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The impact of Bitcoin on portfolio diversification: a comparison using the Diversification Ratio

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Introduction

Historically, one of the main challenges in investment management has been diversification. A well-diversified portfolio is theoretically capable of eliminating idiosyncratic risk, the risk associated with a single asset. Since the development of Markowitz's Modern Portfolio Theory (Markowitz, 1952), investors have sought to develop solutions that combine assets with different risk and return characteristics to reduce overall portfolio risk without impacting expected returns. The underlying principle is that by selecting financial assets whose expected returns are not correlated with each other, portfolio risk can be reduced and lower than the risk-weighted average.

Diversification is often simplified as the distribution of investable capital across different assets such as stocks, bonds, commodities, etc. However, measuring diversification is not as simple and straightforward as it seems. Indeed, its study is often associated with minimizing variance or the number and type of assets held in the portfolio. The problem is that these indicators are often unable to accurately explain the risk of interactions between assets.

To overcome these limitations, Choueifaty and Coignard studied the concept of the Diversification Ratio (Choueifaty & Coignard, 2008), a metric capable of measuring the level of diversification of an investable universe by comparing the weighted average volatility of individual assets with the overall volatility. This approach allows not only to define the weights assigned to each asset, but also to understand the correlation between them. Furthermore, the Diversification Ratio can be used as a method for building diversified portfolios when constrained, such as the so-called Most Diversified Portfolio, which is the portfolio capable of maximizing the Diversification Ratio.

With the entry of cryptocurrencies into the financial market, new opportunities have emerged for building investment portfolios. Bitcoin is the most popular, traded, and academically analyzed cryptocurrency, characterized by high volatility and independence from traditional financial markets.

Considering the above, the objective of this research is to analyze Bitcoin's impact on portfolio diversification, using the Diversification Ratio as the primary metric.

Chapter 1

Diversification Ratio

The primary objective in portfolio management is diversification, that is, attempting to combine assets with dissimilar behaviors to minimize the overall portfolio risk compared to the average risk of each asset in the portfolio. In practice, it is treated because of variance minimization, or as a qualitative objective, without a measure capable of summarizing the level of portfolio diversification.

The Diversification Ratio ("DR") is the idea proposed by Choueifaty and Coignard . They place diversification at the heart of the problem and propose this metric, explaining that a well-diversified portfolio is one that maximizes the Diversification Ratio while respecting given constraints (Choueifaty & Coignard, 2008).

This idea has been useful for assessing the correlation between different assets, but also for evaluating existing portfolios, clarifying how the level of diversification depends jointly on the concentration of weights and the correlation between assets.

1.1 Mathematical preliminaries

For the subsequent definition and explanation of the details of the Diversification Ratio, it is important to consider the following notations.

In a universe of *n* risky assets, the following are defined:

$w = (w_1, \dots, w_n)^T$ the portfolio weight vector;

$\Sigma \in \mathbb{R}^{n \times n}$ return covariance matrix;

$\sigma = (\sigma_1, \dots, \sigma_n)^T$ marginal volatility vector.

So, the portfolio volatility will be represented as follows:

$$\sigma_p = \sqrt{w^T \Sigma w}$$

Furthermore, we will assume that the portfolio is fully invested, so: $\sum_{i=1}^{n} w_i = 1$

And in the long-only case an additional constraint will be:

$w_i \geq 0, \forall i.$

1.2 Definition of Diversification Ratio

Choueifaty and Coignard define the Diversification Ratio of a portfolio P in this way:

$$DR(w) = \frac{w^T \sigma}{\sqrt{w^T \Sigma w}}$$

The definition can be described as the ratio between the weighted average volatility of the assets and the volatility of the portfolio.

The numerator is defined as the weighted average of the individual volatilities, while the denominator represents the actual volatility of the portfolio under consideration, which also includes correlation terms.

If the correlations between assets were low, the aggregate volatility would in turn be low compared to the numerator, and consequently the DR would increase. Conversely, the DR would be relatively low if the assets were perfectly correlated with each other, meaning that the portfolio volatility would be similar, if not identical, to the average volatility of each individual asset.

Weights and correlation: Choueifaty and Coignard explain that the Diversification Ratio is a function of these two elements. It has been shown that DR decreases if the weights are highly concentrated, and similarly, the metric decreases if the correlation between assets is significant. This clarification explains why a portfolio composed of many assets will not necessarily be highly diversified, while a portfolio with few assets with low correlations and justified and unconcentrated weights would most likely obtain a higher Diversification Ratio.

1.3 From measurement to construction: Most Diversified Portfolio

The Diversification Ratio is not only a descriptive metric but can also be used as a construction metric when constraints are included in its construction. The Most Diversified Portfolio ("MDP") is defined as the portfolio that maximizes the DR given some linear constraints T on the weights (Choueifaty et al.,2013). In the previous paragraph, the Diversification Ratio was defined as follows:

$$DR(w) = \frac{w^T \sigma}{\sqrt{w^T \sum w}}$$

Where $w \in \mathbb{R}^n$ represents the weight vector and $\sum$ the covariance matrix.

In the long-only and fully invested case the MDP solves this problem:

$$w_{MDP} = \arg\max_{w} \frac{\sigma^T w}{\sqrt{w^T \Sigma w}} \quad s.t. \quad \sum_{i=1}^{n} w_i = 1$$

With $w_i \geq 0 \ (i = 1, \ldots, n)$

In this case, T introduces positivity constraints on the weights, which must add up to 1.

It is important to emphasize that the MDP does not maximize diversification in general but maximizes diversification only with respect to the Diversification Ratio.

Choueifaty and Coignard provide two very simple examples to illustrate how the MDP behaves. In the first example, a group of assets with different volatilities is considered, or a group of assets with high correlations and a partially decorrelated asset. When two assets have a correlation less than 1, the authors demonstrate that the weights are inversely proportional to volatility. In the other example, three assets are considered, two highly correlated with each other and one asset with a low correlation with both. In this example, the MDP tends to assign a greater weight to the less correlated, "diversifying" asset.

Chapter 2

Beyond the Diversification Ratio: alternative measures

The Diversification Ratio is not the only measure of a portfolio's level of diversification. There are many other alternative measures that focus on aspects other than the Diversification Ratio. These measures can be divided into three categories:

- Weight-based
- Risk-based
- Factor-based

2.1 Weight-based measures

Weight-based measures focus on how capital is distributed among the assets in a portfolio.

Shannon Entropy is one of the most widely used weight-based measures (Shannon, 1948). It is primarily used to assess the dispersion of weights in a portfolio (Philippatos and Wilson, 1972).

Entropy, given a weight distribution $w = (w_1, \ldots, w_n)$, is defined by the following formula:

$$H(w) = -\sum_{i=1}^{N} w_i \log(w_i)$$

The measure reaches a maximum value when the portfolio is equally weighted, while it reaches zero or close to zero if the invested capital is associated with a single asset in the portfolio.

Shannon Entropy, therefore, measures the dispersion of the capital invested in the portfolio.

A measure like Shannon Entropy presents a significant risk, as an equally weighted portfolio may be exposed to only a few risk factors, thus being poorly diversified from an economic standpoint. Furthermore, it does not consider the statistics of individual assets.

2.2 Risk-based measures

The goal of risk-based measures is to understand how much each asset contributes to the overall risk of the portfolio being studied. To understand the portfolio's risk structure and break down risk, an important concept is Marginal Risk Contributions (MRC), which indicates the change in portfolio risk as an asset's weight varies.

This concept is very important for Risk Parity strategies, which construct portfolios by attempting to ensure each asset contributes a balanced and carefully considered amount to the portfolio's total risk. With this type of measure, the focus shifts from where capital is allocated within the portfolio to how risk is distributed within the portfolio (Meucci, 2009).

2.3 Factor-based measures

These measures represent a further evolution in measuring portfolio diversification; they are based on the idea of exposure to underlying risk factors. The idea is that a portfolio composed of many assets is not necessarily highly diversified because it may be characterized by common factors.

One measure based on this idea is the Effective Number of Bets (ENB), which aims to understand and quantify the number of risk sources to which the portfolio under consideration is exposed. The ENP is based on the decomposition and analysis of the main risk components, allowing the different underlying risk factors to be distinguished, offering a more realistic representation of the risk structure (Meucci, 2009).

2.4 Why the Diversification Ratio?

In summary, weight-based measures focus on the distribution of capital within the portfolio; risk-based measures take volatility and correlations into account but do not provide a synthetic indicator; and risk-factor-based measures are more complex and difficult to apply.

The Diversification Ratio (Choueifaty and Coignard, 2008) represents the perfect compromise between theoretical accuracy and applicability. The Diversification Ratio addresses correlations and volatility, provides a synthetic indicator that allows comparisons between different portfolios, and is easily applicable and easy to study.

Chapter 3

Bitcoin and portfolio diversification

With the spread of cryptocurrencies, investors and the academic community have increasingly focused on this type of asset, which appears uncorrelated with all traditional assets and follows its own unique trend. Bitcoin is the precursor asset of cryptocurrencies and is also economically significant. For these reasons, the main studies focus on Bitcoin.

The first researcher who studied and analyzed cryptocurrencies and in particular the characteristics of Bitcoin was Nakamoto (Nakamoto, 2008). In his research, Nakamoto defined Bitcoin as a decentralized payment system based on a peer-to-peer network. Bitcoin's goal, Nakamoto explains, is to avoid financial intermediaries to increase trust in encrypted mechanisms rather than centralized institutions. Over time, Bitcoin has transformed into a true alternative investment, characterized by unique risk-return dynamics that are difficult to interpret and predict, and whose value appears unaffected by central bank monetary policy dynamics. Unconstrained by specific constraints such as interest rates or corporate earnings, its value is primarily determined by speculative dynamics and investor expectations. This lack of ties to central banks, and thus the decentralized nature of cryptocurrencies in general, makes them assets with independent returns compared to traditional assets. Having established itself over time as an alternative asset, questions began to be asked about how it could help diversify a portfolio and which traditional assets it was most correlated with. Over time, therefore, Bitcoin has acquired a dual meaning: on the one hand, it is studied as a technological and monetary innovation; on the other, it is considered an asset with unique risk-return characteristics.

3.1 Statistical properties

Obviously, Bitcoin's statistical properties are highly relevant to studying the asset, and numerous studies demonstrate that the cryptocurrency has very different characteristics from traditional assets.

For example, Brière, Oosterlinck, and Szafar, explained how Bitcoin has much higher average returns and volatility than traditional assets (Brière et al.,2015). The three scholars decided to analyze a short period between 2010 and 2013, where they found that Bitcoin had an average annualized return of 400% and annualized volatility greater than 170%. Furthermore, they found that Bitcoin has a non-normal distribution of returns, explained by the presence of extreme events that appear frequent, making forecasting future returns and risk complex, as traditional strategies typically applied to study risk and return may be useless.

Another key aspect is correlations. Indeed, studies state that Bitcoin has low, if any, correlations with most traditional assets. Brière, Oosterlinck, and Szafarz show almost zero correlations between Bitcoin and other assets; on the other hand, Platanakis and Urquhart show how correlations with stocks and bonds are weak and unstable over time (Platanakis & Urquhart, 2020). For example, the correlation between Bitcoin and the S&P 500 is positive but limited, while that with bonds is often negative. Scholars are keen to point out that correlations are not static but vary or can vary over time. These statistics just highlighted give an idea of why Bitcoin cannot be compared to any traditional asset class but represents a new asset class with its own properties.

3.2 Bitcoin as a diversification tool

There have been many studies examining Bitcoin as a diversification tool. Brière, Oosterlinck, and Szafarz analyzed the impact of Bitcoin on a portfolio composed of both traditional and alternative assets. They chose to analyze diversification using a mean-variance approach (Brière et al, 2015).

The main result shows how even a small amount of Bitcoin can cause changes in diversification, increasing it and improving portfolio performance. These findings are partly justified by Bitcoin's low correlation with other assets in the portfolio.

Despite the results obtained, the authors recommend interpreting the results with caution, as they used a three-year analysis period, coinciding with Bitcoin's initial period of emergence and therefore characterized by strong growth and speculative activity.

Despite the limitations just mentioned, the study represents a first approach to the topic of diversification regarding a cryptocurrency and opens a line of studies that concern this very topic.

3.3 Recent empirical evidence: an out-of-sample approach

Subsequently, Platanakis and Urquhart approach the topic using a different approach. To analyze Bitcoin's contribution to portfolio diversification, they adopt an out-of-sample approach and consider different allocation strategies (Platanakis & Urquhart, 2020).

They decide to estimate the weights to allocate to the assets in the portfolio and then evaluate their performance over subsequent periods. To analyze the performance of the Bitcoin portfolio, they use several portfolio optimization techniques, including the Markowitz model, the Black-Litterman model, and the minimum variance portfolio. Their results are consistent, showing that the inclusion of Bitcoin increases risk-adjusted performance.

Their results reinforce the theory that Bitcoin can play an important role in portfolio diversification and performance, even if future returns and risks are difficult to interpret.

3.4 Limitations of the literature analyzed

The literature reviewed so far offers significant results but has a significant limitation. Both studies analyzed a short period, lacking dynamics that change over time or moments of crisis, such as the global crisis caused by the Covid-19 pandemic. It is important to consider a longer time period to also consider changes in the correlation between Bitcoin and other assets in a portfolio. Furthermore, the studies discussed above address the topic of diversification with Bitcoin using traditional methods such as mean-variance analysis. Obviously, these tools are still very valid and used by the academic community today, but it would be interesting to analyze a modern and unique asset like Bitcoin and cryptocurrencies in general with more modern methods or analyses.

3.5 Implications for this research

Since Nakamoto's definition and important analysis of Bitcoin, Bitcoin has evolved, and a small portion of the literature on Bitcoin and the benefits it can provide for diversifying an investable universe has been examined.

In this well-researched and interesting context, the following research will seek to contribute to the existing literature by offering an alternative analysis focused on the Diversification ratio and the Most Diversified Portfolio, seeking to provide a further analytical perspective regarding the diversification that Bitcoin offers to a universe of traditional assets.

