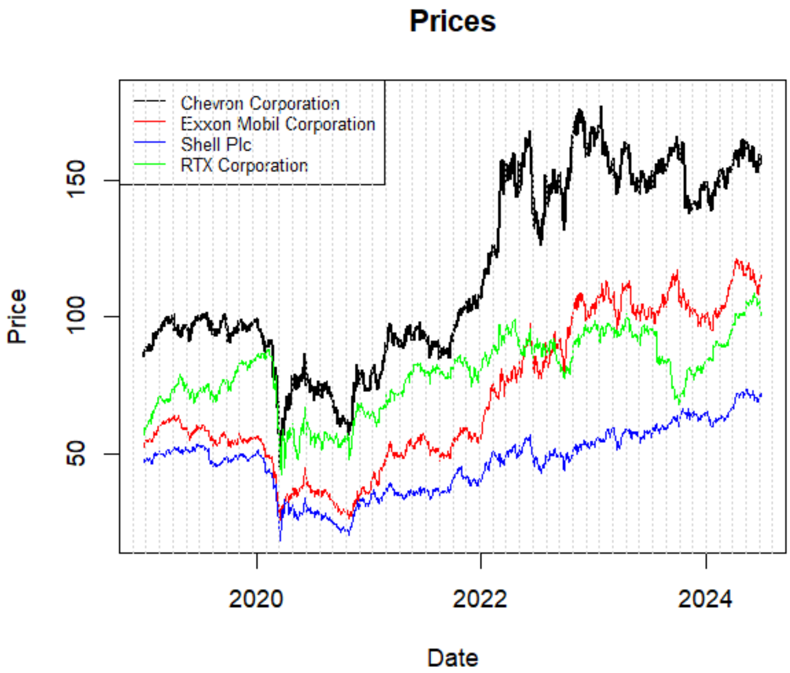


Figure 22: Prices of Exxon, Chevron, Shell and RTX from 2019/01/01 to 2024/06/30

Example From now on, we consider the case of four companies operating within the energy sector:

- Exxon Mobil Corporation (XOM),
- Chevron Corporation (CVX),
- Shell Plc (SHEL),
- RTX Corporation (RTX).

The example shows how to compute the properties of a minimum-risk mean-variance portfolio. This portfolio has a quadratic objective function defined by the covariance matrix of the financial assets and a fixed target return. For practical purposes, we consider the case of long-only constraint. Firstly, we define a feasible portfolio the equal-weight portfolio to obtain a target return that we will use in a minimum-risk portfolio. Then we compute the portfolio weights that minimize the risk given the target return. See the results in figure (23).

The minimum variance portfolio

The so-called global minimum variance portfolio is the one with the smallest risk on the efficient frontier. The minimum variance portfolio represents just the minimum risk point on the efficient frontier with a set of weights:

$$w_* = \frac{\Sigma^{-1}1}{1^T\Sigma^{-1}1}$$

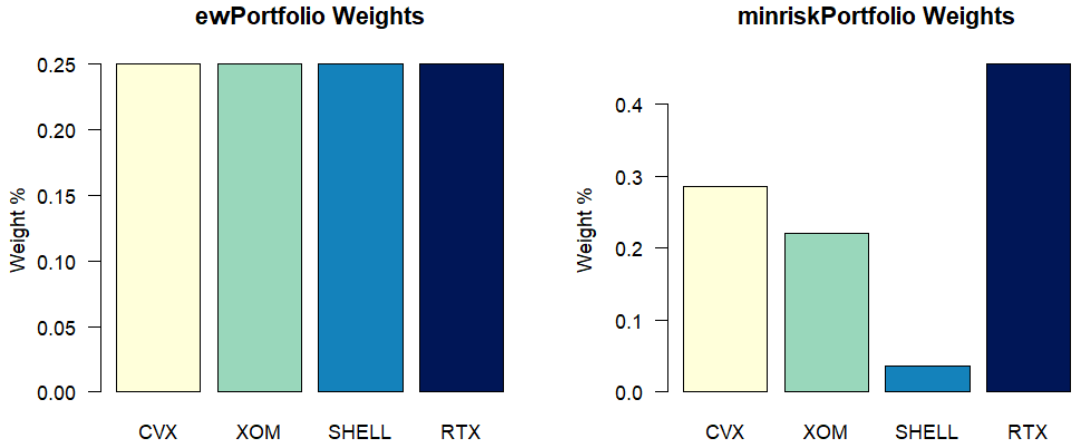


Figure 23: Equally weighted portfolio on the left and the minimum risk MV portfolio on the right

Tangency portfolio

Reward/risk profiles of different combinations of a risky portfolio with a riskless asset r_f, with expected return $\overline{r}$, can be represented as a straight line, the so-called capital market line, CML. The point where the CML touches the efficient frontier corresponds to the optimal risky portfolio. Mathematically, this can be expressed as the portfolio that maximizes the quantity

$$\begin{aligned} \max_{w} \quad & h(w) = \frac{\hat{\mu}^T w - r_f}{w^T \hat{\Sigma} w} \\ \text{s.t.} \quad & w^T \hat{\mu} = \overline{r} \\ & w^T 1 = 1 \end{aligned} \tag{23}$$

among all w. The quantity is the Sharpe ratio, (Sharpe,1994). Back to the example. As we did before, we can now compute the global minimum variance portfolio, the one with the lowest possible risk, and the tangency portfolio. For practical purposes, we consider a zero risk-free rate. Results are shown in figure (24).

Graphically we can represent these portfolios on the efficient frontier, in figure (25). Furthermore, is interesting to explore in more detail the feasible set of the long-only constrained mean-variance portfolio. For this, we plot the frontier, add randomly generated portfolios from a Monte Carlo simulation, and add the frontier lines of all two-asset portfolios, represented in figure (26).

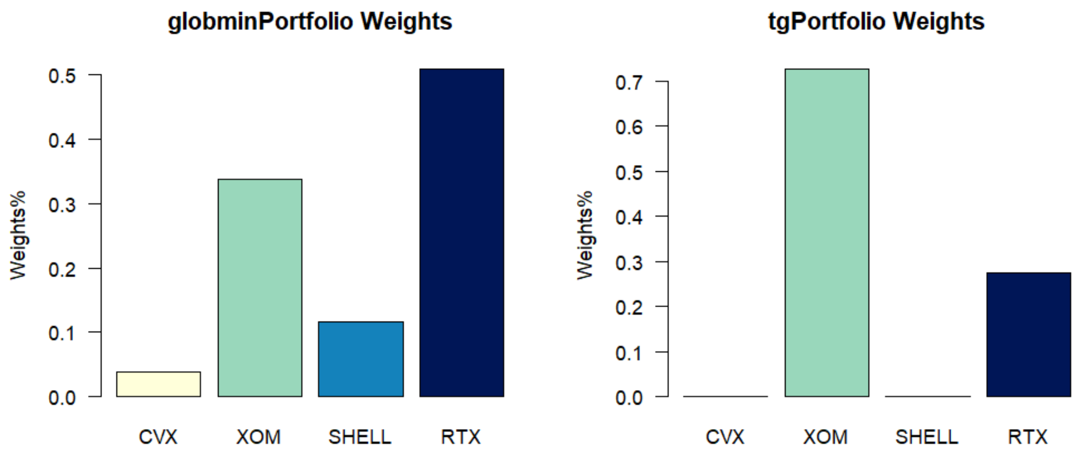


Figure 24: Global minimum portfolio and tangency portfolio with zero risk-free rate.

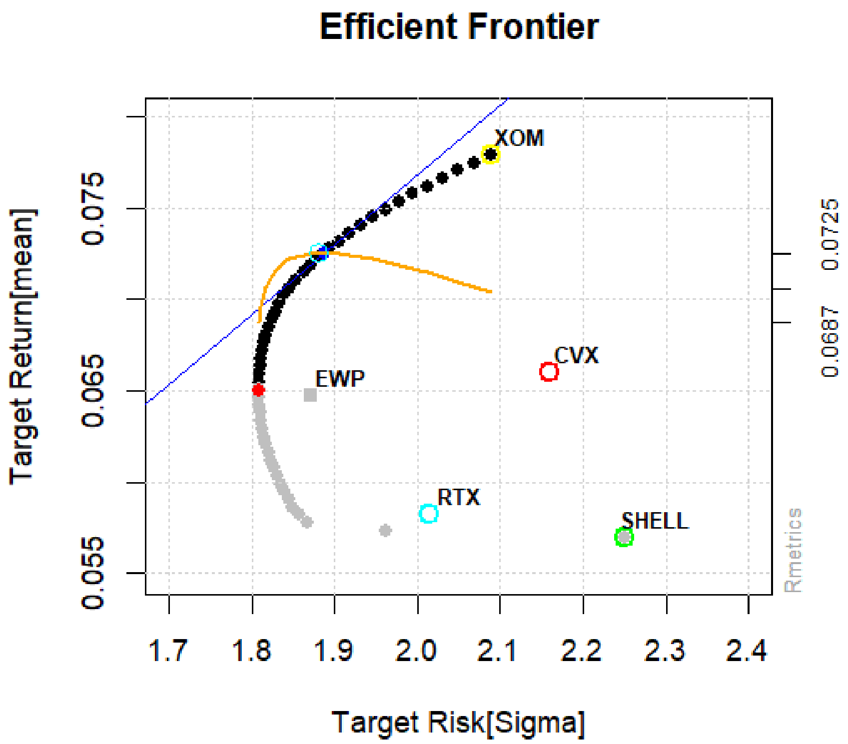


Figure 25: Efficient frontier of a long-only constrained mean-variance portfolio: The plot includes the efficient frontier, the tangency line, and the tangency point for a zero risk-free rate. Additionally, it displays the equal weights portfolio (EWP), the risk vs. return points for all individual assets, and the Sharpe ratio line, with its maximum at the tangency portfolio point. The range of the Sharpe ratio is shown on the right-hand side axis of the plot.

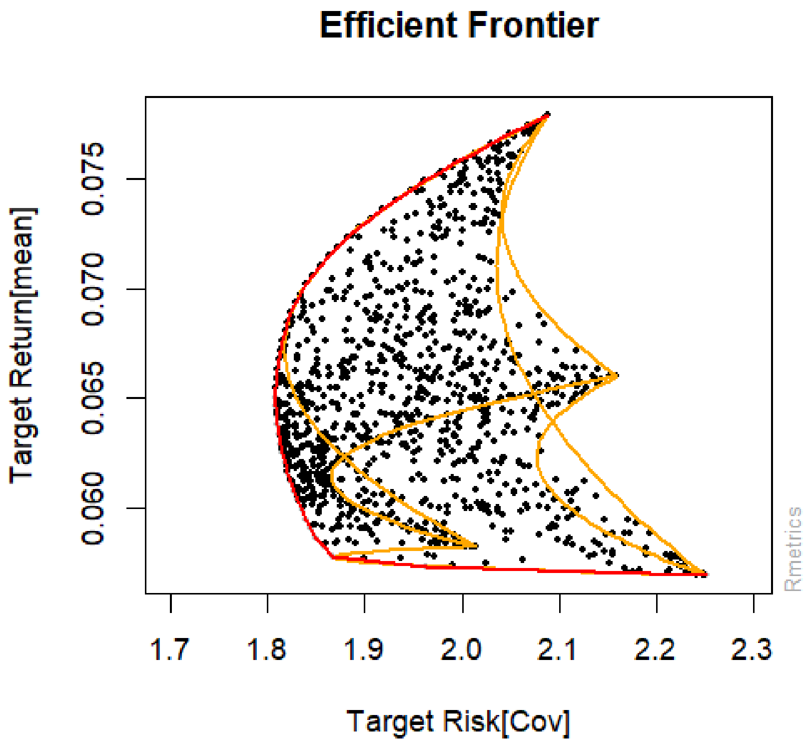


Figure 26: The feasible set for long-only constrained mean-variance portfolios: The graph displays the risk/return plot for 1000 randomly generated mean-variance portfolios under long-only constraints. The plot includes the efficient frontier, the minimum variance locus, and the pairwise frontier lines for all two-asset portfolio combinations. The endpoints of these lines coincide with the risk/return values for the four individual assets.

3.2 Mean-CVaR Portfolio

Variance as the risk measure has its weaknesses. Controlling the variance not only leads to low deviation from the expected return on the downside but also on the upside. Hence, alternative risk measures have been suggested to replace the variance such as Value at Risk (VaR) that manage and control risk in terms of percentiles of loss distribution.

Instead of regarding both the upside and downside of the expected return, VaR considers only the downside of the expected return as risk and represents the predicted maximum loss with a specified confidence level (e.g., 95%) over a certain period (e.g., one day). As we have already seen, however, VaR may have drawbacks and undesirable properties that limit its use, such as lack of subadditivity; that is, the VaR of two different investment portfolios may be greater than the sum of the individual VaRs. Also, VaR is nonconvex and nonsmooth and has multiple local minimums, while we seek the global minimum.

A new approach was introduced by R. Tyrrell Rockafellar and Stanislav Uryasev (1999) to optimizing or hedging a portfolio of financial instruments to reduce risk. It focuses on minimizing the Conditional Value-at-Risk (CVaR) rather than minimizing Value-at-Risk (VaR).

Consider assets $S_1, ..., S_n$, $n \geq 2$, with random returns. Suppose μ_i denotes the expected return of asset S_i, and also consider x_i as the proportion of holding in the ith asset. We can represent the expected return of the resulting portfolio x as follows:

$$E[\mathbf{x}] = \mu_1 x_1 + ... + \mu_n x_n = \mu^T \mathbf{x} \tag{24}$$

Also, we will assume that the set of a feasible portfolio is nonempty and represent that as $\Omega = \{\mathbf{x} | \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{C}\mathbf{x} \geq \mathbf{d}\}$ where $\mathbf{A}$ is a $m \times n$ matrix, $\mathbf{b}$ is a m-dimensional vector, $\mathbf{C}$ is a $p \times n$ matrix and $\mathbf{d}$ is a p-dimensional vector. In particular, one of the constraints in the set Ω is $\sum_{i=1}^{n} x_i = 1$. Let $f(\mathbf{x}, \mathbf{y})$ denote the loss function when we choose the portfolio x from a set of feasible portfolios, and $\mathbf{y}$ is the realization of the random events (the vector of the asset returns of n assets).

We define the portfolio return loss that is the negative of the portfolio return as:

$$f(\mathbf{x}, \mathbf{y}) = -\mathbf{y}^T \mathbf{x} = -[y_1 x_1 + ... + y_n x_n] \tag{25}$$

The cumulative distribution function of the loss associated with the random vector $\mathbf{y}$ with density function $p(\mathbf{y})$,i.e. the probability that the loss $f(\mathbf{x}, \mathbf{y})$ does not exceed some threshold value γ, is computed as follows:

$$\Psi(\mathbf{x}, \gamma) = \int_{f(\mathbf{x},\mathbf{y}) \leq \gamma} p(\mathbf{y}) d\mathbf{y} \tag{26}$$

Now, setting a confidence level α, we can define the α-VaR associated with portfolio x as:

$$\mathrm{VaR}_\alpha(x) = \min\{\gamma \in \mathbb{R} | \Psi(\mathbf{x}, \gamma) \geq \alpha\} \tag{27}$$

In the same way we can compute the αCVaR associated with portfolio x as:

$$\mathrm{CVaR}_\alpha(x) = \frac{1}{(1-\alpha)} \int_{f(\mathbf{x},\mathbf{y}) \geq \mathrm{VaR}_\alpha(x)} f(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) d\mathbf{y} \tag{28}$$

Since CVaR is defined in terms of the VaR function, it is difficult to handle CVaR and is difficult to manipulate and optimize it. The idea is to define a simpler function as an alternative:

$$F_\alpha(\mathbf{x}, \gamma) = \gamma + \frac{1}{(1-\alpha)} \int_{f(\mathbf{x},\mathbf{y}) \geq \gamma} (f(\mathbf{x}, \mathbf{y}) - \gamma) p(\mathbf{y}) d\mathbf{y} \tag{29}$$

The function possesses several important properties that make it useful for computing both VaR and CVaR, such as:

- F_α is a convex function of γ
- VaR_α is a minimizer over γ of F_α
- The minimum value over γ of the function F_α is CVaR_α

It can be seen that, as a consequence of the properties, to minimize $\text{CVaR}_\alpha(x)$ over $\mathbf{x}$, we need to minimize the function $F_\alpha(\mathbf{x}, \gamma)$ with respect both to $\mathbf{x}$ and γ. In that way we can optimize CVaR directly, overcoming the VaR computation

$$\min_{\mathbf{x}} \text{CVaR}_\alpha(x) = \min_{\mathbf{x},\gamma} F_\alpha(\mathbf{x}, \gamma) \tag{30}$$

3.2.1 Optimization problem with constraints on risk

Banks, investment firms, and other businesses tolerate varying levels of risk based on their objectives and available capital. Effectively representing and managing risk is crucial for business success.

A common strategy in risk management is to estimate and control Value at Risk (VaR) at specified confidence levels, such as 0.95, 0.99, or 0.999. VaR can be estimated over different periods depending on risk management goals short-term VaR is typically calculated for one day or two weeks, while long-term VaR might span one, two, or five years.

Controlling VaR can be formulated as a mathematical programming problem with VaR constraints. However, this approach is challenging due to VaR's non-convex nature relative to portfolio positions, resulting in numerous local minima.

In this section, we explore what Uryasev (2000) did, unlike VaR constraints, Conditional Value at Risk (CVaR) constraints can be more easily managed using formal optimization techniques. Furthermore, imposing CVaR constraints indirectly restricts VaR since CVaR provides an upper bound to VaR. Thus, VaR constraints can be effectively replaced by more conservative CVaR constraints [21].

In the same way as CVaR minimization, we include CVaR in constraints and replace it by the function $F_\alpha(\mathbf{x}, \gamma)$. Now, consider the problem of minimizing the mean loss $\mu(\mathbf{x}) = Ef(\mathbf{x}, \mathbf{y})$, subject to some balance constraints $\mathbf{x} \in \mathbf{X}$ and two CVaR constraints with confidence level α and λ. From now on, we define $\text{CVaR}_\alpha(x)$ as $\phi_\alpha(\mathbf{x})$. In this framework, the optimization problem can be stated as:

$$\begin{aligned} \min \quad & \mu(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{x} \in \mathbf{X}, \\ & \phi_\alpha(\mathbf{x}) \leq C_\alpha, \\ & \phi_\lambda(\mathbf{x}) \leq C_\lambda, \end{aligned}$$

where C_α and C_λ are some constants constraining CVaR at different confidence levels, that can be replaced by:

$$\begin{aligned} F_\alpha(\mathbf{x}, \gamma_1) &\leq C_\beta, \\ F_\lambda(\mathbf{x}, \gamma_2) &\leq C_\lambda \end{aligned}$$

So, if these constraints are satisfied for some γ_1 and γ_2, they are satisfied for the minimal values $\min_{\gamma_1} F_\alpha(\mathbf{x}, \gamma_1) = \phi_\beta(\mathbf{x})$ and $\min_{\gamma_2} F_\lambda(\mathbf{x}, \gamma_2) = \phi_\lambda(\mathbf{x})$.

Let $(\mathbf{x}^*, \gamma^*)$ be a solution, then $F_\alpha(\mathbf{x}^*, \gamma^*)$ equals an optimal CVaR, the optimal portfolio equals $\mathbf{x}^*$ and the corresponding VaR equals $\mathbf{gamma}^*$. Optimization with the above constraints assures that the CVaR values are properly restricted. Indeed, if a risk constraint is active, e.g. $F_\alpha(\mathbf{x}^*, \gamma_1 *) \leq C_\beta$, then the optimal value γ_1^* equals α-VaR.

3.2.2 Finite number of Scenario

In some cases, where the analytical representation of the density function $p(\mathbf{y})$ is not available or undesirable to compute, we can use several J scenarios in the names of $\mathbf{y}_i$ for $i = 1, ..., J$, sampled from the density $p(\mathbf{y})$. For instance, we may have historical observations of prices for instruments of the portfolio or we may use Monte Carlo simulations to price instruments. In this case, the 29 can be approximated as:

$$\overline{F}_\alpha(\mathbf{x}, \gamma) = \gamma + \frac{1}{(1-\alpha)J} \sum_{i=1}^{J} (f(\mathbf{x}, \mathbf{y}_i) - \gamma) \tag{31}$$

In the minimization problem 30, we now replace $F_\alpha(\mathbf{x}, \gamma)$ whit $\overline{F}_\alpha(\mathbf{x}, \gamma)$:

$$\min_{\mathbf{x},\gamma} \gamma + \frac{1}{(1-\alpha)J} \sum_{i=1}^{J} (f(\mathbf{x}, \mathbf{y}_i) - \gamma) \tag{32}$$

The next step is to introduce artificial variables v_i to replace $(f(\mathbf{x}, \mathbf{y}_i) - \gamma)$ to solve the optimization problem. We add constraints $v_i \geq 0$ and $f(\mathbf{x}, \mathbf{y}_i) - \gamma$ [21]:

$$\begin{aligned} \min_{\mathbf{x},v,\gamma} \quad & \gamma + \frac{1}{(1-\alpha)J} \sum_{i=1}^{J} v_i, \\ \text{s.t.} \quad & v_i \geq 0, \quad i = 1, \ldots, J, \\ & v_i \geq f(\mathbf{x}, \mathbf{y}_i) - \gamma, \quad i = 1, \ldots, J, \\ & \mathbf{x} \in \Omega. \end{aligned} \tag{33}$$

In the real world, risk managers try to optimize risk measures when while expected return is more than a threshold value. In this framework, we can define the mean-CVaR model as [19]:

$$\begin{aligned} \min_{\mathbf{x},v,\gamma} \quad & \gamma + \frac{1}{(1-\alpha)J} \sum_{i=1}^{J} v_i \\ \text{s.t.} \quad & \mu^T \mathbf{x} \geq R, \\ & v_i \geq 0, \quad i = 1, \ldots, J, \\ & v_i \geq f(\mathbf{x}, \mathbf{y}_i) - \gamma, \quad i = 1, \ldots, J, \\ & \mathbf{x} \in \Omega. \end{aligned} \tag{34}$$

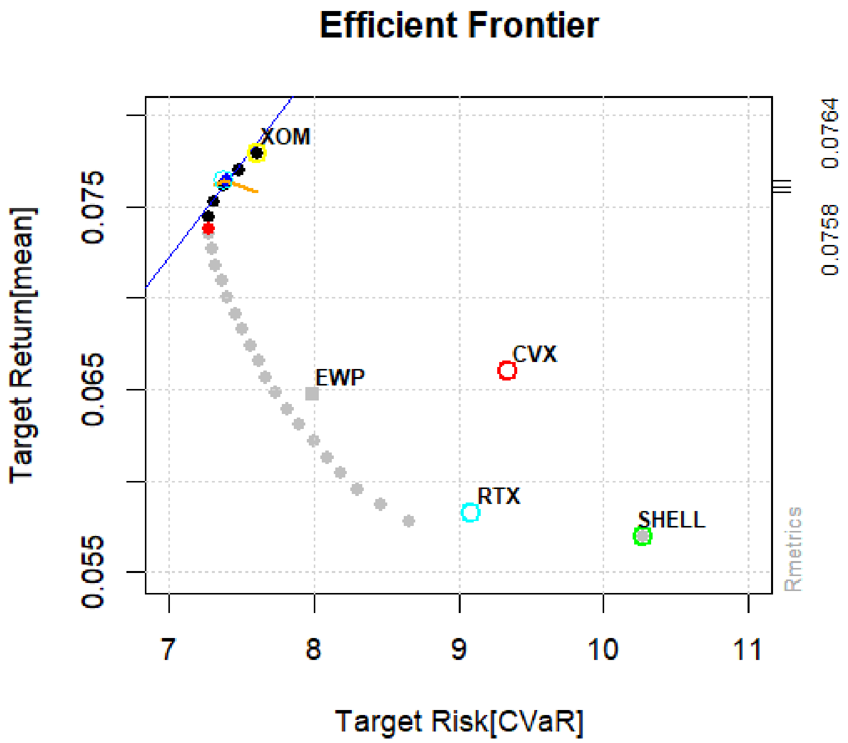


Figure 27: Mean-CVaR efficient frontier

or

$$
\begin{aligned}
\min_{\mathbf{x}, v, \gamma} \; & -\mu^T \mathbf{x} + \theta \left(\gamma + \frac{1}{(1-\alpha)J} \sum_{i=1}^{J} v_i \right) \\
\text{s.t.} \quad & v_i \geq 0, \quad i = 1, \ldots, J, \\
& v_i \geq f(\mathbf{x}, \mathbf{y}_i) - \gamma, \quad i = 1, \ldots, J, \\
& \theta \geq 0, \\
& \mathbf{x} \in \Omega.
\end{aligned}
\tag{35}
$$

where $\theta \geq 0$ is the risk aversion parameter that balances the expected return and $\mathrm{CVaR}_\alpha(\mathbf{x})$.

Example Going back to the previous example, we can do the same thing, as we did in the MV portfolio cases, for the Mean-CVaR ($\alpha = 0.01$) portfolio. As before, we consider the long-only constraint and in the case of the tangency portfolio, we take into account the zero risk-free rate. In figure (28) are represented the weights of the different M-CVaR portfolios while in figure (27) the efficient frontier.

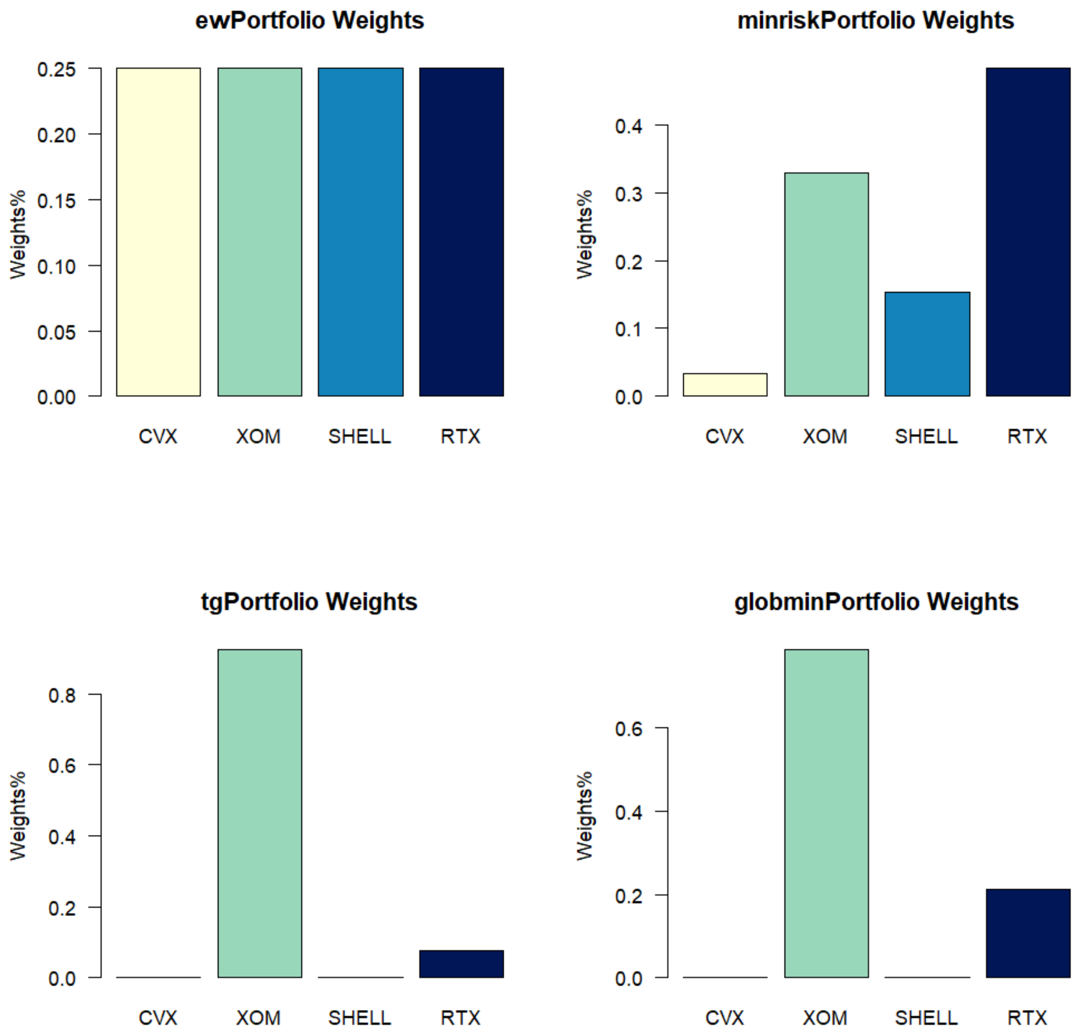


Figure 28: Weights for different Mean-CVaR($\alpha = 0.01$) portfolios

4 Empirical study

Financial institutions are increasingly launching new or repackaged products and services aimed at various sectors, including companies (such as new sustainability-backed investment funds and innovative mortgage options), investors (focused on green and sustainable assets), and policymakers (offering green bonus premiums and tax incentives). The rise of eco-friendly companies and green investments has sparked a higher demand for environmentally sustainable products and services. As a result, green investments are appealing to a broader audience beyond just those motivated by personal values. A growing consensus is forming around the idea that green financial products and services can provide favorable risk-return trade-offs.

In 2021, sustainable and responsible investments saw a 15% increase compared to 2019, reaching a total of 35.3 trillion US dollars. This amount represents 36% of all professionally managed assets globally, suggesting that investors are incorporating more green financial assets into their portfolios[3]. The shift towards green investment can provide an alternative source of investment, reduce environmental costs, and promote sustainability.

Investors are focusing on decarbonizing their portfolios and increasing their investments in green assets. This trend highlights emerging challenges in risk management and asset allocation. A new challenge in portfolio management is to view green and sustainable information as supplementary data that can offer insights into future performance.

A recent body of literature has explored the significance of green assets in portfolio diversification and risk management. Giglio, Kelly, and Stroebel (2021) conducted a review of existing studies regarding the effects of climate change on investor behavior. They emphasize that green investments can serve as an effective hedge against climate-related risks. However, while these investments demonstrate environmental resilience and align with sustainability objectives, they may also introduce unique downside risks (Viviani, Fall, and Revelli, 2019), which remains a significant concern for investors.

Green investments can help alleviate this risk by diversifying portfolios and minimizing exposure to downside risk. Recent empirical studies indicate that green investments generally have a lower correlation with traditional asset classes, particularly stocks and bonds.

Hoepner, Oikonomou, Sautner, Starks, and Zhou (2023) emphasize that ESG practices diminish firms' downside risk, as indicated by value at risk (VaR). Similarly, Zhang, Zhang, and Zong (2023) identify a positive correlation between ESG performance and downside risk, noting that this relationship became less pronounced during the COVID-19 crisis.

Multiple studies have examined the portfolio performance of conventional bonds in comparison to green bonds (Fatica and Panzica, 2021). Although the findings are varied, the majority of the studies indicate that green bond yields are generally lower than those of conventional bonds, suggesting a negative premium associated with green investments.

While these studies offer valuable insights for investors and hedgers in analyzing spillover effects and diversification potential of various asset types, the risk management and portfolio diversification associated with green and sustainable assets remain underexplored.

During this chapter, we try to analyze the potential diversification advantages and safe-haven characteristics of sustainable investments, green bonds, and cleaner energy assets in comparison to traditional stock markets. Furthermore, using a portfolio optimization framework, we assess these assets based on their effectiveness in minimizing a portfolio's extreme risk.

To assess downside risk, we will use the CVaR approach instead of the VaR criterion. This method enables us to evaluate the downside tail risk of a portfolio by calculating the weighted average of extreme losses that occur in the tail of the return distribution, beyond the VaR threshold.

In addition, we will consider an empirical framework for portfolio optimization utilizing the CVaR approach to address the shortcomings of optimization strategies that rely on VaR, particularly issues related to subadditivity and convexity. Our optimization model enables the identification of a portfolio composition focused on green and sustainable assets, which effectively mitigates downside risk while enhancing overall performance.

4.1 Data and preliminary analysis

The idea of the analysis is to fit in and try to fill the space of the literature about risk management and portfolio diversification through the integration of green finance. The approach is based on the analysis made by Ameur, Hachmi Ben and Ftiti, Zied and Louhichi, Waël and Yousfi, Mohamed (2024) which used conventional indices, emerging markets indices and three types of green assets and based on the work made by Ali, Fahad and Khurram, Muhammad Usman and Sensoy, Ahmet and Vo, Xuan Vinh (2024) which compared green and non-green indexes instead.

Table 2: Add caption

Data

Index	**Category**	**Type**	**Ticker**
Standard & Poor 500 (S&P500)	Index		SPX
Deutsche Aktienindex 30 (DAX30)	Index		DAX
CAC 40	Index		FCHI
FTSE100	Index		FTSE100
FTSEMIB	Index		FTSEMIB
Dow Jones U.S. Oil & Gas Index	Equity	Non-Green	Oil._e_gas
S&P 500 Energy	Equity	Non-Green	SP_Energy
SPDR MSCI Europe Energy ETF (STNX.MI)	Equity	Non-Green	Eur_Energy
iShares Core Total USD Bond Market ETF (IUSB)	Bond	Non-Green	Bond_mkt
Bitcoin	Crypto	Non-Green	Bitcoin
Europe_350_carbon_efficient_Index	Equity	Green	g_Carbon
Renewable energy and clean technology Index (TXCT)	Equity	Green	Ren_Energy
S&P 500 fossil fuel free (ŜP5F3UP)	Equity	Green	Fossil_Free
Green Bond Index	Bond	Green	Green_Bond
Stellar	Crypto	Green	Stellar

We selected five conventional indices of developed countries (S&P500, DAX, CAC40, FT-

Table 3: Descriptive statistics of indexes

Descriptive statistics

Index	Median	Mean	SD	Min	Max
FCHI	0,00088	0,00026	0,01235	-0,13098	0,08056
DAX	0,00081	0,00021	0,01267	-0,13055	0,10414
SPX	0,00084	0,00043	0,01284	-0,12765	0,08968
FTSE100	0,00066	0,00005	0,01062	-0,11512	0,08667
FTSEMIB	0,00122	0,00027	0,01390	-0,18541	0,08549
Bitcoin	0,00138	0,00146	0,04580	-0,46473	0,22512
Oil._e_Gas	0,00065	0,00021	0,02152	-0,23179	0,14871
Bond_mkt	0,00019	0,00003	0,00356	-0,03808	0,02190
SP_Energy	0,00077	0,00023	0,02157	-0,22417	0,15111
Eur_Energy	0,00054	0,00034	0,01854	-0,19010	0,14907
g_Carbon	0,00070	0,00011	0,01222	-0,14205	0,08705
Green_Bond	0,00000	-0,00005	0,00409	-0,02410	0,02272
Ren_Energy	-0,00008	-0,00018	0,01480	-0,13290	0,09990
Fossil_Free	0,00074	0,00044	0,01288	-0,12737	0,08916
Stellar	-0,00077	0,00088	0,06993	-0,43581	0,55593

SEMIB, FTSE100). Then, we consider five indices: the Dow Jones U.S. Oil & Gas Index replicates the performance of U.S. companies in the oil and gas sector, the S&P 500 Energy comprises those companies included in the S&P 500 that are classified as members of the GICS[8] energy sector, the SPDR MSCI Europe Energy ETF (STNX.MI) replicates the European energy sector, the iShares Core Total USD Bond Market ETF (IUSB) seeks to track the investment results of an index composed of U.S. dollar-denominated bonds that are rated either investment grade or high-yield and Bitcoin.

