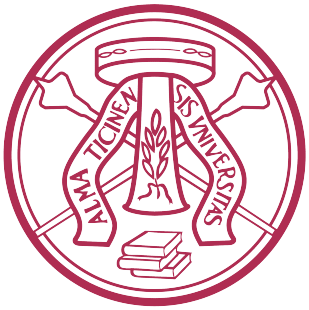

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Towards Fermionic Resource Theories of Computational Advantage

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Alla Pina
La mia prima maestra

A Matilde
Luce dei miei occhi

Ai miei genitori
Per aver sempre creduto in me

Abstract

The stabilizer formalism relies on the Gottesman-Knill theorem: every quantum computer composed of Clifford unitaries and Pauli measurements is efficiently simulable by a classical computer. Non-stabilizerness, or magic, is the resource associated with the departure from this classically simulable regime, and is necessary for universal quantum computation. In this thesis, magic is studied within the framework of Quantum Resource Theories (QRTs), where stabilizer states are the free states and stabilizer operations (SO) are the free operations. We address the quantification of magic through suitable resource-measuring functions, called monotones. In particular, we show that the α-Stabilizer Rényi Entropies fail to completely characterize magic in the single-shot regime. We also generalize the theory by enlarging the free operations to completely stabilizer-preserving operations (CSPO) and by introducing a quotient structure. The second part of the thesis focuses on non-classical simulability in fermionic systems, showing that the fermionic Gaussian simulability paradigm is intrinsically different from the stabilizer one. This motivates a QRT of fermionic non-Gaussianity, where fermionic Gaussian states and operations (FGO) are free. We introduce a novel monotone, the entropy of non-Gaussianity, defined as the quantum relative entropy between a fermionic state and its Gaussianization, and establish its key properties.

Il formalismo stabilizer si basa sul teorema di Gottesman-Knill, in base al quale ogni computer quantistico composto da unitarie di Clifford e misure di Pauli è efficientemente simulabile da un computer classico. La non-stabilizerness, o magic, è la risorsa quantistica che descrive l'allontanamento da tale regime di simulabilità classica, ed è necessaria per la computazione quantistica universale. In questo lavoro di tesi, la magic viene studiata nell'ambito delle Quantum Resource Theories (QRT), in cui gli stati stabilizer fungono da stati liberi e le operazioni stabilizer (SO) da operazioni libere. Viene affrontato il problema della quantificazione della magia attraverso opportune funzioni dette monotones. In particolare, viene mostrato che la famiglia delle α-Stabilizer Rényi Entropies (SRE) non sono in grado di caratterizzare completamente gli stati magic nel regime di convertibilità single-shot. Inoltre, la teoria

viene generalizzata estendendo le operazioni libere tramite mappe completamente stabilizer-preserving (CSPO) e introducendo un'apposita struttura di quoziente. La seconda parte della tesi si concentra sulla simulabilità classica nei sistemi fermionici, mostrando che il paradigma di simulabilità gaussiana è intrinsecamente diverso da quello stabilizer. Ciò motiva la costruzione di una QRT della non gaussianità fermionica. Viene introdotta una nuova monotone, l'entropia di non gaussianità, definita come la quantum relative entropy tra uno stato fermionico e la sua gaussianizzazione, e ne vengono stabilite le proprietà fondamentali.

Contents

Introduction

Since the early days of quantum computation, a central question has driven the field: *what makes a quantum computer more powerful than a classical one?* Answering this question is not only a matter of theoretical inquiry, but a practical necessity. Indeed, identifying the precise resource needed for quantum advantage allows one to understand which physical systems and which operations are worth considering in the race towards scalable quantum devices, and which, instead, could be efficiently reproduced by a classical machine at no computational cost.

A first, fundamental answer came from the foundational works in [1, 20, 21]: quantum circuits composed only of Clifford unitaries and Pauli measurements can be efficiently simulated by a classical computer. This result, known as the Gottesman-Knill theorem, singles out an entire class of quantum states, called the stabilizer states, that show no computational advantage despite having quantum features. It follows that the necessary quantum resource is non-stabilizerness, also known as magic ([10]).

Such a resource can be described and formalized under the lenses of Quantum Resource Theories (QRTs) — see [11, 13, 14] for exhaustive, general reviews. A QRT is built upon free states and free operations. In the magic setting, stabilizer states and stabilizer operations play these roles, respectively. The question of *how one can measure the magic content of an arbitrary state?* is addressed by the construction of suitable resource-quantifying functions, called monotones ([26, 43, 44, 55]). Among them, the Stabilizer Rényi Entropies (SREs) have been introduced by recent efforts ([34]) and deeply analyzed as computable and experimentally accessible ([23, 33]). However, Datta et al. recently showed in [16] the following no-go theorem: no finite set of monotones that are both faithful and continuous can fully characterize a QRT, including the QRT of magic. This naturally raises the question of whether the theory can be generalized, *e.g.*, by enlarging the free operations or by introducing an appropriate quotient structure.

Switching to fermionic systems not only requires additional algebraic structures, but also provides two separate simulability paradigms. On one hand, the stabilizer formalism is completely preserved within the framework of fermionic magic, yielding a fermionic analogue of the Gottesman-Knill theorem ([4, 35]). On the other hand, the fermionic Gaussian paradigm ([8, 28, 29, 51, 54]) is intrinsically different from the sta-

bilizer one, and this motivates the construction of an independent QRT of fermionic non-Gaussianity. Therefore, the quantification of non-Gaussianity naturally arises as a significant challenge, and dedicated monotones have been introduced in recent works (see [46, 50]). In this thesis work, a novel monotone — the entropy of non-Gaussianity — is introduced as the quantum relative entropy between a fermionic Gaussian state and its Gaussianization, providing a new method for the quantification of the resource of non-Gaussianity.