Chapter 4

Research method and design of empirical analysis

4.1 Objective of the research

The following research examines and analyzes how portfolio diversification changes when a cryptocurrency is introduced. Specifically, it seeks to understand whether Bitcoin can increase portfolio diversification.

Diversification is a very important concept in portfolio management as it allows for overall risk reduction by combining assets with different behaviors, exploiting imperfect correlations. Various analyses have been used over time, essentially based on portfolio variance, number of assets, or weight distribution. However, the problem with these measures is that they fail to precisely capture the interactions between volatility and correlations between assets.

Through their studies, Choueifaty and Coignard were able to construct a metric that correlates the weighted average volatility of assets with the overall volatility of the portfolio: the Diversification Ratio. Furthermore, the Diversification Ratio, when conditioned by constraints such as the Most Diversified Portfolio, allows for optimization of portfolio diversification.

With the advent of cryptocurrencies, many investors have decided to focus on this type of asset, characterized by high volatility and very low correlation with the dynamics of traditional assets. For this reason, it is interesting to study whether introducing Bitcoin into a portfolio of traditional assets actually improves diversification or increases the latter's risk. The ex-ante results obtained with the Diversification Ratio will be further analyzed to verify whether they translate into observable ex-post benefits in portfolio performance.

4.2 Datasets used

To implement the empirical analysis, it was decided to construct two separate datasets, representing the two reference portfolios. A sector-specific portfolio includes indices from the GICS (Global Industry Classification Standard) classification, and a geographically diversified portfolio constructed using indices representing selected global stock markets. The analysis of these two datasets provides a more comprehensive view of the potential contribution cryptocurrency can make.

4.2.1 Sectoral Dataset

The first dataset used includes indices belonging to the Global Industry Classification Standard (GICS), developed by MSCI and Standard & Poor's, which allows listed companies to be identified according to their main economic activity. Building a portfolio using GICS indices responds to the need to represent the traditional stock market as a set of interconnected assets. Indeed, sectors reflect key macroeconomic dynamics, being influenced by common factors such as inflation and monetary policy. At the same time, each index reflects its ability to respond to economic trends, growth opportunities, and cost structure. The portfolio is composed of eleven sector indices, listed below: Information Technology, Health Care, Consumer Discretionary, Industrials, Financials, Utilities, Consumer Staples, Energy, Communication Services, Materials, and Real Estate. Each of these sectors includes listed companies that leverage a particular sector as their strength. Bitcoin is added to this set of traditional assets. The data analyzed are the daily prices of each index and were obtained using the Bloomberg platform. The sample spans a period from January 2015 to October 2025, a decade characterized by economic expansion and periods of volatility and uncertainty, especially during the Covid-19 pandemic crisis, allowing us to evaluate the behavior of assets in different contexts. The image below shows the daily price movements of the sector indices and Bitcoin over the period considered.

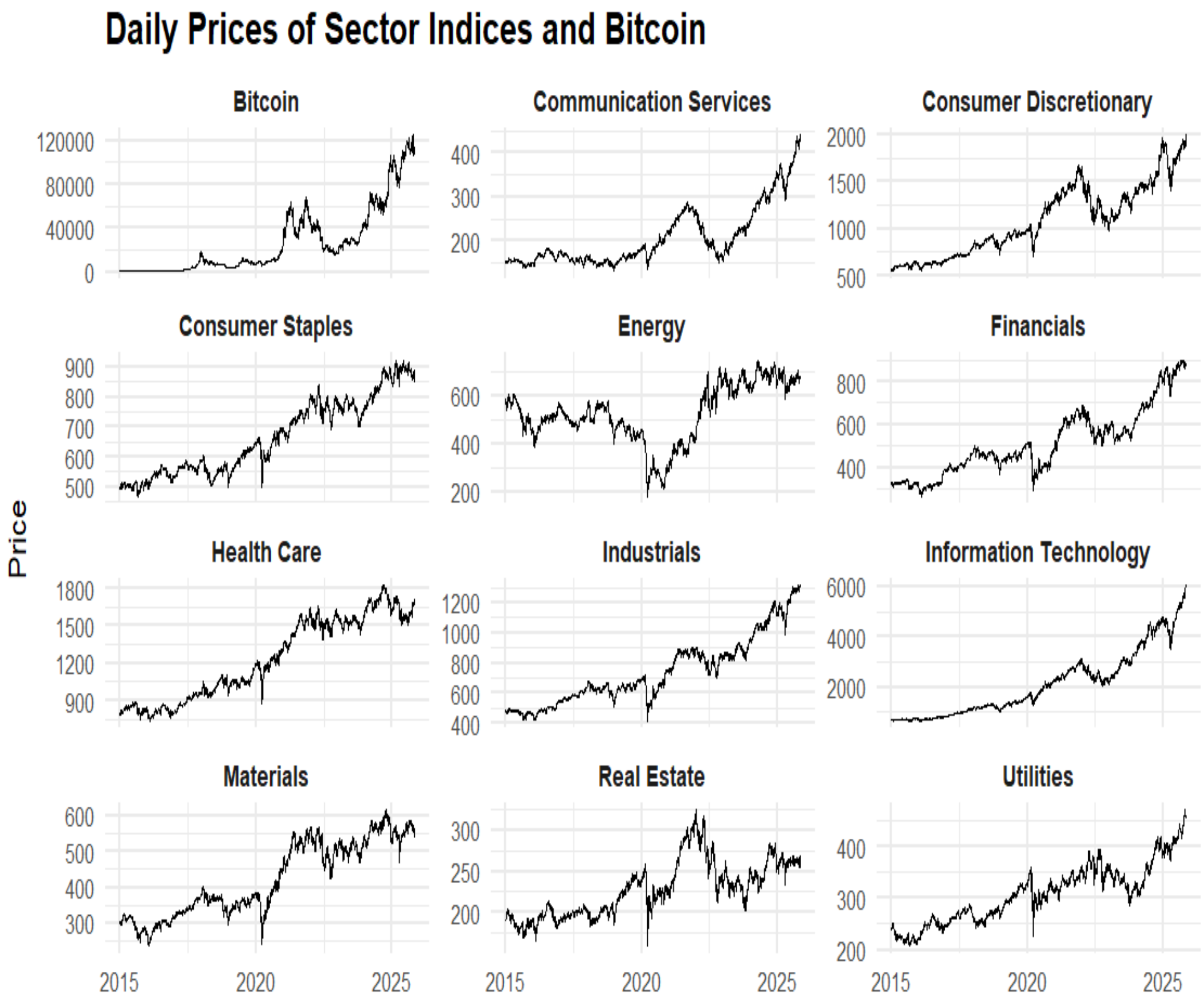

Figure 1

Even just looking at these charts, it's clear how Bitcoin's price reaches very high levels in a very short time, characterized by a different dynamic compared to traditional assets. Specifically, Bitcoin grows very rapidly over the decade under consideration, accompanied by significantly higher levels of volatility. In contrast, sector indices show long-term growth, but it is more stable and less steep than the cryptocurrency, despite moments of discontinuity corresponding to macroeconomic shocks.

To further analyze the indices' characteristic statistics, a descriptive analysis of the annualized returns of each asset in the sector portfolio was conducted. The results are presented in the following table.

Asset	Mean	Variance	Standard Deviation	Min	Max
S5INFT Index	19,25%	5,68%	23,84%	-14,98%	13,23%
S5HLTH Index	6,68%	2,75%	16,58%	-10,53%	7,31%
S5COND Index	11,20%	4,68%	21,63%	-12,88%	10,76%
S5INDU Index	8,82%	3,74%	19,33%	-12,16%	12,00%
S5FINL Index	8,52%	4,76%	21,81%	-15,07%	12,43%
S5UTIL Index	5,60%	3,57%	18,90%	-12,27%	12,32%
S5CONS Index	4,72%	2,08%	14,42%	-9,69%	8,07%
S5ENRS Index	1,19%	8,44%	29,05%	-22,42%	15,11%
S5TELS Index	9,16%	4,51%	21,24%	-11,03%	9,52%
S5MATR Index	5,05%	4,12%	20,30%	-12,15%	11,00%
S5RLST Index	2,68%	4,08%	20,20%	-18,09%	8,28%
BTC	52,18%	41,09%	64,10%	-31,73%	22,32%

Figure 2

The statistics reflect the findings of the cryptocurrency literature, which identifies Bitcoin as a particularly volatile asset characterized by highly variable returns. On the other hand, sector indices exhibit lower and more similar volatility, reflecting greater stability in traditional markets. Looking at average annualized returns, the descriptive analysis suggests that some sectors offer higher returns than others over the long term. Further interesting findings include the minimum and maximum returns between two consecutive days (not annualized). As expected, Bitcoin exhibits both higher minimum and maximum values in absolute terms, indicating a distribution with more frequent extreme events.

Finally, it is important to verify the correlations between assets in the sector portfolio. To better visualize the correlations, this graph was produced.

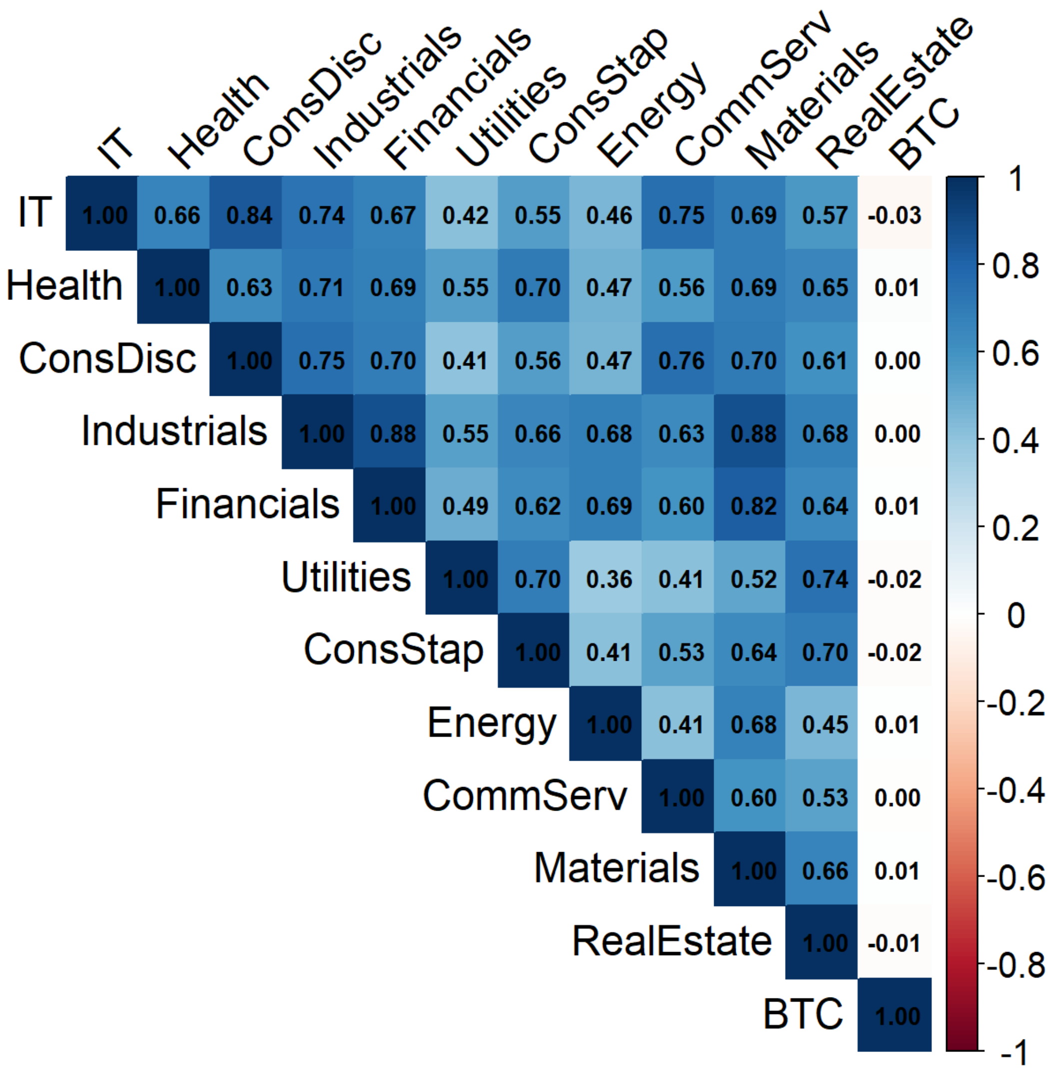
IT
Health
ConsDisc
Industrials
Financials
Utilities
ConsStap
Energy
CommServ
Materials
RealEstate
BTC
IT 1.00 0.66 0.84 0.74 0.67 0.42 0.55 0.46 0.75 0.69 0.57 -0.03
Health 1.00 0.63 0.71 0.69 0.55 0.70 0.47 0.56 0.69 0.65 0.01
ConsDisc 1.00 0.75 0.70 0.41 0.56 0.47 0.76 0.70 0.61 0.00
Industrials 1.00 0.88 0.55 0.66 0.68 0.63 0.88 0.68 0.00
Financials 1.00 0.49 0.62 0.69 0.60 0.82 0.64 0.01
Utilities 1.00 0.70 0.36 0.41 0.52 0.74 -0.02
ConsStap 1.00 0.41 0.53 0.64 0.70 -0.02
Energy 1.00 0.41 0.68 0.45 0.01
CommServ 1.00 0.60 0.53 0.00
Materials 1.00 0.66 0.01
RealEstate 1.00 -0.01
BTC 1.00
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1

Figure 3

Correlation analysis is very important for verifying which assets share a common trend. The correlation matrix presented provides important evidence. Sector indices are characterized by a high average correlation between them; the observed values range between 0.4 and 0.9. However, this also reflects the possibility of common factors simultaneously influencing the sectors. Bitcoin, on the other hand, exhibits correlations very close to zero, and sometimes even negative. This is important because it indicates that Bitcoin is essentially unrelated to the dynamics of sector indices and follows a trend independent of the rest of the portfolio. This makes it a potential diversification tool, as, theoretically, a low correlation between assets should reduce portfolio volatility.