As green indices we consider: the Europe 350 Carbon Efficient Index designed to measure the performance of companies in the S&P Europe 350, while overweighting those companies that have lower levels of carbon emissions per unit of revenue, the Renewable Energy and Clean Technology Index (TXCT) measures the performance of companies listed on TSX whose core business is the development of green technologies and sustainable infrastructure solutions, the S&P 500 fossil fuel free (SP5F3UP) is designed to measure the performance of companies in the S&P 500 that do not own fossil fuel reserves[9], the Green Bond Index measures the market value-weighted performance of globally issued, green-labeled bonds and cryptocurrency Stellar. All indices are resumed in the table (2) and a graphical representation of Indices' prices is provided in figure (29). The daily data for all price indices ranged from November 2017 to April 2024. For each Index the logarithmic returns have been computed, shown in figure (30), and the summary statistics of daily returns for conventional indices, markets, and green investments are shown in table (3).

[8]The Global Industry Classification Standard (GICS) is a method for assigning companies to a specific economic sector and industry group that best defines its business operations

[9]Fossil fuel reserves are defined as economically and technically recoverable sources of crude oil, natural gas, and thermal coal

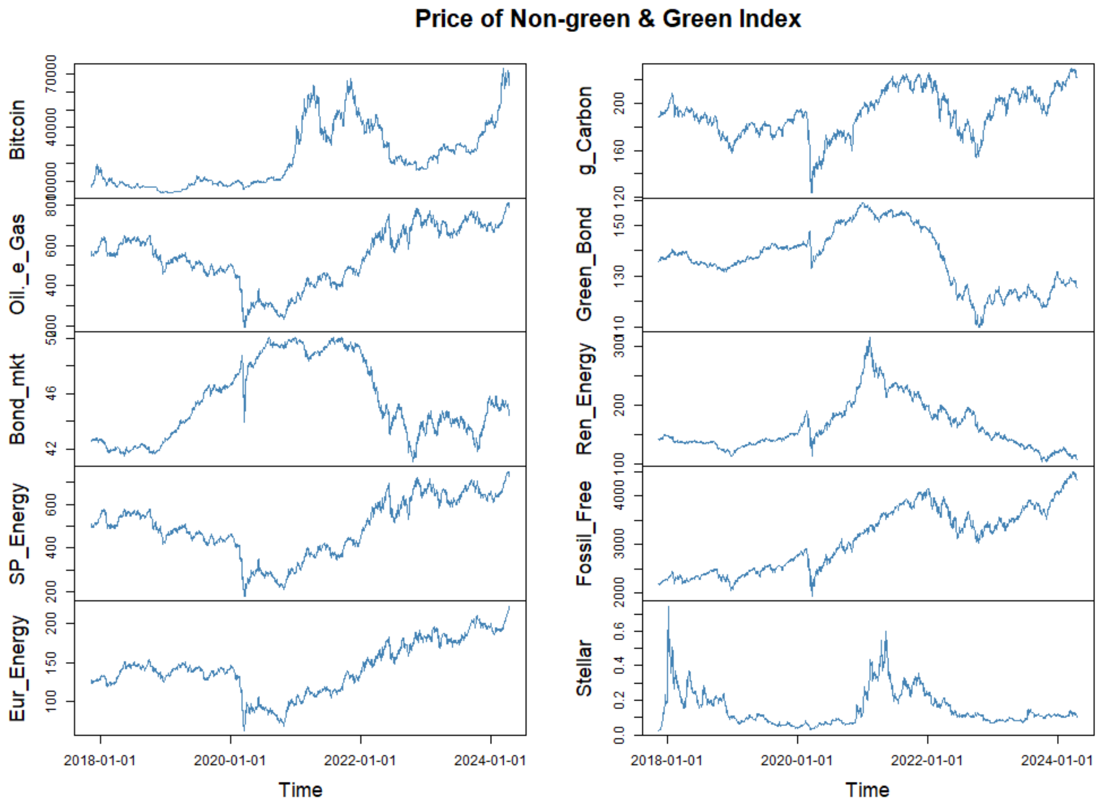


Figure 29

It can be seen that, based on the descriptive statistics, all the indices are pretty similar with means near zero. Seeing the SD column, as we can expect, Bitcoin and Stellar have the highest values being more volatile. Indeed, the maximum and the minimum daily returns of the two cryptocurrencies are the highest. Looking at the returns plot, for all the indices there are mainly two periods of high volatility caused by the COVID-19 pandemic and the Ukraine war.

Table 4: Correlation matrix

Correlation matrix

	Bitcoin	Oil_e_Gas	g_Carbon	Green_Bond	Bond_mkt	Ren_Energy	SP_Energy	Fossil_Free	Eur_Energy	Stellar	FCHI	DAX	SPX	FTSE100	FTSEMIB
Bitcoin	1,000														
Oil_e_Gas	0,178	1,000													
g_Carbon	0,236	0,501	1,000												
Green_Bond	0,130	0,057	0,468	1,000											
Bond_mkt	0,134	0,033	0,199	0,616	1,000										
Ren_Energy	0,264	0,477	0,508	0,202	0,231	1,000									
SP_Energy	0,175	0,999	0,498	0,053	0,027	0,466	1,000								
Fossil_Free	0,276	0,603	0,573	0,190	0,192	0,646	0,594	1,000							
Eur_Energy	0,145	0,717	0,619	0,020	-0,011	0,335	0,720	0,371	1,000						
Stellar	0,569	0,156	0,167	0,052	0,059	0,179	0,154	0,213	0,141	1,000					
FCHI	0,216	0,531	0,908	0,207	0,119	0,499	0,528	0,580	0,666	0,158	1,000				
DAX	0,213	0,491	0,900	0,220	0,127	0,488	0,487	0,586	0,613	0,165	0,937	1,000			
SPX	0,278	0,630	0,579	0,187	0,188	0,651	0,622	0,999	0,392	0,215	0,587	0,590	1,000		
FTSE100	0,187	0,562	0,846	0,129	0,072	0,483	0,561	0,524	0,742	0,170	0,866	0,831	0,534	1,000	
FTSEMIB	0,234	0,508	0,851	0,189	0,113	0,475	0,505	0,561	0,639	0,161	0,884	0,879	0,567	0,796	1,000

The correlation matrix is provided in table (4). All the pairs' correlations are positive except for the correlation between the Bond Market and Eur Energy. It exhibits some highly paired correlation, between the conventional indices and g_Carbon. Although it seems there isn't a general rule of correlation within the conventional, non-green, and green indices, Bitcoin, Stellar, and the two Bond indices are going to play a key role in portfolio optimization since they are very less correlated with other indices.

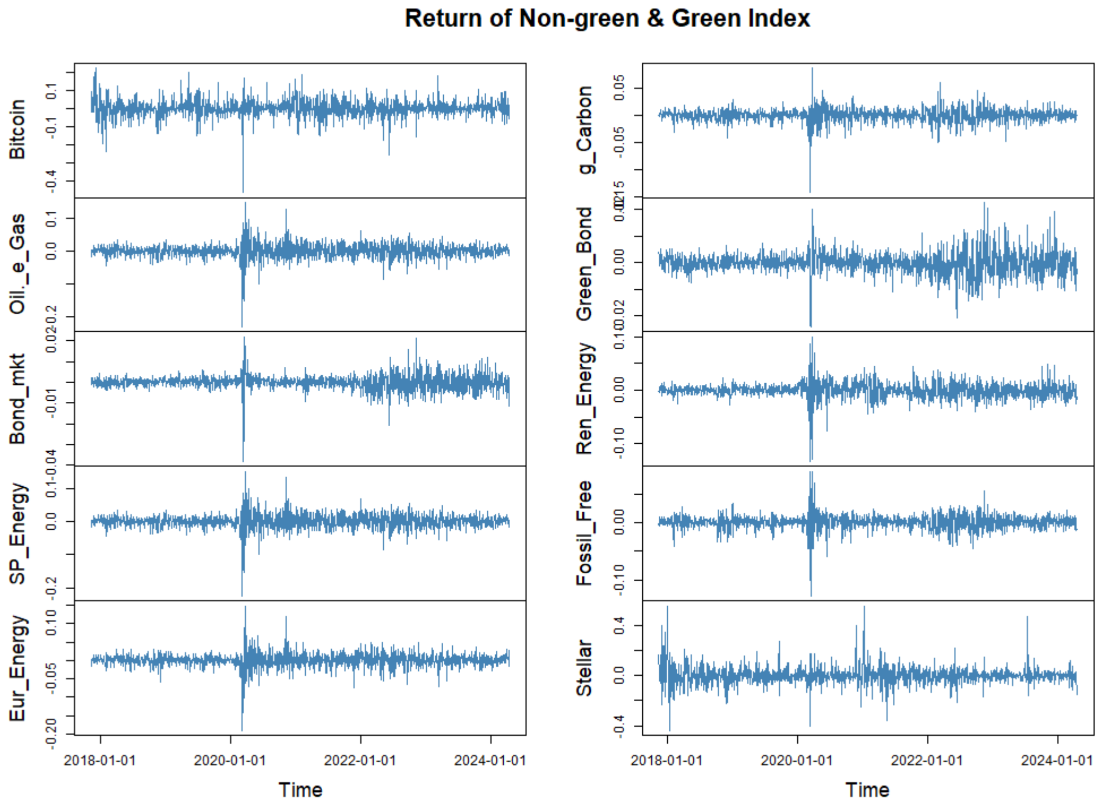


Figure 30

4.2 Full sample

In this section, we are going to explore the results using the full sample. First, the VaR and CVaR of the conventional indices, green and non-green indices (un-mixed portfolio, i.e. 100 % allocation to the index) for different confidence level α are estimated. Then, the VaR and CVaR of the mixed portfolio, allocating 80% to the conventional index and 20% to the green/non-green index, are computed and compared with the un-mixed portfolio. In table 5 we can see the results. For each confidence level of VaR and CVaR, the un-mixed portfolio is classified based on a scale of color from blue, the lowest, to red the highest. It can be seen that, within conventional indexes, FTSEMIB is the one that has the highest downfall risk and expected shortfall for each confidence level. Regarding green and non-green un-mixed portfolios, we can notice that in general the un-mixed green portfolios have lower VaR and CVaR, except for Stellar cryptocurrency which has the highest values. Furthermore, the colorful mixed portfolios are the ones that have higher Value at Risk or CVaR than the un-mixed portfolio of conventional indexes. For instance, the mixed portfolios between FTSEE100 and G_Carbon have a CVaR(1%) higher than the un-mixed portfolio FTSE100 CVaR(1%). It can be seen that adding non-green indices to the unmixed conventional indices, in most cases, increases the downside risk and expected shortfall. On the contrary, adding green indices decreases the risk, except for Stellar and other isolated cases.

Table 5: VaR & CVaR for unmixed and mixed portfolios

VaR and CVaR, full sample										
	FCHI		**DAX**		**SPX**		**FTSE100**		**FTSEMIB**	
Unmixed portfolios										
VaR(1%)	0,0387		0,0392		0,0361		0,0342		0,0411	
VaR(5%)	0,0179		0,0188		0,0191		0,0156		0,0199	
CVaR(1%)	0,0539		0,0519		0,0552		0,0471		0,0619	
CVaR(5%)	0,0306		0,0313		0,0317		0,0272		0,0344	
	G carbon	**Fossil free**	**Ren energy**	**Green bond**	**Stellar**	**Energy**	**EU Energy**	**Oil & Gas**	**Bond mkt**	**Bitcoin**
Unmixed portfolios										
VaR(1%)	0,0375	0,0366	0,0355	0,0118	0,1782	0,0559	0,0540	0,0545	0,0091	0,1365
VaR(5%)	0,0180	0,0190	0,0217	0,0068	0,0971	0,0317	0,0269	0,0315	0,0052	0,0685
CVaR(1%)	0,0509	0,0550	0,0639	0,0150	0,2469	0,0907	0,0786	0,0909	0,0146	0,1816
CVaR(5%)	0,0301	0,0318	0,0350	0,0099	0,1518	0,0498	0,0446	0,0497	0,0085	0,1114
Mixed portfolios: FCHI (80%)										
VaR(1%)	0,0387	0,0344	0,0358	0,0309	0,0658	0,0379	0,0387	0,0378	0,0298	0,0496
VaR(5%)	0,0178	0,0166	0,0168	0,0146	0,0313	0,0176	0,0184	0,0175	0,0143	0,0241
CVaR(1%)	0,0517	0,0517	0,0534	0,0518	0,1021	0,0568	0,0574	0,0569	0,0414	0,0802
CVaR(5%)	0,0299	0,0281	0,0281	0,0247	0,0466	0,0317	0,0317	0,0318	0,0246	0,0376
Mixed portfolios: DAX (80%)										
VaR(1%)	0,0381	0,0341	0,0354	0,0300	0,0667	0,0376	0,0405	0,0377	0,0291	0,0482
VaR(5%)	0,0186	0,0178	0,0173	0,0153	0,0318	0,0185	0,0183	0,0185	0,0149	0,0256
CVaR(1%)	0,0514	0,0482	0,0504	0,0406	0,1011	0,0552	0,0555	0,0563	0,0397	0,0801
CVaR(5%)	0,0301	0,0291	0,0296	0,0249	0,0467	0,0323	0,0316	0,0320	0,0248	0,0379
Mixed portfolios: SPX (80%)										
VaR(1%)	0,0341	0,0366	0,0347	0,0306	0,0641	0,0367	0,0346	0,0367	0,0320	0,0441
VaR(5%)	0,0179	0,0191	0,0186	0,0162	0,0299	0,0194	0,0180	0,0193	0,0159	0,0243
CVaR(1%)	0,0527	0,0552	0,0577	0,0475	0,0952	0,0582	0,0572	0,0562	0,0455	0,0719
CVaR(5%)	0,0300	0,0317	0,0306	0,0267	0,0470	0,0321	0,0321	0,0320	0,0261	0,0382
Mixed portfolios: FTSE100 (80%)										
VaR(1%)	0,0342	0,0302	0,0320	0,0259	0,0679	0,0353	0,0352	0,0353	0,0254	0,0480
VaR(5%)	0,0154	0,0145	0,0142	0,0125	0,0296	0,0169	0,0164	0,0169	0,0123	0,0241
CVaR(1%)	0,0473	0,0434	0,0485	0,0363	0,1044	0,0517	0,0512	0,0518	0,0351	0,0773
CVaR(5%)	0,0272	0,0255	0,0255	0,0214	0,0449	0,0289	0,0293	0,0289	0,0208	0,0389
Mixed portfolio: FTSEMIB (80%)										
VaR(1%)	0,0386	0,0336	0,0344	0,0318	0,0689	0,0380	0,0402	0,0389	0,0311	0,0489
VaR(5%)	0,0194	0,0186	0,0183	0,0161	0,0329	0,0198	0,0198	0,0200	0,0157	0,0263
CVaR(1%)	0,0588	0,0567	0,0554	0,0502	0,1124	0,0581	0,0635	0,0581	0,0487	0,0849
CVaR(5%)	0,0329	0,0309	0,0312	0,0279	0,0515	0,0332	0,0342	0,0334	0,0269	0,0413

4.2.1 Relative VaR & CVaR

To better understand the effects of adding green and nongreen indices to conventional indices, it is interesting to use the relative VaR and CVaR following the method of Ali et al. (2024). Relative portfolio risk is calculated as a function of the VaR or CVaR of a mixed portfolio scaled by the VaR or CVaR of the corresponding unmixed conventional index-only portfolio. This measure will help investors understand whether diversifying their equity investments with green or non-green indices lowers the downside risk and expected shortfall they were exposed to by holding conventional index-only portfolios. Following that, relative value at risk (RVaR) and relative conditional value at risk (RCVaR) can be written as follows:

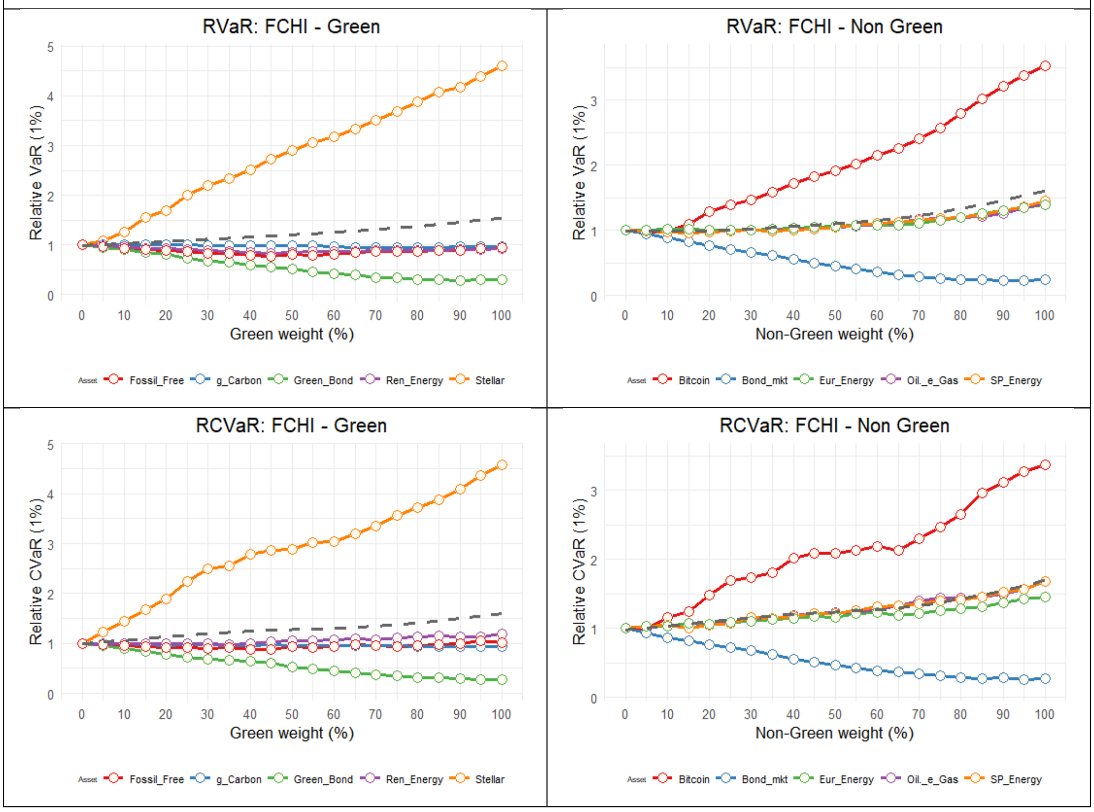
$$\mathrm{RVaR}_p = \frac{\mathrm{VaR}_{I-G/N}}{\mathrm{VaR}_{I-O}} \tag{36}$$

$$\mathrm{RCVaR}_p = \frac{\mathrm{CVaR}_{I-G/N}}{\mathrm{CVaR}_{I-O}} \tag{37}$$

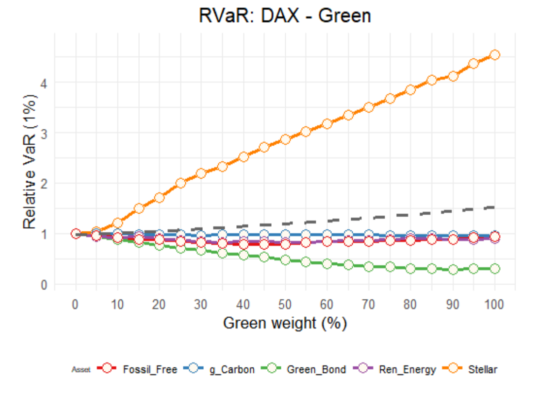
where $\mathrm{VaR}_{I-G/N}$ ($\mathrm{CVaR}_{I-G/N}$) indicates the downside risk (expected short-fall) of an index-green/nongreen mixed portfolio and VaR_{I-O} (CVaR_{I-O}) indicates the downside risk (expected shortfall) of the conventional index unmixed portfolio. Note that if a value of RVaR or RCVaR is lower than 1.00 for a mixed portfolio containing any two assets (e.g., Stellar and SPX) for a specific proportion of wealth allocated to them (e.g., 5% to Stellar and 95% to SPX), then pairing these two assets with the given proportion, provides diversification benefits in terms of reducing the downside risk or expected shortfall. In this framework, a wide range of proportions among all possible pairs is tested [0.05, 0.10, 0.15, . . . , 1.00]. The results are graphically provided below.

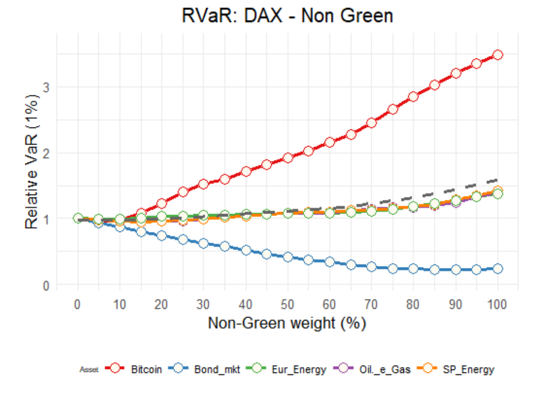
Table 6: Relative CVaR and VaR at 1% confidence interval

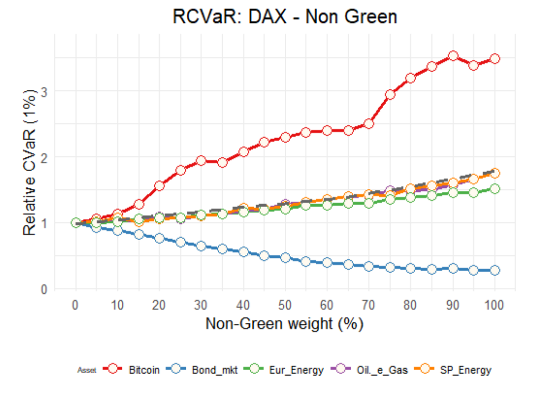
Panel A: Relative VaR and CVaR for FCHI

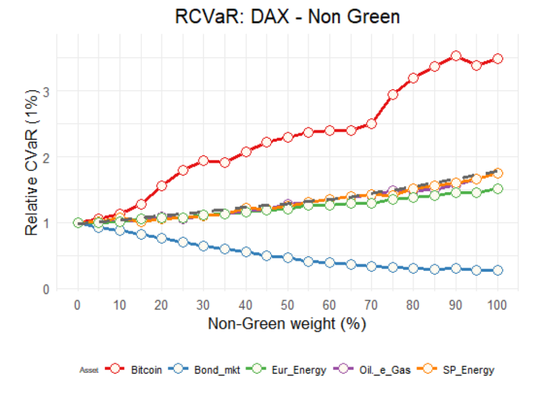


Panel B: Relative VaR and CVaR for DAX

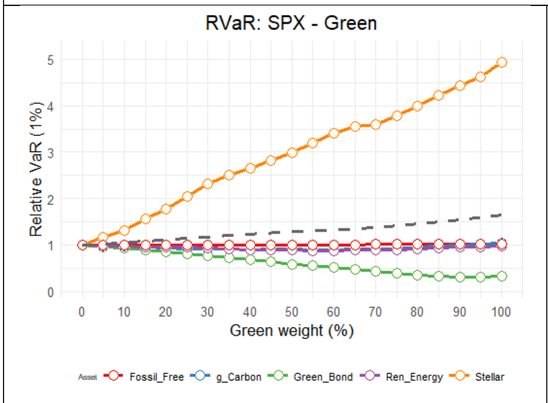


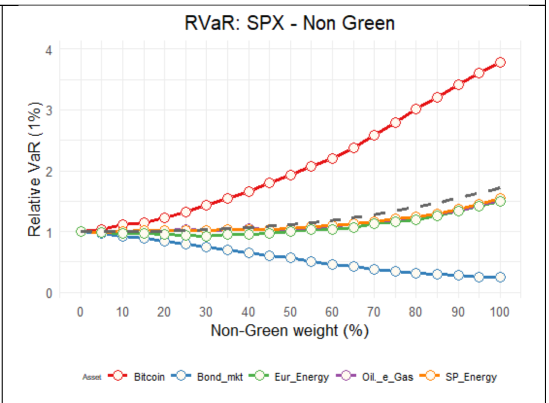


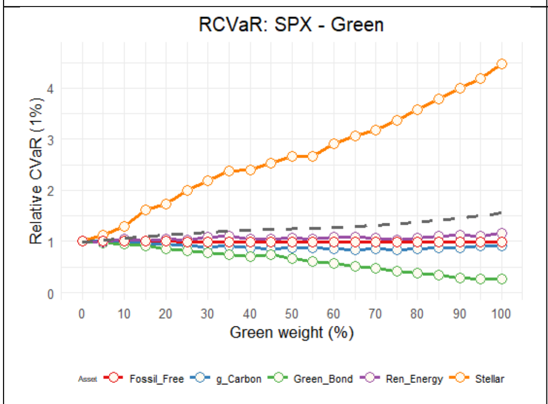


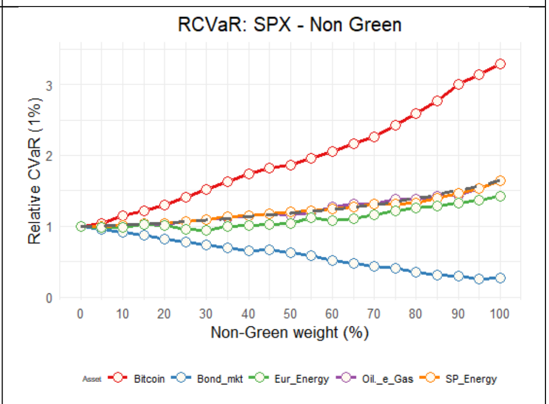


Panel C: Relative VaR and CVaR for SPX

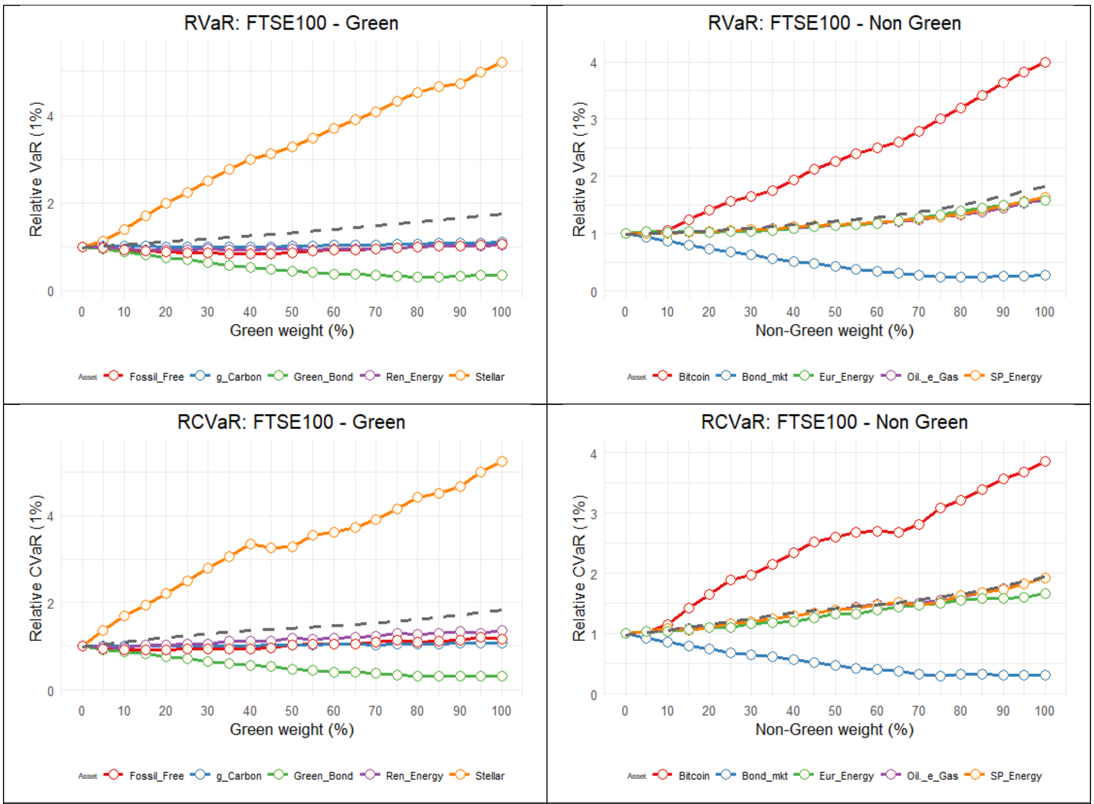




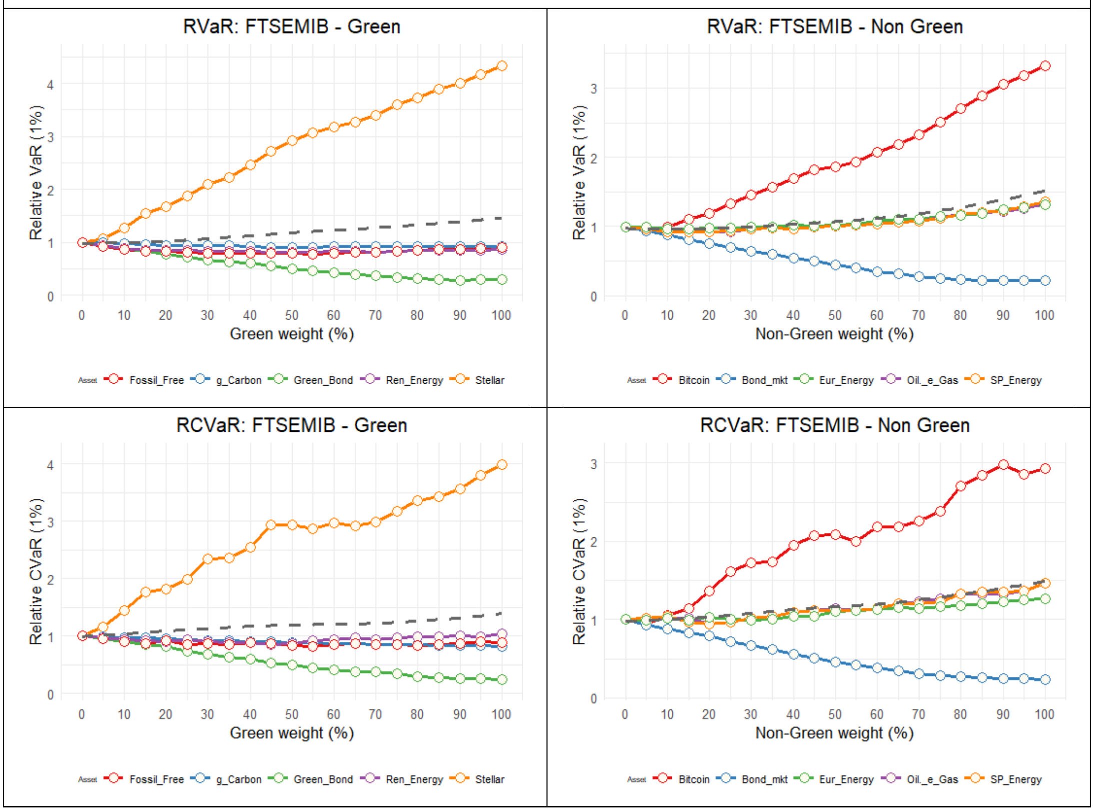




Panel D: Relative VaR and CVaR for FTSE100



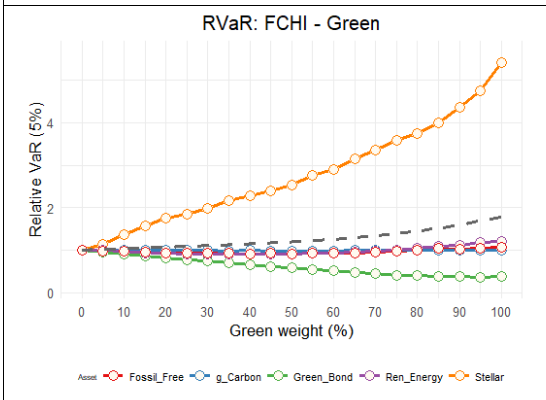
Panel E: Relative VaR and CVaR for FTSEMIB

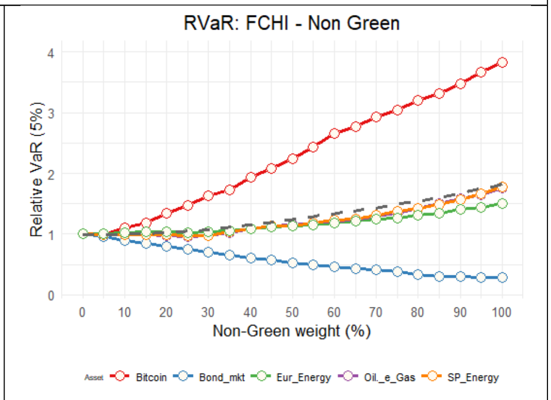


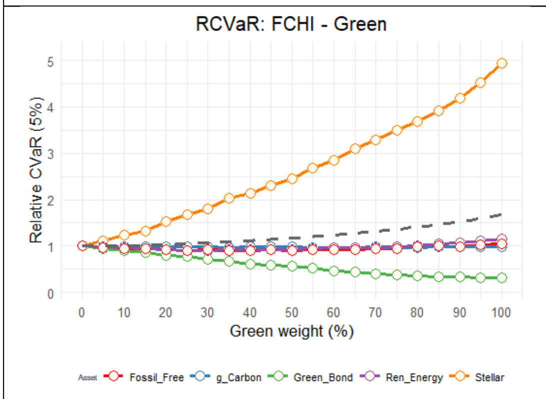
These graphs show how the relative VaR and CVaR change as the weight of green/non-green indices in the portfolio increases. For instance, see Panel A. Both RVaR plots indicate an increase in relative risk (VaR) with the increasing weight of indices, but non-green indexes tend to have higher risk compared to green indexes. Indeed, looking at the relative CVaR graphs, it can be seen that, similar to VaR, CVaR increases with the rising weight of indexes, with non-green indexes presenting higher tail risk than green indexes. Their results can help investors better understand the different impacts in terms of risk of including green and non-green indexes. In general, across all indices examined, non-green indexes tend to increase the relative risk (both VaR and CVaR) more than green ones. This suggests that from a risk perspective, green indexes might be a better choice for diversifying an equity portfolio. The table below illustrates VaR and CVaR at the 5% confidence interval for robustness and shows that the results are largely similar to those of the 1% confidence interval. The results show that the addition of green indices up to 30/40% of the portfolio weight decreases VaR and CVaR, except for Stellar. Furthermore, by increasing the weight VaR and CVaR remain substantially unchanged making the curves flat. To better understand the effect of the indices in "one shot", it is useful to look at the black dotted trend line. It can be seen how compared to those of the non-green indices, they are flatter and increase less as the weights increase.