This thesis studies both QRTs — magic and fermionic non-Gaussianity — exploring their common structure, their differences, and open problems. This thesis work is organized as follows:

- Chapter 1 lays the mathematical groundwork for the stabilizer formalism, focusing in particular on the Pauli group, the stabilizer subgroup, and the Clifford group;
- Chapter 2 builds the QRT of magic, addresses quantification through monotones, discusses the no-go theorem, shows the incompleteness of the SREs in the single-shot convertibility regime, and generalizes the theory via completely stabilizer-preserving operations and a quotient structure;
- Chapter 3 introduces the fermionic Gaussian formalism, providing a description of the local fermionic modes via Majorana operators, and the relevant algebraic structures;
- Chapter 4 builds the QRT of fermionic non-Gaussianity, introduces the entropy of non-Gaussianity as a novel monotone and discusses its main properties and its informational meaning; in addition, the property of fermionic magic, understood as the fermionic extension of qubit magic, is analyzed.

Chapter 1

Introduction to Stabilizer Formalism

The concept of stabilizer computation, understood as a paradigm for classical simulability, is rooted in the pioneering work of D. Gottesman ([20, 21]). The aim of this chapter is to lay the mathematical and physical groundwork necessary for a rigorous introduction to the stabilizer formalism.

1.1 Quantum Information Preliminaries

In the following section, the notation adopted is largely based on [11], though it has been partially adapted for the sake of consistency and clarity. Moreover, it is fully compatible with the following postulates of Quantum Mechanics:

- **Quantum systems.** Any quantum system A is associated with a Hilbert space $\mathcal{H}_A$. We will use the notation $\mathcal{H}$ without the subscript to denote a general Hilbert space; the dimension of $\mathcal{H}$ will be denoted by d;

- **Bounded operators.** The set of the bounded operators acting on $\mathcal{H}_A$ will be denoted by $\mathcal{B}(\mathcal{H}_A)$;

- **Quantum states.** Any quantum state will be associated with a density matrix ρ. In particular, the set of quantum states will be denoted by $\mathrm{St}(A) := \{\rho \in \mathcal{B}(\mathcal{H}_A) \mid \rho \geq 0 \text{ and } \mathrm{Tr}[\rho] = 1\}$;

- **Linear maps.** A linear map acting from $\mathcal{B}(\mathcal{H}_A)$ to $\mathcal{B}(\mathcal{H}_B)$ will be denoted by $\Phi : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$. The identity map is $I_A : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_A)$.

Let us introduce some useful notions. We would like to construct a *quantum channel*, that is, a map between density matrices.

Definition 1.1. A map $\Phi : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ is linear if

$$\Phi(\rho + \sigma) = \Phi(\rho) + \Phi(\sigma), \forall \rho, \sigma \in \mathcal{B}(\mathcal{H}_A) \text{ and}$$
$$\Phi(\lambda\rho) = \lambda\Phi(\rho), \forall \lambda \in \mathbb{C}.$$

We need this map to be completely positive (CP), in order to map positive semi-definite operators into positive semi-definite operators.

Definition 1.2. A map $\Phi : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ is positive if $\Phi(\rho) \geq 0$, $\forall \rho \in \mathcal{B}(\mathcal{H}_A)$ with $\rho \geq 0$.

Definition 1.3. Let $\mathcal{H}_C$ be a Hilbert space with dimension $d_C = k$ associated with a quantum system C. A map $\Phi : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ is $k-$positive if $\Phi \otimes I_C$ is positive.

Definition 1.4. A map $\Phi : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ is Completely Positive (CP) if it is $k-$positive $\forall k$.

Finally, we need this map to be Trace-Preserving (TP), in order to map trace-one operators into trace-one operators.

Definition 1.5. A map $\Phi : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ is Trace-Preserving (TP) if $\mathrm{Tr}[\Phi(\rho)] = \mathrm{Tr}[\rho]$, $\forall \rho \in \mathcal{B}(\mathcal{H}_A)$.

Thus, we are able to define a quantum channel.

Definition 1.6. A map $\Phi : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ is called a quantum channel if it is a linear CPTP map.

It is useful to introduce three possible representations for quantum channels.

Theorem 1.1 (Stinespring's Dilation)**.** *Let $\Phi : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ be a quantum channel and let $\rho \in \mathcal{B}(\mathcal{H}_A)$. Then, there exist*

- *An ancilla system E, called environment, associated with a Hilbert space $\mathcal{H}_E$ such that $\mathcal{H}_A \otimes \mathcal{H}_E \simeq \mathcal{H}_B \otimes \mathcal{H}_E$;*
- *A unitary $U : \mathcal{H}_A \otimes \mathcal{H}_E \to \mathcal{H}_B \otimes \mathcal{H}_E$;*
- *A pure state ρ_E for the environment system E;*

such that Φ can be dilated as follows:

$$\Phi(\rho) = \mathrm{Tr}_E \left[U \left(\rho \otimes \rho_E \right) U^\dagger \right].$$

Proof. We refer the reader to the original work of W. F. Stinespring in [48]. □

Theorem 1.2 (Kraus Decomposition). *Let $\Phi : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ be a quantum channel, with $\dim(\mathcal{H}_A) = d_A$ and $\dim(\mathcal{H}_B) = d_B$. Then, there exists a set of operators $\{K_i\}_{i=1}^{d_A d_B}$, called Kraus operators, such that*

$$\Phi(\rho) = \sum_{i=1}^{d_A d_B} K_i \rho K_i^\dagger,$$

where $\sum_{i=1}^{d_A d_B} K_i^\dagger K_i = \mathbb{I}$.