4.2.2 Geographic Dataset

A second portfolio was constructed with a greater focus on geographical differentiation. This portfolio selected stock indices from various regions of the world.

The geographical dataset is composed of a set of stock indices representing some of the world's major economies and emerging economies. Specifically, the following indices were considered: S&P 500 for the United States, Euro Stoxx 50 for the Eurozone, FTSE 100 for the United Kingdom, Nikkei 225 for Japan, Kospi for South Korea, NIFTY 50 for India, Ibovespa for Brazil, CSI 300 for China, and TWSE for Taiwan.

These indices were chosen because they allowed for the creation of a globally diversified portfolio capable of capturing the dynamics of international stock markets. The portfolio includes both developed and emerging markets, which are characterized by greater volatility. This combination allows us to analyze how geographical diversification can contribute to reducing the portfolio's overall risk. It's also important to remember that each index reflects the movements of the companies represented by the index and how they respond to events and situations that impact them.

The data here also comes from the Bloomberg platform and presents the daily prices of each index over the period from January 2015 to October 2025, again including economic expansion and the crisis caused by the Covid-19 pandemic.

The figure below shows the daily price trends of the indices and Bitcoin.

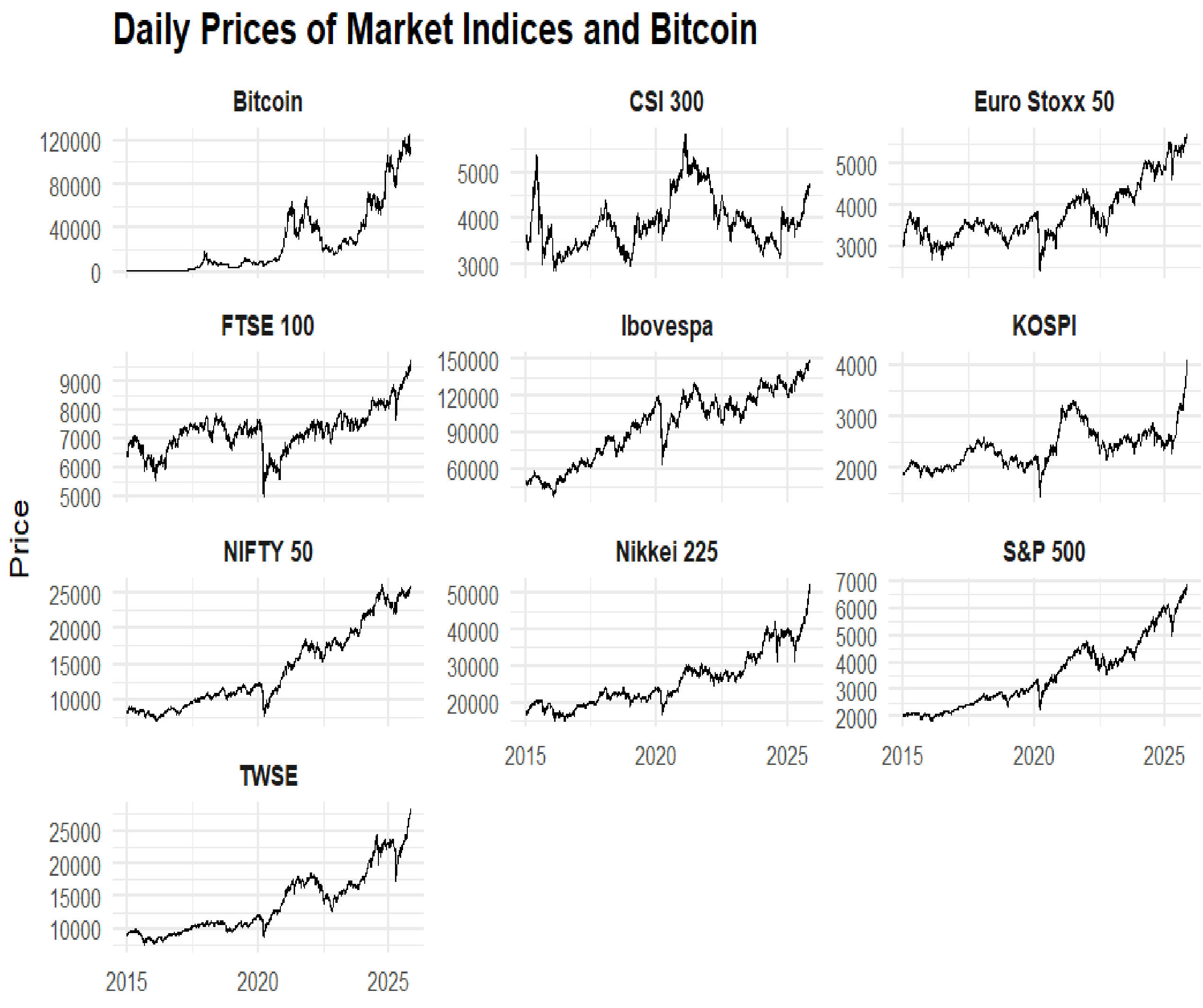

Figure 4

Looking at the charts, we can see similar trends in developed markets like the United States and Europe, which have followed a stable and growing trend over the long term, halted by the temporary shock of 2020. Emerging markets, on the other hand, have followed more erratic trends, as they are characterized by greater uncertainty and volatility, with periods of stagnation. Bitcoin, on the other hand, as already noted for the sector portfolio, exhibits a different trend from traditional assets, showing steady growth over the

Asset	Mean_ann	Variance_ann	Std_Dev_ann	Min	Max
SPX Index	10,71%	3,13%	17,70%	-12,77%	9,09%
SX5E Index	5,24%	3,54%	18,80%	-13,24%	8,83%
UKX Index	3,50%	2,34%	15,28%	-11,51%	8,67%
NKY Index	9,81%	4,12%	20,30%	-13,23%	9,74%
SHSZ300 Index	2,43%	4,46%	21,11%	-9,15%	8,14%
IBOV Index	9,77%	5,28%	22,98%	-15,99%	13,02%
TWSE Index	9,90%	2,72%	16,49%	-10,20%	8,84%
KOSPI Index	6,80%	2,74%	16,54%	-9,18%	8,25%
NIFTY Index	10,10%	2,55%	15,98%	-13,90%	8,40%
BTC	52,18%	41,09%	64,10%	-31,73%	22,32%

Figure 5

period under consideration but with extreme events that indicate the cryptocurrency's high volatility. This difference in dynamics between Bitcoin and market indices is better described in the following table.

Annualized returns were used for descriptive analysis. As previously explained, Bitcoin exhibits very high and abnormal volatility. Emerging markets exhibit higher levels of volatility than developed markets. This difference reflects classic risk factors for emerging countries, such as political instability and less stable macroeconomic conditions. Developed markets, such as the United States, Europe, and the United Kingdom, exhibit more stable levels of volatility and average returns over time. However, differences can also be found among developed markets. Extreme values (highs and lows) were calculated based on daily returns. Bitcoin, of course, exhibits much higher extreme events than other assets.

Finally, the correlation between the assets in the geographical portfolio was analyzed.

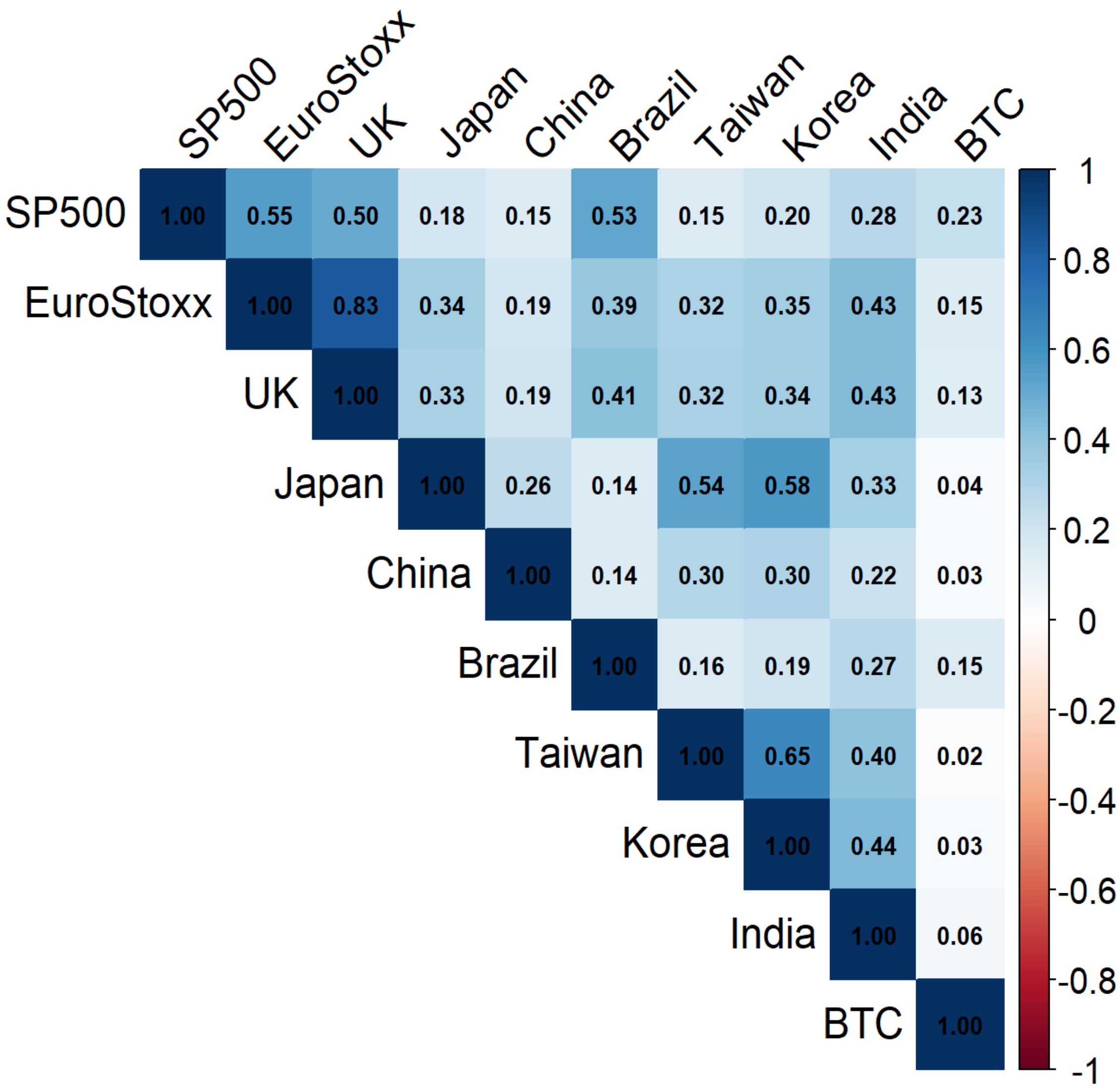

Figure 6

The correlation matrix shows how correlations between market indices are generally lower than those previously highlighted by the GICS indices. The graph shows a strong correlation between the European markets, the Euro Stoxx 50 and the FTSE 100. Correlations between developed and emerging markets are more moderate. Interestingly, the CSI 300 Index, representing the Chinese economy, has low correlations with almost all other markets, highlighting how the Chinese market is highly independent from the other markets in the portfolio. Integration between Asian markets is also evident; Taiwan and South Korea show a high correlation, reflecting how the two economies collaborate and are commercially linked. However, they are also exposed to the same factors, such as trends in global demand for technology. Focusing on Bitcoin's correlations, we see low but consistently positive correlations with global markets, suggesting that Bitcoin is not entirely unrelated to global financial market movements. Compared to the sector portfolio, which had close to zero or negative correlations, this portfolio shows greater Bitcoin integration.

If we compare the correlation matrix of the sector portfolio with that of the portfolio composed of market indices, we note that the sector dataset has higher average correlation levels among assets, reflecting the presence of factors that simultaneously influence all assets. Furthermore, the presence of Bitcoin and its role differ in the two contexts. In the sector dataset, it is almost completely decorrelated with other assets, while in the geographical dataset, it has slightly positive correlations.

It is important to understand that Bitcoin's role in diversifying the portfolio composed of global market indices may be less significant than in the portfolio sector, as the geographical dataset is already more diversified.

4.3 Portfolio construction methodology

4.3.1 Ex-ante portfolio construction

To undertake and construct the empirical analysis, I needed a methodology that would allow me to obtain comparable results, transforming price data into variables used to optimize diversification. Specifically, returns were constructed, the covariance matrix was estimated, the Most Diversified Portfolio was optimized, a rolling window procedure was used, and finally the portfolio's Diversification Ratio was calculated.

As just mentioned, the first step was to transform the price series into returns. This procedure is quite common in time series analysis because it allows for a more stable representation of market dynamics. Returns were calculated using a logarithmic transformation, i.e., the natural logarithm of the ratio between the price at time t and the price at time $t - 1$. Using log returns allows for better comparison and calculation of returns, making them more suitable for empirical analysis.

Next, the covariance matrix is estimated, which is central to the subsequent analyses. It represents the dependence between assets and is a key element in assessing the overall risk of the portfolio. The covariance is estimated using the log returns observed over a specific period. The estimation of the covariance matrix is crucial for applying the subsequent formulas regarding the Most Diversified Portfolio and Diversification Ratio.

The Most Diversified Portfolio optimization is applied, defined as the portfolio that maximizes the Diversification Ratio. The weights of the assets chosen to construct the portfolio are constrained to be positive, and their sum must be 1. The Most Diversified Portfolio constraint is important because it allows us to find the best combination of weights to maximize the Diversification Ratio and allow for a realistic comparison between portfolios.

Another important element is trying to take into account the evolution of the assets considered over time. For this reason, we decided to adopt rolling windows because they allow us to exploit some advantages that would not be available without them. Generally, they allow us to study how data properties evolve over time, considering that financial markets are not stationary over time, but are constantly evolving, just like assets, which change their correlations, risks, and returns. Rolling windows allowed the analysis to adapt to these dynamics. Furthermore, they allow for a more realistic empirical analysis while avoiding the problem of look-ahead bias. The adoption of rolling windows allows us to obtain a time series of the Diversification Ratio and portfolio weights, enabling a more precise analysis of how diversification evolves over time.