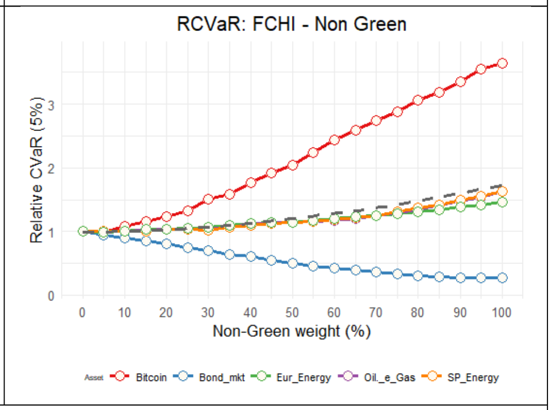
Table 7: Relative CVaR and VaR at 5% confidence interval

Panel A: Relative VaR and CVaR for FCHI

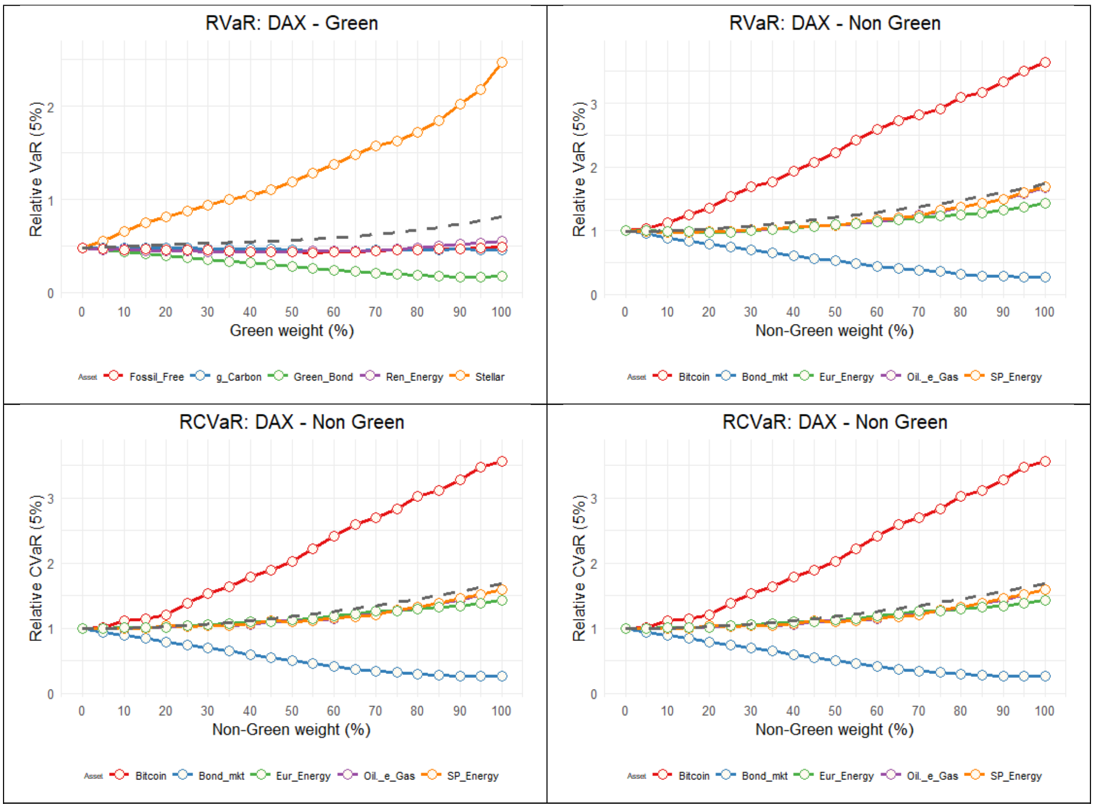




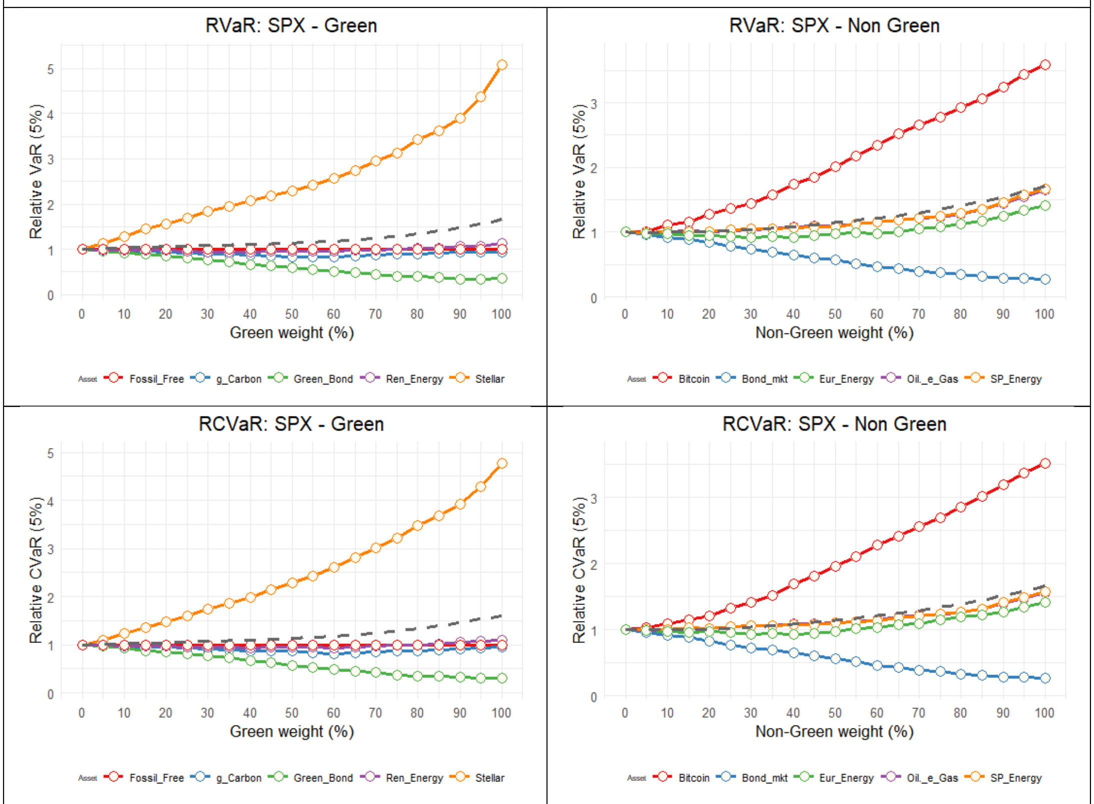




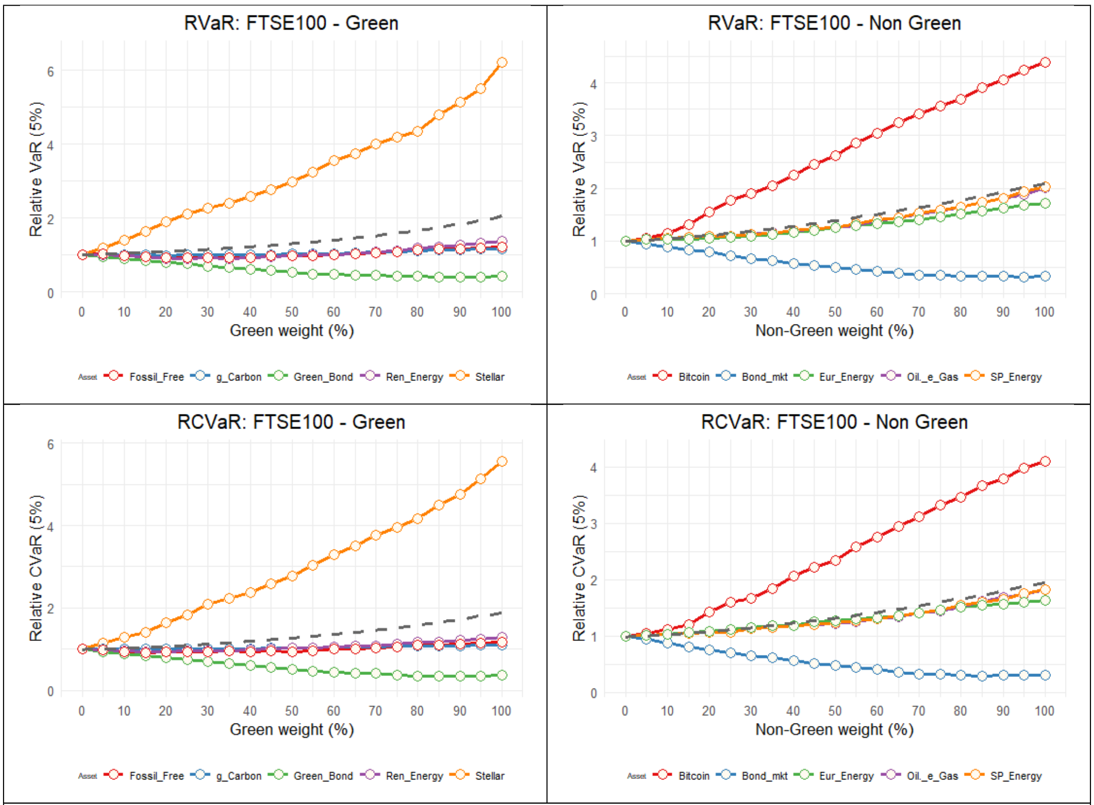
Panel B: Relative VaR and CVaR for DAX



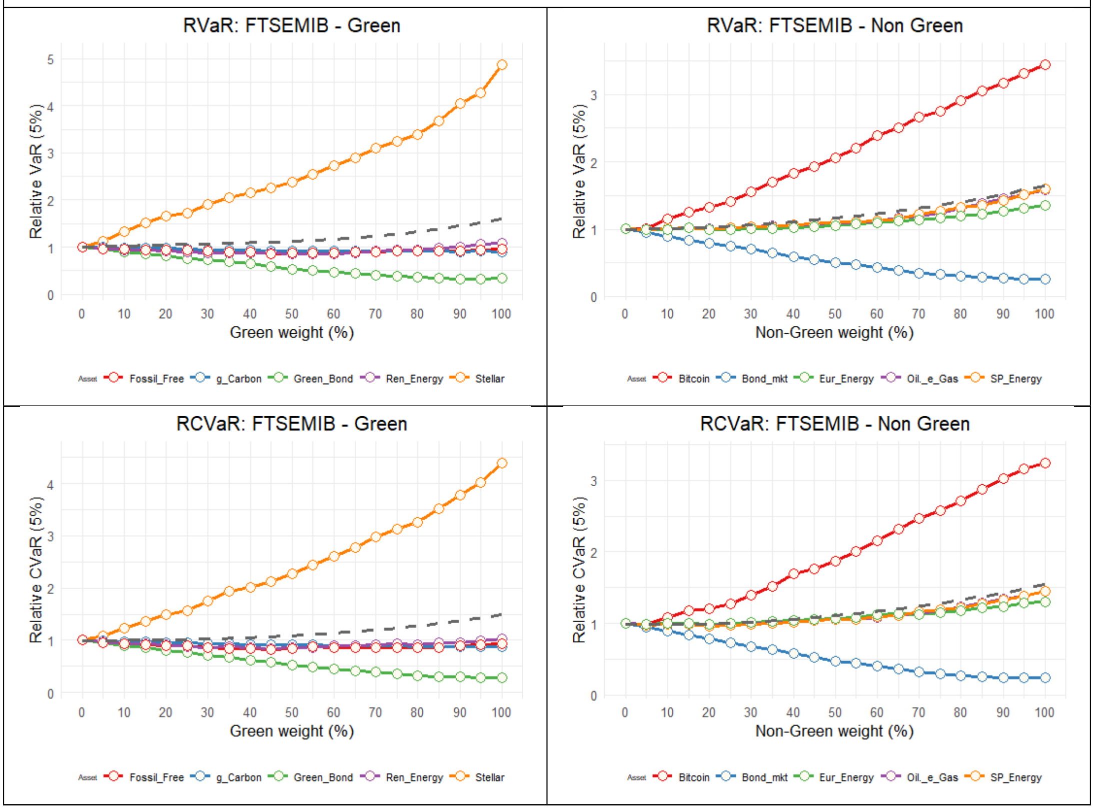
Panel C: Relative VaR and CVaR for SPX



Panel D: Relative VaR and CVaR for FTSE100



Panel E: Relative VaR and CVaR for FTSEMIB



4.2.2 Portfolio optimization

This work utilizes portfolio optimization strategies, as explained in Chapter 3, to examine the economic and practical implications for investors and policymakers interested in green assets. Each of the scenarios, with and without short sales restrictions, applies various weight constraints. In this section, we are going to analyze three different mean-variance and three mean-CVaR strategies. Table 8 provided the results of the minimum risk efficient portfolio optimization strategy. Twenty

Table 8: Minimum risk efficient portfolio

Minimum risk efficient portfolio, full sample

	FCHI	DAX	SPX	FTSE100	FTSEMIB	G carbon	Fossil free	Ren energy	Green bond	Stellar	Energy	EU Energy	Oil & Gas	Bond mkt	Bitcoin
Panel A: Minimum risk efficient portfolio (unrestricted, but positive) between selected Index and green Index (target return: equal weights)															
Weights %	10,58	X	X	X	X	0	50,61	0	37,53	1,28	X	X	X	X	X
Weights %	X	0	X	X	X	0	57,21	0	41,42	1,37	X	X	X	X	X
Weights %	X	X	0	X	X	0	57,21	0	41,42	1,37	X	X	X	X	X
Weights %	X	X	X	0	X	0	57,21	0	41,42	1,37	X	X	X	X	X
Weights %	X	X	X	X	4,72	0	54,25	0	39,71	1,32	X	X	X	X	X
Panel B: Minimum risk efficient portfolio (unrestricted, but positive) between selected Index and non-green Index (target return: equal weights)															
Weights %	14,47	X	X	X	X	X	X	X	X	X	0	9,57	0	65,44	10,52
Weights %	X	2,66	X	X	X	X	X	X	X	X	0	15,3	0	70,75	11,29
Weights %	X	X	25,7	X	X	X	X	X	X	X	0	6,16	0	61,75	6,39
Weights %	X	X	X	0	X	X	X	X	X	X	0	16,41	0	72,21	11,38
Weights %	X	X	X	X	6,71	X	X	X	X	X	0	13,13	0	69,17	10,99
Panel C: Minimum risk efficient portfolio (restricted: 50%-25%, but positive) between selected Index and green Index (target return: equal weights)															
Weights %	47,78	X	X	X	X	0	25	0	25	2,22	X	X	X	X	X
Weights %	X	47,8	X	X	X	0	25	0	22,07	5,12	X	X	X	X	X
Weights %	X	X	28,1	X	X	21,79	25	0	25	0,11	X	X	X	X	X
Weights %	X	X	X	20,05	X	25	25	0	17,94	12,02	X	X	X	X	X
Weights %	X	X	X	X	48,02	0	25	0	25	1,98	X	X	X	X	X
Panel D: Minimum risk efficient portfolio (restricted: 50%-25%, but positive) between selected Index and non-green Index (target return: equal weights)															
Weights %	50	X	X	X	X	X	X	X	X	X	1,47	15,33	5,74	25	2,46
Weights %	X	50	X	X	X	X	X	X	X	X	3,91	16,52	0	25	4,57
Weights %	X	X	25,6	X	X	X	X	X	X	X	9,86	14,54	25	25	0
Weights %	X	X	X	50	X	X	X	X	X	X	0	13,64	0	25	11,36
Weights %	X	X	X	X	50	X	X	X	X	X	7,51	15,23	0	25	2,26

different portfolios are computed between the selected indexes (the ones with X are not selected), each of which has its target return the one of the equally-weighted conventional-only indexes portfolio. Portfolios in panels C and D, included a constraint whereby the conventional selected index could weigh a maximum of 50%, while each of the other indexes could weigh a maximum of 25%. The unrestricted portfolios (Panels A and B) tend to allocate heavily towards specific indices, particularly traditional and green bonds, indicating their lower risk and higher efficiency in these scenarios. The restricted portfolios (Panels C and D) demonstrate a more balanced allocation due to the constraints, which promote diversification but still show significant allocations to key indices like Green_Bond and the Bond market. However, based on these results, we can't yet say that green indices have positive effects, as there are no substantial differences between portfolios with green indices and those with non-green indices.

Table 9 provides results for optimal-variance optimization strategy (tangent portfolio). As before we take into account the conventional indexes one by one with green and non-green indexes. It can be seen that in panel A, the unrestricted portfolios are strongly composed of Fossil Free. Panel B shows portfolios with non-green indexes. We can see how the weights are now more balanced as the strategy selected three non-green indexes out of five (EU Energy, Bond mkt, and Bitcoin). An interesting aspect is that Bitcoin is heavily selected in both the strategies faced so far, which could have been unexpected given the nature of the asset.

Table 9: Optimal-variance optimization (tangent portfolio)

Optimal-variance optimization (tangent portfolio), full sample

	FCHI	**DAX**	**SPX**	**FTSE100**	**FTSEMIB**	**G carbon**	**Fossil free**	**Ren energy**	**Green bond**	**Stellar**	**Energy**	**EU Energy**	**Oil & Gas**	**Bond mkt**	**Bitcoin**
Panel A: Optimal variance optimization (unrestricted, but positive) between selected Index and green Index															
Weights %	6,07	X	X	X	X	0	90,99	0	0	2,94	X	X	X	X	X
Weights %	X	0	X	X	X	0	96,94	0	0	3,06	X	X	X	X	X
Weights %	X	X	0	X	X	0	96,94	0	0	3,06	X	X	X	X	X
Weights %	X	X	X	0	X	0	96,94	0	0	3,06	X	X	X	X	X
Weights %	X	X	X	X	0	0	96,94	0	0	3,06	X	X	X	X	X
Panel B: Optimal variance optimization (unrestricted, but positive) between selected Index and non-green Index															
Weights %	28,03	X	X	X	X	X	X	X	X	X	0	16,89	0	32,09	22,98
Weights %	X	3,65	X	X	X	X	X	X	X	X	0	28,68	0	43,06	24,6
Weights %	X	X	70,25	X	X	X	X	X	X	X	0	10,55	0	0,02	19,17
Weights %	X	X	X	0	X	X	X	X	X	X	0	30,33	0	44,83	24,84
Weights %	X	X	X	X	12,99	X	X	X	X	X	0	23,91	0	39,08	24,01
Panel C: Optimal variance optimization (restricted: 50%-25%, but positive) between selected Index and green Index															
Weights %	50	X	X	X	X	0	25	0	20,41	4,59	X	X	X	X	X
Weights %	X	50	X	X	X	0	25	0	19,08	5,92	X	X	X	X	X
Weights %	X	X	50	X	X	0	25	0	21,55	3,45	X	X	X	X	X
Weights %	X	X	X	16,76	X	25	25	0	25	8,24	X	X	X	X	X
Weights %	X	X	X	X	50	0	25	0	19,51	5,49	X	X	X	X	X
Panel D: Optimal variance optimization (restricted: 50%-25%, but positive) between selected Index and non green Index															
Weights %	31,92	X	X	X	X	X	X	X	X	X	0	18,08	0	25	25
Weights %*	X	10,77	X	X	X	X	X	X	X	X	0	30	0	30	29,23
Weights %	X	X	50	X	X	X	X	X	X	X	0	10,12	0	25	14,88
Weights %*	X	X	X	4,12	X	X	X	X	X	X	5,88	30	0	30	30
Weights %	25	X	X	X	X	X	X	X	X	X	0	25	0	25	25

*DAX,FTSE100(50%-30%)

The restricted portfolios (Panels C and D) showed a more balanced allocation due to the constraints, promoting diversification. Now, in panel C, the portfolios are composed of higher weights of conventional indexes instead. How we can expect based on panel A, Fossil Free takes all the available quantity of portfolio. See that in portfolios between DAX, FTSE100, and selected indexes, optimization works only with a 30% maximum weight constraint of green and non-green indexes.

Table 10 shows portfolios resulting from the global minimum variance strategy which aims exclusively to minimize risk, without considering the expected return. This time we consider even the portfolios with short-selling, particularly portfolios of green and non-green indexes only. Panel A and B present similar results for tangency strategy in terms of weight distributions. Indeed what has changed is the choice of Green Bond as the main index in the portfolio over Fossil Free.

Table 10: Global minimum variance optimization

Global minimum variance, full sample

	FCHI	DAX	SPX	FTSE100	FTSEMIB	G carbon	Fossil free	Ren energy	Green bond	Stellar	Energy	EU Energy	Oil & Gas	Bond mkt	Bitcoin
Panel A: Global minimum variance (unrestricted, but positive) between selected Index and green Index															
Weights %	2,6	X	X	X	X	0	2,72	0	94,67	0	X	X	X	X	X
Weights %	X	1,54	X	X	X	0	3,28	0	95,18	0	X	X	X	X	X
Weights %	X	X	4,27	X	X	0	0	0	95,73	0	X	X	X	X	X
Weights %	X	X	X	9,4	X	0	0	0	90,6	0	X	X	X	X	X
Weights %	X	X	X	X	1,52	0	3,24	0	95,24	0	X	X	X	X	X
Panel B: Global minimum variance (unrestricted, but positive) between selected Index and non green Index															
Weights %	12,99	X	X	X	X	X	X	X	X	X	0	23,91	0	39,08	24,01
Weights %	X	3,65	X	X	X	X	X	X	X	X	0	28,68	0	43,06	24,6
Weights %	X	X	70,25	X	X	X	X	X	X	X	0	10,55	0	0,02	19,17
Weights %	X	X	X	0	X	X	X	X	X	X	0	30,33	0	44,83	24,84
Weights %	X	X	X	X	12,99	X	X	X	X	X	0	23,91	0	39,08	24,01
Panel C: Global minimum variance (restricted: 50%-25%, but positive) between selected Index and green Index															
Weights %	27,56	X	X	X	X	11,52	25	10,92	25	0	X	X	X	X	X
Weights %	X	17,79	X	X	X	20,48	25	11,73	25	0	X	X	X	X	X
Weights %	X	X	10,37	X	X	25	25	14,63	25	0	X	X	X	X	X
Weights %	X	X	X	50	X	0	19,18	5,82	25	0	X	X	X	X	X
Weights %	X	X	X	X	10,7	25	25	14,3	25	0	X	X	X	X	X
Panel D: Global minimum variance(restricted: 50%-25%, but positive) between selected Index and non green Index															
Weights %	50	X	X	X	X	X	X	X	X	X	7,02	15,37	0	25	2,61
Weights %	X	50	X	X	X	X	X	X	X	X	6,92	15,69	0	25	2,39
Weights %	X	X	50	X	X	X	X	X	X	X	0	24,33	0	25	0,67
Weights %	X	X	X	50	X	X	X	X	X	X	5,88	24,37	14,29	1,59	3,11
Weights %	X	X	X	X	50	X	X	X	X	X	7,53	15,23	0	25	2,24
Panel E: Global minimum variance(without short-selling, restricted: 50%) between selected green and non green															
Weights %	X	X	X	X	X	20,4	21,53	8,07	50	0	0	X	X	X	X
Weights %	X	X	X	X	X	14,59	21,33	7,86	50	0	X	6,22	X	X	X
Weights %	X	X	X	X	X	20,4	21,53	8,07	50	0	X	X	0	X	X
Weights %	X	X	X	X	X	0	2,12	0	47,88	0	X	X	X	50	X
Weights %	X	X	X	X	X	20,4	21,53	8,07	50	0	X	X	X	X	0
Weights %	X	X	X	X	X	45,14	X	X	X	X	2,32	2,52	0	50	0,03
Weights %	X	X	X	X	X	X	36,95	X	X	X	0	13,05	0	50	0
Weights %	X	X	X	X	X	X	X	31,45	X	X	0	18,44	0	50	0,11
Weights %	X	X	X	X	X	X	X	X	46,51	X	0	3,49	0	50	0
Weights %	X	X	X	X	X	X	X	X	X	0	0	36,28	9,69	50	4,03
Panel F: Global minimum variance(with short-selling, restricted: -25%) between selected green and non green															
Weights %	X	X	X	X	X	-11,43	6,8	1,45	100	-0,08	3,26	X	X	X	X
Weights %	X	X	X	X	X	-19,88	8,55	1,46	100	-0,15	X	100,1	X	X	X
Weights %	X	X	X	X	X	-11,35	6,87	1,42	100	-0,08	X	X	3,14	X	X
Weights %	X	X	X	X	X	-3,56	4,13	-1,21	36,56	-0,08	X	X	X	64,16	X
Weights %	X	X	X	X	X	-10,4	8,96	1,97	100	0,2	X	X	X	X	-0,73
Weights %	X	X	X	X	X	-0,89	X	X	X	X	24,95	4,18	-25	97,36	-0,6
Weights %	X	X	X	X	X	X	1,77	X	X	X	24,23	3,99	-25	95,71	-0,7
Weights %	X	X	X	X	X	X	X	-0,83	X	X	25,16	3,86	-25	97,4	-0,58
Weights %	X	X	X	X	X	X	X	X	30,63	X	24,74	3,51	-25	66,79	-0,68
Weights %	X	X	X	X	X	X	X	X	X	0,09	24,9	3,84	-25	96,86	-0,7
Panel G: Global minimum variance(with short-selling: restricted:max 50%,min-25%) between selected Index and green Index															
Weights %	32,81	X	X	X	X	-8,67	18,65	7,68	50	-0,47	X	X	X	X	X
Weights %	X	23,58	X	X	X	-0,47	19,07	8,33	50	-0,51	X	X	X	X	X
Weights %	X	X	16,3	X	X	20,35	5,8	8,07	50	-0,51	X	X	X	X	X
Weights %	X	X	X	50	X	-20,48	16,61	4,41	50	-0,55	X	X	X	X	X
Weights %	X	X	X	X	9,08	12,41	20,88	8,15	50	-0,52	X	X	X	X	X
Panel H: Global minimum variance(with short-selling: restricted:max 50%,min-25%) between selected Index and non green Index															
Weights %	47,91	X	X	X	X	X	X	X	X	X	27,21	-0,42	-25	50	0,3
Weights %	X	44	X	X	X	X	X	X	X	X	27,72	2,86	-25	50	0,42
Weights %	X	X	44,04	X	X	X	X	X	X	X	11,24	20,32	-25	50	-0,6
Weights %	X	X	X	50	X	X	X	X	X	X	24,43	-3,54	-21,54	-50	0,65
Weights %	X	X	X	X	40,57	X	X	X	X	X	28,74	5,1	-25	50	0,59
Panel I: Global minimum variance(restricted: max 50%, min -25%) between selected Index and all green and non green Index															
Weights %	18,99	X	X	X	X	-25	2,05	-1,54	50	-0,09	0	4,45	0	51,15	0
Weights %	X	16,86	X	X	X	-25	1,95	-1,15	50	-0,13	0	5,76	0	51,7	0
Weights %	X	X	-17,39	X	X	-12,17	21,38	-1,08	46,67	-0,13	0	7,46	0	55,26	0
Weights %	X	X	X	26,77	X	-25	3,07	-2,19	50	-0,15	0	1,17	0	46,32	0
Weights %	X	X	X	X	10,87	-21,25	3,07	-1,17	50	-0,13	0	5,87	0	52,74	0

Panel E shows ten portfolios divided into five portfolios of green indices and non-green indices selected one at a time and five portfolios of non-green indices and green indices selected one at a time with the constraint of 50% as the maximum possible weight. First, it is evident how Stellar and Bitcoin are not selected, differently from before. It can be seen how green indexes are heavily selected even in portfolios of non-green indices and green indices are selected one at a time. For instance, the portfolio between non-green indexes and G Carbon, attributed 45% of the overall portfolio weight to the green index. Furthermore, the Bond mkt index is the only one within non-green indexes that is heavily selected both when the non-green indices are all selected and when it is the only one selected. The important thing we can say from panel E is that in risk minimization green indices are preferred to non-green ones except for the Bond mkt.

We analyze now the strategy including short-shelling. The decision to short depends on the investor's specific objectives, market conditions, and return and risk expectations. When optimizing a portfolio, shorting an index can help minimize the portfolio's overall variance. For instance, if an index is positively correlated with other assets in the portfolio, shorting it can reduce overall risk. In panel F the strategy imposes a constraint of -25% as the minimum weight for a single asset. In the first four portfolios with all green indexes selected, we see a similar behavior where the main indexes are Green Bond, G Carbon, and Fossil Free. The strong allocation to green bonds suggests that the asset is seen as extremely safe and low variance, ideal for portfolio stabilization, except for portfolio when the non-green index selected is Bond mkt. Interesting is the fact that the short position on G_carbon probably indicates expectations of decline for green carbon assets, or better a strategic use to hedge risks. In the other four portfolios, we can see how the results are the same but mirrored, where Bond mkt has the main role with a decrease of its weight in a portfolio with Green Bond as the selected green index. Oil & Gas is heavily shorted in all portfolios, which probably suggests caution towards oil and gas sectors.

In panel G, minimum variance optimization is performed by allowing short selling, with a maximum constraint of 50% and a minimum of -25% on individual asset weights between selected indices and green indices. Here, Green Bonds emerge as the dominant component in all portfolios, achieving a maximum weight of 50%. This consistent allocation reflects high confidence in the stability and low variance of green assets, making them critical to minimizing overall risk. At the same time, green carbon-related assets are frequently shorted. The presence of Fossil Free and Ren Energy with positive weights shows a balanced diversification between traditional and green assets, maintaining a focus on risk reduction.

Panel H shows that Bond Markets hold significant weight, reaching 50% in all portfolios, reflecting a conservative strategy aimed at minimizing overall portfolio risk. Oil & Gas assets are constantly shorted, suggesting a hedge against the risks associated with the gas sector. The presence of positive weights in the Energy index demonstrates confidence in the energy sector. The small weights attributed to Bitcoin indicate an attempt at diversification without significant exposure.

Panel I analyzes the optimization of the minimum variance between selected indices, green and non-green, with short selling, with maximum constraints of 50% and minimum constraints of -25% on the weights of individual assets. As before, portfolio optimization confirms the importance of Green Bond and Bond mkt. The particular thing is that in the third portfolio, SPX is shorted by around -17% while the G_Carbon index continues to be shorted in all portfolios.

We can say that the results indicate that green indices, mainly represented by Green Bonds, are considered less risky than traditional indices. This makes green indices particularly attractive to investors looking to minimize risk in their portfolios. The trend towards shorting carbon assets and holding significant long positions in green bonds reflects a shift towards more sustainable and less risky investments.

Table 11: Minimum risk efficient portfolio (CVaR)

Minimum risk efficient portfolio (target return: equal weights portfolio), full sample

	CVaR $\alpha = 1\%$			CVaR $\alpha = 2,5\%$			CVaR $\alpha = 5\%$			CVaR $\alpha = 10\%$		
Weight %	**G**	**N**	**G+N**	**G**	**N**	**G+N**	**G**	**N**	**G+N**	**G**	**N**	**G+N**
FCHI	2,27	0	0	10,11	2,16	1,34	12,03	0	0	11,59	0	0
DAX	0	0	0	0	0	0	0	0	0	0	0	0
SPX	0	18,04	0	0	23,65	0	0	23,16	0	0	23,67	0
FTSE100	0	0	0	0	0	0	0	0	0	0	0	0
FTSEMIB	0	0	0	0	0	0	0	0	0	0	0	0
G carbon	0	X	0	0	X	0	0	X	0	0	X	0
Fossil free	57,53	X	20,73	46,31	X	23,44	48,84	X	22,73	49,05	X	23,02
Ren energy	0	X	0	0	X	0	0	X	0	0	X	0
Green bond	39,76	X	0	39,88	X	0	37,41	X	0	37,6	X	0
Stellar	0,44	X	0	3,67	X	0	1,72	X	0	1,75	X	0
Energy	X	0	0	X	0	0	X	0	0	X	0	0
EU Energy	X	0	0	X	4,27	5,11	X	5,89	6,24	X	7,67	8,17
Oil & Gas	X	0	0	X	0	0	X	0	0	X	0	0
Bond mkt	X	72,03	70,21	X	62,89	63,18	X	63,78	63,93	X	62,02	62,21
Bitcoin	X	9,92	9,06	X	7,04	6,93	X	7,17	7,11	X	6,63	6,59

As we said in chapter 3, considering mean-CVaR optimization instead of mean-variance is preferable in many financial contexts, especially because it offers a more accurate assessment of the risk of extreme loss. While the mean-variance approach focuses on total variance as a measure of risk, the mean-CVaR focuses specifically on losses beyond a predetermined threshold, better capturing the tail risks that can cause significant damage during turbulent market periods. This approach is particularly useful for risk-averse investors, as it allows them to manage and limit the most potential losses, improving the robustness of the portfolio in situations of financial stress. As a result, mean-CVaR optimization provides more effective protection against extreme and sudden events, enabling more prudent and targeted risk management.