Proof. We refer the reader to the original work of K. Kraus in [30]. □

Theorem 1.3 (Choi Representation). *Let $\Phi : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ be a quantum channel. Let $|\Omega\rangle := \frac{1}{\sqrt{d_A}} \sum_{i=0}^{d_A-1} |i\rangle_A \otimes |i\rangle_{A'}$ denote the normalized maximally entangled vector state acting on $\mathcal{H}_A \otimes \mathcal{H}_{A'}$, with $\mathcal{H}_A \simeq \mathcal{H}_{A'}$. Then, the Choi matrix $J_\Phi \in \mathcal{B}(\mathcal{H}_B \otimes \mathcal{H}_{A'})$ associated with Φ is given by*

$$J_\Phi = \frac{1}{d_A} \Phi \otimes \mathrm{id}_{A'} \left(|\Omega\rangle \langle\Omega| \right),$$

with $\mathrm{Tr}\left[J_\Phi\right] = 1$, $J_\Phi \geq 0$, and $\mathrm{Tr}_B[J_\Phi] = \frac{\mathrm{id}_{A'}}{d_A}$.

Proof. We refer the reader to the original work of M.D. Choi in [12]. □

1.2 Basics of Group Theory

For a comprehensive introduction to group theory, we refer the reader to [3].

Definition 1.7. Let $\mathcal{G}$ be a set and let $M : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ be a composition rule defined by

$$(g, h) \to M(g, h) := gh.$$

The set $\mathcal{G}$ is called a group if the following properties hold:

- (Associativity) $\forall g, h, k \in \mathcal{G}, (gh)k = g(hk) = ghk$;
- (Trivial Element) $\exists e \in \mathcal{G} \mid g = ge = eg, \forall g \in \mathcal{G}$;
- (Inverse Element) $\forall g \in \mathcal{G}, \exists g^{-1} \in \mathcal{G} \mid gg^{-1} = g^{-1}g = e$.

It is possible to prove the uniqueness of both trivial element and inverse element, but we will omit that for simplicity. Furthermore, we will work with finite and/or abelian groups.

Definition 1.8. Let $\mathcal{G}$ be a group as above with order $|\mathcal{G}|$. Then,

- $\mathcal{G}$ is called a finite group if $|\mathcal{G}| < \infty$;
- $\mathcal{G}$ is called an abelian (or commutative) group if $gh = hg,\ \forall g, h \in \mathcal{G}$.

We also need the notion of subgroup.

Definition 1.9. Let $\mathcal{G}$ be a group and $\mathcal{W}$ a subset of $\mathcal{G}$. We call $\mathcal{W}$ a subgroup of $\mathcal{G}$ if $\mathcal{W}$ is a group with respect to the composition rule of $\mathcal{G}$, and we write $\mathcal{W} \leq \mathcal{G}$.

Definition 1.10. Let $\mathcal{G}$ be a group and $\mathcal{W} \neq \mathcal{G}$ be a subgroup of $\mathcal{G}$. We call $\mathcal{W}$ a maximal subgroup if, $\forall \mathcal{K}$ such that $\mathcal{W} \leq \mathcal{K} \leq \mathcal{G}$, it holds either $\mathcal{K} = \mathcal{W}$ or $\mathcal{K} = \mathcal{G}$. This is equivalent to saying that $\mathcal{W}$ is a maximal subgroup of $\mathcal{G}$ if we cannot find a subgroup "bigger" than $\mathcal{W}$ except for $\mathcal{G}$ itself.

Furthermore, we would like to know how to generate a group.

Definition 1.11. Let $\mathcal{G}$ be a group and $G = \{g_1, g_2, \dots, g_k\}$ a subset of $\mathcal{G}$. Then, G is called a generating set for $\mathcal{G}$ if any element of $\mathcal{G}$ can be written as a composition (under the group operation) of finitely many elements of G and their inverses. We write:

$$\mathcal{G} = \langle g_1, g_2, \dots, g_k \rangle.$$

For practical purposes, we adapt the definition above for finite groups.

Definition 1.12. Let $\mathcal{G}$ be a finite group with a generating set G. Then, $\mathcal{G}$ is called a finitely generated group if G is a finite set. It is possible to show that every finite group is also a finitely generated group.

Finally, we need the following theorem.

Theorem 1.4 (Lagrange)**.** *Given a finite group $\mathcal{G}$ and its subgroup $\mathcal{Q}$, then $|\mathcal{Q}|$ is a divisor of $|\mathcal{G}|$.*

Proof. Since this theorem is a standard result of group theory, we omit the proof for simplicity, but it can be found in [3]. □

Let us introduce a few notions concerning normality and normalizer.

Definition 1.13. Let $\mathcal{G}$ be a group and $\mathcal{W}$ be a subgroup of $\mathcal{G}$. We call $\mathcal{W}$ a normal subgroup of $\mathcal{G}$ if $\mathcal{W}$ is invariant under conjugation, *i.e.*, $ghg^{-1} \in \mathcal{W}$, $\forall g \in \mathcal{G}$ and $\forall h \in \mathcal{W}$, and we write $\mathcal{W} \trianglelefteq \mathcal{G}$. Analogously, we can write $g\mathcal{W}g^{-1} = \mathcal{W}$, $\forall g \in \mathcal{G}$.