Once the optimal weights have been calculated using the Most Diversified Portfolio, the Diversification Ratio is finally calculated, allowing us to quantify and subsequently compare the level of diversification over time. Higher values indicate greater diversification, while lower values indicate greater risk concentration.

Once the Diversification Ratio results were obtained, they were compared between the sector or geographic portfolio without Bitcoin and the sector or geographic portfolio with Bitcoin. The goal of this phase is to compare the contribution that Bitcoin has made to an asset universe in terms of diversification, comparing the Diversification Ratio values obtained in the two configurations.

The results obtained were initially aligned over the same time frames to allow for precise comparisons between the same periods and to ensure a consistent comparison. In other words, the Diversification Ratio of the portfolio without Bitcoin is compared with the Diversification Ratio of the portfolio with Bitcoin over the same period.

Once the data were aligned, $\Delta DR_t$was calculated, which is the difference between the two Diversification Ratio values over time. This measure allows us to understand how much Bitcoin marginally contributes to portfolio diversification over time. Positive values of the measure indicate an increase in the level of diversification, while negative values indicate a decrease in the level of diversification. Analyzing ΔDR_t allows us to more precisely compare the distribution of Bitcoin's contribution to the portfolio, allowing us to visualize how Bitcoin influences the Diversification Ratio over time, as well as other values in the same portfolio related to diversification, and consequently understand how the cryptocurrency's effect changes over time.

Next, a descriptive analysis of the Diversification Ratio values was performed for both portfolios. The descriptive analysis allowed us to summarize the characteristics of the Diversification Ratio distribution with and without Bitcoin, highlighting several results.

Several graphs were constructed to better visualize and discuss the results obtained from the comparisons. First, graphs were used to compare the Diversification Ratio over time across the portfolios, and then the dynamics of ΔDR_t were visualized to identify the periods in which Bitcoin's contribution to diversification is significant.

The comparison between the results obtained for the two portfolios allowed for an in-depth analysis of Bitcoin's contribution to diversification using the Diversification Ratio.

Next, the objective was to statistically validate the results obtained in the previous steps. The goal was to understand whether the differences found between the two portfolios are statistically significant. Statistical inference allows us to assess whether the results are robust and not due to random variations in the data. With this new objective, a new dataset was constructed containing, for each time window, the Diversification Ratio values for the portfolio without Bitcoin and the portfolio with Bitcoin. First, we decided to use the *paired t-test*, commonly used to compare the means of two samples. This test allows us to test the following hypotheses:

$$H_0: E\ (DR^{BTC} - DR^{NoBTC}) = 0$$

$$H_1: E\ (DR^{BTC} - DR^{NoBTC}) \neq 0$$

The null hypothesis H_0 states that the inclusion of Bitcoin has no effect on the level of diversification since the mean difference between the two values is zero.

Conversely, hypothesis H_1 indicates the presence of a noteworthy effect. An assumption of the paired t-test is that the distribution of differences between observations should be approximately normal. However, in financial markets, return distributions often exhibit significant deviations from normality due to the presence of extreme values. For this reason, it was decided to also use another test, the *Wilcoxon signed-rank test*, as it allows us to verify the significance of differences without assuming normality. Its hypotheses are as follows:

$$H_0: Median\ (DR^{BTC} - DR^{NoBTC}) = 0$$

$$H_1: Median\ (DR^{BTC} - DR^{NoBTC}) \neq 0$$

In this case, the null hypothesis H_0 states that the median of the differences is zero, while the hypothesis H_1 states that it is different from zero. Given the non-normal nature of financial data, particular importance was given to the results of the *Wilcoxon signed-rank test*.

The main difference between the two tests lies in the type of information they use. Using these two tests together allowed for a more precise and robust assessment of the empirical analysis conducted thus far.

4.3.2 Out-of-sample performance and risk evaluation

After analyzing the ex-ante effect of Bitcoin on diversification, it was necessary to determine whether these results were observable ex-post. The objective of this subsequent analysis is to understand whether the improvement in diversification observed during portfolio construction can be observed in future returns.

In this analysis, it was also decided to use log-returns; this approach allows for consistency in the ex-post evaluation.

Subsequently, the optimal weights obtained with the Most Diversified Portfolio were used. These weights were calculated during the ex-ante portfolio construction phase and estimated in each rolling window. To assess the economic relevance of the estimated weights over time, an out-of-sample methodology was adopted. This methodology applies the weights estimated at time t to the returns observed in the subsequent period, $t+1$. Using this system allowed us to simulate how an investor would behave over time based on the information available. He or she then assigns weights to the assets in his or her portfolio and observes how the portfolio's performance evolves in the subsequent period.

The result of this methodology is the out-of-sample logarithmic returns achieved by the portfolio. The out-of-sample logarithmic returns allow for the analysis and comparison of the portfolio's performance without Bitcoin and that of the portfolio with Bitcoin. In particular, the introduction of out-of-sample logarithmic returns allows us to verify whether the ex-ante results obtained regarding diversification can be correlated with the ex-post behavior of the portfolio.

Following the development of out-of-sample logarithmic returns, various metrics were applied to analyze and compare the performance and risk between the non-Bitcoin and Bitcoin-infused portfolios. The first metric considered is cumulative out-of-sample growth, used to understand how a single investment grows over time. It allows us not only to assess the final level of a single investment but also to understand how it evolves over time. The second metric considered is annualized out-of-sample volatility, which allows us to assess how returns vary over time. Comparing volatility between the two portfolios allows us to assess whether improved diversification increases or decreases the overall portfolio risk. Analyzing cumulative growth and volatility is essential to assessing the risk-return trade-off.

The distributional characteristics of returns are then analyzed using skewness and kurtosis. Skewness allows us to analyze the degree of symmetry of the distribution of returns with respect to the mean. Kurtosis, on the other hand, allows us to measure the degree of concentration of the distribution in the tails compared to a normal distribution.

To improve the analysis, the Sharpe Ratio is introduced to evaluate the performance of portfolios without and with Bitcoin in risk-adjusted terms. The Sharpe Ratio was introduced by William F. Sharpe (1966) and is one of the most widely used measures for evaluating a portfolio's performance in relation to the total risk assumed. It is defined as the ratio between the excess return compared to a risk-free asset and the standard deviation of the portfolio's returns:

$$SR = \frac{E(R_\rho) - R_f}{\sigma_\rho}$$

The Sharpe Ratio was calculated and compared for both portfolios with and without Bitcoin. The higher the Sharpe Ratio, the greater the ability to generate returns per unit of risk assumed.

The performance analysis is then further explored using the Sortino Ratio to consider a risk measure closer to the investor's perception. While the Sharpe Ratio uses standard deviation as a risk measure, the Sortino Ratio is based on a measure that takes into account negative deviations in returns. The concept of downside risk was explained by Frank A. Sortino and Van Der Meer (1991), who explained that standard deviation is unable to distinguish positive and negative variations in returns. Therefore, Sortino and Van Der Meer show that volatility associated with higher-than-expected returns does not constitute a significant risk, while the possibility of achieving returns below a minimum acceptable level, in this case called the Minimum Acceptable Return (MAR), is more important and relevant. Therefore, the Sortino Ratio is based on a risk measure called downside deviation, which quantifies the dispersion of returns below the MAR, and is expressed as:

$$Sortino = \frac{E(R_p) - MAR}{\sigma_d}$$

The application of the Sortino Ratio allows us to evaluate the efficiency of the strategy in relation to the risk of loss, completing the analysis started with the Sharpe Ratio.

Chapter 5

Empirical analysis of the GICS sector portfolio

5.1 Diversification Ratio of the GICS and GICS+Bitcoin portfolio.

The empirical analysis is obtained by comparing development of the Diversification Ratio with the constraints of the Most Diversified Portfolio in two similar portfolios: a portfolio built with GICS sectors and a portfolio that also includes Bitcoin. We will try to understand if and how important the differences between the two portfolios will be, and how much the inclusion of the cryptocurrency influences the Diversification Ratio.

The following table represents the results obtained that are going to be discussed in this section.

Portfolio	Mean	Median	St.Deviation
GICS	1.73	1.69	0.45
GICS+BTC	1.86	1.79	0.45

Figure 7

The analysis demonstrates that the value of DR is systematically bigger in the portfolio with Bitcoin. This is verified by the mean and median in the two portfolios. The average goes from a value of 1.73 in the GICS portfolio to a value of approximately 1.86 in the GICS+Bitcoin portfolio, increasing by approximately 0.12. Similarly, the median also increases from 1.70 to 1.80. These very important results, although not significantly increased because the main block of the two portfolios is essentially the same, indicate that the increase is not due to extreme and sporadic events but rather to the distribution of values in the two sets.

It is important to ask whether dispersion has also increased in the cryptocurrency portfolio. The research shows that the standard deviation remains substantially similar (approximately 0.45), indicating that the inclusion of a volatile asset like Bitcoin in this portfolio does not increase instability. This result reinforces the idea that the inclusion of Bitcoin produces a structural, rather than episodic, effect on diversification. The graph below provides an excellent overview to support this conclusion.

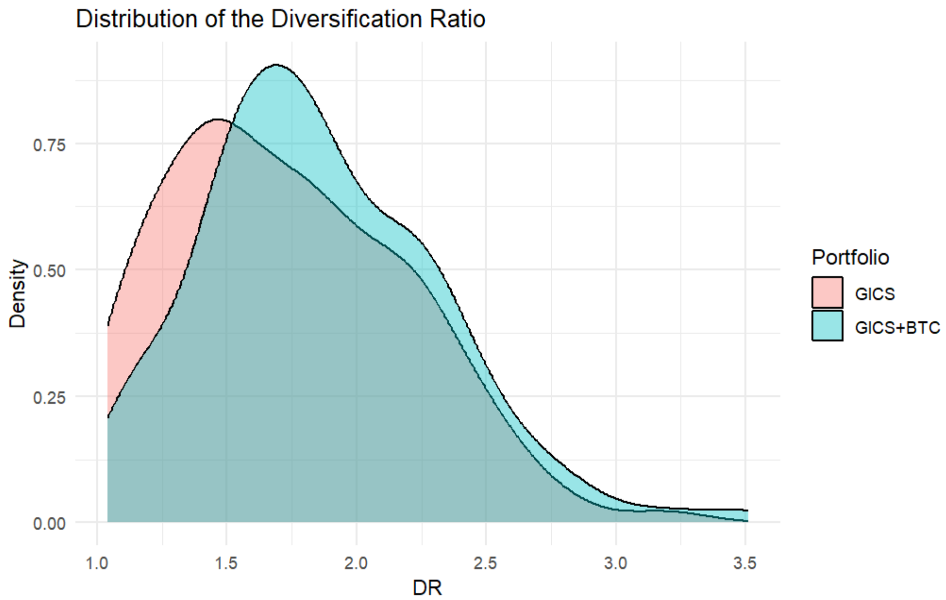

Figure 8

We see how the distribution of the Bitcoin portfolio is shifted to the right with respect to a concentration of values between 1.6 and 2.2. On the other hand, we see how the values of the GICS portfolio distribution are lower. Although the two distributions are partially on top of each other, it is much more likely to observe higher DR values in the extended portfolio.

Below, we can also analyze this box plot, which allows us to summarize these differences.

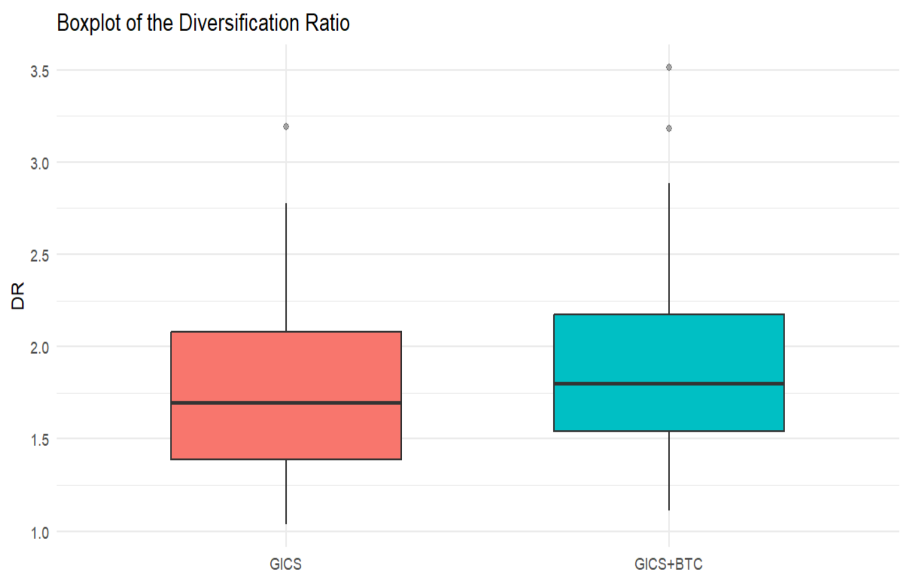

Figure 9

The median of the portfolio without Bitcoin is lower, as are both the lower and upper quartiles. This is an important observation as it shows that the increase in our metric affects not only the central observations but also most of the distribution. Furthermore, the maximum values observed in the "basic" portfolio are lower, while in the extended portfolio we can observe maximums exceeding 3.5. This result indicates that there are specific moments in which the addition of Bitcoin allows us to achieve much higher levels of diversification.

It is extremely important to observe and compare the temporal dynamics of the Diversification Ratio in the two portfolios. The following graph shows precisely how the DR varies in the two portfolios.