We will analyze three mean-car optimization strategies for four different confidence levels. For each level, we will see three portfolios: the one between conventional indexes and green indexes

(G), conventional indexes and non-green indexes (N), and between all indexes (G+N).

In table 11, as in the mean-variance case, we consider as target return the one from the equally weighted portfolio between conventional indexes. It can be immediately seen that as the CVaR threshold increases (from 1% to 10%), allocations do not change dramatically, suggesting the robustness of the portfolio strategy against various levels of risk, selecting the same indexes for all portfolios. For instance, in G portfolios, the strategy selects FCHI, Fossil Free, Green Bond, and Stellar. The particular thing is that in N+G portfolios, the conventional indexes are not selected, leaving weight to Fossil Free, Bond mkt, and Bitcoin. Even by increasing the confidence level, new green indices are not selected on the contrary, space is made for EU Energy and this is in contrast with what we saw and what we expected.

Table 12: Minimum CVaR portfolio

Global minimum portfolio, full sample

	Cvar $\alpha = 1\%$			CVaR $\alpha = 2,5\%$			CVaR $\alpha = 5\%$			CVaR $\alpha = 10\%$		
Weight %	**G**	**N**	**G+N**	**G**	**N**	**G+N**	**G**	**N**	**G+N**	**G**	**N**	**G+N**
FCHI	0	0	0	0	0	0	0	0	0	0	0	0
DAX	0	0	0	0	0	0	0	0	0	0	0	0
SPX	0	0	0	0	0	0	0	0	0	1,27	0	0
FTSE100	2,98	5,38	8,45	10,94	13,23	12,05	11,62	11,89	5,55	10,33	6,35	3,66
FTSEMIB	0	0	0	0	0	0	0	0	0	0	0	0
G carbon	0	X	0	0	X	0	0	X	0	0	X	0
Fossil free	0	X	0	0	X	0	0	X	0	0	X	0
Ren energy	0	X	0	0	X	0	0	X	0	0	X	0
Green bond	97,02	X	34,65	89,06	X	19	88,38	X	26,41	88,4	X	25,98
Stellar	0	X	0	0	X	0	0	X	0	0	X	0
Energy	X	0	0	X	0	0	X	0	0	X	0	0
EU Energy	X	0	0	X	0	0	X	0,12	2,79	X	3,62	3,73
Oil & Gas	X	0	0	X	0	0	X	0	0	X	0	0
Bond mkt	X	94,62	56,9	X	86,77	68,94	X	87,98	65,25	X	90,03	66,62
Bitcoin	X	0	0	X	0	0	X	0	0	X	0	0

As we can expect from seeing table (5), the global minimum CVaR portfolios are made by FTSE100, Green Bond, and Bond mkt. Indeed, their values of CVaR are the lowest within all indexes and therefore the composition of the portfolio is a direct consequence.

The last strategy we consider is the Max Return/Risk portfolio which is calculated by the minimization of the 'Sortino Ratio' for a given risk-free rate. The Sortino ratio is the ratio of the target return lowered by the risk-free rate and the CVaR risk. For our purpose, we don't consider a risk-free rate. Table 13 shows the results of this strategy. Green (G) portfolios show a strong predominance of Fossil Free, suggesting that this index is perceived as an optimal combination of return and risk within the green context. Non-green (N) portfolios are heavily dominated by SPX, indicating strong confidence in the stock market as a source of high return. The inclusion of Bitcoin in significant percentages reflects an aggressive approach to risk diversification. In combined portfolios (G+N), Fossil Free and Bitcoin consistently dominate, suggesting that a combination of green and non-green assets offers an optimal balance between return and risk. With the increase of the CVaR confidence level from 1% to 10%, a decrease in the equity component (SPX) is observed in the non-green and combined portfolios, while increasing the allocation towards EU Energy, remaining pretty stable for the other indexes. In this case, the interesting thing is that the strategy portfolio prefers to allocate to Fossil Free instead of Green Bond. Non-green assets like SPX and Bitcoin are used to increase returns, but with greater risk exposure, while the selected green indexes try to stabilize the trade-off between risk and return.

Table 13: Max Return/Risk Ratio Mean-CVaR portfolio

Max Return/Risk Ratio Mean-CVaR Portfolio, full sample

	Cvar $\alpha = 1\%$			CVaR $\alpha = 2,5\%$			CVaR $\alpha = 5\%$			CVaR $\alpha = 10\%$		
Weight %	**G**	**N**	**G+N**	**G**	**N**	**G+N**	**G**	**N**	**G+N**	**G**	**N**	**G+N**
FCHI	4,57	0	0	4,05	1,59	0	12,41	0	0	7,6	0	0
DAX	0	0	0	0	0	0	0	0	0	0	0	0
SPX	0	75,79	0	0	73,98	0	0	55,95	0	0	55,97	0
FTSE100	0	0	0	0	0	0	0	0	0	0	0	0
FTSEMIB	0	0	0	0	0	0	0	0	0	0	0	0
G carbon	0	X	0	0	X	0	0	X	0	0	X	0
Fossil free	91,85	X	80,44	90,3	X	73,33	83,58	X	52,51	88,56	X	56,56
Ren energy	0	X	0	0	X	0	0	X	0	0	X	0
Green bond	3,58	X	0	0	X	0	0	X	0	0	X	0
Stellar	X	X	0	5,65	X	0	4,02	X	0	3,83	X	0
Energy	X	0	0	X	0	0	X	0	0	X	0	0
EU Energy	X	0	0	X	3,25	5,04	X	9,08	9,6	X	10,83	11,01
Oil & Gas	X	0	0	X	0	0	X	0	0	X	0	0
Bond mkt	X	0	0	X	0	0	X	16,4	19,5	X	16,94	16,33
Bitcoin	X	24,3	19,56	X	21,18	21,62	X	18,57	18,39	X	16,26	16,1

4.3 Rolling window analysis

Rolling window analysis is a crucial tool for assessing the volatility and risk associated with investment portfolios over varying periods. This approach helps capture dynamic changes in financial risk, making it particularly useful during periods of high economic and geopolitical uncertainty. In this section, we will apply a rolling window to examine the behavior of VaR and CVaR over time. We will also apply the strategy to the portfolio optimization seen in the previous section to study their behavior, especially during two periods of strong stress and volatility: the COVID-19 pandemic and the Ukraine War.

Exploded in early 2020, the COVID-19 pandemic has caused unprecedented volatility in financial markets. Drastic lockdown measures, disruptions in supply chains and economic uncertainties have induced significant movements in asset prices, increasing the risk perceived by investors.

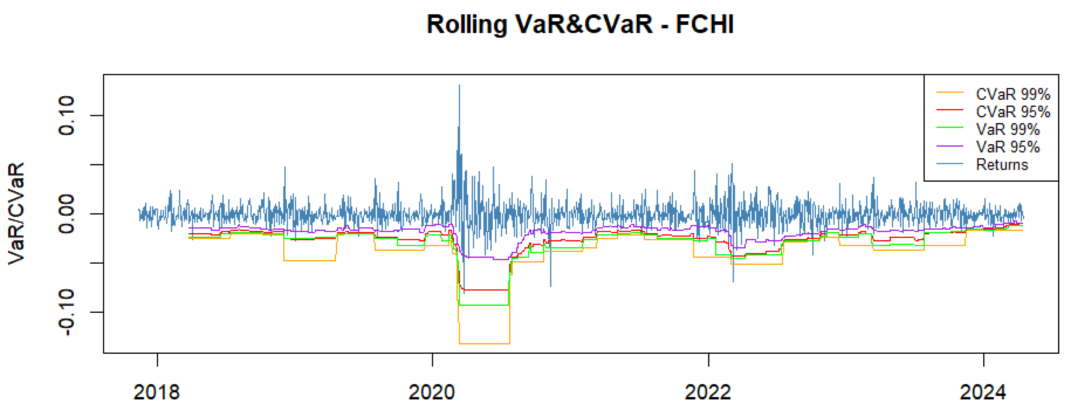
Ukraine War, which began in February 2022, has further triggered new tensions in global markets. This conflict has had direct impacts not only on energy and commodity markets but has also increased geopolitical uncertainty globally.

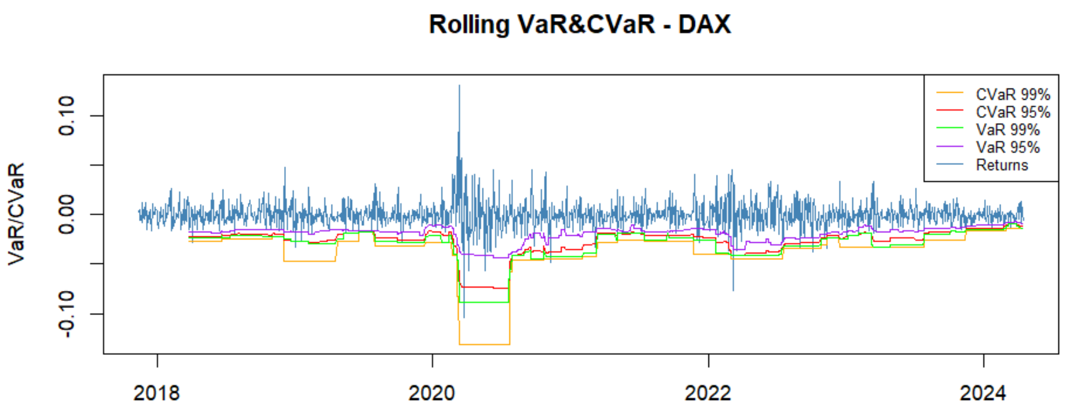
Through the use of moving windows, we can provide a detailed and temporal view of the risk associated with investment portfolios, highlighting how extraordinary events influence volatility and risk management. This analysis is critical for investors seeking to better understand the evolution of risk over time and adopt more informed and resilient management strategies.

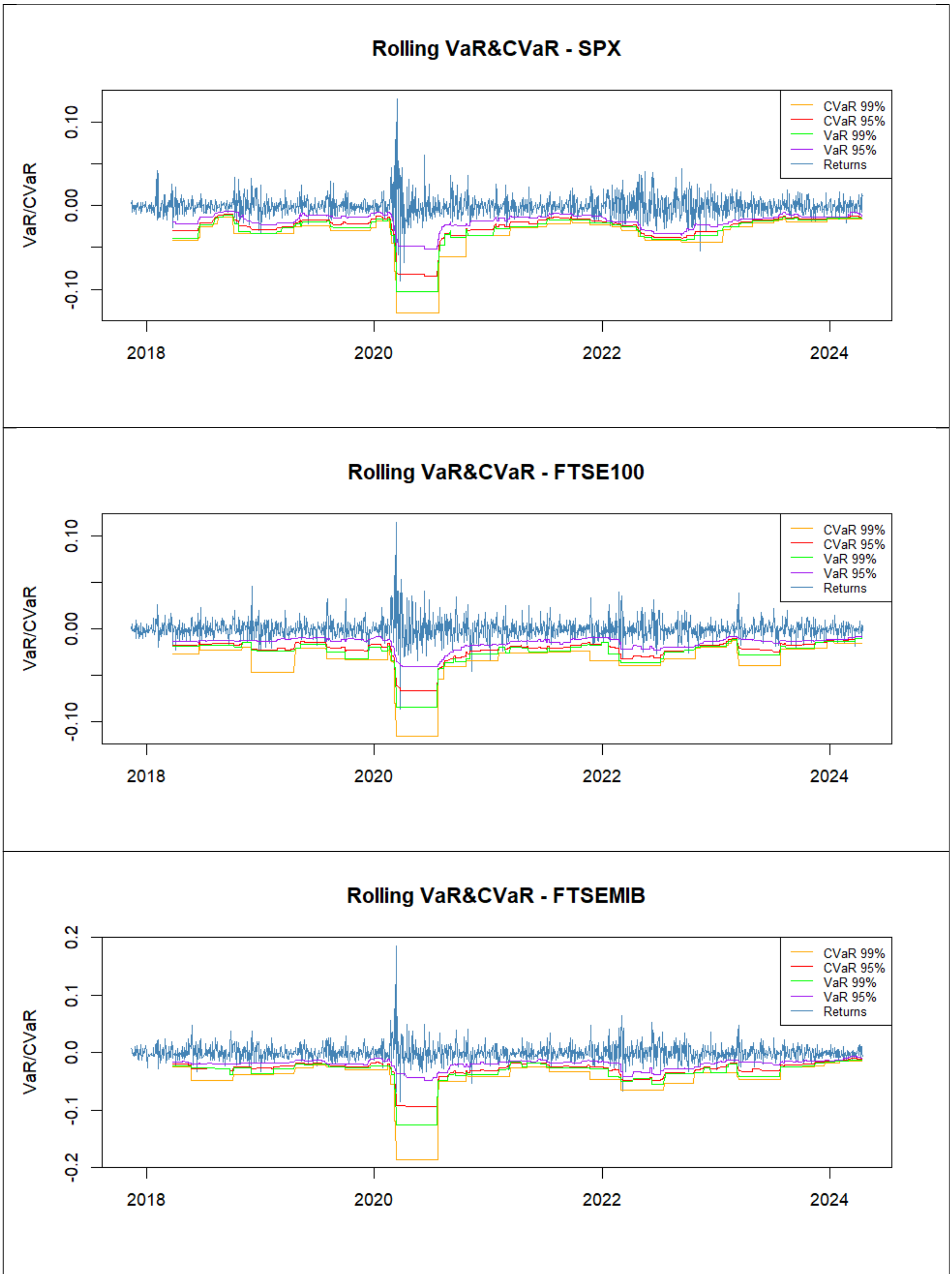
During the analysis, we will use a 90-day window since it is a good trade-off between the need to detect changes in market conditions and the need to avoid excessive statistical fluctuations. Indeed, in the context of financial markets, quarters (approximately 90 days) are commonly used for reporting and analysis, reflecting seasonal and quarterly changes in business performance and economic conditions. Events such as the COVID-19 pandemic and the war in Ukraine have significant and persistent impacts on financial markets. A 90-day window allows us to analyze how risk evolves during and after such events, providing a deep understanding of how markets react over the medium term. The table below shows rolling VaR and CVaR for all indexes.

Table 14: Rolling CVaR and VaR of Indexes

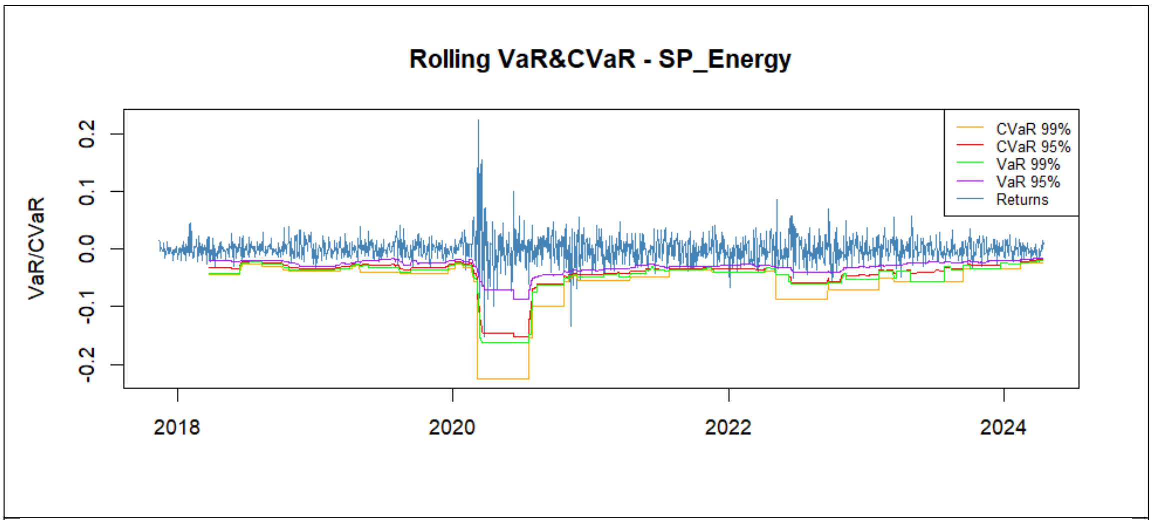
Panel A: Rolling VaR and CVaR for conventional indexes

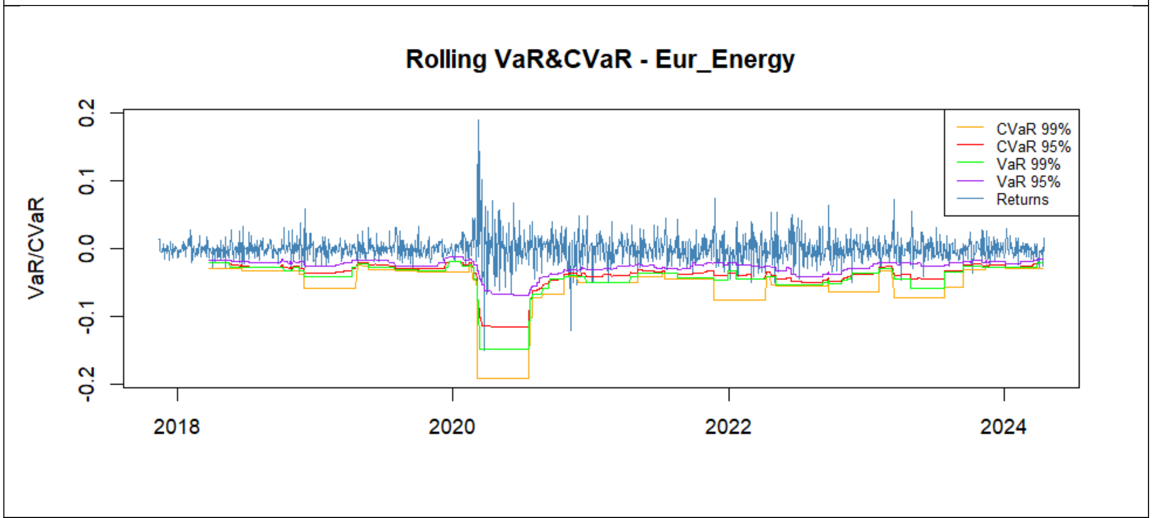


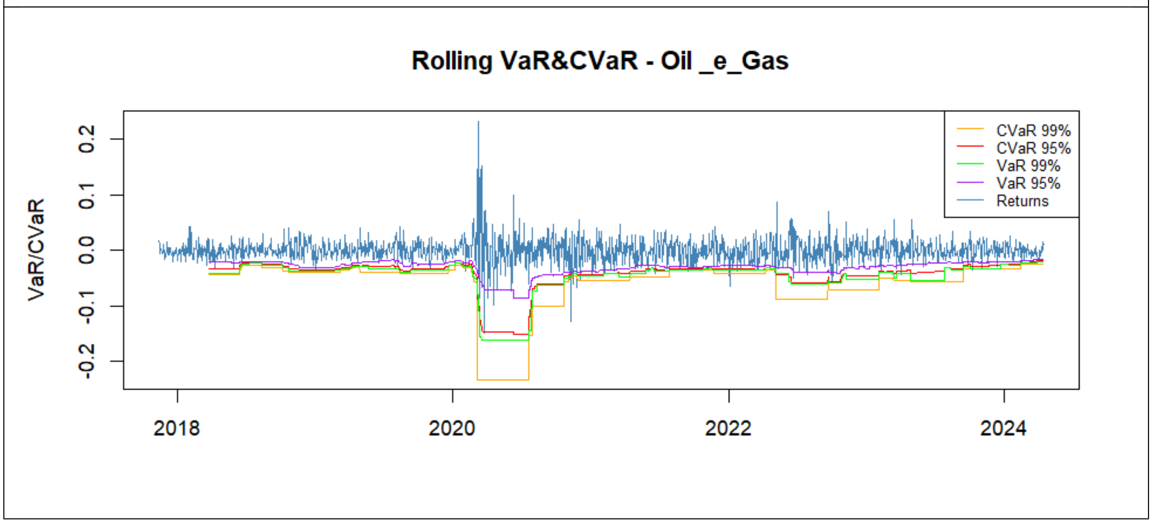


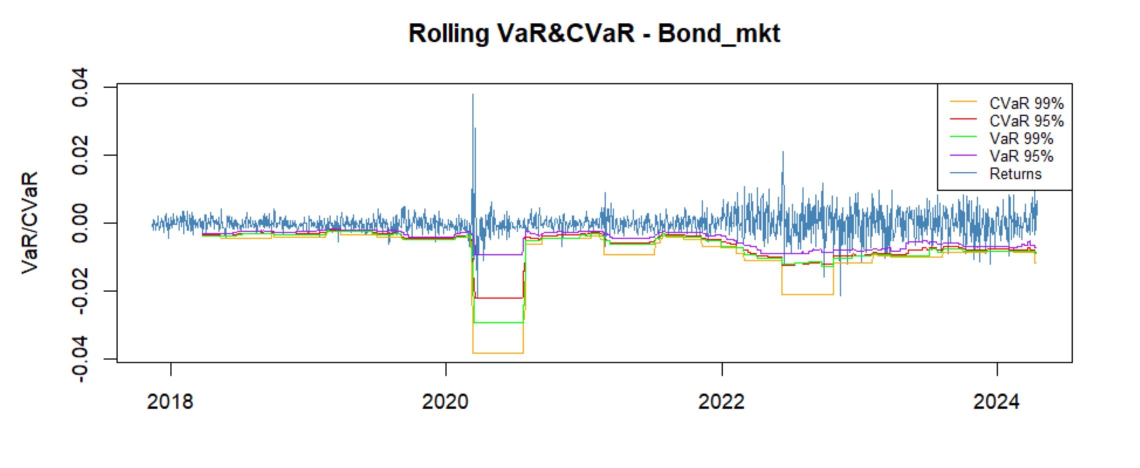


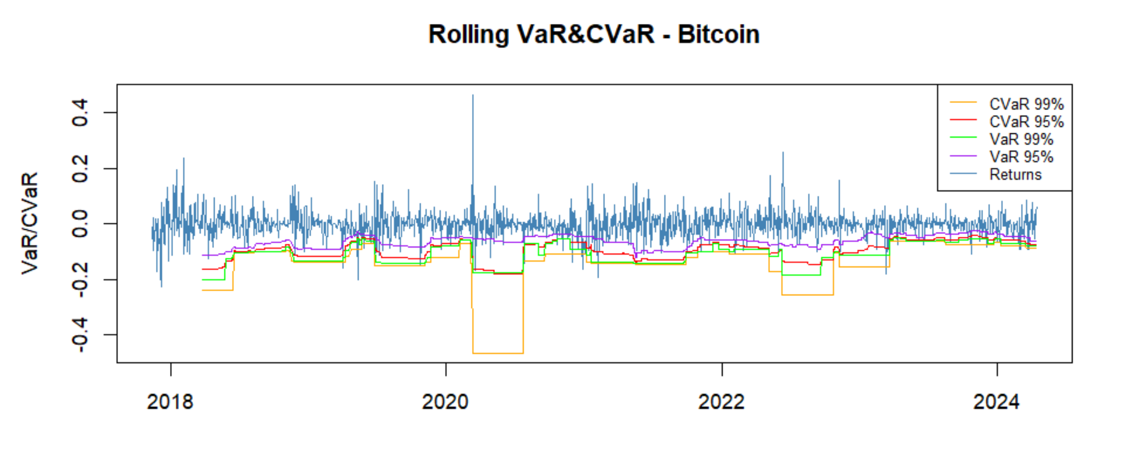
Panel B: Rolling VaR and CVaR for non-green indexes


Rolling VaR&CVaR - SP_Energy
VaR/CVaR
0.2
0.1
0.0
-0.1
-0.2
2018
2020
2022
2024
CVaR 99%
CVaR 95%
VaR 99%
VaR 95%
Returns

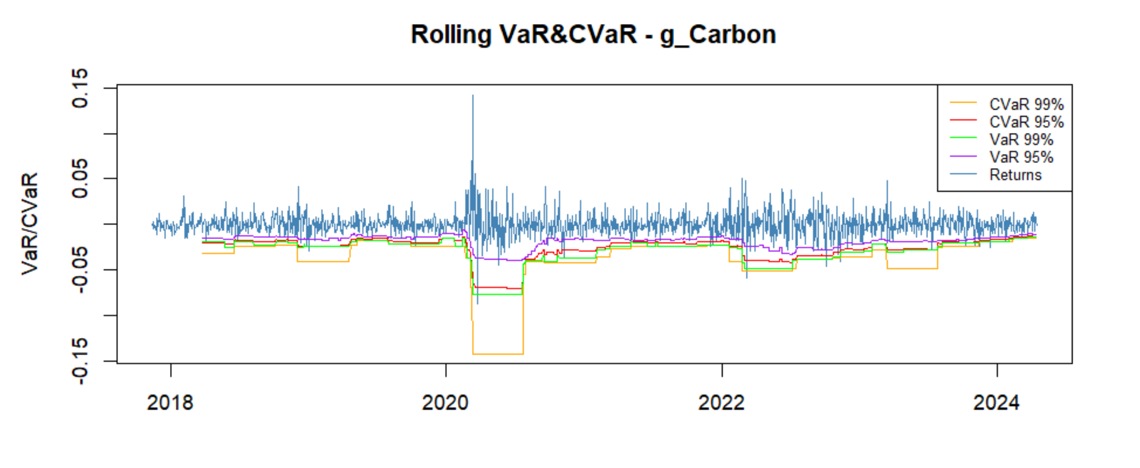

Rolling VaR&CVaR - Eur_Energy
VaR/CVaR
0.2
0.1
0.0
-0.1
-0.2
2018
2020
2022
2024
CVaR 99%
CVaR 95%
VaR 99%
VaR 95%
Returns

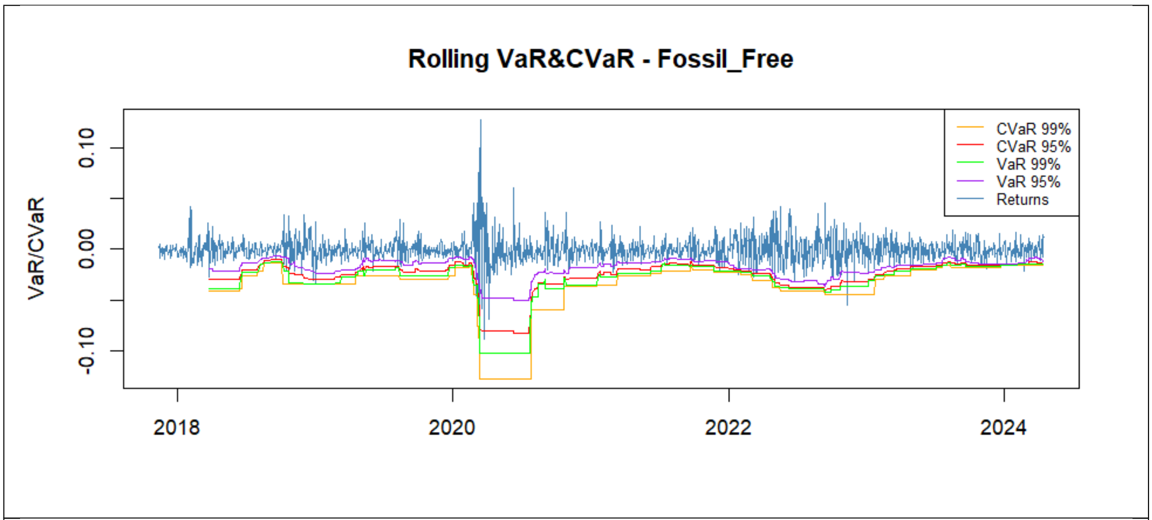

Rolling VaR&CVaR - Oil _e_Gas
VaR/CVaR
0.2
0.1
0.0
-0.1
-0.2
2018
2020
2022
2024
CVaR 99%
CVaR 95%
VaR 99%
VaR 95%
Returns

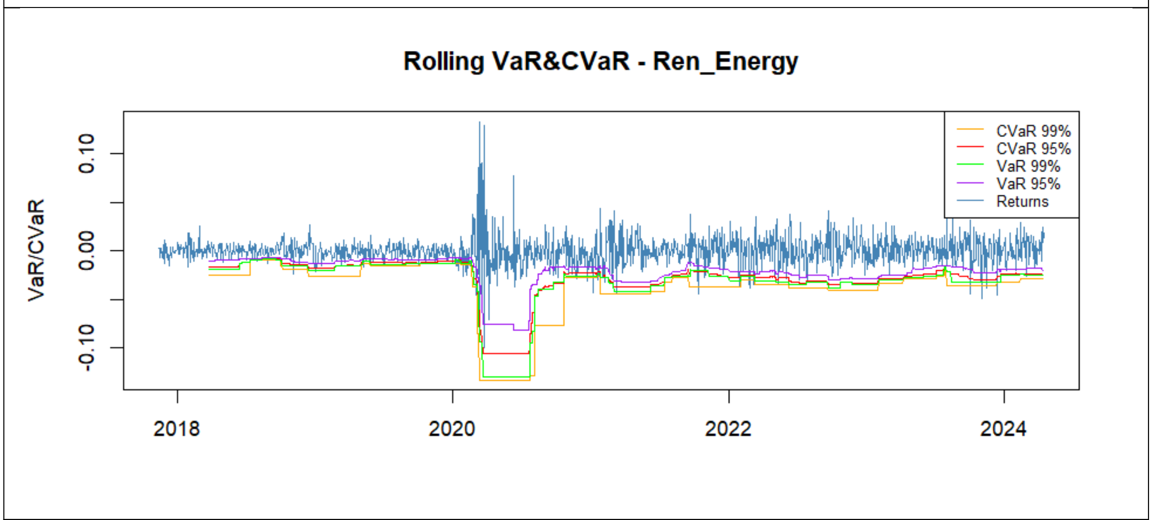


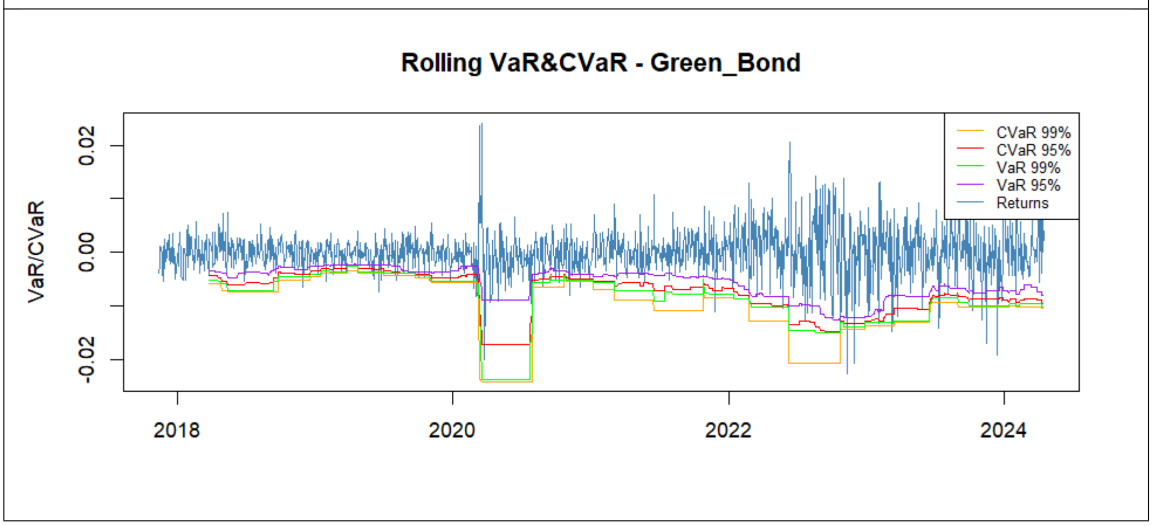


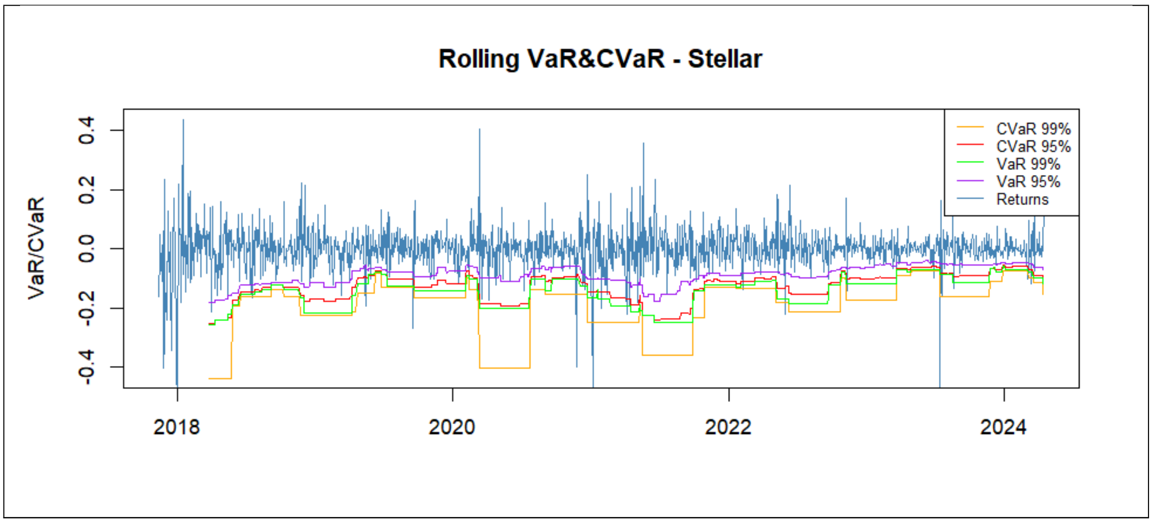
Panel C: Rolling VaR and CVaR for green indexes




Rolling VaR&CVaR - Fossil_Free
VaR/CVaR
0.10
0.00
-0.10
2018
2020
2022
2024
CVaR 99%
CVaR 95%
VaR 99%
VaR 95%
Returns


Rolling VaR&CVaR - Ren_Energy
VaR/CVaR
0.10
0.00
-0.10
2018
2020
2022
2024
CVaR 99%
CVaR 95%
VaR 99%
VaR 95%
Returns


Rolling VaR&CVaR - Green_Bond
VaR/CVaR
0.02
0.00
-0.02
2018
2020
2022
2024
CVaR 99%
CVaR 95%
VaR 99%
VaR 95%
Returns



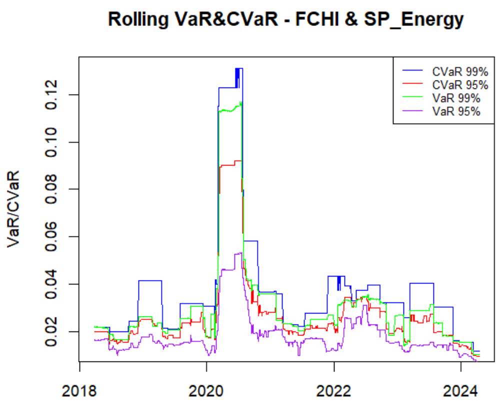
The results obtained show how extraordinary events such as the COVID-19 pandemic and the war in Ukraine have significantly increased the volatility and risk associated with several financial indices. The 90-day rolling window allows us to capture these dynamic changes, providing a detailed view of how risk evolves. Spikes in VaR and CVaR values during these periods indicate an increase in expected losses and market risk, underlining the importance of dynamic and adaptive risk management. It can be seen that, during the COVID and Ukraine War period, the CVaR(1%) had a higher absolute than VaR. This finding emphasizes that a limitation of VaR is its unresponsiveness to large losses beyond a certain threshold. In other words, VaR tells us the chance of a certain loss, but it can't measure losses beyond that point. CVaR, on the other hand, looks at the average loss when things go worse than expected. This means two investments might have the same VaR but very different potential losses. Also, VaR has some math problems that make it bad for choosing the best mix of investments, while CVaR doesn't have these issues as we saw in section 1.6. During the pandemic period traditional indices such as FCHI, DAX, SPX, and FTSE100 were significantly impacted. A marked increase in volatility can be seen in these graphs, evidenced by the spikes in VaR and CVaR values. Notably, both 95% and 99% VaR dropped dramatically, indicating higher expected losses under stress scenarios. These results clearly show how the pandemic hit traditional markets hard, increasing tail risk and thus potential losses. Similarly, non-green indices, such as SP_Energy, Eur_Energy, and Oil_Gas, were even more heavily impacted. Their VaR and CVaR values fell further, especially for the energy sector, signaling very high expected losses during the pandemic. This indicates that the energy sector was one of the most vulnerable during this period, with significantly higher volatility and risk of loss. On the other hand, green indices such as g_Carbon, Fossil_Free, Ren_Energy, and Green_Bond have shown greater resilience than both traditional and non-green indices. Although these indices also experienced increased volatility, the impact was relatively minor. Their VaR and CVaR values did not fall as dramatically, suggesting that green investments were less exposed to particularly vulnerable sectors such as energy. Turning to the period of the war in Ukraine, which began in 2022, we see that traditional indexes again experienced an increase in VaR and CVaR values, although the impact was less pronounced than that observed during the pandemic. The war certainly generated uncertainty in the markets, but the expected losses did not reach the dramatic levels of 2020. Non-green sectors, particularly energy-related sectors, were also affected by the war, with increases in VaR and CVaR values. However, these increases were more moderate than during the pandemic, probably due to higher energy prices, which mitigated some of the expected losses. Once again, green indexes proved resilient even during the war in Ukraine. Their VaR and CVaR values remained relatively stable, confirming a lower vulnerability to global shocks than other indices. This behavior suggests that green investments may offer better protection against extreme global events.