Definition 1.14. Let $\mathcal{G}$ be a group and $\mathcal{W}$ be a subgroup of $\mathcal{G}$. We define the normalizer of $\mathcal{W}$ in $\mathcal{G}$ as the largest subgroup of $\mathcal{G}$ in which $\mathcal{W}$ is normal, namely

$$\mathcal{N}_{\mathcal{G}}(\mathcal{W}) := \{g \in \mathcal{G} \mid g\mathcal{W}g^{-1} = \mathcal{W}\}.$$

We make the following statements:

Proposition 1.1. *It follows straightforwardly from Def. 1.13 and Def. 1.14 that $\mathcal{W}$ is a normal subgroup of $\mathcal{N}_{\mathcal{G}}(\mathcal{W})$, but not necessarily of $\mathcal{G}$, namely*

$$\mathcal{W} \trianglelefteq \mathcal{N}_{\mathcal{G}}(\mathcal{W}) \leq \mathcal{G}.$$

Proposition 1.2. *$\mathcal{W}$ is a normal subgroup of $\mathcal{G}$ if and only if $\mathcal{N}_{\mathcal{G}}(\mathcal{W}) = \mathcal{G}$, namely*

$$\mathcal{W} \trianglelefteq \mathcal{G} \iff \mathcal{N}_{\mathcal{G}}(\mathcal{W}) = \mathcal{G}.$$

Finally, let us turn to the center of a group and some related concepts.

Definition 1.15. Let $\mathcal{G}$ be a group. The center $\mathcal{Z}(\mathcal{G})$ of $\mathcal{G}$ is the set of elements that commute with every element of $\mathcal{G}$, namely

$$\mathcal{Z}(\mathcal{G}) := \{z \in \mathcal{G} \mid zg = gz, \ \forall g \in \mathcal{G}\}.$$

Note that $\mathcal{Z}(\mathcal{G})$ is a subgroup of $\mathcal{G}$.

Proposition 1.3. *$\mathcal{Z}(\mathcal{G})$ is a normal subgroup of $\mathcal{G}$.*

Proof. The proof follows directly from Def. 1.15. Given $z \in \mathcal{Z}(\mathcal{G})$ and $\forall g \in \mathcal{G}$, it holds that

$$gz = zg, \quad gzg^{-1} = zgg^{-1}, \quad gzg^{-1} = z$$

This means that $\mathcal{Z}(\mathcal{G})$ is a normal subgroup. □

This property allows us to introduce another fundamental notion from group theory.

Definition 1.16. Let $\mathcal{G}$ be a group and $\mathcal{W}$ be a normal subgroup of $\mathcal{G}$. For every pair $g, g' \in \mathcal{G}$, we say that $g \sim g'$ if

$$g'g^{-1} \in \mathcal{W} \iff g' = hg, \text{ for } h \in \mathcal{W}.$$

We define the equivalence class as follows:

$$[g] := \{hg \mid h \in \mathcal{W}\}.$$

Definition 1.17. The quotient group between $\mathcal{G}$ and $\mathcal{W}$ is

$$\mathcal{G}/\mathcal{W} := \{[g] \mid g \in \mathcal{G}\}.$$

It is both useful and natural to define a map between a group and a quotient group.

Definition 1.18. The canonical projection $\pi : \mathcal{G} \to \mathcal{G}/\mathcal{W}$ is such that

$$\pi(g) = [g].$$

In other words, we are able to efficiently map all the elements in $\mathcal{G}$ that "have something in common" to a single equivalence class.

1.3 The Pauli Group

The following sections about the Pauli, stabilizer, and Clifford groups are based on standard results, as presented in [20, 21, 37, 39]. Whenever necessary, they have been elaborated for the purposes of this analysis. Let us suppose that we are working with a number n of qubits, which live in a Hilbert space $\mathcal{H}$ of dimension $d = 2^n$ – let us keep in mind that d is always an even number. We express the three Pauli matrices as follows:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Pauli matrices and the identity operator form the Pauli group:

Definition 1.19. The n-qubit Pauli group $\mathcal{P}_n$ is defined as

$$\mathcal{P}_n := \{\phi P_1 \otimes P_2 \otimes \cdots \otimes P_n\},$$

where $\phi \in \{\pm 1, \pm i\}$ and $P_i \in \{\mathbb{I}, X, Y, Z\}$, for each $i = 1, \ldots, n$.

Let us make some relevant statements.

Proposition 1.4. *The Pauli group $\mathcal{P}_n$ is finite, with $|\mathcal{P}_n| = 4^{n+1}$.*

Proof. Given n qubits, it is possible to choose between four operators ($\mathbb{I}$, X, Y, Z) for each qubit; then, we have 4^n possible combinations for the tensor products. Since each operator has a phase of ± 1 or $\pm i$, we have $4\ 4^n = 4^{n+1}$ possibilities, *i.e.* $|\mathcal{P}_n| = 4^{n+1}$. □

Proposition 1.5. *Let $n = 1$. Then, $\{i\mathbb{I}, X, Z\}$ is a generating set for $\mathcal{P}_1$, i.e., $\mathcal{P}_1 = \langle i\mathbb{I}, X, Z\rangle$.*

Proof. Since $n = 1$, the order $|\mathcal{P}_1|$ is equal to $4^2 = 16$, that is,

$$\mathcal{P}_1 = \{\pm\mathbb{I}, \pm i\mathbb{I}, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}.$$

The four phases (± 1 and $\pm i$) can be generated as follows:

$$(i\mathbb{I})^q = \begin{cases} +\mathbb{I} & \text{for } q = 0 \\ +i\mathbb{I} & \text{for } q = 1 \\ -\mathbb{I} & \text{for } q = 2 \\ -i\mathbb{I} & \text{for } q = 3 \end{cases}.$$