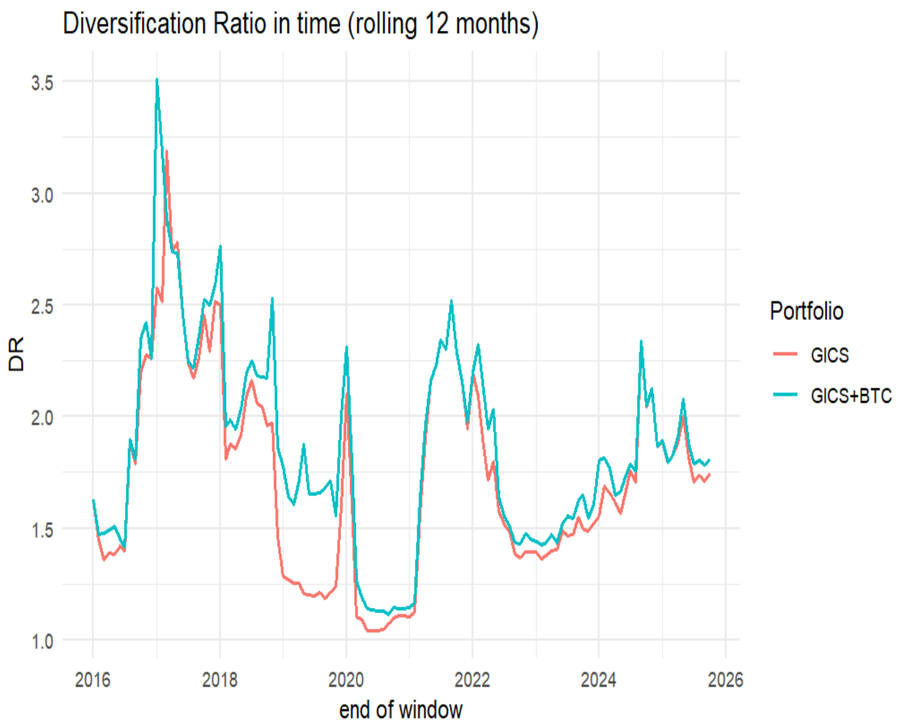

Figure 10

This 12-month rolling chart shows how the diversification measure for both portfolios fluctuates significantly over time, ranging from values close to 1.1 in some periods to values greater than 3 in more favorable phases. As we can see, the blue line is above the red line for most of the observation period, despite essentially the same performance and experiencing the same economic cycles. The chart also clearly shows how Bitcoin's contribution depends on the state of the covariance system in each time window, explaining why the gap between the two lines changes over time.

The focus then shifted to analyzing the differences between the two DR values over time. This analysis yields a significant result: the difference in DR across the time windows between the two portfolios is positive for 92% of the windows considered. Furthermore, the average difference is approximately +0.13, while the median is approximately +0.7. These results indicate once again that Bitcoin's contribution to diversification is positive but variable over time.

The graph below allows you to better visualize the DR time difference.

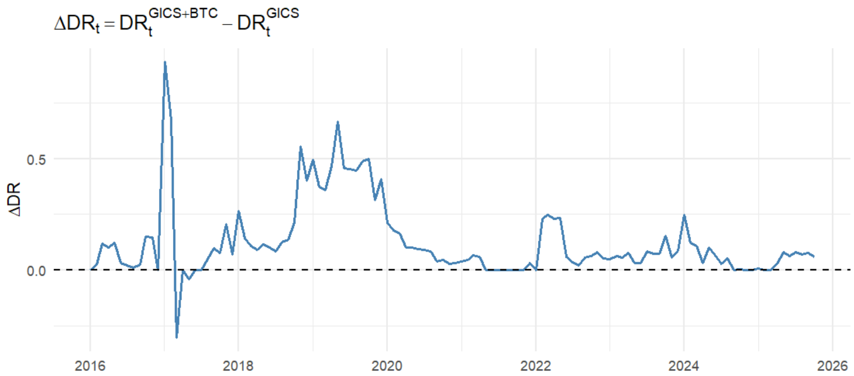

Figure 11

Looking at this graph, it's much easier to see how Bitcoin's contribution to the GICS portfolio's diversification is positive for almost the entire period analyzed. The series is predominantly above zero for most of the time period considered. There are rare negative excursions, while peaks are much more frequent and larger than negative peaks. This asymmetric profile is consistent with what we might define as Bitcoin's nature: an asset disconnected from the rest, characterized by high volatility and unstable correlations.

We essentially analyzed how the inclusion of a highly volatile asset in our investable universe increases the ex-ante diversification of the MDP portfolio. As previously explained, this is demonstrated by an average increase in DR, an upward shift in the distribution, and a prevalence of time windows in which Bitcoin's contribution is positive.

The ex-ante measured DR leaves us with an open question. While research thus far shows that Bitcoin increases portfolio diversification when building a portfolio, we must ask whether this improvement translates into an ex-post increase in performance and realized risk.

It is important to understand the statistical significance between the two levels of Diversification Ratio. To determine this, a paired t-test was applied to verify whether the mean difference between the Diversification Ratios of the Bitcoin portfolio and the non-Bitcoin portfolio was statistically different from 0. The paired t-test yielded significant results, with a t-statistic of 8.02 and a p-value of 9.145e-13. This indicates that the inclusion of Bitcoin changes the metric statistically significantly, as the null hypothesis that the mean difference between the DRs is zero is rejected. The estimated mean difference is 0.13, suggesting that the inclusion of Bitcoin increases portfolio diversification.

The paired t-test is a widely used tool in empirical analysis. In financial datasets, however, it is good practice to combine the paired t-test with the Wilcoxon signed-rank test, as many financial variables can have strong outliers and marked skewness, thus not requiring normal distribution of differences. Even with this test, the results support the findings from the paired t-test. Indeed, the test returns a V of 6809 and a p-value less than 2.2e-16. Therefore, this test also rejects the null hypothesis.

The findings from these two tests strengthen and confirm the idea that including Bitcoin in an investment universe increases diversification according to the Diversification Ratio theory, with a Most Diversified Portfolio constraint.

5.2 Out-of-sample log-return analysis.

To analyze the ex-post degree of diversification of portfolios constructed using the Most Diversified Portfolio strategy, we decided to study the out-of-sample log returns, which were calculated using the optimal ex-ante weights and applied to the returns observed in the subsequent period. The portfolio weights calculated at time t were then used to describe the returns obtained at time t+1. Specifically, while the Diversification Ratio allows us to measure the ex-ante degree of diversification based on the risk structure and estimated covariances, the out-of-sample analysis allows us to understand whether the achieved diversification can translate into observable ex-post effects in terms of the behavior of realized returns. This analysis allows us to observe how the risk-covariance properties we use to identify the Diversification Ratio influence the quality of the analyzed returns.

For the same reasons, the performance, risk, and risk-return measures discussed below, again comparing the portfolio composed of GICS sectors and the one that adds Bitcoin, should not be viewed as alternatives to diversification, but rather as measures that allow us to understand how ex-ante diversification can produce ex-post benefits.

The analysis discussed will be interpreted as the bridge between the diversification measured when deciding how to construct the portfolio and the results an investor could achieve over time.

5.2.1 Out-of-sample log-return analysis: performance and distribution characteristics.

Thanks to this approach, we were able to avoid the risk of overfitting that can be present in strictly in-sample measures and verify how the portfolios perform under conditions not used to measure the weights. This section will discuss the differences observed in the two portfolios analyzed. Specifically, we will compare cumulative performance and the statistical properties of the return distribution.

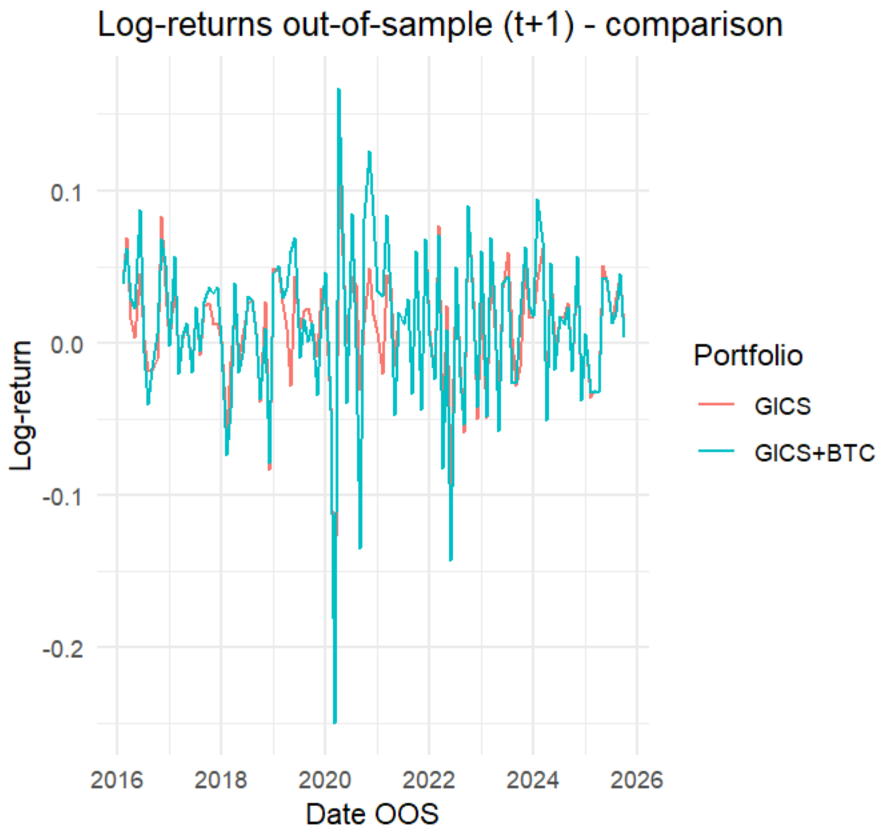

Figure 12

Before comparing the analyzed measures, this graph shows the temporal dynamics of the out-of-sample monthly returns. It's clear that the two portfolios follow the same "direction" for most of the time period considered. This is consistent because the predominant component in both portfolios remains sector-specific equities. Despite this, the line that includes Bitcoin increases the fluctuations, which is why both positive and negative peaks are more pronounced.

First, the cumulative dynamics were compared. As can be seen from the graph below, the comparison is important.

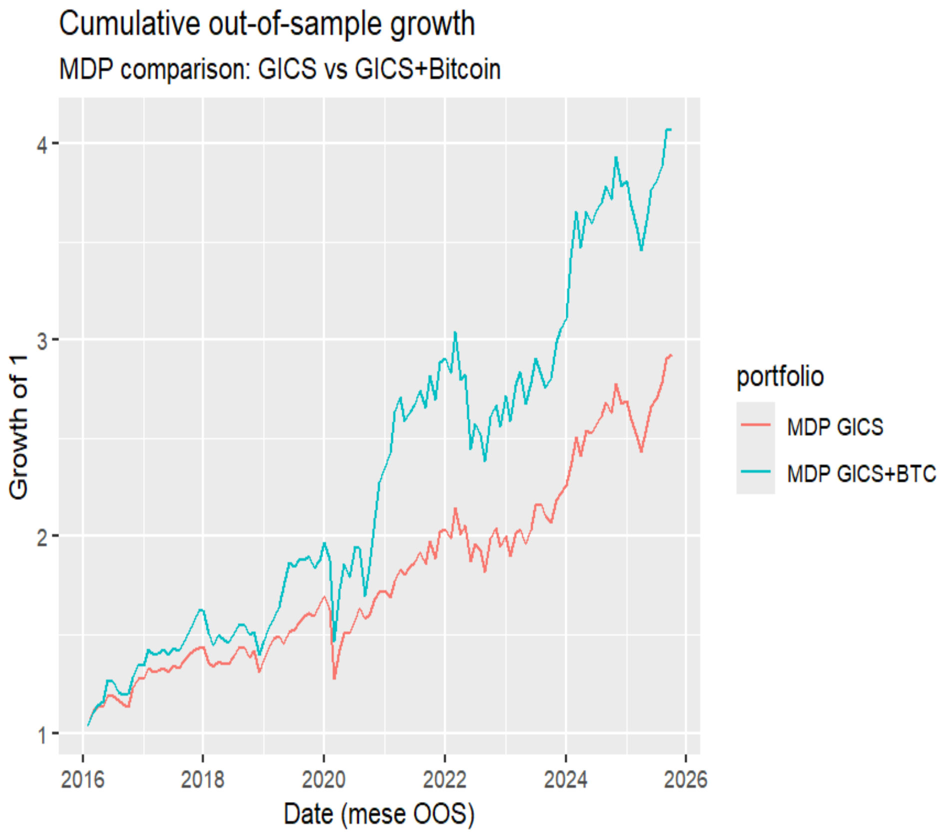

Figure 13

The graph clearly shows how the cumulative out-of-sample growth of the two portfolios follows an increasing trend, with a sharp surge in the portfolio that includes the cryptocurrency. Indeed, the cumulative growth of the GICS portfolio is 2.93, corresponding to a return of +193% over the time horizon we consider. The cumulative growth of the portfolio that includes Bitcoin is much more significant, at 4.08, with an overall return of +308%. It is clear that the difference between the two lines is not one-off, but rather progressively widens over time. This highlights the fact that Bitcoin has a structural impact on the dynamics of capital accumulation. This result could have been expected, as Bitcoin is a highly volatile asset with a low average correlation with sectors, which leads to an improvement in portfolio efficiency.

Further confirmation of this is provided by the analysis of monthly extreme returns. In this case, we compare the month with the best out-of-sample return and the month with the worst out-of-sample return. For the best month, the GICS+Bitcoin portfolio achieves a value of approximately 18%, while the portfolio composed solely of sectors has a much lower value of 11%. On the other hand, the comparison of the worst month between the two portfolios is essentially the same, standing at around 22%. These parameters highlight the fact that cryptocurrency contributes less to the amplification of the largest losses than it does to the widening of the positive side of the distribution.

The annualized out-of-sample volatility represents the most immediate cost of Bitcoin. The volatility of the GICS+Bitcoin portfolio increases by almost 4% compared to the portfolio sector, reaching 18,80%. This is essentially due to the cryptocurrency's enormous volatility. In this case, however, volatility should not be viewed solely as a negative that drastically increases portfolio risk, given that the primary objective is to analyze the difference in diversification between the two portfolios. Furthermore, in this case, volatility is also partly explained by the increase in cumulative return and by the analysis of skewness and kurtosis.

Regarding skewness, it is negative in both portfolios, indicating a more pronounced left tail. The skewness of the Bitcoin portfolio is less negative (-1.06), indicating that the inclusion of Bitcoin partially reduces the risk profile's bias toward significant losses. Therefore, Bitcoin reduces the effect of negative tails by attempting to balance the distribution, benefiting from a greater frequency of high positive returns.