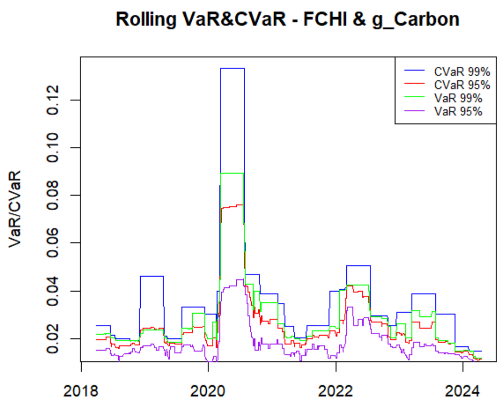
4.3.1 Rolling VaR and CVaR of mixed portfolios

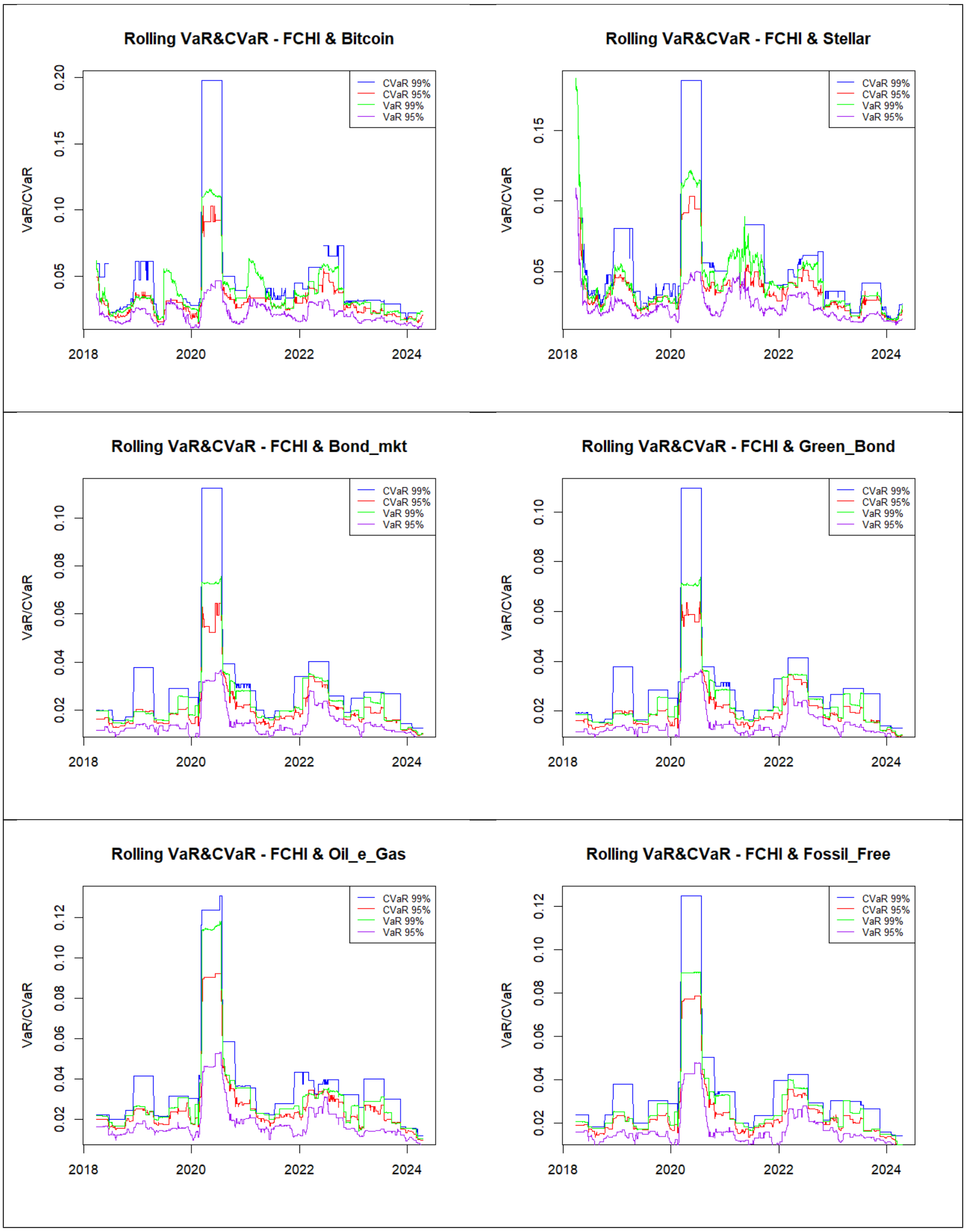
During this section, we will analyze the rolling VaR and CVaR for mixed portfolios. As we did in table 5, we allocate 80% of the portfolio weight to the conventional indexes and 20% to the selected indexes. In order to better capture the differences between green and non-green finance, we will compare indexes in pairs as SP Energy and g Carbon, Bitcoin and Stellar, Bond mkt and Green Bond, Oil&Gas and Fossil Free, Eur Energy and Ren Energy. The table below shows

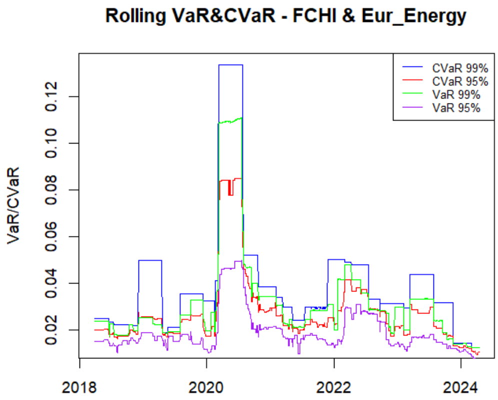
Table 15: Rolling CVaR and VaR of mixed portfolios

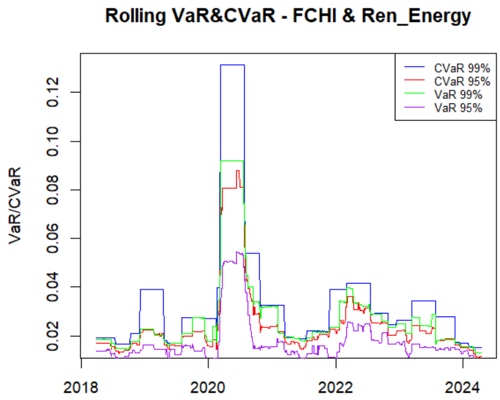
Panel A: Rolling VaR and CVaR for FCHI and selected indexes



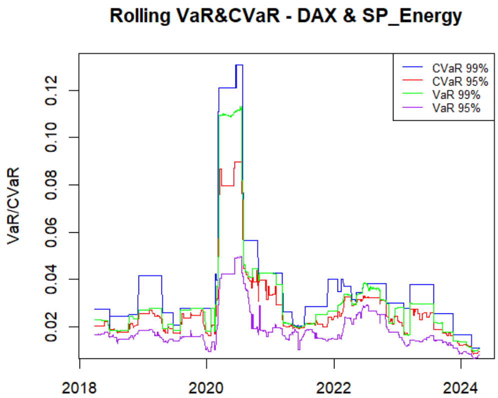


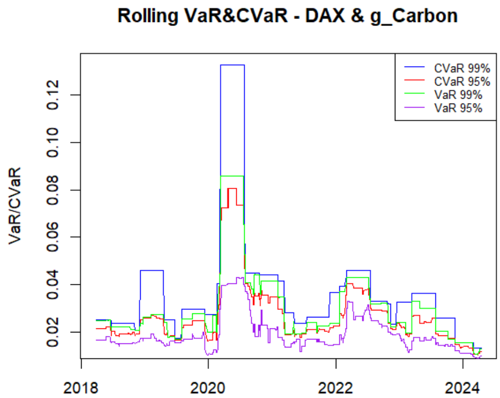

Rolling VaR&CVaR - FCHI & Bitcoin
Rolling VaR&CVaR - FCHI & Stellar
Rolling VaR&CVaR - FCHI & Bond_mkt
Rolling VaR&CVaR - FCHI & Green_Bond
Rolling VaR&CVaR - FCHI & Oil_e_Gas
Rolling VaR&CVaR - FCHI & Fossil_Free
VaR/CVaR
CVaR 99%
CVaR 95%
VaR 99%
VaR 95%
2018
2020
2022
2024

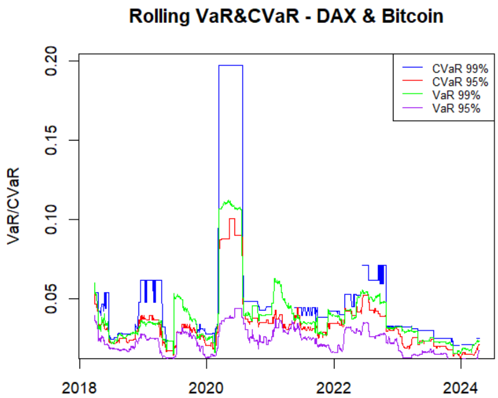


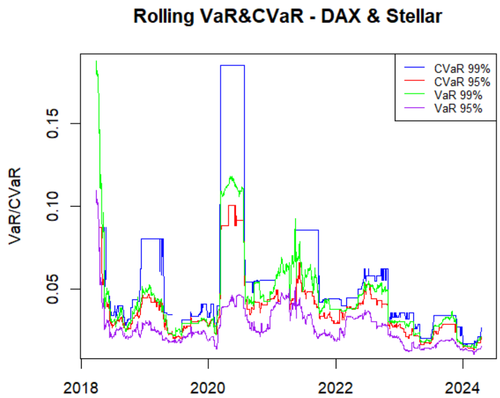


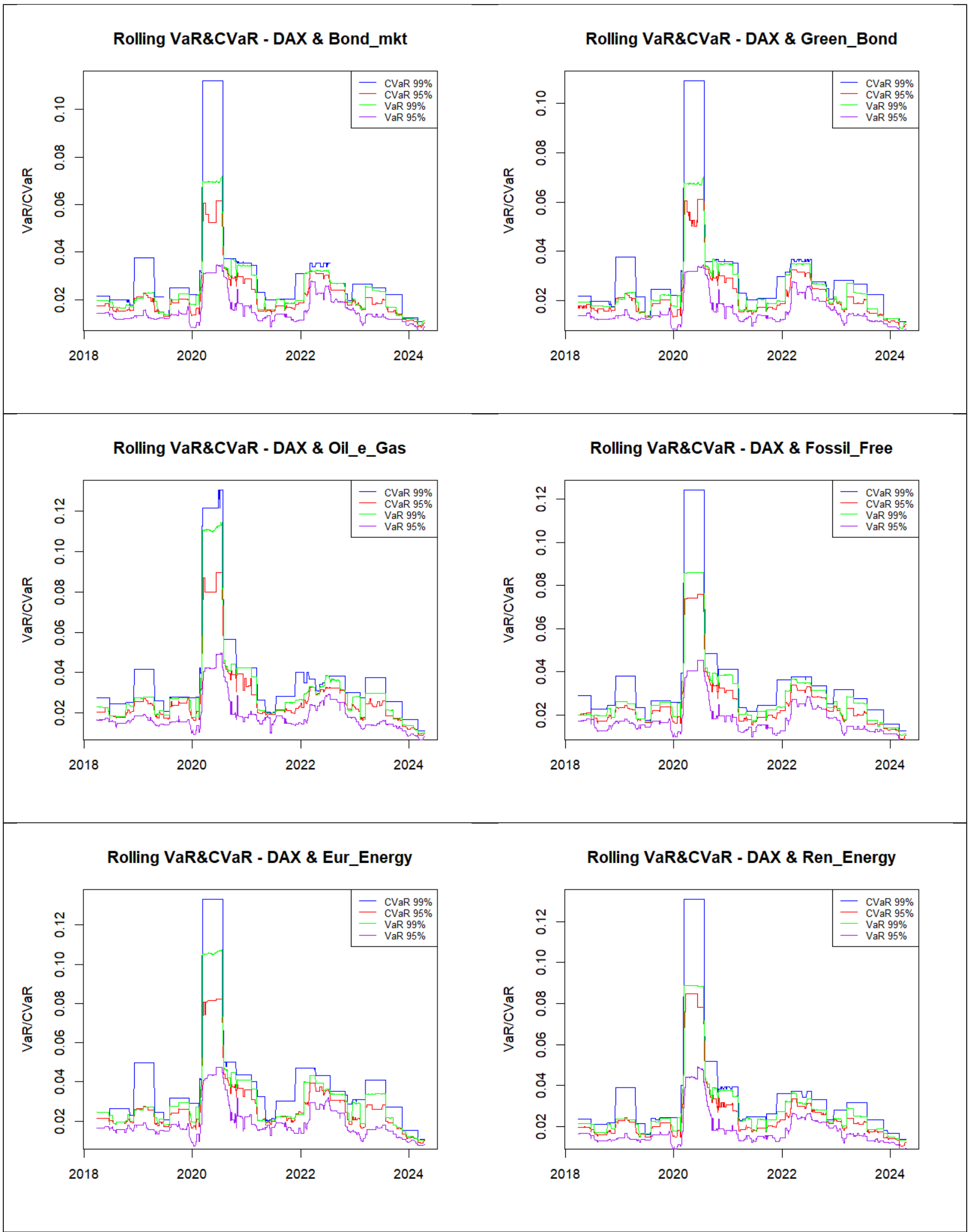
Panel B: Rolling VaR and CVaR for DAX and selected indexes



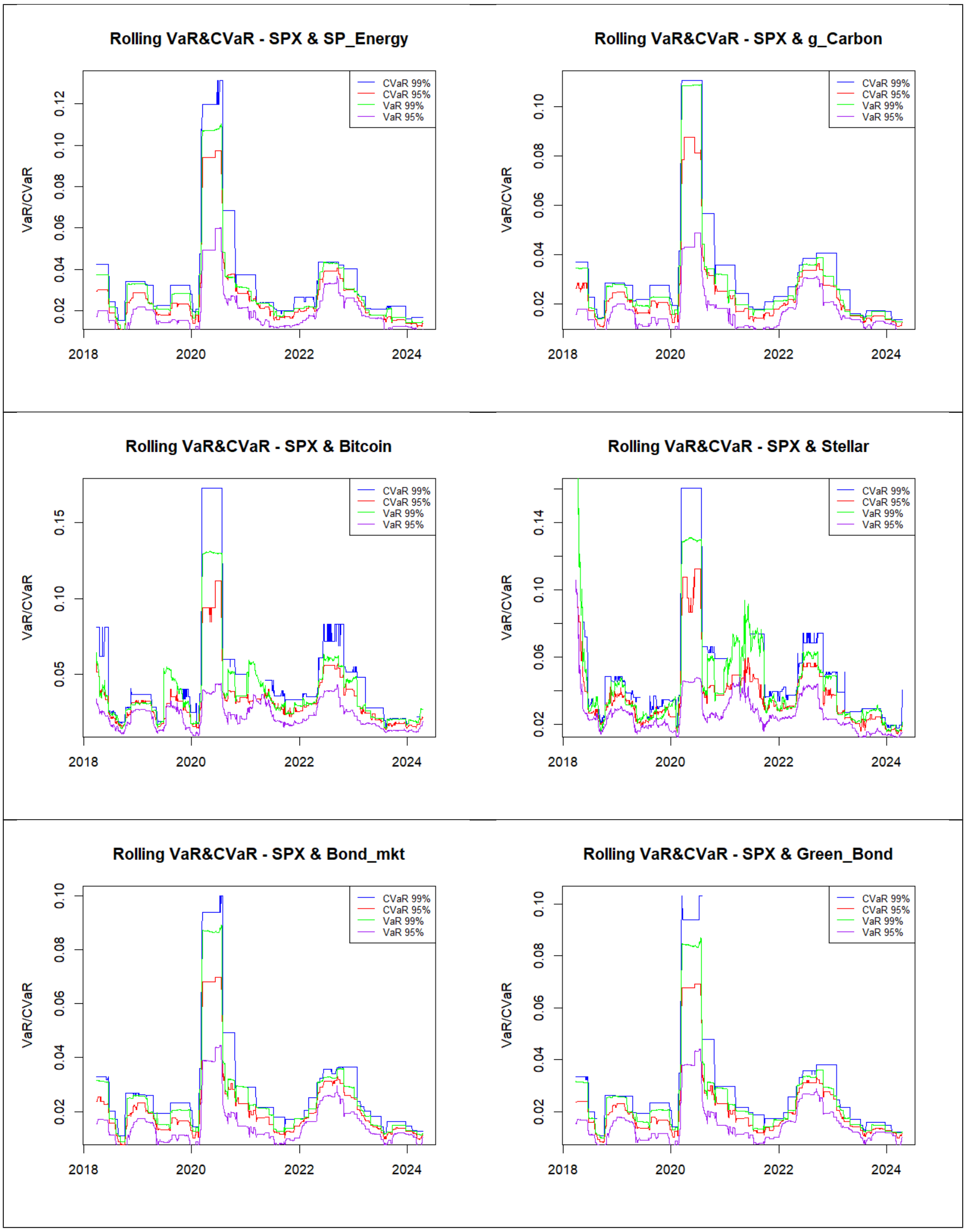


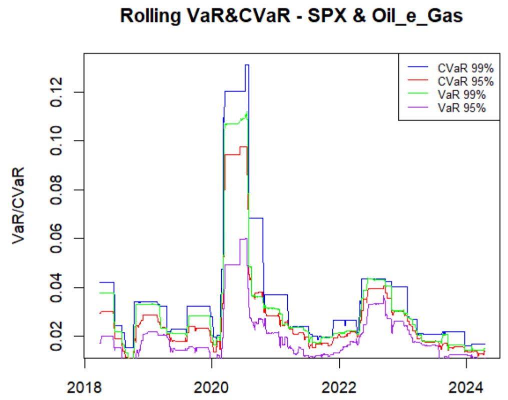


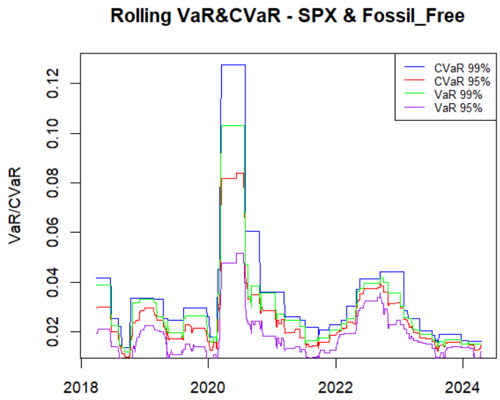


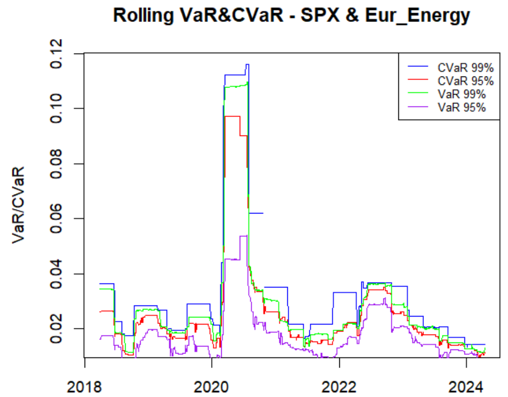


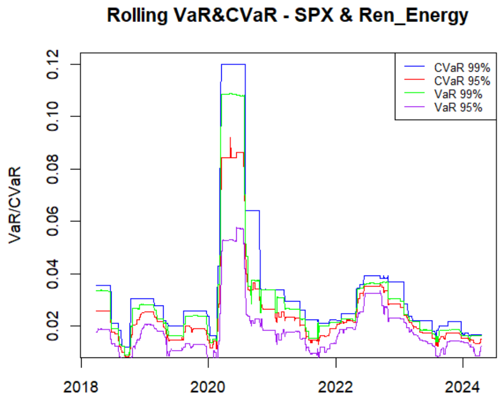
Panel C: Rolling VaR and CVaR for SPX and selected indexes


Rolling VaR&CVaR - SPX & SP_Energy
Rolling VaR&CVaR - SPX & g_Carbon
Rolling VaR&CVaR - SPX & Bitcoin
Rolling VaR&CVaR - SPX & Stellar
Rolling VaR&CVaR - SPX & Bond_mkt
Rolling VaR&CVaR - SPX & Green_Bond
VaR/CVaR
CVaR 99%
CVaR 95%
VaR 99%
VaR 95%
2018
2020
2022
2024

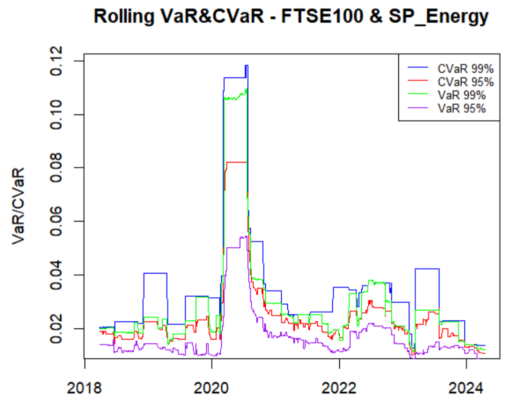


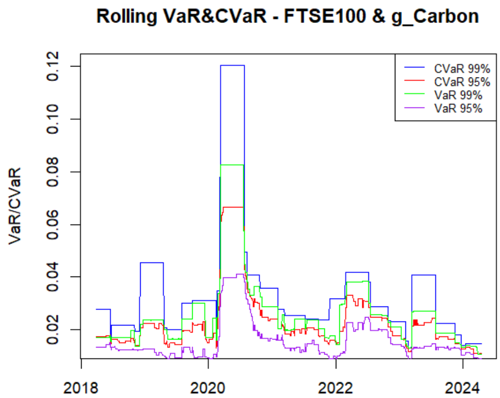


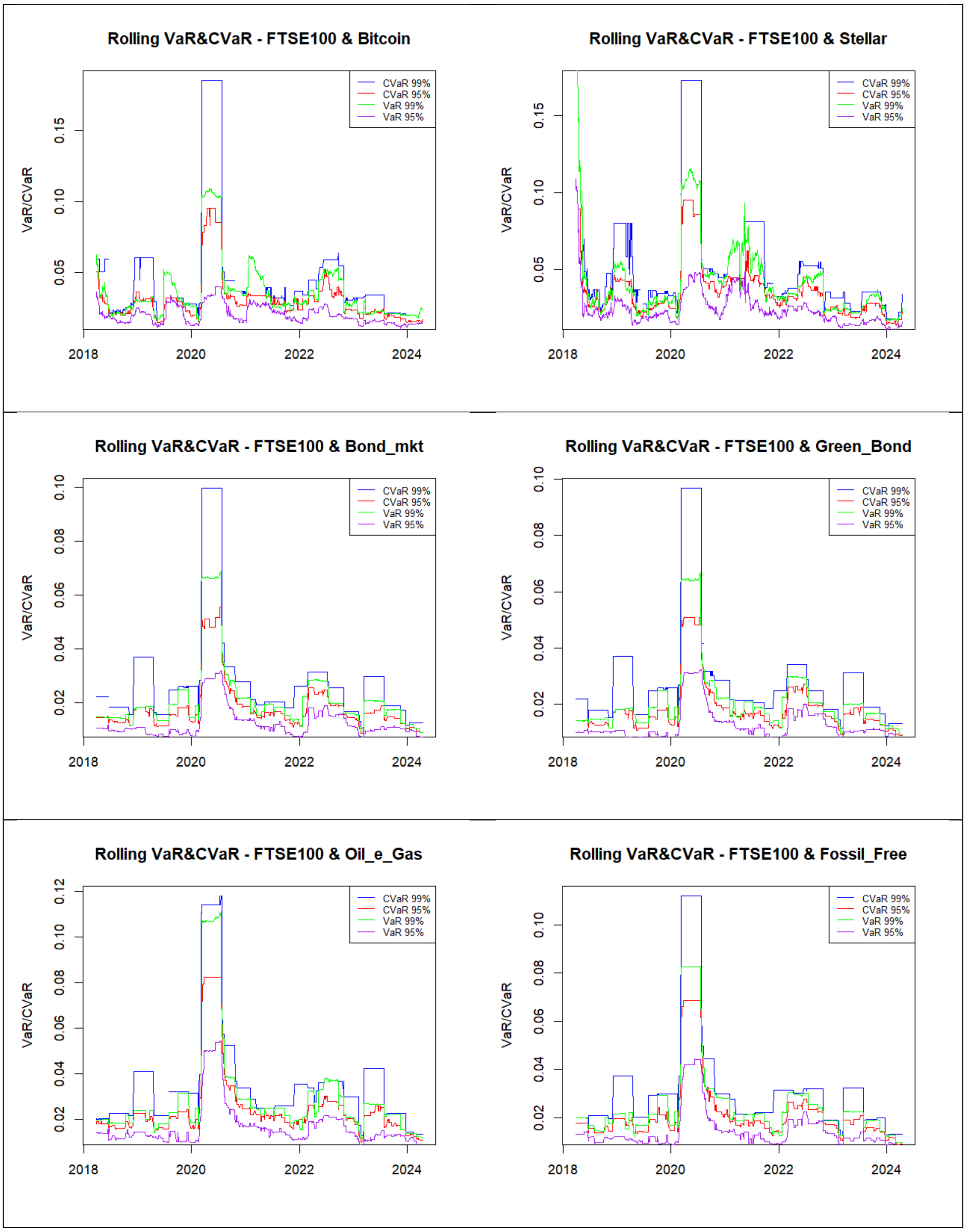


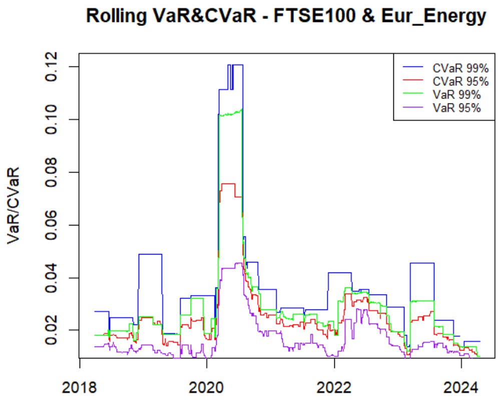


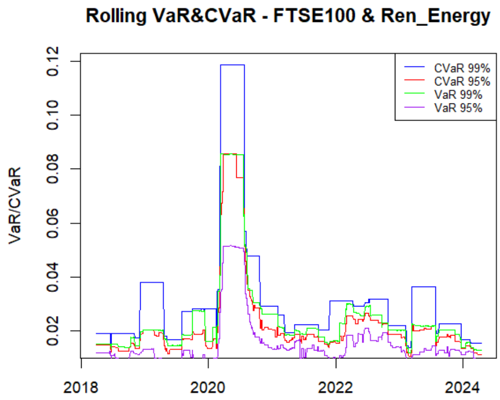
Panel D: Rolling VaR and CVaR for FTSE100 and selected indexes



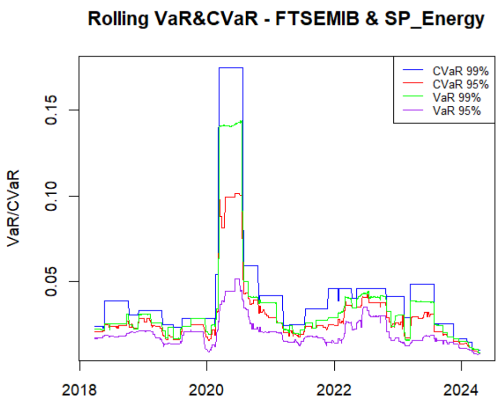


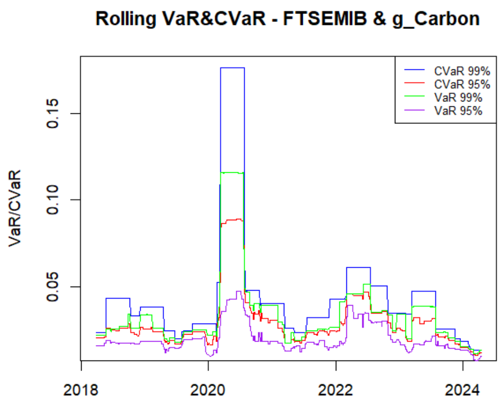

Rolling VaR&CVaR - FTSE100 & Bitcoin
Rolling VaR&CVaR - FTSE100 & Stellar
Rolling VaR&CVaR - FTSE100 & Bond_mkt
Rolling VaR&CVaR - FTSE100 & Green_Bond
Rolling VaR&CVaR - FTSE100 & Oil_e_Gas
Rolling VaR&CVaR - FTSE100 & Fossil_Free
VaR/CVaR
CVaR 99%
CVaR 95%
VaR 99%
VaR 95%
2018
2020
2022
2024