In order to generate Y, it is sufficient to recall that $Y = iXZ$. Furthermore, note that $\mathbb{I}$, X and Z are the generators themselves. Finally, a general element of $\mathcal{P}_1$ can be written as follows:

$$P(q, x, z) = (i\mathbb{I})^q X^x Z^z,$$

where $q \in \{0, 1, 2, 3\}$, $x \in \{0, 1\}$, and $z \in \{0, 1\}$. □

Remark 1.1. Let P be any matrix among X, Y, Z. It is easy to verify what follows:

$$P^p = \begin{cases} P & \text{for odd } p \\ \mathbb{I} & \text{for even } p \end{cases}.$$

As n increases, the order $|\mathcal{P}_n|$ easily diverges: for $n = 2$, $|\mathcal{P}_2| = 64$, but for $n = 8$ $|\mathcal{P}_8| = 262144$. However, due to the structure of $\mathcal{P}_n$, a $(2n+1)$-generating set is sufficient:

$$\{i\mathbb{I}^{\otimes n},\ X_1, \dots,\ X_n,\ Z_1, \dots, Z_n\},$$

where $\mathbb{I}^{\otimes n} := \mathbb{I} \otimes \cdots \otimes \mathbb{I}$ is needed to create the phases ± 1 and $\pm i$; $X_i := \mathbb{I} \otimes \cdots \otimes X \otimes \cdots \otimes \mathbb{I}$, that is, X acting on the i-th qubit; and $Z_i := \mathbb{I} \otimes \cdots \otimes Z \otimes \cdots \otimes \mathbb{I}$, that is, Z acting on the i-th qubit. Thus, any element of $\mathcal{P}_n$ can be expressed as follows:

$$P(q, \bar{x}, \bar{z}) = (i\mathbb{I}^{\otimes n})^q \bigotimes_{j=1}^{n} X_j^{x_j} Z_j^{z_j}, \tag{1.1}$$

where $q \in \{0, 1, 2, 3\}$ and $x_j, z_j \in \{0, 1\}\ \forall j \in \{1, \dots, n\}$.

Proposition 1.6. *Every element of $\mathcal{P}_n$ with ± 1 is self-adjoint and has ± 1 eigenvalues.*

Proof. The proof is trivial, since the Pauli matrices are self-adjoint and have ± 1 eigenvalues. □

Proposition 1.7. *Every two elements of $\mathcal{P}_n$ either commute or anticommute.*

Proof. The proof is trivial, since: a) every operator commutes with both the identity and itself; b) $[\sigma_a; \sigma_b] = 2i\varepsilon_{abc}\sigma_c$, *i.e.*, two different Pauli matrices do not commute; c) $\{\sigma_a, \sigma_b\} = 2\delta_{ab}\mathbb{I}$, *i.e.*, two different Pauli matrices anti-commute. □

1.3.1 The Projective Pauli Group

Since quotient group structures allow us to factor out irrelevant global phases, they prove particularly useful in the framework of Quantum Mechanics. Building on this observation, we define a new version of the Pauli group, following [37].

Definition 1.20. Let $\mathcal{G}$ be a group and $\mathcal{O}$ a subgroup of $\mathcal{G}$. We call $\mathcal{O}$ cyclic if it can be generated by a single element $o \in \mathcal{O}$, *i.e.* $\mathcal{O} = \langle o \rangle = \{o^k \mid k \in \mathbb{Z}_n\}$, where $\mathbb{Z}_n$ is the integers modulo n.

Proposition 1.8. *Let $\mathcal{O}$ be a cyclic subgroup of a group $\mathcal{G}$. Then, $\mathcal{O}$ is abelian.*

Proof. Let $r, s \in \mathcal{O}$, *i.e.* $r = o^h$ and $s = o^k$, where o is the generator of $\mathcal{O}$ and $h, k \in \mathbb{Z}_n$. On one hand, it holds that

$$rs = o^h o^k = o^{h+k}.$$

On the other hand, it holds that

$$sr = o^k o^h = o^{k+h} = o^{h+k} = rs.$$

This means that the cyclic subgroup $\mathcal{O}$ is abelian. □

Proposition 1.9. *The center of $\mathcal{P}_n$ is the cyclic subgroup $\mathcal{O}_4$.*

Proof. Given the $n-$qubit Pauli group $\mathcal{P}_n$, it is possible to define the 4−order cyclic subgroup as follows:

$$\mathcal{O}_4 = \langle i\mathbb{I}^{\otimes n} \rangle = \{(i\mathbb{I}^{\otimes n})^q \mid q \in \mathbb{Z}_4\}.$$

Let us recall Def. 1.15. Given $P \in \mathcal{P}_n$, it is straightforward to verify that

$$(i\mathbb{I}^{\otimes n})^q P = P(i\mathbb{I}^{\otimes n})^q,$$

with $q = 0, 1, 2, 3$. □

Let us construct an equivalence class.

Proposition 1.10. *An equivalence class for $P \in \mathcal{P}_n$ with respect to $\mathcal{O}_4$ is*

$$[P] = \{(i\mathbb{I}^{\otimes n})^q P \mid (i\mathbb{I}^{\otimes n})^q \in \mathcal{O}_4\},$$

that is, $[P]$ is the equivalence class consisting of all the Pauli strings that differ only by a phase in $\{\pm 1, \pm i\}$.