A similar result is obtained with kurtosis, which allows us to observe how concentrated the observations are in the tails compared to a normal distribution. Both portfolios have very high values of kurtosis. The kurtosis of the GICS portfolio is 11.52, while that of the GICS+Bitcoin portfolio is 7.17, lower than that of the sector portfolio. In both cases, they are leptokurtic distributions. This result is important because it contradicts the belief that highly volatile assets increase the tail risk of the overall portfolio.

5.2.2 Ex-post risk-adjusted performance evaluation: Sharpe Ratio and Sortino Ratio.

In this section, we complete the analysis of out-of-sample log returns by comparing risk-adjusted performance measures. The measures we consider are the Sharpe Ratio and the Sortino Ratio, which allow us to conduct a further in-depth analysis of the two portfolios.

The Sharpe ratio, calculated over the entire out-of-sample period, shows positive values for both strategies. This means that both portfolios generated a positive return per unit of total risk. This finding is significant when compared with the results previously analyzed, especially regarding volatility. Indeed, the results show that the annualized Sharpe Ratio of the portfolio composed only of sectors is 0.73, while that of the portfolio that includes the cryptocurrency is slightly higher, equal to 0.77. This indicates a relative increase of 5% for the GICS+Bitcoin portfolio. The introduction of Bitcoin, therefore, certainly increases the overall risk, but on the other hand, this risk is justified by an increase in overall efficiency. Without forgetting our goal of diversification, the addition of the cryptocurrency, which has very different characteristics from traditional sectors, can increase the risk-return ratio despite increasing overall risk. Subsequently, it was possible to implement a more dynamic analysis, as shown in the graph below.

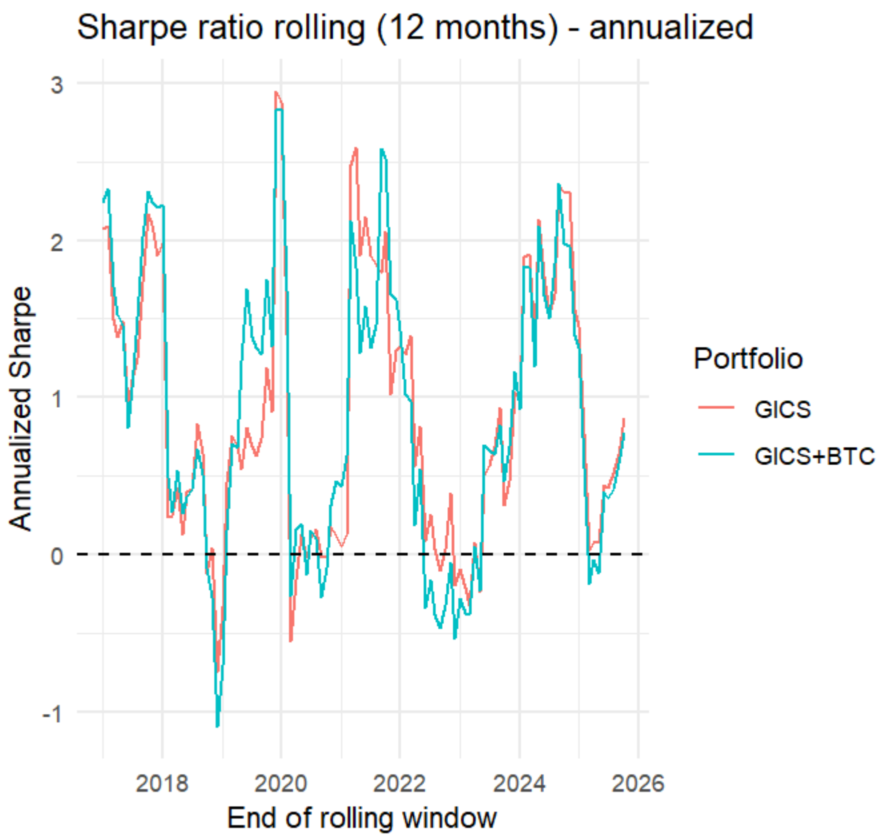

Figure 14

This graph also clearly shows how the annualized Sharpe ratio lines of the two portfolios follow the same fluctuations and are often superimposable. However, in favorable market phases, the portfolio composed exclusively of traditional sectors has lower Sharpe ratio values, indicating that in these environments, Bitcoin amplifies favorable or recovering market moments, while also amplifying periods of high uncertainty or shock. This is due to volatility, which is not always accompanied by sufficiently strong returns. It can therefore be said that Bitcoin's influence on the Sharpe ratio is not uniform across the entire period considered, as it is highly dependent on market performance over a given period. On the other hand, the inclusion of Bitcoin is justified by the fact that there are more periods in which portfolio efficiency increases than in which it decreases or penalizes it.

Subsequently, thanks to the Sortino Ratio, we were able to focus on measuring the average excess return with respect to a minimum acceptable threshold, which was set equal to 0 (MAR) in relation to the standard deviation of negative returns only. Both portfolios have relatively high positive values, but the portfolio that includes Bitcoin has higher values.

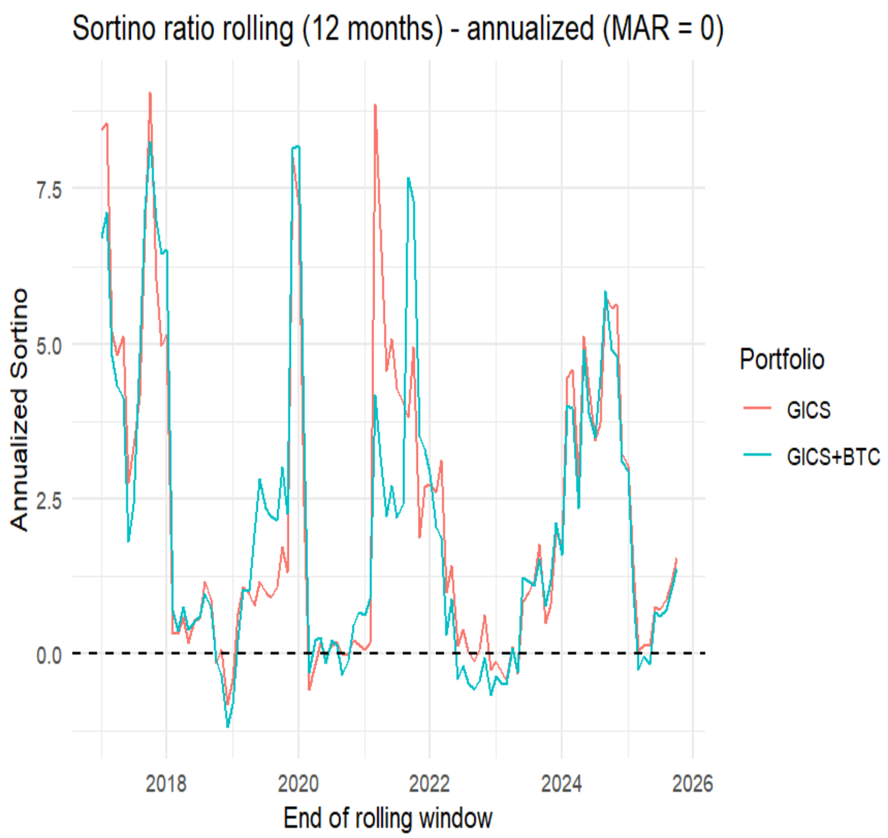

Figure 15

The graph shows how the Sortino ratio of the Gics+Bitcoin portfolio is equal to, and often higher than, the sector portfolio. However, there are also cases where the Sharpe ratio of the Bitcoin portfolio is lower than that of the other portfolio, especially during periods of high volatility. However, the graph essentially shows that Bitcoin improves the downside risk-return ratio. Looking at this graph and comparing it to the Sharpe ratio, however, we note how the differences in this case are more significant and evident. This indicates that the improvements associated with Bitcoin are not only attributable to returns relative to total risks, but also to a change in the risk structure, as the cryptocurrency increases the average return over time without correspondingly increasing the frequency of significant negative returns.

Analyzing the Sharpe Ratio and Sortino Ratio together allows us to create a set of information consistent with our research objective. The Sharpe ratio slightly improves the portfolio's efficiency relative to total risk; on the other hand, the Sortino ratio shows a more marked improvement when the focus is more on downside risk. Despite the increased volatility of the GICS+Bitcoin portfolio, these two measures suggest that including Bitcoin in the MDP portfolio is an efficient choice when considering the risk-return trade-off.

Chapter 6

Empirical Analysis of the geographic portfolio

6.1 Diversification Ratio of the geographic and geographic + Bitcoin portfolio

This chapter describes the empirical analysis conducted to compare the Diversification Ratio obtained in two portfolios: a first portfolio composed of a selection of market indices (INDEXES) and a second portfolio consisting of the same set as the first but with the addition of Bitcoin (INDEXES+BTC). Comparing the two DR series allows us to assess whether the introduction of Bitcoin into the original portfolio increases diversification ex-ante.

We will discuss the results presented in the following table:

Portfolio	Mean	Median	St.Dev
INDEXES	1.95	1.76	0.75
INDEXES+BTC	2.22	1.92	0.86

Figure 16

First, it is important to analyze the central part of the Diversification Ratio distribution in the two portfolios, looking at the mean and median. In the INDEXES+BTC portfolio, there is a 13.7% increase in the mean, suggesting that the introduction of Bitcoin allows us to leverage the benefits of combining assets that are not perfectly correlated. The result obtained from the median is consistent with the increase in the mean. Indeed, the median Diversification Ratio increases from a value of 1.72 in the INDEXES portfolio to 1.92 in the portfolio that includes Bitcoin. This result is important because it allows us to affirm that the previously described improvement in the mean is not due solely to a limited number of extreme observations but is the result of a shift in the entire Diversification Ratio distribution toward higher values.

In addition to the mean and median, it is also important to observe how the standard deviation changes to understand how variable the metric is over time. The portfolio composed exclusively of market indices has gone from a value of 0.75 to a value of 0.86 for the INDEXES+BTC portfolio. This result can be better understood in the graph:

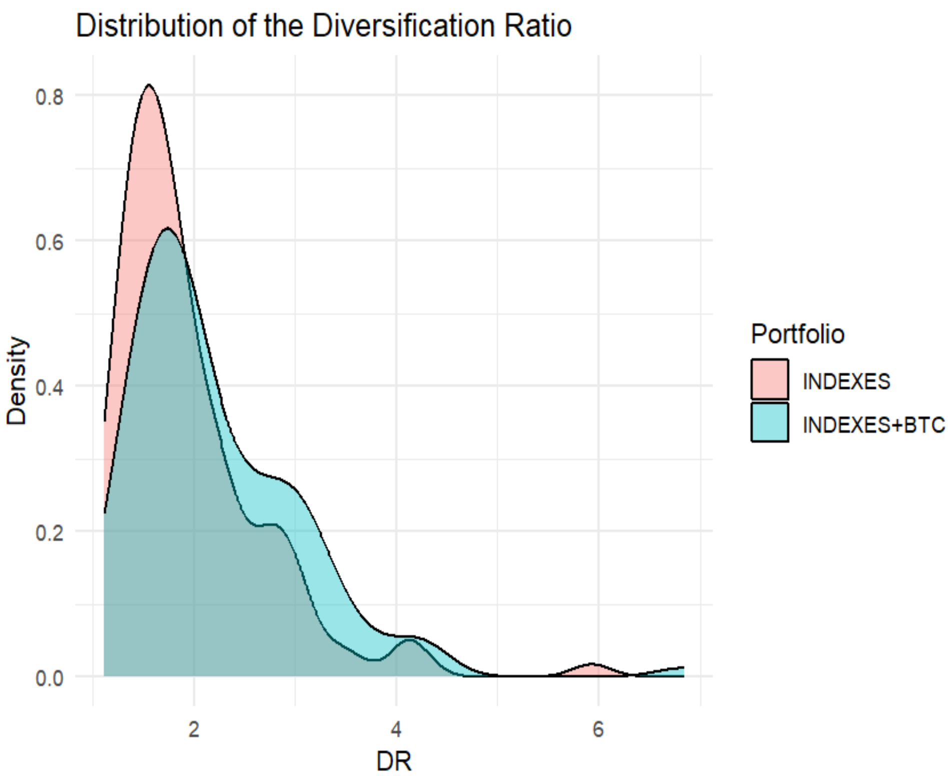

Figure 17

The graph clearly shows how the distribution of the Diversification Ratio in the Bitcoin portfolio is shifted to the right, visually demonstrating how higher values of the metric are more frequent when the cryptocurrency is included in the investable universe. The blue distribution also appears broader, reflecting how Bitcoin introduces new possible combinations into the portfolio. This greater dispersion need not necessarily be viewed as a negative. From an economic perspective, it can be seen as an opportunity to expand one's investable universe, increasing the possible combinations of assets to achieve high levels of diversification.

To compare the distribution of the metric, it is also useful to observe the differences in the boxplot.