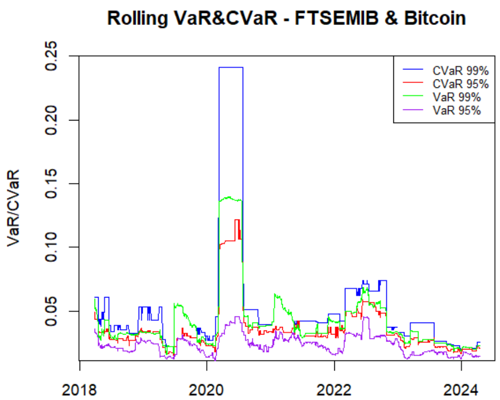


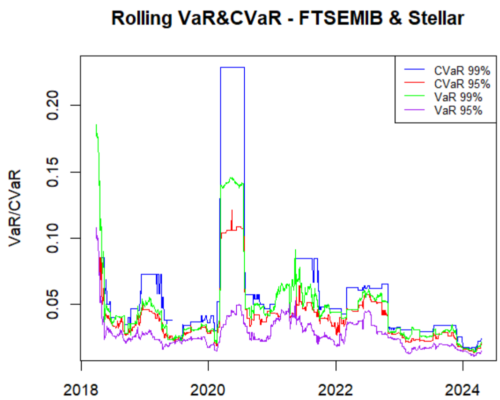


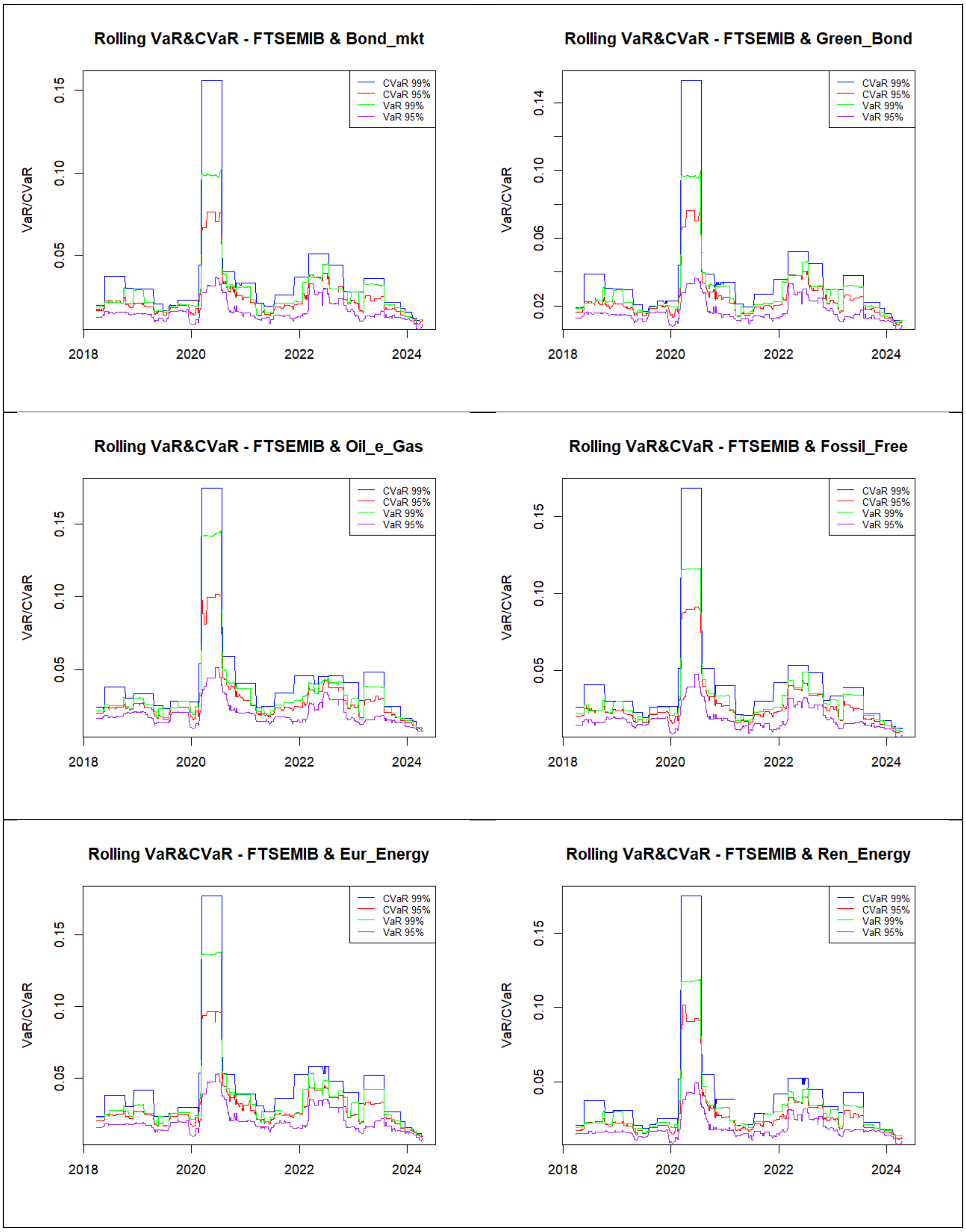
Panel E: Rolling VaR and CVaR for FTSEMIB and selected indexes











Looking at Panel A, it can be seen that the integration of the g_Carbon index showed an increase in risk during 2020. However, it is interesting to note that, compared to SP Energy, the g_Carbon presented a faster stabilization. This suggests that, although not immune, the green index has limited expected losses in the medium term. In general, the different mixed portfolios behave in the same way within the different conventional indexes. For instance, panel B shows that the integration of Green_Bond to Dax index, as well as of Bond mkt, leads to the same results

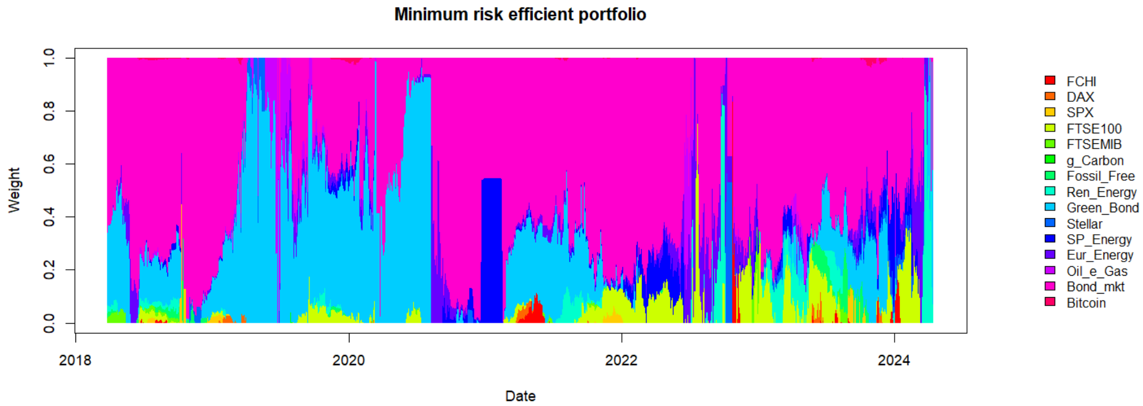


Figure 31: Minimum risk efficient portfolio between I+N+G (unrestricted)

in terms of expected losses. On the contrary, it seems that adding Bitcoin to conventional indices, compared to Stellar, could be the best choice for building a portfolio that does not face exponential losses in periods of severe crisis.

However, while green indices do show increased risk during crises, they tend to recover and stabilize faster than traditional indices, particularly those focused on specific sectors like oil and gas. On the other hand, adding bond indices to a portfolio can help reduce overall risk, making it more resilient during economic uncertainties. These observations emphasize the importance of diversification in investment strategies. They suggest that including both green and bond indices in a portfolio can lead to more effective risk management and increased resilience to market shocks.

4.3.2 Rolling portfolio optimization

In this section, we will explore the strategies faced in the previous chapters using the rolling window analysis to capture the dynamic of the different portfolios through time, focusing on stressful periods. Portfolios will be provided graphically and numerically utilizing tables to see in more detail the immediate periods after the collapse and after the stabilization.

Minimum risk efficient portfolio As before, we use as the target return the one from the equally weighted portfolio between the conventional indexes. We will see the portfolios between green, non-green, and conventional indexes, even the restricted case. In figure (31) we can see how the portfolio has changed over the last six years. It can be seen how the two bond indexes have been the main actors for a long time in the first half of the chart. Indeed, table 16 shows that the portfolios attribute to Bond mkt and Green_Bond, both in the first and second periods analyzed, all the weight. During the Ukraine War, things change. Indeed, we can see more uncertainty with respect COVID crisis. In both periods green indexes are not considered, leaving room for non-green indexes and this was unexpected since the war in Ukraine caused a strong crisis in the energy sector.

Table 16: The weights of rolling portfolio at the beginning and a month later of the COVID pandemic and the war in Ukraine

I+G+N

COVID-19	FCHI	DAX	SPX	FTSE100	FTSEMIB	g_Carbon	Fossil_Free	Ren_Energy	Green_Bond	Stellar	SP_Energy	Eur_Energy	Oil_e_Gas	Bond_mkt	Bitcoin
09/03/2020	0,000	0,025	0,000	0,072	0,000	0,000	0,000	0,000	0,633	0,000	0,022	0,000	0,000	0,249	0,000
10/03/2020	0,000	0,000	0,000	0,081	0,000	0,000	0,000	0,000	0,551	0,000	0,017	0,000	0,000	0,351	0,000
11/03/2020	0,000	0,019	0,002	0,069	0,000	0,000	0,000	0,000	0,597	0,000	0,005	0,000	0,000	0,309	0,000
12/03/2020	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,986	0,000	0,000	0,000	0,014	0,000	0,000
13/03/2020	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,986	0,000	0,000	0,000	0,014	0,000	0,000
16/03/2020	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,412	0,000	0,000	0,000	0,000	0,588	0,000
17/03/2020	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,412	0,000	0,000	0,000	0,000	0,588	0,000
...															
09/04/2020	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,485	0,000	0,000	0,000	0,000	0,515	0,000
14/04/2020	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,524	0,000	0,000	0,000	0,000	0,476	0,000
15/04/2020	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,553	0,000	0,000	0,000	0,000	0,447	0,000
16/04/2020	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,509	0,000	0,000	0,000	0,000	0,491	0,000
17/04/2020	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,569	0,000	0,000	0,000	0,000	0,431	0,000
20/04/2020	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,487	0,000	0,000	0,000	0,000	0,513	0,000
21/04/2020	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,503	0,000	0,000	0,000	0,000	0,497	0,000
Ukraine War	**FCHI**	**DAX**	**SPX**	**FTSE100**	**FTSEMIB**	**g_Carbon**	**Fossil_Free**	**Ren_Energy**	**Green_Bond**	**Stellar**	**SP_Energy**	**Eur_Energy**	**Oil_e_Gas**	**Bond_mkt**	**Bitcoin**
24/02/2022	0,000	0,000	0,000	0,084	0,000	0,000	0,000	0,000	0,000	0,000	0,165	0,000	0,000	0,751	0,000
25/02/2022	0,000	0,000	0,000	0,098	0,000	0,000	0,000	0,000	0,000	0,000	0,123	0,000	0,000	0,778	0,000
28/02/2022	0,000	0,000	0,000	0,087	0,000	0,000	0,000	0,000	0,000	0,000	0,105	0,000	0,000	0,808	0,000
01/03/2022	0,000	0,000	0,000	0,062	0,000	0,000	0,000	0,000	0,000	0,000	0,115	0,000	0,000	0,823	0,000
02/03/2022	0,000	0,000	0,000	0,071	0,000	0,000	0,000	0,000	0,000	0,000	0,123	0,011	0,000	0,794	0,000
03/03/2022	0,000	0,000	0,000	0,059	0,000	0,000	0,000	0,000	0,000	0,000	0,125	0,014	0,000	0,815	0,000
04/03/2022	0,000	0,000	0,000	0,039	0,000	0,000	0,000	0,000	0,000	0,000	0,110	0,008	0,000	0,842	0,000
...															
24/03/2022	0,000	0,000	0,000	0,104	0,000	0,000	0,000	0,000	0,000	0,000	0,157	0,026	0,000	0,713	0,000
25/03/2022	0,000	0,000	0,000	0,111	0,000	0,000	0,000	0,000	0,000	0,000	0,161	0,026	0,000	0,702	0,000
28/03/2022	0,000	0,000	0,000	0,083	0,000	0,000	0,014	0,000	0,000	0,000	0,159	0,015	0,000	0,729	0,000
29/03/2022	0,000	0,000	0,000	0,086	0,000	0,000	0,018	0,000	0,000	0,000	0,153	0,065	0,000	0,736	0,000
30/03/2022	0,000	0,000	0,000	0,077	0,000	0,000	0,008	0,000	0,000	0,000	0,130	0,025	0,000	0,760	0,000
31/03/2022	0,000	0,000	0,000	0,073	0,000	0,000	0,000	0,000	0,000	0,000	0,134	0,026	0,000	0,766	0,000
01/04/2022	0,000	0,000	0,000	0,088	0,000	0,000	0,000	0,000	0,000	0,000	0,137	0,015	0,000	0,760	0,000

In the chart above, the portfolio was constructed with no restrictions on asset weights, allowing for complete flexibility in the distribution of investments. Setting limits on how much of each asset can be in a portfolio is crucial for managing risk. These limits help spread investments around, which can lower overall volatility. During tough times, like the COVID-19 pandemic or the Ukraine war, these restrictions help keep the portfolio stable by preventing too much money from being put into risky assets. These limits also make sure the portfolio follows investment rules and regulations. They help manage cash flow and can reduce the costs of buying and selling assets. Additionally, these constraints can encourage sustainable investing. They can help align the portfolio with ethical goals and social responsibility, which can protect investors in the long run. We now impose in figure (32) that the conventional indices can weigh a maximum of 50% each and that the other indices 25% each.

With the onset of the pandemic, we can see how the weight distribution tends strongly towards the green indices. In table 17, all the green indices are taken into consideration, remaining strongly present in the portfolio even in the second period considered in which Stellar also has an important part in the portfolio. Differently, regarding the war in Ukraine, both initially and in the stabilization period, only Green Bond is taken among the green indices, leaving room for Bond mkt, SP_Energy, and the conventional indices FTSE100 and SPX. In the second period, we can see how Ren_Energy begins to appear in the portfolio, a sign of how a trend toward green indices can begin, as we can see in the figure (17).

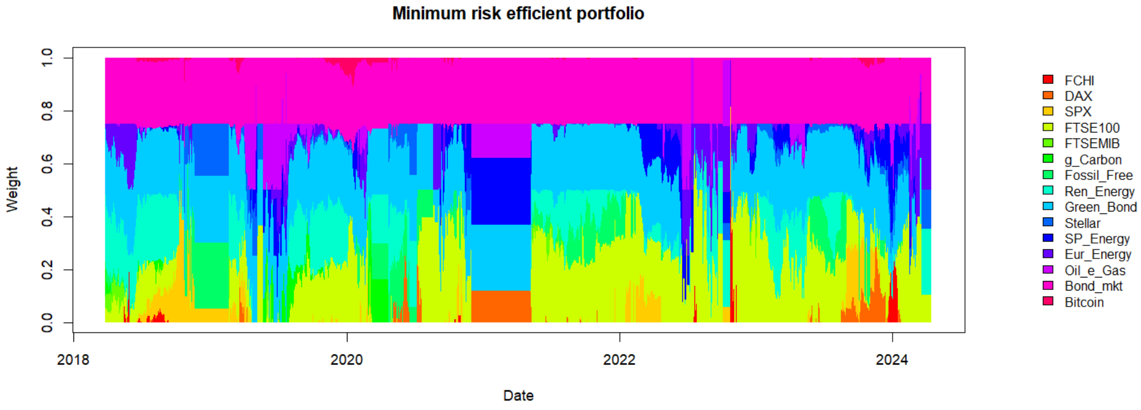


Figure 32: Minimum risk efficient portfolio between I+N+G (restricted)

Table 17: The weights of rolling portfolio at the beginning and a month later of the COVID pandemic and the war in Ukraine

I+G+N (Restricted)

COVID-19	**FCHI**	**DAX**	**SPX**	**FTSE100**	**FTSEMIB**	**g_Carbon**	**Fossil_Free**	**Ren_Energy**	**Green_Bond**	**Stellar**	**SP_Energy**	**Eur_Energy**	**Oil_e_Gas**	**Bond_mkt**	**Bitcoin**
09/03/2020	0,000	0,000	0,000	0,000	0,000	0,184	0,076	0,220	0,250	0,000	0,000	0,000	0,000	0,250	0,020
10/03/2020	0,000	0,000	0,000	0,000	0,000	0,161	0,136	0,185	0,250	0,000	0,000	0,000	0,000	0,250	0,018
11/03/2020	0,000	0,000	0,000	0,000	0,000	0,161	0,136	0,185	0,250	0,000	0,000	0,000	0,000	0,250	0,018
12/03/2020	0,000	0,000	0,000	0,000	0,000	0,161	0,136	0,185	0,250	0,000	0,000	0,000	0,000	0,250	0,018
13/03/2020	0,000	0,000	0,000	0,000	0,000	0,161	0,136	0,185	0,250	0,000	0,000	0,000	0,000	0,250	0,018
16/03/2020	0,000	0,000	0,000	0,000	0,000	0,161	0,136	0,185	0,250	0,000	0,000	0,000	0,000	0,250	0,018
17/03/2020	0,000	0,000	0,000	0,000	0,000	0,161	0,136	0,185	0,250	0,000	0,000	0,000	0,000	0,250	0,018
...															
09/04/2020	0,000	0,000	0,000	0,000	0,000	0,000	0,106	0,250	0,250	0,144	0,000	0,000	0,000	0,250	0,000
14/04/2020	0,000	0,000	0,000	0,000	0,000	0,000	0,119	0,250	0,250	0,131	0,000	0,000	0,000	0,250	0,000
15/04/2020	0,000	0,000	0,000	0,000	0,000	0,000	0,197	0,250	0,250	0,053	0,000	0,000	0,000	0,250	0,000
16/04/2020	0,000	0,000	0,000	0,038	0,000	0,000	0,155	0,250	0,250	0,057	0,000	0,000	0,000	0,250	0,000
17/04/2020	0,000	0,000	0,000	0,103	0,000	0,000	0,116	0,240	0,250	0,041	0,000	0,000	0,000	0,250	0,000
20/04/2020	0,000	0,000	0,000	0,000	0,000	0,000	0,199	0,250	0,250	0,000	0,000	0,000	0,000	0,250	0,051
21/04/2020	0,000	0,000	0,000	0,015	0,000	0,000	0,215	0,198	0,250	0,060	0,000	0,000	0,000	0,250	0,011
Ukraine War	**FCHI**	**DAX**	**SPX**	**FTSE100**	**FTSEMIB**	**g_Carbon**	**Fossil_Free**	**Ren_Energy**	**Green_Bond**	**Stellar**	**SP_Energy**	**Eur_Energy**	**Oil_e_Gas**	**Bond_mkt**	**Bitcoin**
24/02/2022	0,000	0,000	0,100	0,249	0,000	0,000	0,000	0,000	0,250	0,000	0,151	0,000	0,000	0,250	0,000
25/02/2022	0,000	0,000	0,183	0,261	0,000	0,000	0,000	0,005	0,250	0,000	0,051	0,000	0,000	0,250	0,000
28/02/2022	0,000	0,000	0,176	0,257	0,000	0,000	0,000	0,010	0,250	0,000	0,058	0,000	0,000	0,250	0,000
01/03/2022	0,000	0,000	0,160	0,235	0,000	0,000	0,000	0,000	0,250	0,000	0,105	0,000	0,000	0,250	0,000
02/03/2022	0,000	0,000	0,164	0,235	0,000	0,000	0,000	0,000	0,250	0,000	0,101	0,000	0,000	0,250	0,000
03/03/2022	0,000	0,000	0,169	0,207	0,000	0,000	0,000	0,000	0,250	0,000	0,124	0,000	0,000	0,250	0,000
04/03/2022	0,000	0,000	0,179	0,169	0,000	0,000	0,000	0,000	0,250	0,000	0,152	0,000	0,000	0,250	0,000
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24/03/2022	0,000	0,000	0,115	0,211	0,000	0,000	0,000	0,012	0,250	0,000	0,162	0,000	0,000	0,250	0,000
25/03/2022	0,000	0,000	0,117	0,209	0,000	0,000	0,000	0,015	0,250	0,000	0,159	0,000	0,000	0,250	0,000
28/03/2022	0,000	0,000	0,126	0,184	0,000	0,000	0,000	0,036	0,250	0,000	0,155	0,000	0,000	0,250	0,000
29/03/2022	0,000	0,000	0,128	0,187	0,000	0,000	0,000	0,045	0,250	0,000	0,139	0,000	0,000	0,250	0,000
30/03/2022	0,000	0,000	0,122	0,195	0,000	0,000	0,000	0,054	0,250	0,000	0,130	0,000	0,000	0,250	0,000
31/03/2022	0,000	0,000	0,111	0,197	0,000	0,000	0,000	0,050	0,250	0,000	0,142	0,000	0,000	0,250	0,000
01/04/2022	0,000	0,000	0,107	0,201	0,000	0,000	0,000	0,054	0,250	0,000	0,138	0,000	0,000	0,250	0,000

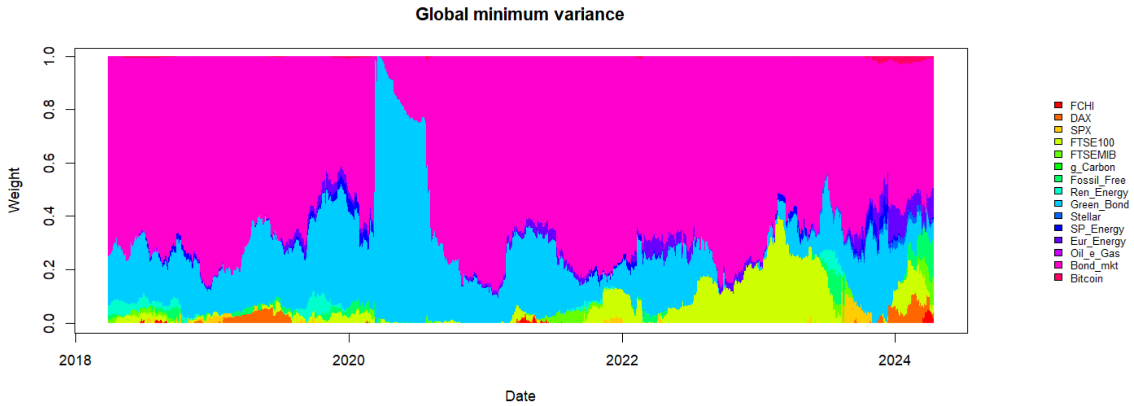

Figure 33: Global minimum variance portfolio between I+N+G (unrestricted)

Global minimum portfolio Based on this strategy, we are going to see the unrestricted and restricted rolling portfolio between all indexes, and the green/non-green restricted portfolio. From figure (33), the initial COVID pandemic portfolios are dominated by Green Bond, leaving space for Bond mkt. Regarding the Ukraine War, table 18 shows that portfolios are mainly made by the two bond indexes with small percentages of Fossil_Free and Eur_Energy. The situation one month after remains stable as SP_Energy slowly increases. Until now, even considering previous examples, we see that portfolio composition differs between pandemic and war. Indeed, while the COVID period tends to allocate to green indexes, war portfolios are mostly composed of non-green indexes.

Table 18: The weights of rolling portfolio at the beginning and a month later of the war in Ukraine

I+G+N

Ukraine War	FCHI	DAX	SPX	FTSE100	FTSEMIB	g_Carbon	Fossil_Free	Ren_Energy	Green_Bond	Stellar	SP_Energy	Eur_Energy	Oil_e_Gas	Bond_mkt	Bitcoin
24/02/2022	0,000	0,000	0,000	0,047	0,000	0,000	0,010	0,007	0,197	0,000	0,016	0,023	0,000	0,692	0,008
25/02/2022	0,000	0,000	0,000	0,005	0,000	0,000	0,001	0,013	0,199	0,000	0,010	0,044	0,000	0,721	0,007
28/02/2022	0,000	0,000	0,000	0,000	0,000	0,000	0,020	0,004	0,217	0,001	0,000	0,055	0,000	0,701	0,002
01/03/2022	0,000	0,000	0,000	0,009	0,000	0,000	0,033	0,000	0,211	0,001	0,000	0,052	0,000	0,693	0,000
02/03/2022	0,000	0,000	0,000	0,002	0,000	0,000	0,045	0,000	0,219	0,000	0,000	0,063	0,000	0,671	0,000
03/03/2022	0,000	0,000	0,000	0,000	0,000	0,000	0,045	0,000	0,218	0,000	0,000	0,063	0,000	0,673	0,000
04/03/2022	0,000	0,000	0,000	0,000	0,000	0,000	0,044	0,000	0,213	0,000	0,000	0,062	0,000	0,680	0,000
...															
24/03/2022	0,000	0,000	0,000	0,004	0,000	0,000	0,026	0,000	0,218	0,000	0,001	0,069	0,000	0,682	0,000
25/03/2022	0,000	0,000	0,000	0,002	0,000	0,000	0,027	0,000	0,227	0,000	0,004	0,068	0,000	0,671	0,000
28/03/2022	0,000	0,000	0,000	0,000	0,000	0,000	0,028	0,000	0,235	0,000	0,012	0,063	0,000	0,663	0,000
29/03/2022	0,000	0,000	0,000	0,000	0,000	0,000	0,025	0,000	0,227	0,000	0,011	0,067	0,000	0,670	0,000
30/03/2022	0,000	0,000	0,000	0,000	0,000	0,000	0,030	0,000	0,218	0,000	0,014	0,062	0,000	0,676	0,000
31/03/2022	0,000	0,000	0,000	0,000	0,000	0,000	0,030	0,000	0,216	0,000	0,014	0,061	0,000	0,679	0,000
01/04/2022	0,000	0,000	0,000	0,000	0,000	0,000	0,031	0,000	0,208	0,000	0,014	0,062	0,000	0,685	0,000

As before, we use constraints to try to improve the diversification. We see in figure (34) that the two considered periods are now similar as portfolios are composed by FTSE100, Bond mkt, and Green Bond. During COVID period stabilization, FTSE100 decreases, leaving room for Fossil_Free.

To see the differences between Green and non-Green indexes we try to construct the rolling portfolio without conventional indexes. It can be seen that in both periods green indexes have the main role with the difference that the Eur_Energy index gains weight during the Ukraine war stabilization period. Indeed, table 19 shows how Eur_Energy persists in both periods, balancing

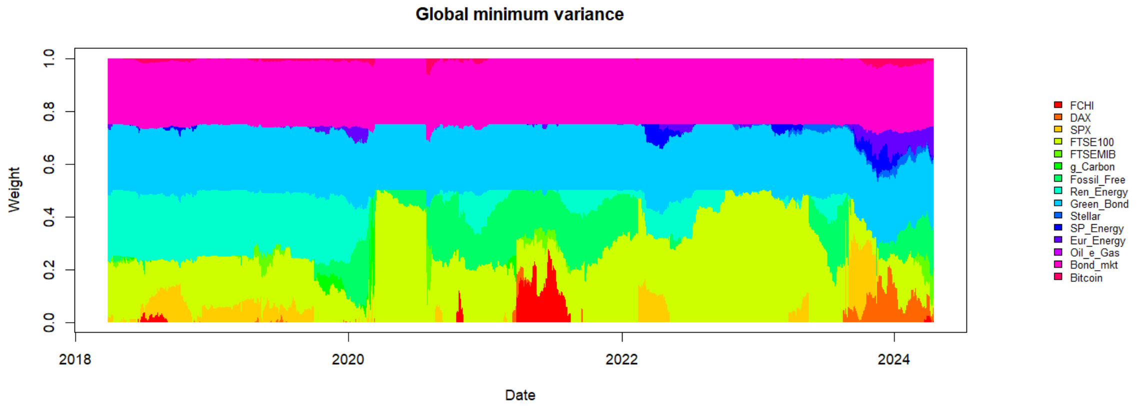

Figure 34: Global minimum variance portfolio between I+N+G (restricted)

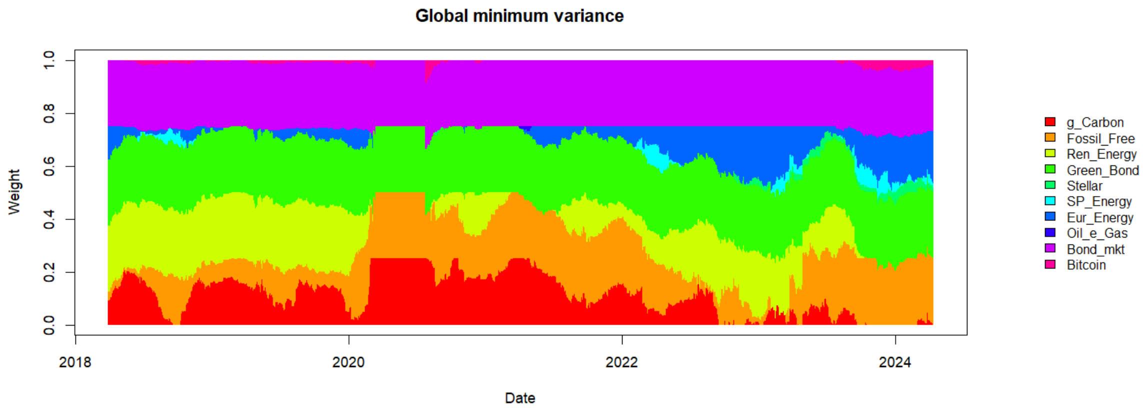

Figure 35: Global minimum variance portfolio between N+G (restricted)

the weight between green and non-green indexes. However, this strategy, by attributing almost the entire portfolio to green indices, highlights the ability to limit risk during periods of stress. The shift towards green indices during turbulent times suggests their potential as a stabilizing force in portfolio management. This trend may be attributed to increasing investor awareness of environmental concerns and the growing perception of green investments as a haven during economic uncertainties.

Table 19: The weights of rolling portfolio at the beginning and a month later of the COVID and the war in Ukraine

G+N (restricted)

COVID-19	g_Carbon	Fossil_Free	Ren_Energy	Green_Bond	Stellar	SP_Energy	Eur_Energy	Oil_e_Gas	Bond_mkt	Bitcoin
09/03/2020	0,250	0,233	0,000	0,250	0,000	0,000	0,000	0,000	0,250	0,017
10/03/2020	0,250	0,219	0,000	0,250	0,000	0,000	0,000	0,000	0,250	0,031
11/03/2020	0,250	0,219	0,000	0,250	0,000	0,000	0,000	0,000	0,250	0,031
12/03/2020	0,250	0,250	0,000	0,250	0,000	0,000	0,000	0,000	0,250	0,000
13/03/2020	0,250	0,250	0,000	0,250	0,000	0,000	0,000	0,000	0,250	0,000
16/03/2020	0,250	0,250	0,000	0,250	0,000	0,000	0,000	0,000	0,250	0,000
17/03/2020	0,250	0,250	0,000	0,250	0,000	0,000	0,000	0,000	0,250	0,000
...										
14/08/2020	0,213	0,250	0,011	0,250	0,000	0,000	0,000	0,000	0,250	0,026
17/08/2020	0,213	0,250	0,011	0,250	0,000	0,000	0,000	0,000	0,250	0,026
18/08/2020	0,210	0,250	0,015	0,250	0,000	0,000	0,000	0,000	0,250	0,025
19/08/2020	0,163	0,250	0,052	0,250	0,012	0,000	0,000	0,000	0,250	0,023
20/08/2020	0,163	0,250	0,052	0,250	0,012	0,000	0,000	0,000	0,250	0,023
21/08/2020	0,143	0,250	0,073	0,250	0,012	0,000	0,000	0,000	0,250	0,022
24/08/2020	0,153	0,250	0,067	0,250	0,012	0,000	0,000	0,000	0,250	0,019
Ukraine War	**g_Carbon**	**Fossil_Free**	**Ren_Energy**	**Green_Bond**	**Stellar**	**SP_Energy**	**Eur_Energy**	**Oil_e_Gas**	**Bond_mkt**	**Bitcoin**
24/02/2022	0,079	0,204	0,107	0,250	0,000	0,032	0,078	0,000	0,250	0,000
25/02/2022	0,081	0,215	0,099	0,250	0,000	0,021	0,085	0,000	0,250	0,000
28/02/2022	0,081	0,215	0,099	0,250	0,000	0,021	0,085	0,000	0,250	0,000
01/03/2022	0,084	0,220	0,092	0,250	0,000	0,027	0,077	0,000	0,250	0,000
02/03/2022	0,085	0,223	0,090	0,250	0,000	0,037	0,065	0,000	0,250	0,000
03/03/2022	0,073	0,226	0,092	0,250	0,000	0,055	0,053	0,000	0,250	0,000
04/03/2022	0,066	0,202	0,111	0,250	0,000	0,053	0,068	0,000	0,250	0,000
...										
24/03/2022	0,062	0,182	0,108	0,250	0,000	0,079	0,069	0,000	0,250	0,000
25/03/2022	0,062	0,182	0,108	0,250	0,000	0,079	0,069	0,000	0,250	0,000
28/03/2022	0,050	0,183	0,110	0,250	0,000	0,086	0,071	0,000	0,250	0,000
29/03/2022	0,051	0,185	0,111	0,250	0,000	0,088	0,065	0,000	0,250	0,000
30/03/2022	0,051	0,182	0,115	0,250	0,000	0,086	0,067	0,000	0,250	0,000
31/03/2022	0,051	0,183	0,114	0,250	0,000	0,086	0,067	0,000	0,250	0,000
01/04/2022	0,051	0,181	0,113	0,250	0,000	0,089	0,065	0,000	0,250	0,000

4.3.3 Rolling Mean-CVaR portfolios

Among various risk measures, as we said, Conditional Value at Risk (CVaR) has established itself as an effective tool for capturing potential losses in extreme scenarios, going beyond the limits of traditional Value at Risk (VaR). The CVaR not only considers the threshold value of losses (quantile) but also the extent of losses that exceed this threshold, offering a more complete view of tail risk. To better understand the temporal dynamics of optimal portfolio weights in response to changes in market conditions, we employ rolling window analysis. In this context, we will analyze how the optimal portfolio weights evolve, implementing a mean-CVaR optimization model. This analysis will allow us to observe the portfolio's responsiveness to the COVID-19 pandemic and the war in Ukraine, and evaluate the effectiveness of diversification and risk management strategies during periods of high volatility, as we previously did. The rolling window analysis will be conducted with a 90-day window, chosen to balance the need for statistical stability with the ability to rapidly adapt to market changes.