Proof. It is straightforward to verify that $\sim$ is an equivalence relation, since reflexivity, symmetry and transitivity hold. Furthermore, let $P, P' \in \mathcal{P}_n$. It is true that

$$P'P^{-1} \in \mathcal{O}_4 \implies P' = (i\mathbb{I}^{\otimes n})^q P, \text{ for } (i\mathbb{I}^{\otimes n})^q \in \mathcal{O}_4.$$

Recalling Eq. 1.1, a Pauli string can be expressed as

$$P(q, \bar{x}, \bar{z}) = (i\mathbb{I}^{\otimes n})^q \bigotimes_{j=1}^{n} X_j^{x_j} Z_j^{z_j}.$$

Then if $P'P^{-1} \in \mathcal{O}_4$, P and P' must differ only by $(i\mathbb{I}^{\otimes n})^q$, for some $q = 0, 1, 2, 3$. Conversely, it is also true that

$$P'P^{-1} \in \mathcal{O}_4 \impliedby P' = (i\mathbb{I}^{\otimes n})^q P, \text{ for } (i\mathbb{I}^{\otimes n})^q \in \mathcal{O}_4.$$

In fact, we have that

$$\begin{aligned} P' &= (i\mathbb{I}^{\otimes n})^q P \\ P'P^{-1} &= (i\mathbb{I}^{\otimes n})^q P P^{-1} \\ P'P^{-1} &= (i\mathbb{I}^{\otimes n})^q \in \mathcal{O}_4. \end{aligned}$$

By Def. 1.16, for every pair $P, P' \in \mathcal{P}_n$ that differ by a phase in $\{\pm 1, \pm i\}$, $P \sim P'$ and $[P]$ is an equivalence class. □

Finally, we are able to introduce a slightly different version of the Pauli group, defined modulo phases.

Definition 1.21. The $n-$qubit projective Pauli group is a quotient group defined as follows:

$$\widetilde{\mathcal{P}}_n := \mathcal{P}_n / \mathcal{O}_4.$$

This is equivalent to saying that the canonical projection $\Pi_{Pauli} : \mathcal{P}_n \longrightarrow \widetilde{\mathcal{P}}_n$ maps a Pauli string into the equivalence class defined in Prop. 1.10.

Proposition 1.11 (Generating Set for the Projective Pauli Group)**.** *The generating set of $\widetilde{\mathcal{P}}_n$ is $\{X_1, \dots, X_n, Z_1, \dots, Z_n\}$, while the one of $\mathcal{P}_n$ is $\{i\mathbb{I}^{\otimes n}, X_1, \dots, X_n, Z_1, \dots, Z_n\}$. Note that $i\mathbb{I}^{\otimes n}$ is not needed anymore, due to the equivalence class previously constructed.*

Proposition 1.12. *The projective Pauli group $\widetilde{\mathcal{P}}_n$ is abelian.*

Proof. Validity of Prop. 1.7 depends on the properties of the Pauli matrices and in particular on the phases ± 1 or $\pm i$ of the Pauli strings, that is, every two Pauli strings either commute or anticommute. Instead, in the framework of the projective Pauli group, every string is defined up to a phase – ± 1, $\pm i$ –, that is, any pair of strings only commute. □

1.4 The Stabilizer Subgroup

We now introduce the notion of stabilizer subgroup, which plays a central role in the stabilizer formalism. In Quantum Mechanics, self-adjoint operators with positive eigenvalues are needed. Thus, we consider only the operators in $\mathcal{P}_n$ that form the following subgroup:

Definition 1.22. The subset $\mathcal{S}$ of $\mathcal{P}_n$ is called a stabilizer group and is defined as:

$$\mathcal{S} := \{S \in \mathcal{P}_n \quad | \quad S|\psi\rangle = |\psi\rangle\},$$

where $|\psi\rangle$ is an allowed state, *i.e.*, a $+1$-eigenvector. Throughout this work, we will use the terms *stabilizer group* and *stabilizer subgroup* interchangeably.

Remark 1.2. In the framework of Quantum Error Correction (see for example [39]), it is often specified that $-\mathbb{I}$ is not an element of the stabilizer group $\mathcal{S}$, for practical purposes. Let us admit that $-\mathbb{I} \in \mathcal{S}$. Then:

$$\begin{aligned} -\mathbb{I}\,|\psi\rangle &= |\psi\rangle\,, \quad \text{but also} \\ -\mathbb{I}\,|\psi\rangle &= -\,|\psi\rangle\,, \end{aligned}$$

so that the only possible state satisfying this relation is the null vector. Thus, $-\mathbb{I} \notin \mathcal{S}$. Throughout this paper, it is sufficient to say that $-\mathbb{I} \notin \mathcal{S}$, because it does not have positive eigenvalues.

Definition 1.23. We define the coding space $\mathcal{H}_\mathcal{S}$ as the set of vectors stabilized by $\mathcal{S}$:

$$\mathcal{H}_\mathcal{S} := \{|\psi\rangle \;\mid\; S\,|\psi\rangle = |\psi\rangle\,,\ \forall S \in \mathcal{S}\}.$$

This is equivalent to saying that $\mathcal{H}_\mathcal{S}$ is the eigenspace associated with the $+1$-eigenvectors of the elements in $\mathcal{S}$, namely

$$\mathcal{H}_\mathcal{S} = \bigcap_{S\in\mathcal{S}} \mathsf{Ker}\,(S - \mathbb{I})\,.$$

Let us provide a full characterization of the stabilizer group.