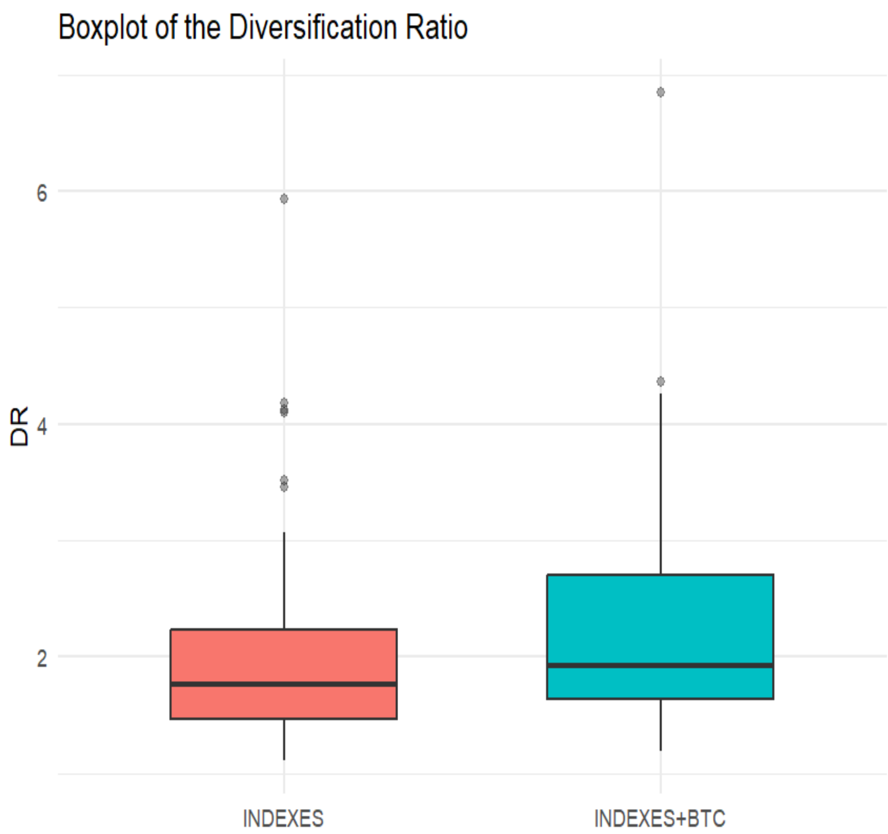

Figure 18

Thanks to this representation, we clearly see how most metrics change with the introduction of cryptocurrency. We can see how the DR distribution of the INDEXES+BTC portfolio is broader and positioned at broader levels. The median of the Bitcoin-included portfolio is higher, and the interquartile range is wider, suggesting what was previously stated regarding a greater dispersion of the central values of the distribution due to the increase in standard deviation. Looking at the boxplot, we see how the maximum values of the Diversification Ratio are higher in the INDEXES+BTC portfolio, indicating that the inclusion of Bitcoin increases diversification at the most favorable times. The boxplot clearly and visually confirms what was previously stated about the descriptive statistics.

Once the descriptive statistics of the two portfolios have been analyzed, it is useful to have a temporal view of how the Diversification Ratio changes over time along the 12-month rolling windows. The time graph allows us to visualize the difference in the metric's evolution over time in the two portfolios.

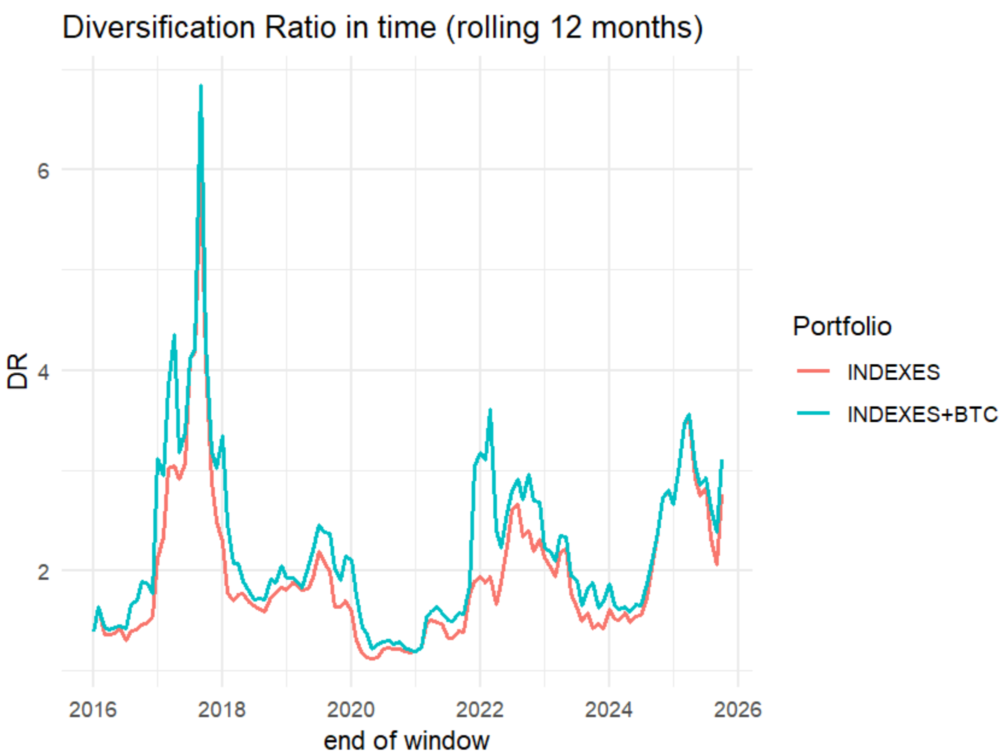

Figure 19

The graph clearly shows how the two lines essentially follow the same trend, obviously since the two portfolios are composed of essentially the same assets. Despite this, the blue line (INDEXES + BTC) remains higher throughout the time horizon, indicating a constant contribution from Bitcoin to the increase in the metric studied. Furthermore, the lines show how the Diversification Ratio is not constant over time but fluctuates significantly over time depending on market conditions and the correlations between the assets comprising the portfolios considered.

The following representation, which displays the difference between the two Diversification Ratios, provides a better understanding of the cryptocurrency's contribution to the total invested in the INDEXES portfolio.

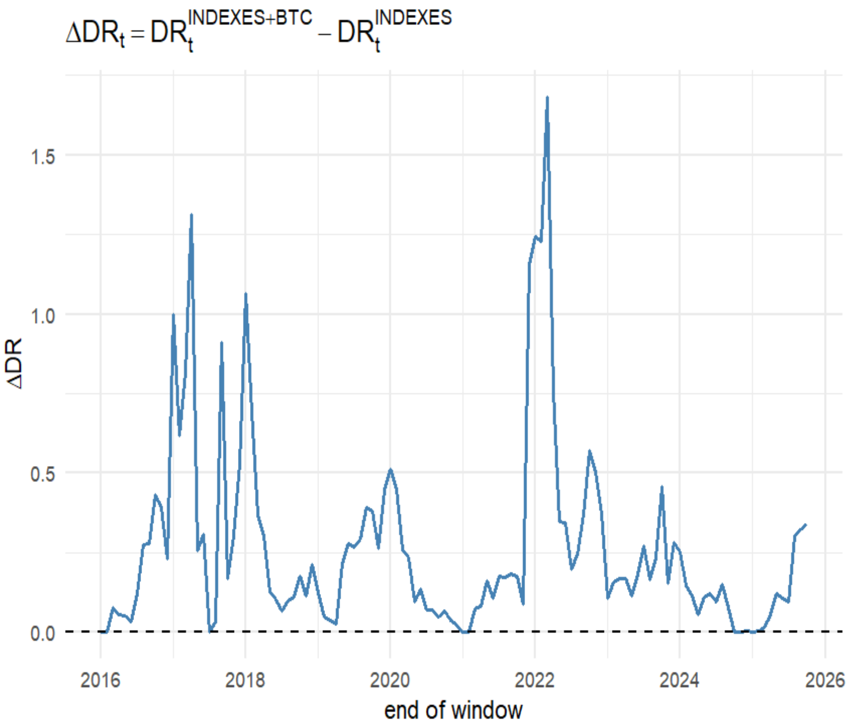

Figure 20

The resulting measure allows us to isolate the marginal effect of including Bitcoin on portfolio optimization. The difference is always positive, indicating that in most time frames, the INDEXES portfolio exhibits a lower level of diversification than the other portfolio. The magnitude of the difference varies over time; there are times when the difference is marked and reaches significant values, while at other times the difference is limited, reaching (albeit rarely) values close to zero.

The analysis explained in this section allows us to discuss and visualize the influence Bitcoin has on portfolio diversification. In conclusion, it can be stated that in the ex-ante analysis, Bitcoin managed to increase the level of diversification of the initial portfolio through the Most Diversified Portfolio constraint.

To verify whether the results obtained were statistically significant, the paired t-test and subsequently the Wilcoxon signed-rank test (which does not require the assumption of normality of the distribution of differences) were used.

The paired t-test provided highly significant results, with a t-statistic of 9.46 and a p-value of 3.99e-16. The p-value was well below the significance thresholds, and for this reason the null hypothesis was rejected. Furthermore, the estimated mean difference in DR between the two portfolios was 0.27, confirming that the presence of Bitcoin increased the diversification of the original portfolio.

The evidence provided by the Wilcoxon signed-rank test confirms the results of the paired t-test, with a V of 7010 and a p-value less than 2.2e-16, another very low value indicating high statistical significance. Here too, the null hypothesis was rejected.

The results obtained with the paired t-test and the Wilcoxon signed-rank test support the findings obtained so far, confirming that cryptocurrency induces a significant increase in the diversification of a portfolio composed only of market indices.

6.2 Out-of-sample log-returns analysis

In this section of the analysis, the research focuses on analyzing the out-of-sample log returns resulting from the two market index portfolios with and without Bitcoin, thus further developing the results obtained in the previous section.

The portfolio weights obtained using the Most Diversified Portfolio strategy are applied at the end of each window to the returns observed in the subsequent period.

This step is crucial for understanding whether the results obtained with the Diversification Ratio during a portfolio construction phase can translate into actual future benefits. In other words, this section allows us to connect the diversification measured ex ante with the behavior of returns over time, providing a more comprehensive assessment of Bitcoin's impact on an investable universe.

6.2.1 Out-of-sample log-returns analysis: performance and distribution characteristics

First, the analysis of the results begins by comparing the out-of-sample log-returns of the two portfolios over the period.

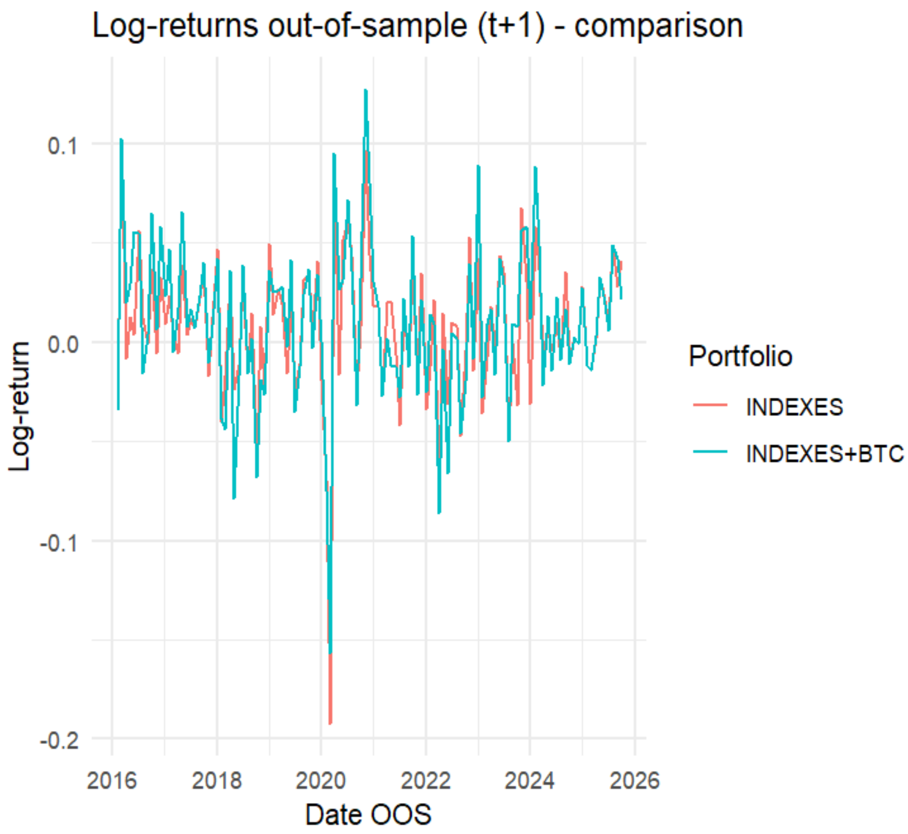

Figure 21

The resulting graph is consistent with the fact that the two portfolios are composed almost entirely of the same assets, which is why they follow a very similar trajectory, with oscillations around zero and a succession of positive and negative phases. Although the two lines essentially follow the same trajectory, some differences are evident. The portfolio that includes Bitcoin experiences higher positive returns in many phases but also faces more pronounced negative returns. This demonstrates that cryptocurrency is a highly volatile asset and can amplify variations in portfolio returns. The most obvious episode occurred immediately after the 2020 financial crisis due to the Covid-19 pandemic, followed by a series of recoveries, positively accentuated by Bitcoin in subsequent periods.

Subsequently, to assess the impact of these dynamics over the long term, it is useful to visualize the cumulative growth of out-of-sample returns using the graph below.

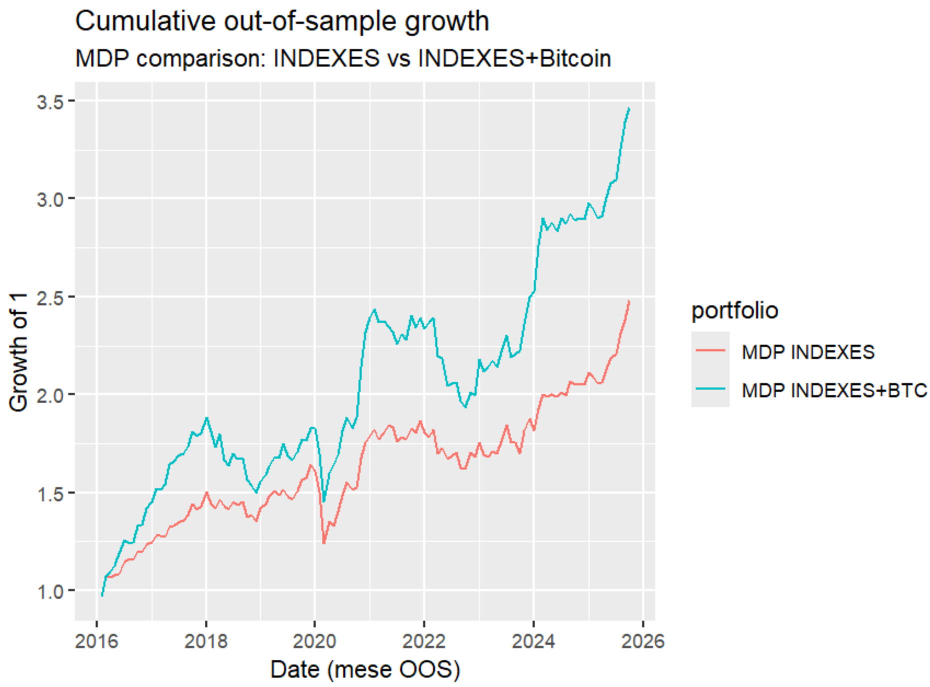

Figure 22

The graph shows a very marked difference between the two strategies. Over the period considered, the strategy that includes Bitcoin shows much more pronounced growth than the portfolio composed solely of market indices. This finding is important because, despite Bitcoin's notorious volatility, over the long term, Bitcoin's contribution allows for very high levels to be reached, benefiting from more pronounced growth phases that the INDEXES portfolio cannot match. It is also noted that the gap between the two lines tends to widen over time and during periods of strong financial growth.