As before, we will analyze the minimum risk efficient portfolio using the target return obtained from the equally weighted portfolio among conventional money indices. Looking at the figure (36),

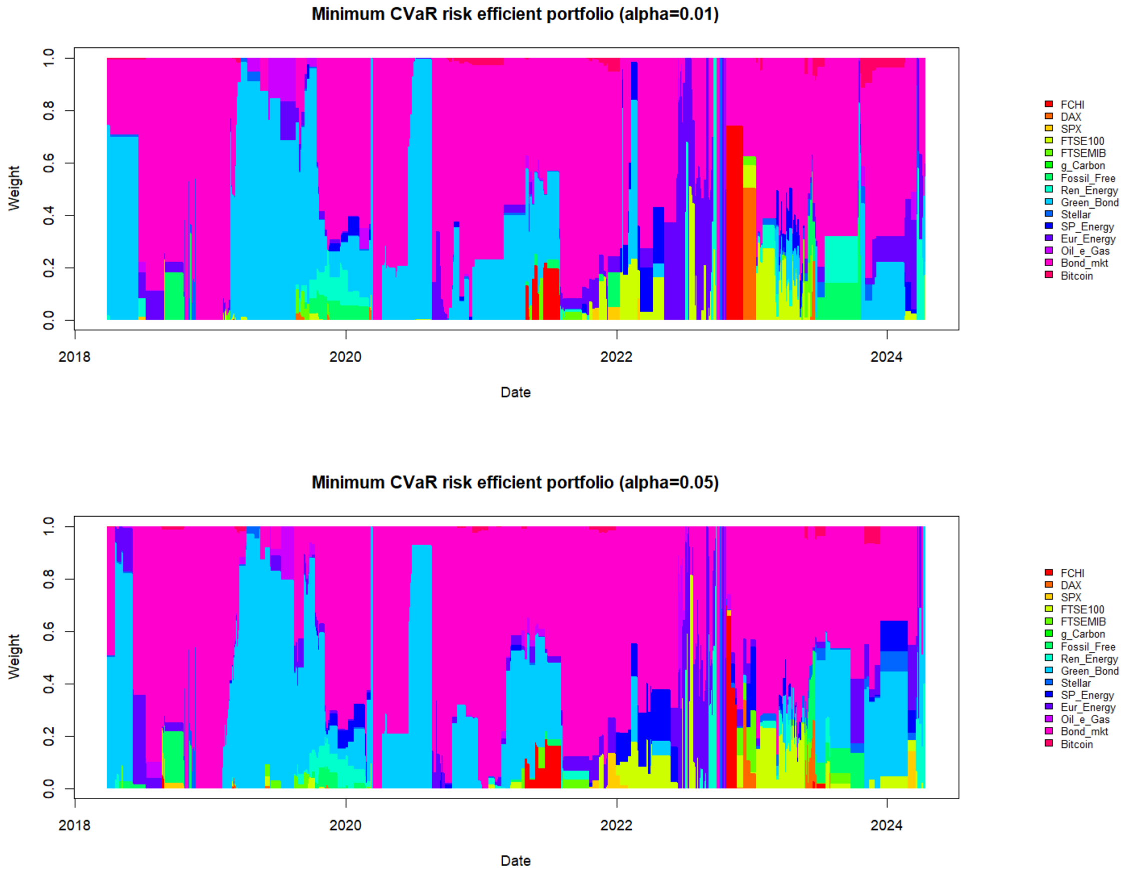

Figure 36: Minimum risk efficient portfolio between I+N+G (unrestricted) for confidence level $\alpha = 0.01$ and $\alpha = 0.05$

during the pandemic both portfolios allocate most of the weight to Green Bond and Bond mkt, persisting for a long period. Differently, the Ukraine War time exhibits more uncertainty even in the months after the beginning of the war. Indeed, table 20 shows a more dynamic portfolio for both two confidence levels, which don't include green indexes at all, allocating toward SP_Energy, Bond mkt, and FTSE100. Interesting is the fact that throughout the period taken into consideration, the green indices have the main role, except for the first year of the war in Ukraine. This can be seen well for $\alpha = 0.05$ where a year after the start of the war the green indices regain ground.

Figure (37) shows how the portfolio changes with constraints. As always, we impose 50% as the maximum weight the portfolio can allocate to conventional indexes and 25% to each green and non-green index. During the peak of the COVID-19 pandemic, the portfolio composition changes with an increase in the weight of more stable assets such as Green Bond and Bond Market, indicating a risk reduction strategy during periods of high volatility. We also see assets like Fossil Free and Ren_Energy gain weight, highlighting a greater emphasis on sustainable investing. With the start of the war in Ukraine in 2022, both portfolios show a weight change. Non-green energy assets gain weight. At the same time, green assets such as Green Bond and Ren Energy continue to play an

Table 20: The weights of the rolling portfolio at the beginning and a month later of the war in Ukraine for confidence level $\alpha = 0.01$ and $\alpha = 0.05$

I+G+N

Ukraine War ($\alpha = 0.01$)	FCHI	DAX	SPX	FTSE100	FTSEMIB	g_Carbon	Fossil_Free	Ren_Energy	Green_Bond	Stellar	SP_Energy	Eur_Energy	Oil_e_Gas	Bond_mkt	Bitcoin
24/02/2022	0,000	0,000	0,000	0,232	0,000	0,000	0,000	0,000	0,606	0,000	0,145	0,000	0,000	0,016	0,000
25/02/2022	0,000	0,000	0,000	0,232	0,000	0,000	0,000	0,000	0,606	0,000	0,145	0,000	0,000	0,016	0,000
28/02/2022	0,000	0,000	0,080	0,108	0,000	0,000	0,000	0,000	0,000	0,000	0,172	0,000	0,000	0,640	0,000
01/03/2022	0,000	0,000	0,080	0,108	0,000	0,000	0,000	0,000	0,000	0,000	0,172	0,000	0,000	0,640	0,000
02/03/2022	0,000	0,000	0,080	0,108	0,000	0,000	0,000	0,000	0,000	0,000	0,172	0,000	0,000	0,640	0,000
03/03/2022	0,000	0,000	0,080	0,108	0,000	0,000	0,000	0,000	0,000	0,000	0,172	0,000	0,000	0,640	0,000
04/03/2022	0,000	0,000	0,080	0,108	0,000	0,000	0,000	0,000	0,000	0,000	0,172	0,000	0,000	0,640	0,000
24/03/2022	0,000	0,000	0,000	0,030	0,000	0,000	0,000	0,000	0,000	0,000	0,169	0,072	0,000	0,729	0,000
25/03/2022	0,000	0,000	0,000	0,030	0,000	0,000	0,000	0,000	0,000	0,000	0,169	0,072	0,000	0,729	0,000
28/03/2022	0,000	0,000	0,000	0,030	0,000	0,000	0,000	0,000	0,000	0,000	0,169	0,072	0,000	0,729	0,000
29/03/2022	0,000	0,000	0,000	0,030	0,000	0,000	0,000	0,000	0,000	0,000	0,169	0,072	0,000	0,729	0,000
30/03/2022	0,000	0,000	0,000	0,030	0,000	0,000	0,000	0,000	0,000	0,000	0,169	0,072	0,000	0,729	0,000
31/03/2022	0,000	0,000	0,000	0,030	0,000	0,000	0,000	0,000	0,000	0,000	0,169	0,072	0,000	0,729	0,000
01/04/2022	0,000	0,000	0,000	0,030	0,000	0,000	0,000	0,000	0,000	0,000	0,169	0,072	0,000	0,729	0,000
Ukraine War ($\alpha = 0.05$)	**FCHI**	**DAX**	**SPX**	**FTSE100**	**FTSEMIB**	**g_Carbon**	**Fossil_Free**	**Ren_Energy**	**Green_Bond**	**Stellar**	**SP_Energy**	**Eur_Energy**	**Oil_e_Gas**	**Bond_mkt**	**Bitcoin**
24/02/2022	0,000	0,000	0,000	0,178	0,000	0,000	0,000	0,000	0,248	0,000	0,128	0,000	0,000	0,447	0,000
25/02/2022	0,000	0,000	0,000	0,178	0,000	0,000	0,000	0,000	0,248	0,000	0,128	0,000	0,000	0,447	0,000
28/02/2022	0,000	0,000	0,000	0,146	0,000	0,000	0,000	0,000	0,000	0,000	0,172	0,000	0,000	0,683	0,000
01/03/2022	0,000	0,000	0,000	0,084	0,000	0,000	0,000	0,000	0,000	0,000	0,196	0,000	0,000	0,720	0,000
02/03/2022	0,000	0,000	0,000	0,083	0,000	0,000	0,000	0,000	0,000	0,000	0,204	0,000	0,000	0,713	0,000
03/03/2022	0,000	0,000	0,000	0,083	0,000	0,000	0,000	0,000	0,000	0,000	0,204	0,000	0,000	0,713	0,000
04/03/2022	0,000	0,000	0,000	0,083	0,000	0,000	0,000	0,000	0,000	0,000	0,204	0,000	0,000	0,713	0,000
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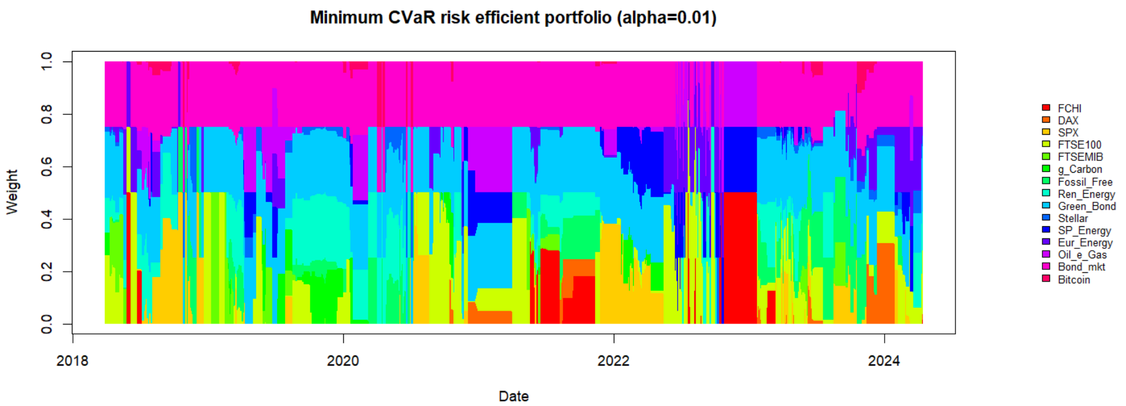
24/03/2022	0,000	0,000	0,000	0,186	0,000	0,000	0,000	0,000	0,000	0,000	0,215	0,000	0,000	0,599	0,000
25/03/2022	0,000	0,000	0,000	0,186	0,000	0,000	0,000	0,000	0,000	0,000	0,216	0,000	0,000	0,598	0,000
28/03/2022	0,000	0,000	0,000	0,054	0,000	0,000	0,000	0,000	0,000	0,000	0,236	0,000	0,000	0,710	0,000
29/03/2022	0,000	0,000	0,000	0,054	0,000	0,000	0,000	0,000	0,000	0,000	0,236	0,000	0,000	0,710	0,000
30/03/2022	0,000	0,000	0,000	0,054	0,000	0,000	0,000	0,000	0,000	0,000	0,236	0,000	0,000	0,710	0,000
31/03/2022	0,000	0,000	0,000	0,054	0,000	0,000	0,000	0,000	0,000	0,000	0,236	0,000	0,000	0,710	0,000
01/04/2022	0,000	0,000	0,000	0,054	0,000	0,000	0,000	0,000	0,000	0,000	0,236	0,000	0,000	0,710	0,000



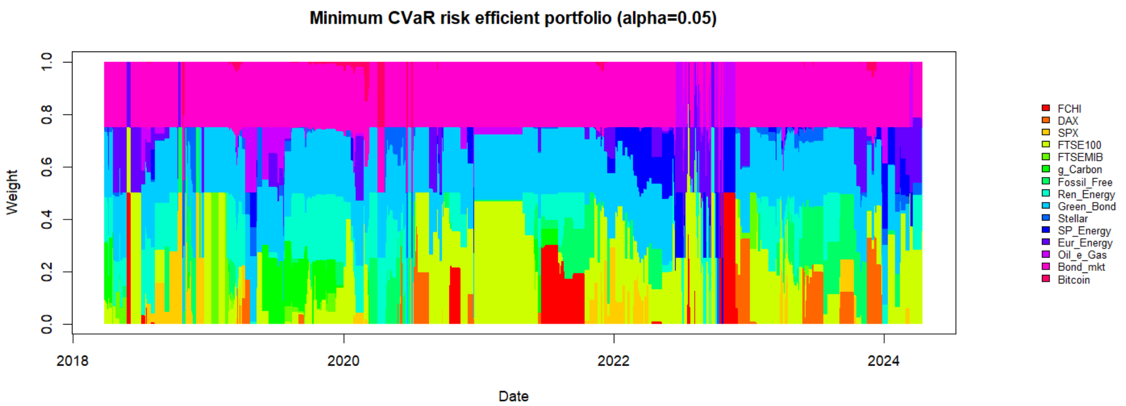


Figure 37: Minimum risk efficient portfolio between I+N+G (restricted) for confidence level $\alpha = 0.01$ and $\alpha = 0.05$

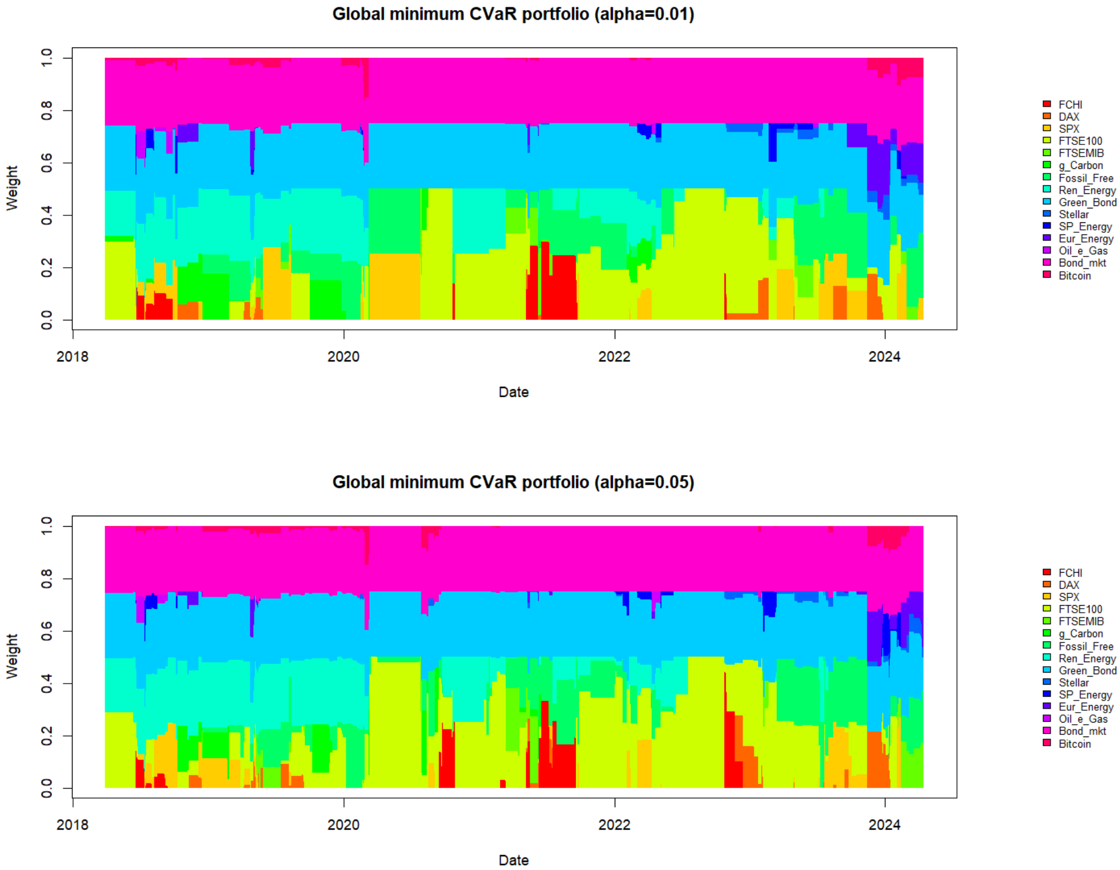


Figure 38: Global minimum CVaR portfolio between I+N+G (restricted) for confidence level $\alpha = 0.01$ and $\alpha = 0.05$

important role, suggesting that the strategy of mitigating risk through sustainable investments has remained relevant even in these times of stress. Table 21 shows how the portfolio behaved during the Pandemic and the Ukraine War. Interesting is the allocation of Bitcoin during the stabilization period after the pandemic.

Lastly, we see the Global minimum CVaR-restricted portfolio in figure (38). Both portfolios at different confidence levels show very slight diversification, tending to allocate the same financial indices. Here, both in times of COVID and war, the optimization strategy attributes the main role to green indices. However, Bond mkt remains significantly present in the portfolio. The main difference that can be seen between the two confidence levels is that with $\alpha = 0.01$, during the pandemic, the portfolio allocates an important portion of the weight to Fossil_Free, while with alpha 0.05 it is attributed to the SPX index. What we saw previously is confirmed: when we apply weight constraints, green indices are chosen to limit financial risk both in the period of the pandemic and the war in Ukraine. In unconstrained portfolios, we see a tendency to prefer non-green indices for risk mitigation during the Ukrainian war.

Table 21: The weights of the rolling portfolio of the COVID and the war in Ukraine for confidence level $\alpha = 0.01$ and $\alpha = 0.05$

I+G+N (restricted, $\alpha = 0.01$)

COVID-19	FCHI	DAX	SPX	FTSE100	FTSEMIB	g_Carbon	Fossil_Free	Ren_Energy	Green_Bond	Stellar	SP_Energy	Eur_Energy	Oil_e_Gas	Bond_mkt	Bitcoin
09/03/2020	0,000	0,000	0,000	0,000	0,000	0,007	0,243	0,250	0,250	0,000	0,000	0,000	0,000	0,250	0,000
10/03/2020	0,000	0,000	0,000	0,000	0,000	0,007	0,243	0,250	0,250	0,000	0,000	0,000	0,000	0,250	0,000
11/03/2020	0,000	0,000	0,000	0,000	0,000	0,000	0,250	0,250	0,250	0,000	0,000	0,000	0,000	0,250	0,000
12/03/2020	0,000	0,000	0,000	0,000	0,000	0,000	0,250	0,250	0,250	0,000	0,000	0,000	0,000	0,250	0,000
13/03/2020	0,000	0,000	0,000	0,000	0,000	0,000	0,250	0,250	0,250	0,000	0,000	0,000	0,000	0,250	0,000
16/03/2020	0,000	0,000	0,000	0,000	0,000	0,000	0,250	0,250	0,250	0,000	0,000	0,000	0,000	0,250	0,000
17/03/2020	0,000	0,000	0,000	0,000	0,000	0,000	0,250	0,250	0,250	0,000	0,000	0,000	0,000	0,250	0,000
...															
09/04/2020	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,250	0,250	0,000	0,000	0,000	0,000	0,250	0,250
14/04/2020	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,250	0,250	0,000	0,000	0,000	0,000	0,250	0,250
15/04/2020	0,000	0,000	0,000	0,000	0,000	0,000	0,250	0,250	0,250	0,000	0,000	0,000	0,000	0,250	0,000
16/04/2020	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,250	0,250	0,000	0,000	0,000	0,000	0,250	0,250
17/04/2020	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,250	0,250	0,000	0,000	0,000	0,000	0,250	0,250
20/04/2020	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,250	0,250	0,000	0,000	0,000	0,000	0,250	0,250
21/04/2020	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,250	0,250	0,000	0,000	0,000	0,000	0,250	0,250
Ukraine War	**FCHI**	**DAX**	**SPX**	**FTSE100**	**FTSEMIB**	**g_Carbon**	**Fossil_Free**	**Ren_Energy**	**Green_Bond**	**Stellar**	**SP_Energy**	**Eur_Energy**	**Oil_e_Gas**	**Bond_mkt**	**Bitcoin**
24/02/2022	0,000	0,000	0,121	0,022	0,000	0,105	0,000	0,000	0,250	0,000	0,250	0,000	0,001	0,250	0,000
25/02/2022	0,000	0,000	0,251	0,039	0,000	0,000	0,000	0,000	0,250	0,000	0,210	0,000	0,000	0,250	0,000
28/02/2022	0,000	0,000	0,251	0,039	0,000	0,000	0,000	0,000	0,250	0,000	0,210	0,000	0,000	0,250	0,000
01/03/2022	0,000	0,000	0,251	0,039	0,000	0,000	0,000	0,000	0,250	0,000	0,210	0,000	0,000	0,250	0,000
02/03/2022	0,000	0,000	0,240	0,040	0,000	0,000	0,000	0,000	0,250	0,000	0,219	0,000	0,000	0,250	0,000
03/03/2022	0,000	0,000	0,240	0,040	0,000	0,000	0,000	0,000	0,250	0,000	0,219	0,000	0,000	0,250	0,000
04/03/2022	0,000	0,000	0,240	0,040	0,000	0,000	0,000	0,000	0,250	0,000	0,219	0,000	0,000	0,250	0,000
...															
24/03/2022	0,000	0,000	0,175	0,047	0,000	0,000	0,000	0,000	0,250	0,000	0,250	0,000	0,028	0,250	0,000
25/03/2022	0,000	0,000	0,191	0,046	0,000	0,000	0,000	0,000	0,250	0,000	0,250	0,000	0,013	0,250	0,000
28/03/2022	0,000	0,000	0,191	0,046	0,000	0,000	0,000	0,000	0,250	0,000	0,250	0,000	0,013	0,250	0,000
29/03/2022	0,000	0,000	0,191	0,046	0,000	0,000	0,000	0,000	0,250	0,000	0,250	0,000	0,013	0,250	0,000
30/03/2022	0,000	0,000	0,193	0,046	0,000	0,000	0,000	0,000	0,250	0,000	0,250	0,000	0,011	0,250	0,000
31/03/2022	0,000	0,000	0,193	0,046	0,000	0,000	0,000	0,000	0,250	0,000	0,250	0,000	0,011	0,250	0,000
01/04/2022	0,000	0,000	0,204	0,045	0,000	0,000	0,000	0,000	0,250	0,000	0,250	0,000	0,001	0,250	0,000

4.4 GARCH and Value-at-Risk

In financial risk management, accurate estimation of risk measures, such as Value at Risk (VaR) and Conditional Value at Risk (CVaR), is essential for assessing a portfolio's potential loss in adverse market conditions. The complexity and variability of financial markets require advanced methods to model volatility, a key component of risk. This section applies GARCH (Generalized Autoregressive Conditional Heteroskedasticity) models to estimate VaR and CVaR with 1% and 5% confidence levels.

GARCH models are particularly well suited to capture the autoregressive nature and conditional variance of the volatility of financial returns. They allow for modeling the evolution of volatility over time by accounting for volatility clusters, a phenomenon commonly observed in financial data, where periods of high volatility tend to follow other periods of high volatility, and vice versa.

Applying GARCH models to estimate VaR and CVaR is justified because these models provide a more realistic representation of volatility dynamics than models that assume constant volatility. The ability of GARCH models to adjust for changes in volatility allows for more accurate and timely risk estimates. This is particularly relevant in modern financial markets, which are characterized by high uncertainty and rapid changes in market conditions.

For the analysis, we will follow the work carried out by Perlin, Marcelo Scherer and Mastella,

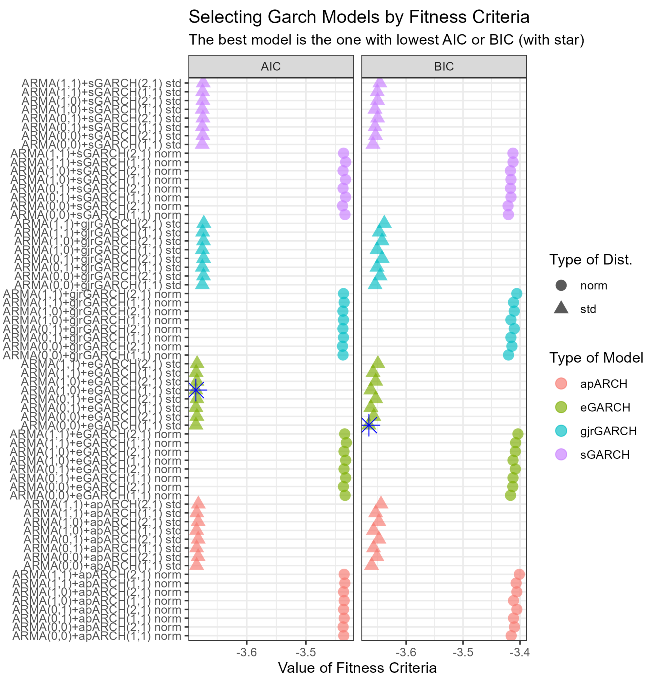
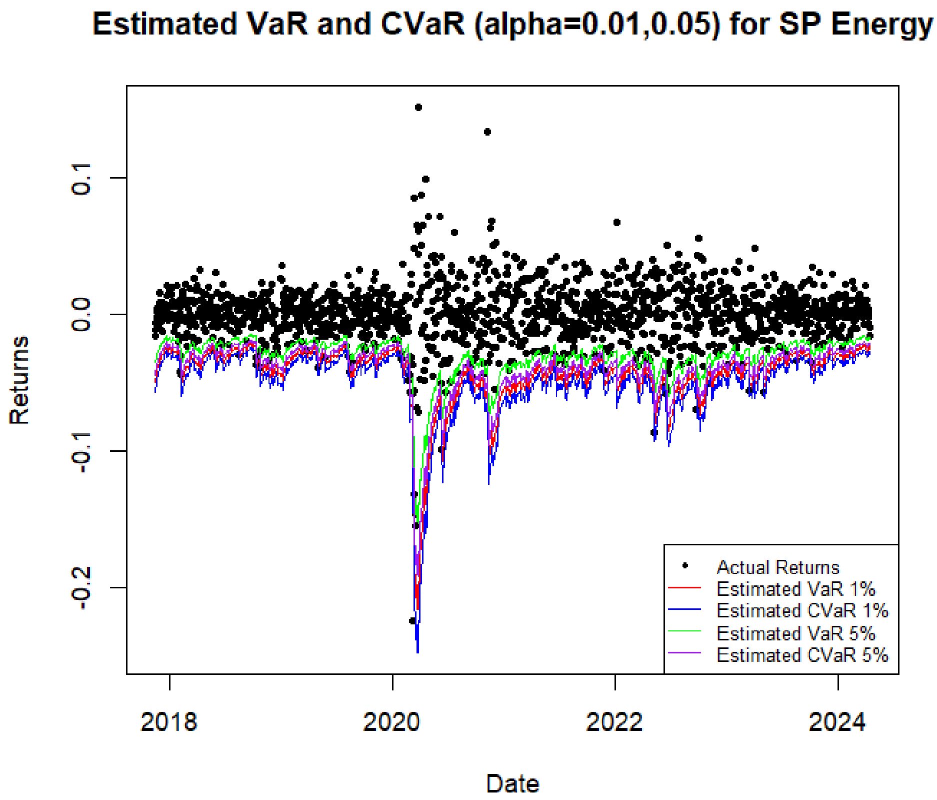


Figure 39: Selection of GARCH models by fitness criteria

Mauro and Vancin, Daniel Francisco and Ramos, Henrique Pinto (2020) in the application of Garch models. We will use R functions developed by Perlin et al. for the selection of the best GARCH model to apply to our financial indexes. The function computed the best model based on AIC and BIC criteria. An example of the results is shown in figure (39). Note that the results report a small number of combinations of lags and distribution parameters. We do so for simplicity and to keep the number of results manageable. Table 22 presents the best model for each index based on AIC and BIC and the figures below show the graphical results.

Table 22: Best GARCH model for each index

Index	**BIC**	**AIC**
Bitcoin	ARMA(1,1)+apARCH(2,1) norm	ARMA(1,1)+eGARCH(1,1) norm
Oil_e_Gas	ARMA(1,1)+sGARCH(2,1) norm	ARMA(1,1)+sGARCH(2,1) norm
g_Carbon	ARMA(1,1)+sGARCH(2,1) norm	ARMA(1,1)+sGARCH(2,1) norm
Green_Bond	ARMA(0,0)+apARCH(2,1) norm	ARMA(0,0)+apARCH(2,1) norm
Bond_mkt	ARMA(1,1)+apARCH(2,1) norm	ARMA(0,1)+apARCH(2,1) norm
Ren_Energy	ARMA(1,1)+gjrGARCH(2,1) norm	ARMA(0,0)+sGARCH(2,1) norm
SP_Energy	ARMA(1,1)+sGARCH(2,1) norm	ARMA(1,1)+sGARCH(2,1) norm
Fossil_Free	ARMA(1,1)+sGARCH(2,1) norm	ARMA(1,1)+sGARCH(2,1) norm
Eur_Energy	ARMA(1,1)+sGARCH(2,1) norm	ARMA(1,1)+sGARCH(2,1) norm
Stellar	ARMA(1,1)+apARCH(2,1) norm	ARMA(1,1)+apARCH(2,1) norm
FCHI	ARMA(1,1)+sGARCH(2,1) norm	ARMA(1,1)+sGARCH(2,1) norm
DAX	ARMA(1,1)+sGARCH(2,1) norm	ARMA(0,1)+sGARCH(2,1) norm
SPX	ARMA(1,1)+sGARCH(2,1) norm	ARMA(1,0)+sGARCH(2,1) norm
FTSE100	ARMA(1,1)+apARCH(2,1) norm	ARMA(1,0)+sGARCH(2,1) norm
FTSEMIB	ARMA(1,1)+sGARCH(2,1) norm	ARMA(1,1)+sGARCH(2,1) norm


Estimated VaR and CVaR (alpha=0.01,0.05) for SP Energy
Returns
0.1
0.0
-0.1
-0.2
Date
2018
2020
2022
2024
Actual Returns
Estimated VaR 1%
Estimated CVaR 1%
Estimated VaR 5%
Estimated CVaR 5%

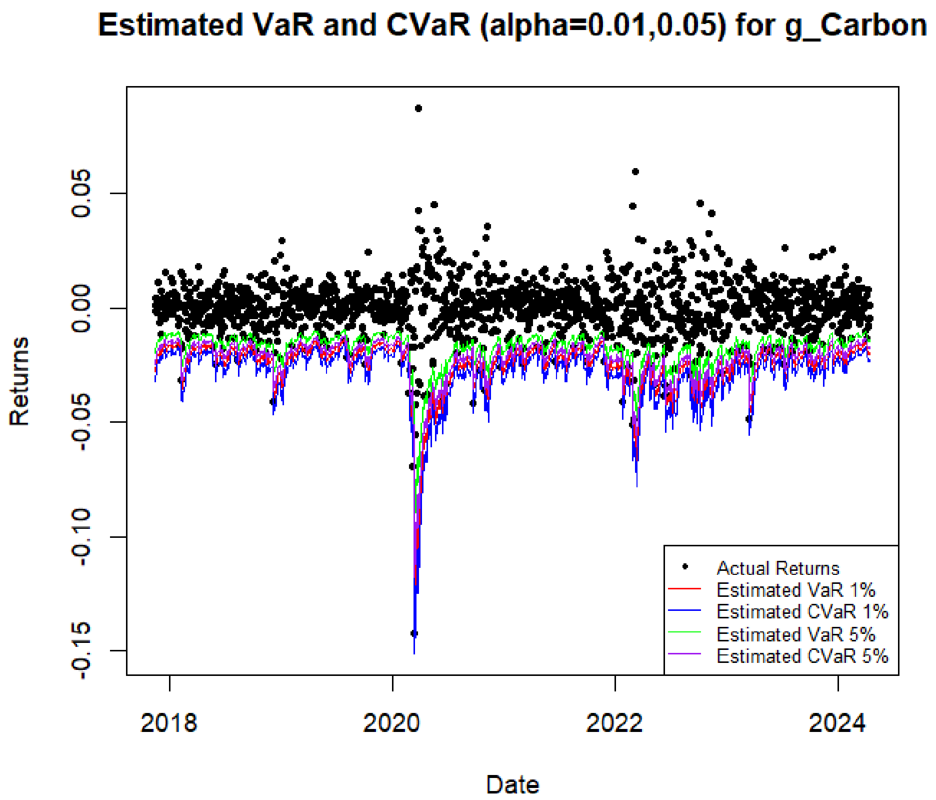
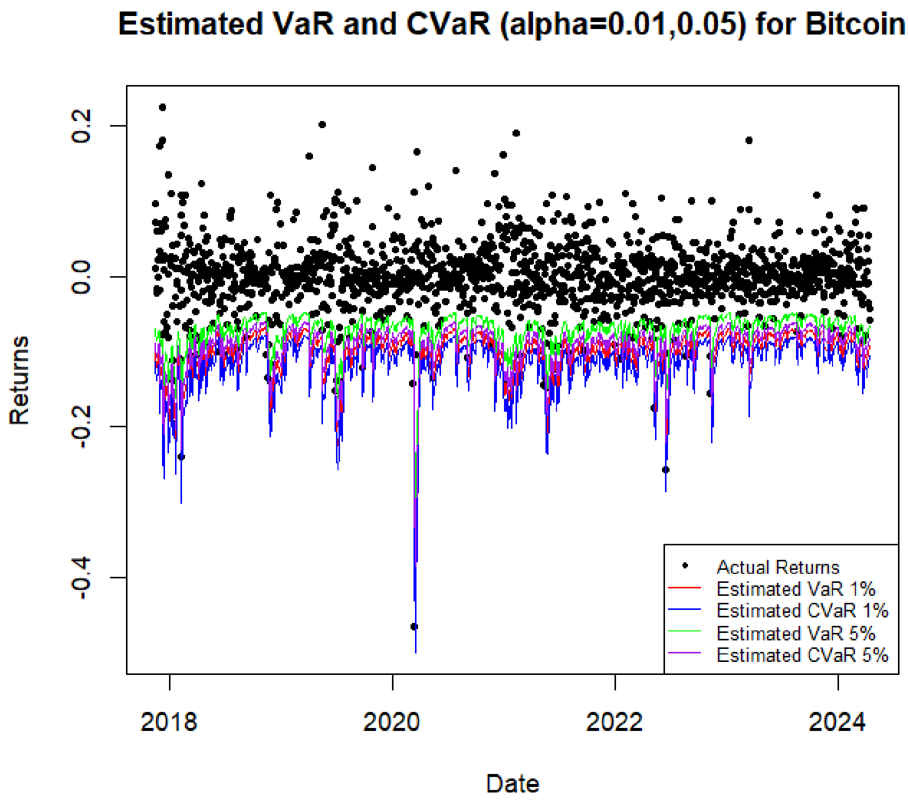