Proposition 1.13. *$\mathcal{S}$ in Def. 1.22 is a subgroup of $\mathcal{P}_n$.*

Proof. $\mathcal{S}$ is a subset of $\mathcal{P}_n$. The composition rule is chosen to be the matrix multiplication M. It is possible to verify that $\mathcal{S}$ is closed under M. Let $S_1, S_2 \in \mathcal{S}$; then:

$$(S_2 S_1)\,|\psi\rangle = S_2(S_1\,|\psi\rangle) = S_2\,|\psi\rangle = |\psi\rangle\,,$$

that is, $S_3 = S_2 S_1$ is an element of $\mathcal{S}$. Moreover, it is evident that $\mathcal{P}_n$ and $\mathcal{S}$ share the same trivial element, *i.e.*, $\mathbb{I}$. Let $|\psi\rangle$ be an allowed state and $S \in \mathcal{S}$; then:

$$\mathbb{I}S\,|\psi\rangle = \mathbb{I}\,|\psi\rangle = |\psi\rangle = S\,|\psi\rangle = S\mathbb{I}\,|\psi\rangle \quad ,$$

that is, $\mathbb{I}S = S\mathbb{I} = S$. They share the inverse element, as well. In fact:

$$\begin{aligned} S\,|\psi\rangle &= |\psi\rangle \\ S^{-1}S\,|\psi\rangle &= S^{-1}\,|\psi\rangle \\ |\psi\rangle &= S^{-1}\,|\psi\rangle \quad , \end{aligned}$$

that is the inverse element stabilizes an allowed state $|\psi\rangle$, so $S^{-1} \in \mathcal{S}$. Hence, $\mathcal{S}$ is a group itself, and it is a subgroup of $\mathcal{P}_n$. □

Proposition 1.14. *$\mathcal{S}$ is an abelian group.*

Proof. Let $R, S \in \mathcal{S}$. Then:

$$SR\,|\psi\rangle = S\,|\psi\rangle = |\psi\rangle \text{ and}$$
$$RS\,|\psi\rangle = R\,|\psi\rangle = |\psi\rangle,$$

so $SR = RS$, that is, $[S, R] = 0, \forall R, S \in \mathcal{S}$. □

Commutativity of the stabilizer group $\mathcal{S}$ is much more than a mathematical property. It is a consequence of the choices we made to work within the *Church* of Quantum Mechanics. Since every element in $\mathcal{S}$ shares the same eigenvalue $+1$, any pair of elements simply commutes.

Example 1.1. Let us construct the stabilizer group for the one-qubit scenario. The order of $\mathcal{P}_1$ is $|\mathcal{P}_1| = 4^{1+1} = 16$. Let $|S|$ be the order of the stabilizer group. From Theorem 1.4, $|S|$ must be a divisor of $|\mathcal{P}_1|$:

$$|\mathcal{S}| = 2, 4, 8, 16.$$

From Def. 1.22, $\mathcal{S}$ contains only self-adjoint elements with positive eigenvalues. Therefore, we rule out $-\mathbb{I}$ and all the elements with $\pm i$ phases. Since the order must be a divisor of 16, only two options remain: 2 and 4. Moreover, since $\mathcal{S}$ is an abelian group and different Pauli matrices do not commute, it is forbidden to construct a stabilizer group with more than one Pauli matrix. Thus, $|\mathcal{S}| = 2$ for $n = 1$. In this scenario, every possible stabilizer group has the following form:

$$\mathcal{S} = \{\mathbb{I}, \Sigma\},$$

where Σ corresponds to one between $+X, +Y, +Z, -X, -Y, -Z$. The generating set is simply $\{\Sigma\}$, which is equivalent to $\mathcal{S} = \langle\Sigma\rangle$, since $\Sigma^2 = \mathbb{I}$. Furthermore, note that it follows from commutativity that $\mathcal{S}$ always contains the identity $\mathbb{I}$.

Example 1.2. Let us generalize for $n \geq 1$. The proof for this statement can be found in [20] and [39]. Let us consider an $[n, k, h]$ code, that is, an $h-$distance code that encodes k qubits into n qubits. Thus, the dimension of $\mathcal{H}_\mathcal{S}$ is 2^k, while the order of $\mathcal{S}$ is 2^{n-k}. Note that $|\mathcal{S}| = 2^n$ when $k = 0$. In this case, $\mathcal{S}$ is maximal and $\mathcal{H}_\mathcal{S}$ is minimal.

Example 1.3. In the previous example, we stated that $|\mathcal{S}| = 2$ when $n = 1$. This is not completely true. In fact, when $k = 1$, $|\mathcal{S}| = 2^0 = 1$, *i.e.*, $\mathcal{S}$ is the trivial stabilizer group, namely $\mathcal{S} = \{\mathbb{I}\}$.

Let us now introduce the stabilizer group's generating set. A more in-depth analysis can be found in [39].

Proposition 1.15. *Let $\mathcal{S}$ be a stabilizer group with $|\mathcal{S}| = 2^{n-k}$. The generating set of $\mathcal{S}$ is composed of $(n-k)$ elements, namely*

$$\mathcal{S} = \langle S_1, S_2, \dots, S_{n-k} \rangle.$$

When $\mathcal{S}$ is maximal, it is straightforward to verify that $\mathcal{S} = \langle s_1, s_2, \dots, s_n \rangle$, since $k = 0$.

From the proposition above, we are now able to introduce a general expression for the elements of $\mathcal{S}$.

Proposition 1.16. *Let $\mathcal{S}$ be a stabilizer group for $\mathcal{P}_1$ and let $s \in \mathcal{S}$. Then:*

$$S(q, x, z) = (i\mathbb{I})^q X^x Z^z,$$

where $q \in \{0, 1, 2, 3\}$, $x \in \{0, 1\}$, and $z \in \{0, 1\}$. However, we need some extra rules with respect to Prop. 1.5. In particular, we have

$$\begin{cases} q = 0, 2 & \text{when } x \neq z \\ q = 0 & \text{when } x = z = 0 \\ q = 1, 3 & \text{when } x = z = 1 \end{cases}.$$

Every other combination is strictly forbidden in the stabilizer group framework.