This is also confirmed by the extreme returns of the two portfolios. Comparing the best and worst months observed in the out-of-sample period, the INDEXES+BTC portfolio shows a higher best month of +13.60%, indicating that Bitcoin helps benefit from favorable market phases. On the other hand, the portfolio without Bitcoin records a worse month in terms of returns, more negative than the strategy that includes Bitcoin. This result is interesting because, despite Bitcoin's strong volatility, the portfolio that includes it does not have the most pronounced worst month, suggesting that the cryptocurrency does not necessarily amplify the portfolio's exposure in negative times.

Regarding the annualized volatility of the out-of-sample log returns, the results highlight what was previously stated regarding differences in the magnitude of returns. The portfolio with only market indices shows lower volatility. This result is consistent with what has been discussed so far, namely that Bitcoin is an asset characterized by a very different volatility from other traditional assets and therefore its presence affects the variability of returns.

Finally, skewness and kurtosis were analyzed. Regarding skewness, both the Bitcoin portfolio and the portfolio composed only of market indices exhibit a more skewness distribution. However, the INDEXES+BTC portfolio has a more positive skewness level of -0.44, indicating that Bitcoin increases the probability of seeing positive extreme returns compared to the portfolio without cryptocurrency. Regarding kurtosis, the results indicate that both portfolios exhibit a kurtosis greater than a normal distribution, which translates into a higher probability of extreme events occurring.

6.2.2 Ex-post risk-adjusted performance evaluation: Sharpe Ratio and Sortino Ratio

To conclude the analysis of out-of-sample log returns, risk-adjusted performance measures were compared. The measures compared were the Sharpe Ratio and the Sortino Ratio, as described in Chapter 4.

The Sharpe Ratio indicates that both portfolios were able to generate positive returns relative to absolute risk, as suggested by the positive Sharpe Ratio values calculated over the entire period considered. Specifically, the annualized Sharpe Ratio of the portfolio that includes Bitcoin is higher than that of the other portfolio, equal to 0.91. This value indicates that the inclusion of Bitcoin allows for a higher return for each unit of total risk incurred, thus highlighting an improvement in the portfolio's performance. Combining this evidence with the results obtained previously, it can be argued that the increased volatility brought by the cryptocurrency in the portfolio is offset by an increase in returns. Looking at the graph below, it is possible to visualize how the relationship between volatility and return changes over time.

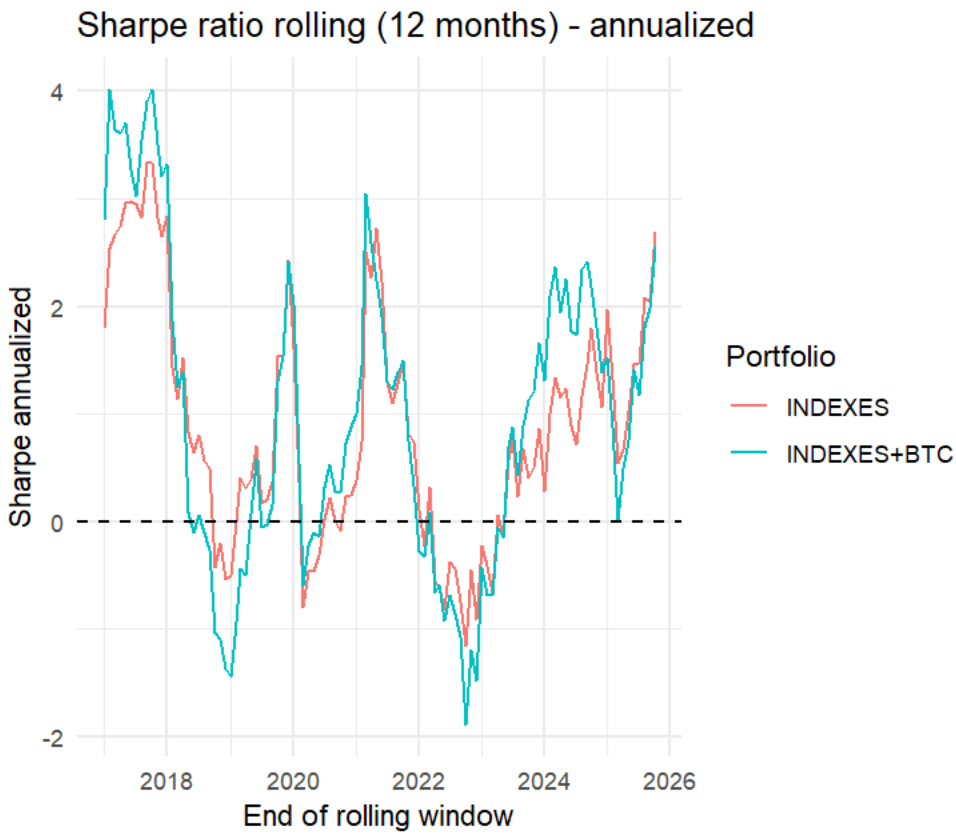

Figure 23

Despite the very similar performance of the two strategies (as also confirmed in the previous graphs), it is clear that the Sharpe Ratio of the portfolio that includes Bitcoin often exceeds the other Sharpe Ratio, demonstrating how Bitcoin can positively impact market growth by increasing portfolio efficiency. Likewise, however, cryptocurrency also amplifies market declines and unfavorable situations. This dynamic is primarily due to the high volatility that characterizes cryptocurrency, which tends to amplify the Sharpe Ratio in both positive and negative movements, thus increasing risk without being accompanied by sufficiently high returns. However, looking at the entire out-of-sample horizon considered, the times when Bitcoin penalizes portfolio efficiency are significantly lower than the times when it improves it, explaining why the Sharpe Ratio in the Bitcoin portfolio is higher.

Then, we study the Sortino Ratio with a minimum acceptable return (MAR) of zero, allowing us to focus on downside risk. Here too (as with the Sharpe Ratio), positive values emerge, indicating that returns are adequate for the risk assumed. The INDEXES+BTC portfolio has a value of 1.48, compared to the portfolio composed solely of market indices, which has a lower value of 1.09. The higher Sortino Ratio value in the Bitcoin portfolio indicates that the benefits of including Bitcoin are very evident when the analysis focuses on losses. The graph below shows how the Sortino Ratio of the two portfolios evolves.

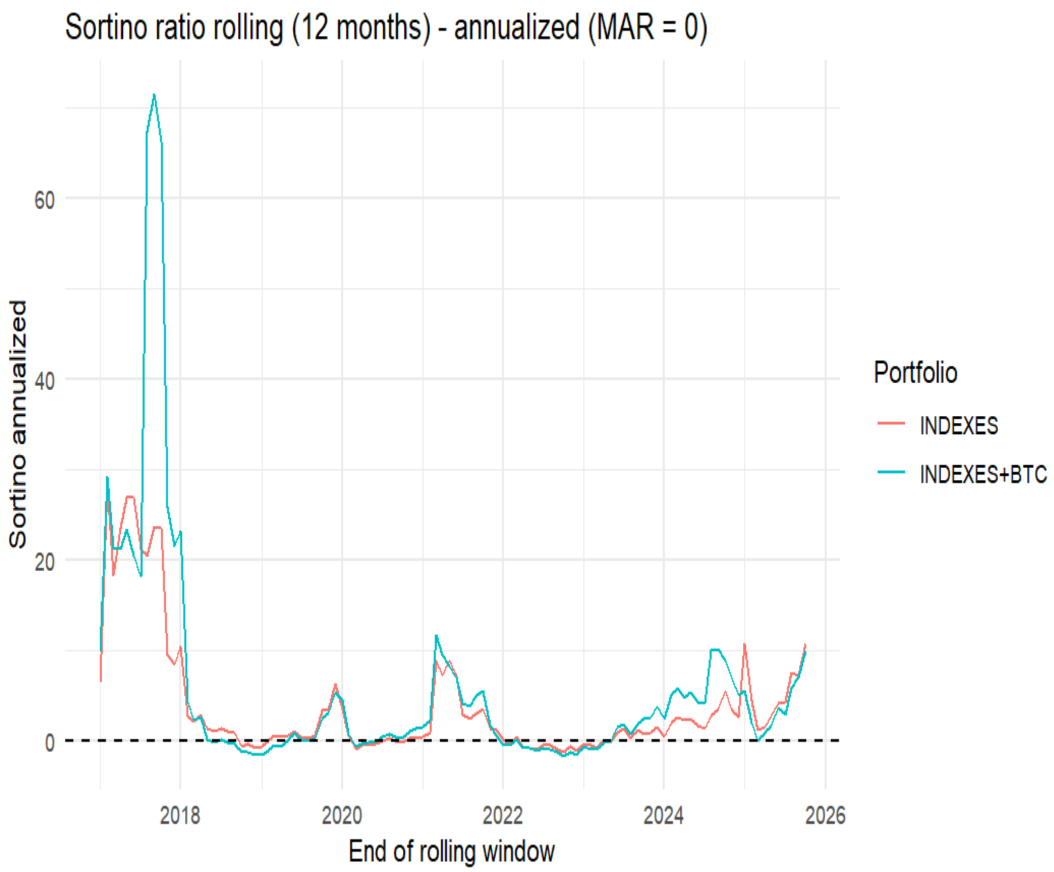

Figure 24

Obviously, in this case too, the two lines follow the same trend, indicating that the two portfolios are essentially composed of the same assets. However, the line indicating the Bitcoin portfolio shows higher values than the red line in many cases. The introduction of the cryptocurrency increases the probability of producing returns without necessarily increasing the risk of negative returns, despite Bitcoin making the portfolio more volatile and sensitive. This result is consistent with the (very similar) values for the downside deviation of the two portfolios, both of which are approximately 8.55%.

The results verified with the Sharpe Ratio and Sortino Ratio are consistent with the analysis sections of the two market index portfolios with and without Bitcoin. This latter verification confirms that the inclusion of Bitcoin in the Most Diversified Portfolio strategy not only increases portfolio diversification and, theoretically, partially reduces aggregate risk, but could also represent a smart choice when considering the risk-return trade-off.

Chapter 7

Conclusions

Numerous studies have focused on optimizing the risk-return tradeoff for a portfolio containing Bitcoin. In this analysis, a different approach was adopted to focus on the issue of diversification. In this context, Bitcoin was used as a diversifying asset capable of impacting the efficiency of a portfolio.

The results explained in the previous chapters clearly demonstrate how Bitcoin increases portfolio diversification, regardless of the assets included. Indeed, the results show an increase in diversification, measured by the Diversification Ratio, in both the sector-specific and geographically defined portfolios. Furthermore, the increase and impact of Bitcoin on the Diversification Ratio also depends on the structure of the starting portfolio. The sector-specific portfolio, characterized by generally high correlations between assets, sees the inclusion of Bitcoin as an asset highly decorrelated from the rest of the investment universe considered. The assets considered in the geographical portfolio exhibit less strong correlation structures, primarily due to the diversity of the economies considered globally and their concentration in different financial markets. In this portfolio too, Bitcoin was able to increase diversification, but to a less radical extent than in the sector-specific portfolio, strengthening the existing diversification without disrupting the portfolio's internal dynamics.

The evidence suggests that Bitcoin, despite its high volatility, can help increase portfolio diversification, as measured by the Diversification Ratio. Therefore, Bitcoin could be considered not only as a speculative asset capable of achieving significant returns, but also as an element for improving the structure of certain portfolios.

Bibliography

Brière, M., Oosterlinck, K., and Szafarz, A. (2015). Virtual Currency, Tangible Return: Portfolio Diversification with Bitcoin. *Journal of Asset Management*, Vol.16(6), 365–373.

Choueifaty, Y., and Coignard, Y. (2008). Toward maximum diversification. *Journal of Portfolio Management*, Vol.35(1), 40-51.

Choueifaty, Y., Froidure, T. and Reynier, J. (2013). Properties of the Most Diversified Portfolio. *Journal of Investment Strategies,* Vol.2(2), 49-70.

Markowitz, H. (1952). Portfolio Selection. *Journal of Finance,* Vol.7, 77-91.

Meucci, A. (2009). Managing Diversification. *Risk,* Vol.22(5), 74-79.

Nakamoto, S. (2008). Bitcoin: A Peer-to-Peer Electronic Cash System. http://www.bitcoin.org/bitcoin.pdf

Philippatos, G.C., and Wilson, C.J. (1972). Entropy, market risk, and the selection of efficient portfolios. *Applied Economics,* Vol.4, 209-220.

Platanakis, E., and Urquhart, A. (2020). Should investors include Bitcoin in their portfolios? A portfolio theory approach. *The British Accounting Review*, Vol.52(4), 100837, pp. 1-42.

Shannon, C.E., (1948). A Mathematical Theory of Communication. *The Bell System Technical Journal,* Vol.27, 379-423.

Sharpe, W.F. (1964). Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk. *Journal of Finance.* Vol.19, 425-442.

Sharpe, W.F. (1966). Mutual Fund Performance. *Journal of Business*, Vol.39(1, Part 2), 119–138.

Sortino, F.A., and Van der Meer, R. (1991). Downside Risk. *Journal of Portfolio Management*, Vol.17(4), 27–31.