Estimated VaR and CVaR (alpha=0.01,0.05) for g_Carbon
Returns
0.05
0.00
-0.05
-0.10
-0.15
Date
2018
2020
2022
2024
Actual Returns
Estimated VaR 1%
Estimated CVaR 1%
Estimated VaR 5%
Estimated CVaR 5%

Figure 40


Estimated VaR and CVaR (alpha=0.01,0.05) for Bitcoin
Returns
0.2
0.0
-0.2
-0.4
2018
2020
2022
2024
Date
Actual Returns
Estimated VaR 1%
Estimated CVaR 1%
Estimated VaR 5%
Estimated CVaR 5%

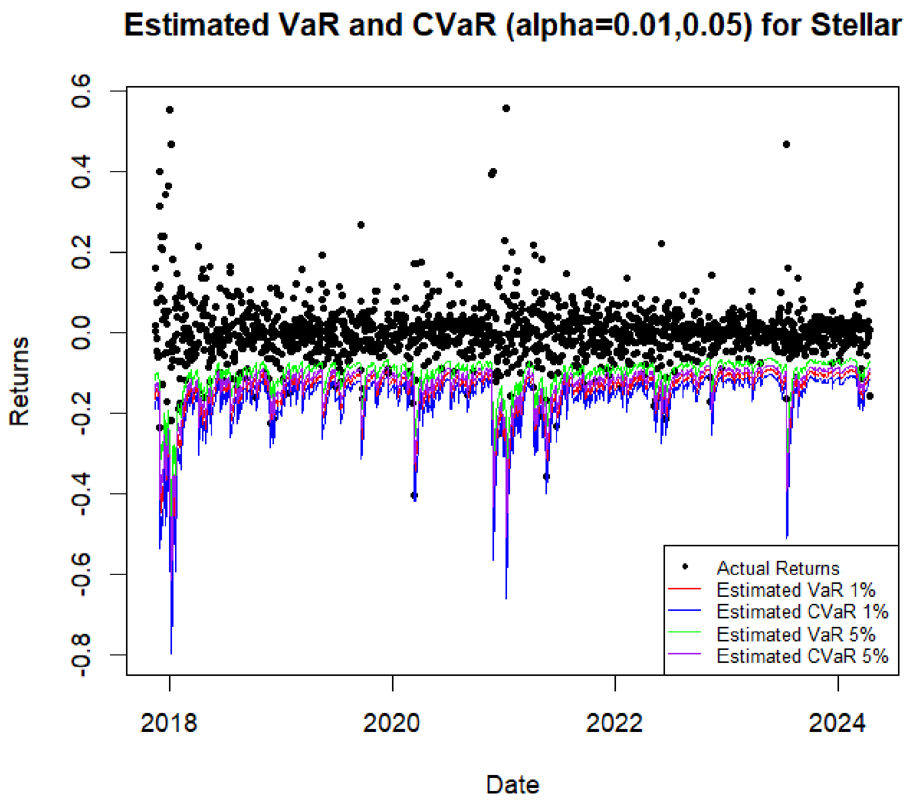
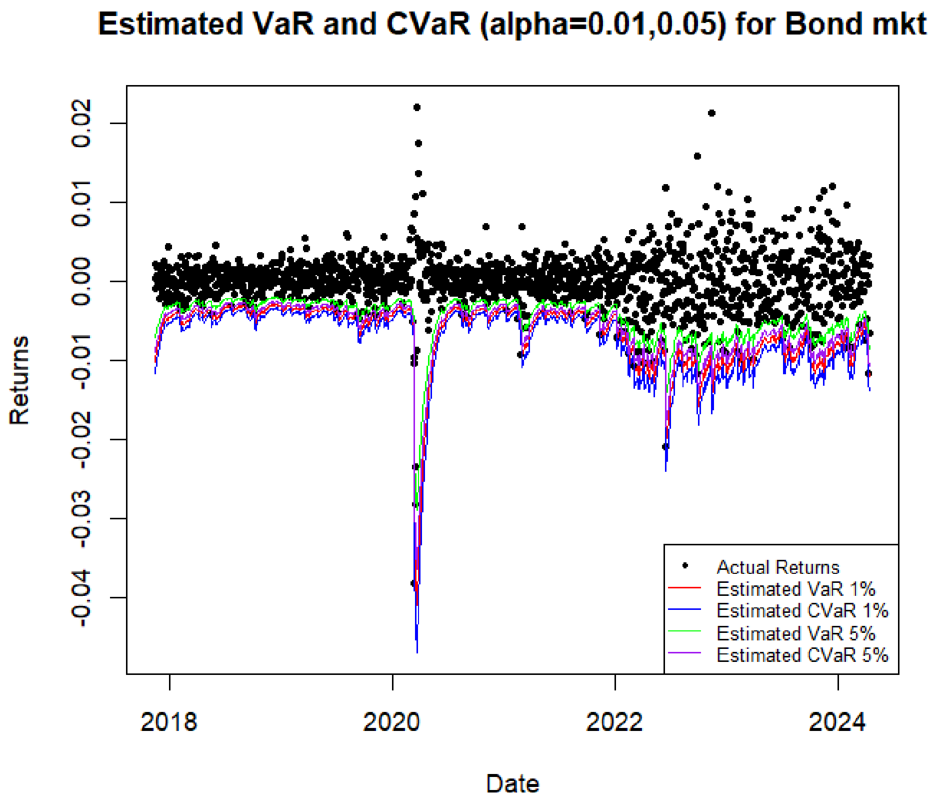
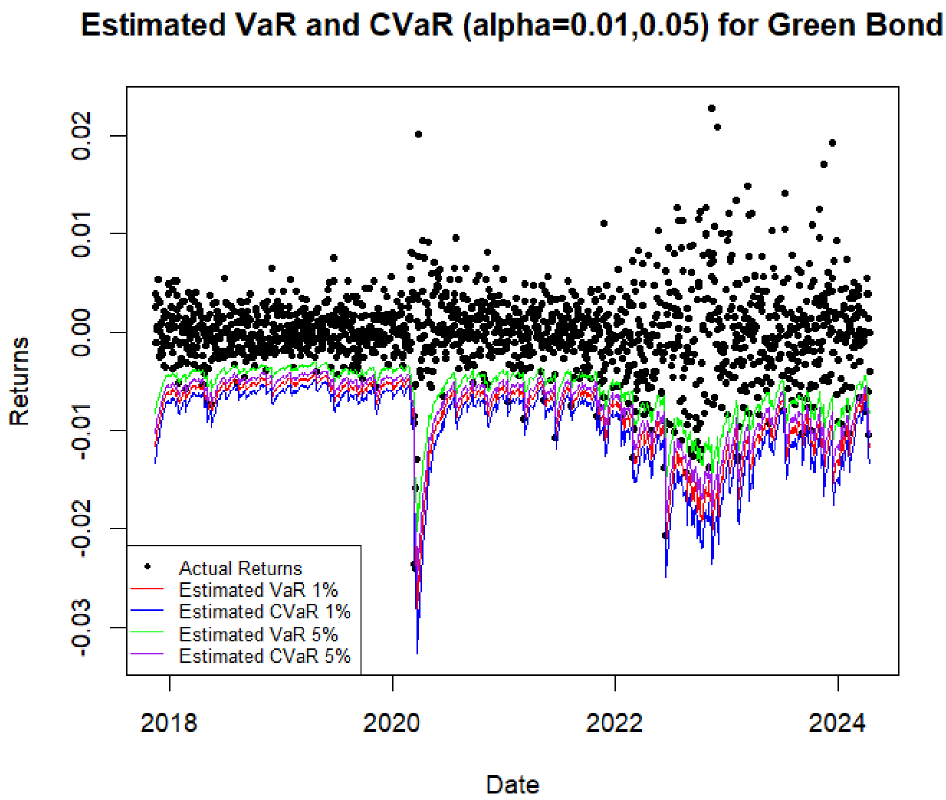

Estimated VaR and CVaR (alpha=0.01,0.05) for Stellar
Returns
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
-0.8
2018
2020
2022
2024
Date
Actual Returns
Estimated VaR 1%
Estimated CVaR 1%
Estimated VaR 5%
Estimated CVaR 5%

Figure 41

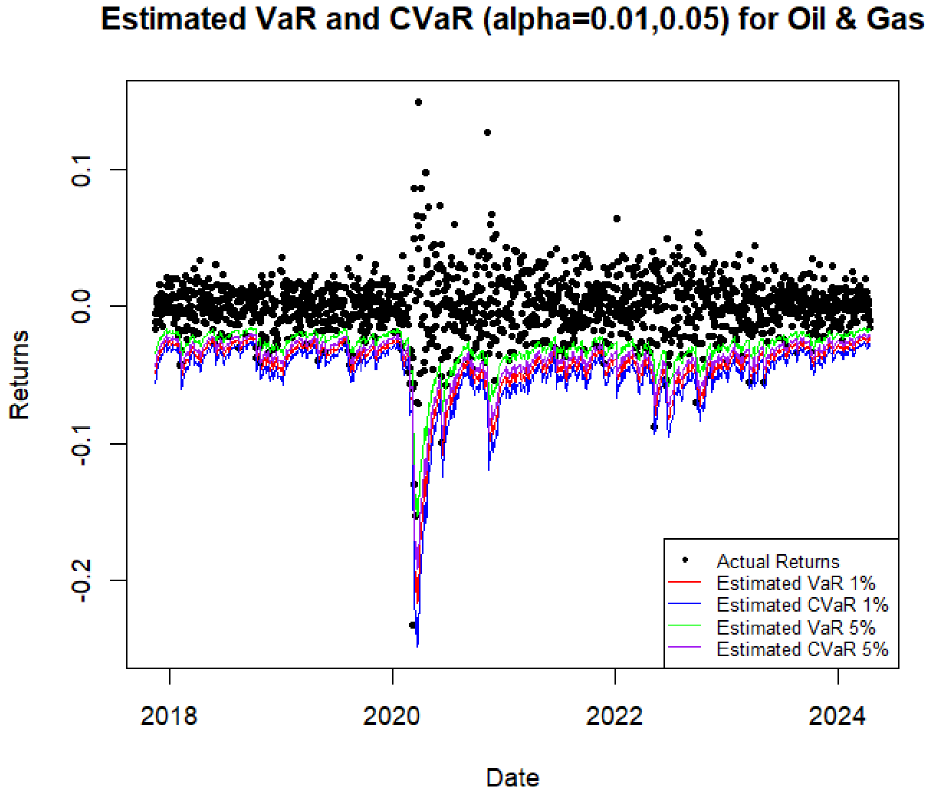


Estimated VaR and CVaR (alpha=0.01,0.05) for Bond mkt
Returns
0.02
0.01
0.00
-0.01
-0.02
-0.03
-0.04
Actual Returns
Estimated VaR 1%
Estimated CVaR 1%
Estimated VaR 5%
Estimated CVaR 5%
2018
2020
2022
2024
Date

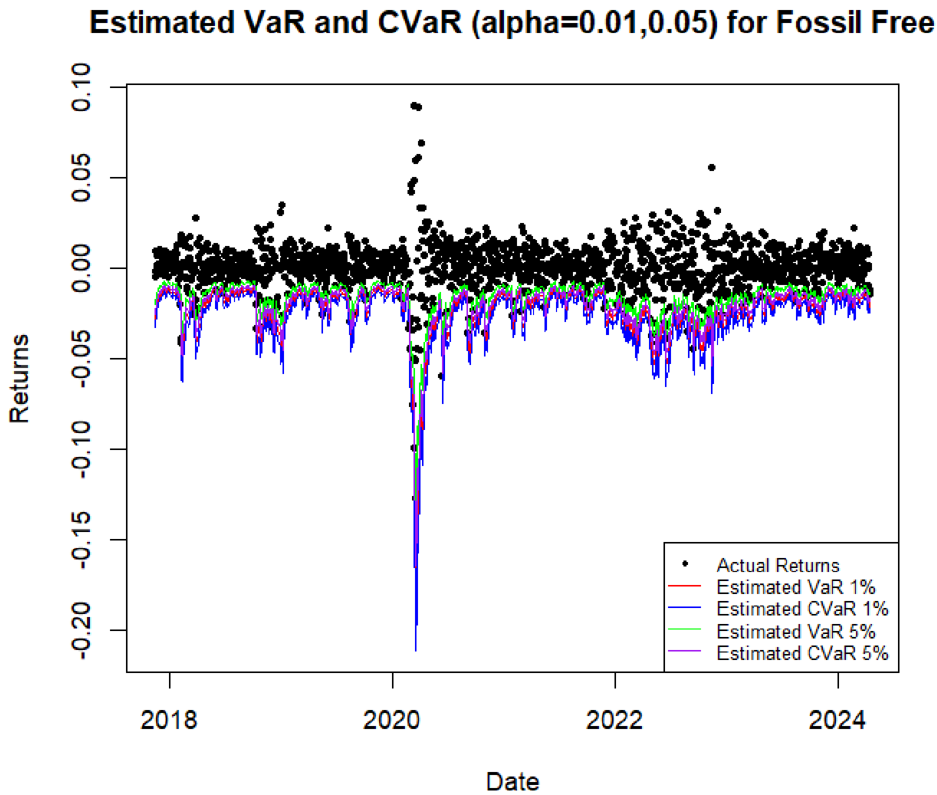


Estimated VaR and CVaR (alpha=0.01,0.05) for Green Bond
Returns
0.02
0.01
0.00
-0.01
-0.02
-0.03
Actual Returns
Estimated VaR 1%
Estimated CVaR 1%
Estimated VaR 5%
Estimated CVaR 5%
2018
2020
2022
2024
Date

Figure 42

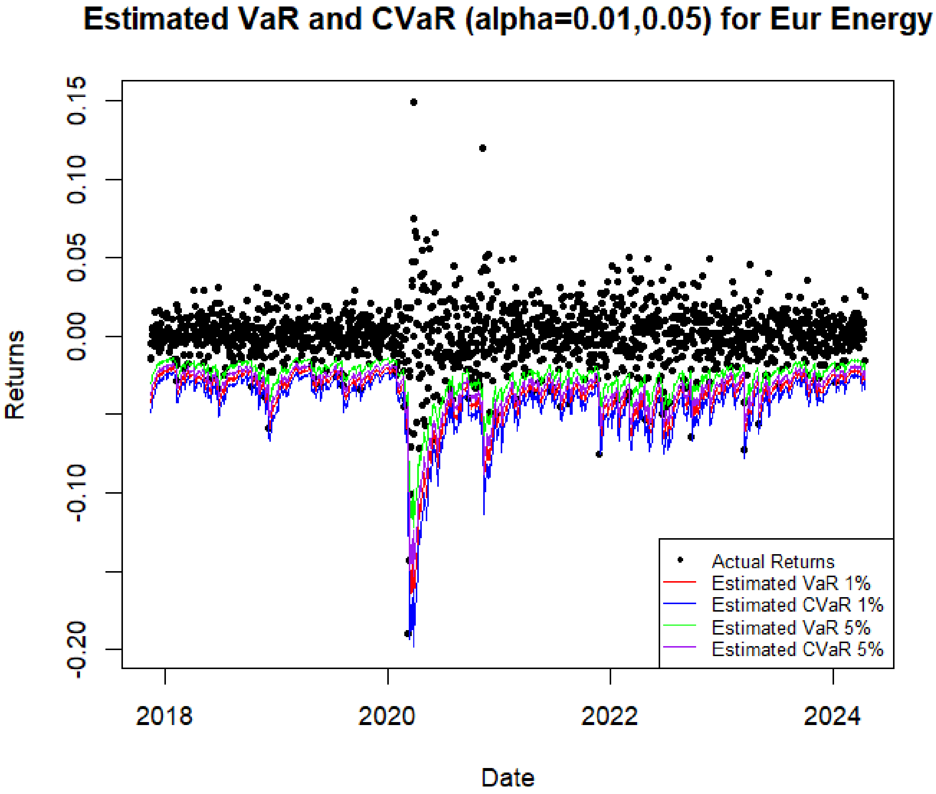


Estimated VaR and CVaR (alpha=0.01,0.05) for Oil & Gas
Returns
0.1
0.0
-0.1
-0.2
2018
2020
2022
2024
Date
Actual Returns
Estimated VaR 1%
Estimated CVaR 1%
Estimated VaR 5%
Estimated CVaR 5%

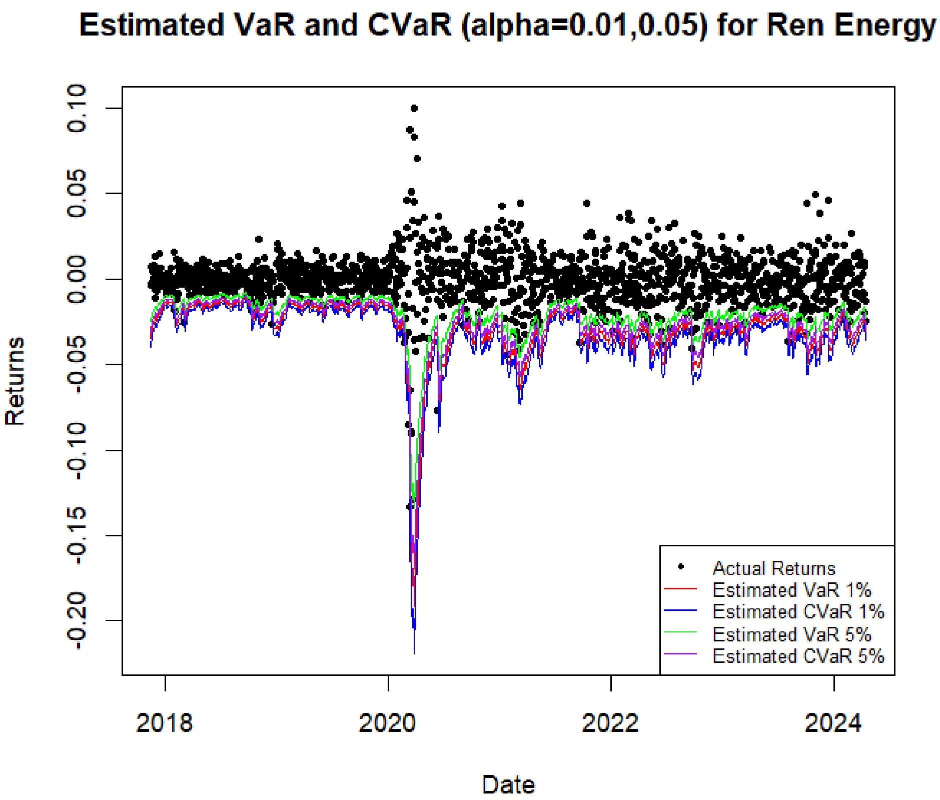


Estimated VaR and CVaR (alpha=0.01,0.05) for Fossil Free
Returns
0.10
0.05
0.00
-0.05
-0.10
-0.15
-0.20
2018
2020
2022
2024
Date
Actual Returns
Estimated VaR 1%
Estimated CVaR 1%
Estimated VaR 5%
Estimated CVaR 5%

Figure 43



Estimated VaR and CVaR (alpha=0.01,0.05) for Eur Energy
Returns
0.15
0.10
0.05
0.00
-0.10
-0.20
2018
2020
2022
2024
Date
Actual Returns
Estimated VaR 1%
Estimated CVaR 1%
Estimated VaR 5%
Estimated CVaR 5%

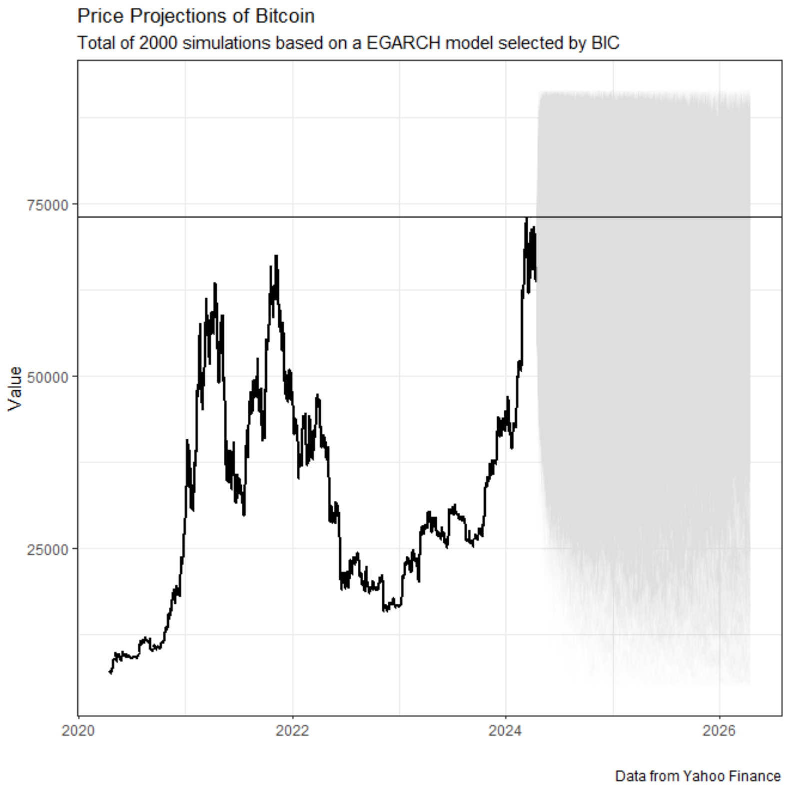


Estimated VaR and CVaR (alpha=0.01,0.05) for Ren Energy
Returns
0.10
0.05
0.00
-0.05
-0.10
-0.15
-0.20
2018
2020
2022
2024
Date
Actual Returns
Estimated VaR 1%
Estimated CVaR 1%
Estimated VaR 5%
Estimated CVaR 5%

Figure 44

Now, are presented the results of applying GARCH models to estimate the probabilities of reaching historical peaks and future price projections for two highly volatile assets: Bitcoin and Stellar. These assets were chosen because, during the preliminary analysis of the portfolio, it was observed that both had the highest Value at Risk (VaR) and Conditional Value at Risk (CVaR) values, making them among the most volatile indices. Furthermore, both Bitcoin and Stellar were not selected in the portfolio optimization process, which is why it was decided to apply GARCH models to further analyze them and evaluate their risk and potential opportunities. Following the work made by Perlin et al., with the optimal GARCH model we've estimated with our dataset, we can now generate simulations. These simulations will project potential future scenarios for both the returns time series and the possible trajectories of the Bitcoin and Stellar index over the next few years. Our goal in this study is to generate numerous simulated time series and potential future trajectories for the Bitcoin and Stellar index. After completing these simulations, we'll analyze the results to assess the probability of the index surpassing its historical high point again. We've conducted 2,000 simulations, and the outcomes of these are presented in figure (45) and (46).

A Monte Carlo simulation based on an EGARCH model was used to project Bitcoin's price. The results show considerable variability, indicating high future volatility. This simulation suggests that Bitcoin may continue to experience wide fluctuations, with the potential to reach new all-time highs or undergo significant declines. Using the same EGARCH model, probabilities were calculated for Bitcoin to reach its all-time high of $73,083. The graph shows these probabilities increasing over time, reaching 50% by August 21, 2024, and 75% by November 25, 2026. This suggests that despite high volatility, there's a substantial chance Bitcoin could return to its peak levels soon. A Monte Carlo simulation using an SGARCH model indicates high volatility in Stellar's price. Given its history of significant fluctuations, the price is likely to continue on this volatile path. The simulation suggests a wide range of potential future outcomes, from substantial increases to significant decreases. The probability chart for Stellar in figure (46) shows that its chances of reaching its historical peak of 0.746234 are considerably lower compared to Bitcoin. The probability approaches 50% only by March 11, 2037. This implies that despite its past significant fluctuations, Stellar may face a more challenging path to recover to its all-time high levels compared to Bitcoin. The study reveals distinct outlooks for Bitcoin and Stellar, despite both cryptocurrencies exhibiting high volatility. By employing GARCH models, we were able to quantify these prospects, offering investors a valuable probabilistic framework for decision-making. The application of Monte Carlo simulations, grounded in GARCH models, provides a comprehensive view of possible price paths. Additionally, it offers insights into the likelihood of these cryptocurrencies reaching their historical highs. This information serves as a crucial resource for both risk management and the development of investment strategies. These analytical tools highlight the nuanced differences in risk profiles and potential future returns between Bitcoin and Stellar, demonstrating how seemingly similar assets in the cryptocurrency market can have markedly different long-term prospects.



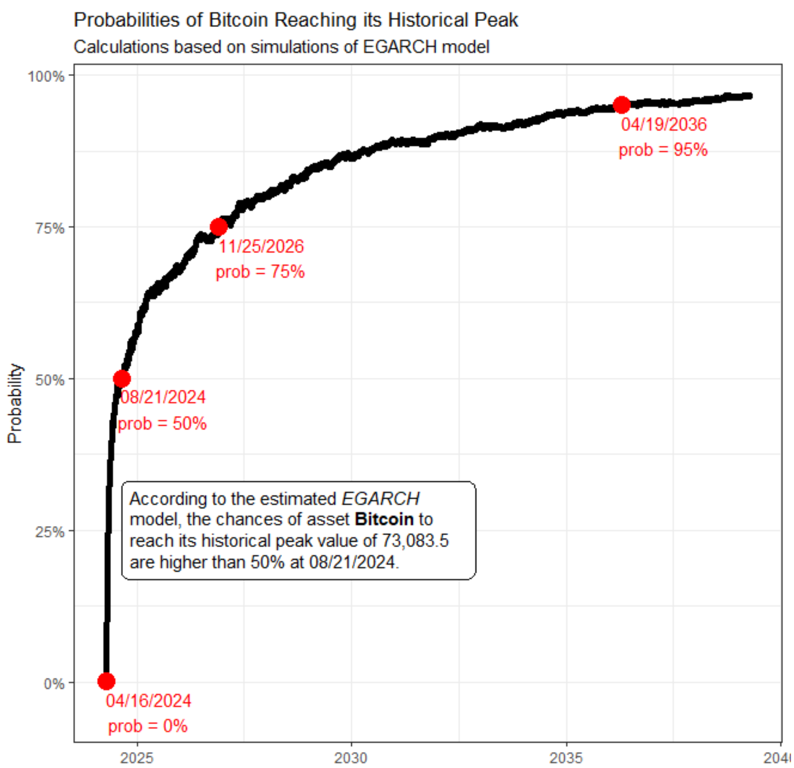
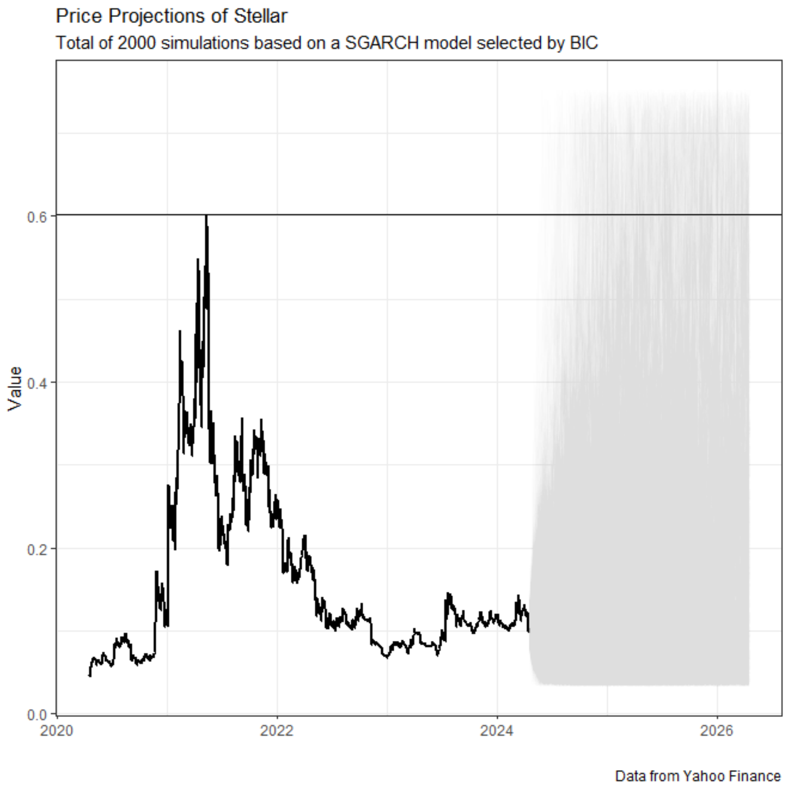


Figure 45: Price simulation and probabilities of Bitcoin reaching its historical peak



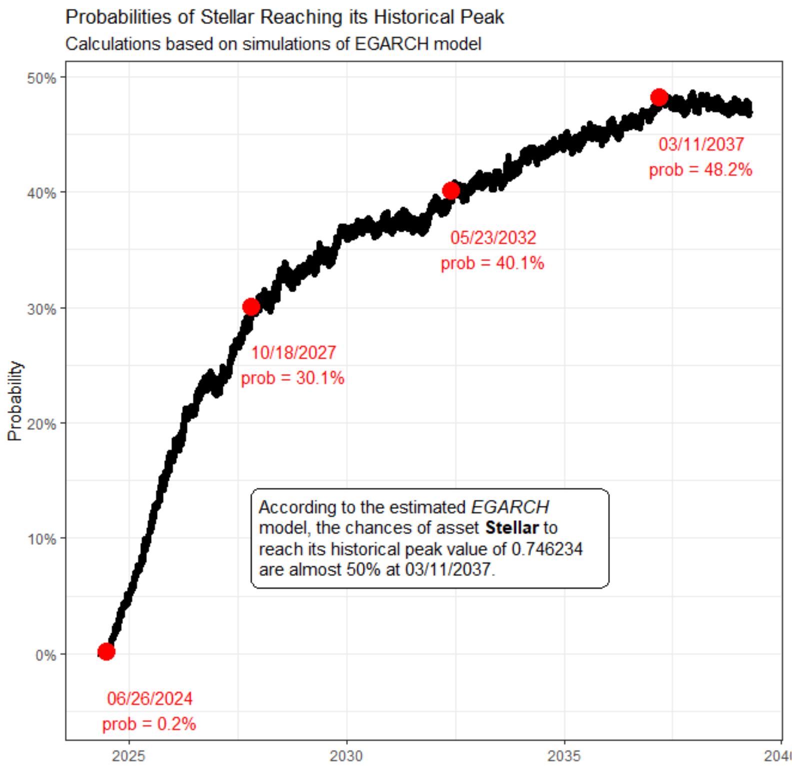


Figure 46: Price simulation and probabilities of Stellar reaching its historical peak

Conclusion

The thesis aimed to analyze the impact of two extraordinary and global events, the COVID-19 pandemic and the war in Ukraine, on the composition and performance of investment portfolios, with a focus on green indexes and traditional financial instruments. Through a quantitative approach based on Value-at-Risk (VaR), Conditional Value-at-Risk (CVaR), and volatility models such as GARCH, the main objective was to understand how these tools can be used to optimize portfolios in times of high volatility and uncertainty.

The empirical study, conducted using a rolling window analysis, revealed how portfolios reacted differently to the two global events examined. During the COVID-19 pandemic, the research showed a clear preference for green indices, which showed lower volatility and relatively low risk compared to traditional assets. This phenomenon can be interpreted as a defensive strategy adopted by investors to protect capital in an economic uncertainty.

In contrast, with the onset of the war in Ukraine, the focus has shifted to non-green assets, particularly those related to the energy sector such as the SP Energy index and the FTSE100 index. This shift reflects the direct impact of the conflict on global energy markets, which has led to increased volatility and, paradoxically, return opportunities for those investors willing to bear higher risks. However, one interesting aspect that emerged from the analysis is that despite the initial flight from green indexes, they recovered ground about a year after the conflict began, signaling a return to a preference for sustainable and resilient long-term investments.

A major contribution of this thesis is to demonstrate the effectiveness of CVaR-based models in managing risk during periods of economic stress. Compared to traditional VaR, CVaR offers a more comprehensive view of tail risk, providing a more accurate measure of potential losses under extreme scenarios. This is particularly relevant in a crisis context such as the one analyzed, where return distributions show non-normal trends and heavy tails.

The use of GARCH models for volatility estimation also made it possible to capture the temporal dynamics of volatility itself, highlighting how volatility is not static but tends to cluster during periods of high uncertainty. This approach made it possible to develop more robust portfolio strategies that can adapt quickly to market changes and manage risk more efficiently.

Another key consideration that emerged from the research concerns the integration of green indices into investment portfolios. The results indicate that although green indexes showed lower performance during the early months of the war, their long-term resilience makes them an essential component for portfolios that aim not only for environmental sustainability but also for financial stability. The analysis showed that diversification between green and non-green assets can improve the risk-return profile of portfolios, especially in the context of increasing focus on sustainable finance.

Investors who maintained a significant proportion of green indexes in their portfolios benefited from greater stability during the pandemic and, subsequently, faster recovery than those who opted exclusively for more traditional assets. This suggests that sustainability is not only an ethical value, but also a viable financial strategy for dealing with periods of crisis.

In conclusion, the thesis has shown that risk management and portfolio optimization in times of crisis require a flexible and adaptive approach capable of integrating advanced tools such as CVaR and volatility models such as GARCH. The differentiated reaction of markets to the events of the pandemic and war in Ukraine underscores the need to diversify portfolios not only by asset class, but also by risk type.

Integrating green indexes has proven to be a winning strategy over the long term, suggesting that sustainability and financial resilience can go hand in hand. This finding is particularly relevant in an era when sustainability is becoming a crucial factor in investment decisions.

Future research perspectives could include analysis of other global events and their impact on portfolios, as well as exploration of new green financial instruments that could further improve risk management. In addition, further exploration of the interaction between sustainability and financial performance could provide additional insights into the development of innovative and resilient investment strategies.

Ultimately, this thesis not only contributes to the existing literature on risk management and portfolio optimization, but also offers practical guidance for investors and portfolio managers operating in an increasingly complex and globally interconnected environment.

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