Proposition 1.17. *Let $\mathcal{S}$ be a stabilizer group for $\mathcal{P}_n$ and let $S \in \mathcal{S}$. Then, we have*

$$S(q, \bar{x}, \bar{z}) = (i\mathbb{I}^{\otimes n})^q \bigotimes_{j=1}^{n} X_j^{x_j} Z_j^{z_j},$$

where rules analogous to the previous ones hold.

1.5 The Clifford Group

We now introduce the Clifford group, another central object in the stabilizer formalism. Let $\mathcal{H}$ be a Hilbert space with dimension $d = 2^n$.

Definition 1.24. The unitary group $\mathcal{U}(2^n)$ is formed by $2^n \times 2^n-$ complex unitary matrices, that is,

$$U^\dagger U = UU^\dagger = \mathbb{I}, \forall U \in \mathcal{U}(2^n).$$

The combination rule is matrix multiplication.

Proposition 1.18. *It is straightforward to verify that $U^\dagger = U^{-1}$, $\forall U \in \mathcal{U}(2^n)$. In fact, the following relations hold at the same time:*

$$UU^\dagger = U^\dagger U = \mathbb{I} \text{ since } U \text{ is unitary;}$$
$$UU^{-1} = U^{-1}U = \mathbb{I} \text{ since } U \text{ is an element of a matrix group.}$$

Definition 1.25. Given n qubits, the Clifford group $\mathcal{C}_n$ is the normalizer of the Pauli group $\mathcal{P}_n$ in the unitary group $\mathcal{U}(2^n)$, namely

$$\mathcal{C}_n := \mathcal{N}_{\mathcal{U}(2^n)}(\mathcal{P}_n) = \{U \in \mathcal{U}(2^n) \mid U\mathcal{P}_n U^\dagger = \mathcal{P}_n\}.$$

Let us introduce the Clifford group's generating set. A general analysis can be found in [21, 39], while a more in-depth algebraic discussion is given in [45].

Proposition 1.19. *Let $\mathcal{C}_n$ be the $n-$qubit Clifford group. Then, $\mathcal{C}_n = \langle \mathrm{H}, \mathrm{S}, \mathrm{CZ} \rangle$, where*

$$\mathrm{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \ \mathrm{S} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \ \mathrm{CZ} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & Z \end{pmatrix}.$$

Analogously, it is possible to choose the CNOT *gate instead of the* CZ *gate, where*

$$\mathrm{CNOT} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & X \end{pmatrix},$$

since $\mathrm{CNOT} = (\mathbb{I} \otimes \mathrm{H})\mathrm{CZ}(\mathbb{I} \otimes \mathrm{H})$.

Proposition 1.20. *In general, it holds that $UPU^\dagger = P' \in \mathcal{P}_n$, $\forall U \in \mathcal{U}(2^n)$ and $\forall P \in \mathcal{P}_n$.*

Proof. The proof follows directly from Def. 1.25. In fact, the Pauli group is only invariant under conjugation by elements of the Clifford group. In other words, given $P \in \mathcal{P}_n$ and $U \in \mathcal{C}_n$, the conjugated element $UPU^\dagger$ is still in $\mathcal{P}_n$, but it may differ from P. □

Example 1.4. Let us provide an example for the previous statement. Given $X \in \mathcal{P}_1$ and $H \in \mathcal{C}_n$, it is straightforward to show that X under conjugation is still an element of $\mathcal{P}_1$.

$$\begin{aligned} HXH &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \\ &= \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z, \end{aligned}$$

where Z is still a Pauli string. This means that X is not invariant under conjugation by Clifford elements, while $\mathcal{P}_1$ as a whole is.

1.5.1 The Projective Clifford Group

Applying the same quotient construction used for the Pauli group, in the following section, based on [37], we obtain the projective Clifford group.

Proposition 1.21. *The center of $\mathcal{C}_n$ is $\mathcal{U}(1)$.*

Proof. The group $\mathcal{U}(1)$ is formed by 1×1 complex unitary matrices, *i.e.*, by complex numbers with modulus one. In fact, the unitary condition for $\mathcal{U}(1)$ corresponds to

$$\bar{c}c = c\bar{c} = |c|^2 = 1, \forall c \in \mathcal{U}(1).$$

This means that $\mathcal{U}(1)$ is formed by complex phases, namely

$$\mathcal{U}(1) = \{c \in \mathbb{C} \mid |c| = 1\}.$$

Finally, note that the product of a complex number with modulus one and a Clifford element satisfies Def. 1.15.

$$cU = Uc, \forall c \in \mathcal{U}(1) \text{ and } \forall U \in \mathcal{C}_n.$$

This means that $\mathcal{U}(1)$ is the center – and also a normal subgroup – of $\mathcal{C}_n$ for an arbitrary number n of qubits. □

We are now able to construct the following equivalence class:

Proposition 1.22. *An equivalence class for $U \in \mathcal{C}_n$ with respect to $\mathcal{U}(1)$ is expressed as*

$$[U] = \{cU \mid c \in \mathcal{U}(1)\},$$

that is, $[U]$ is the equivalence class consisting of all the elements of the Clifford group that differ only by a global phase.

Proof. It is straightforward to verify that $\sim$ is an equivalence relation, since reflexivity, symmetry and transitivity hold. Let $U, U' \in \mathcal{C}_n$. It is true that

$$U'U^\dagger \in \mathcal{U}(1) \implies U' = cU, \text{ for } c \in \mathcal{U}(1).$$

In fact, $U'U^\dagger \in \mathcal{U}(1)$ is equivalent to saying that $U'U^\dagger$ is a phase, *i.e.*, $U'U^\dagger = k'\bar{k} = k'' \in \mathbb{C}$ with $|k''| = 1$. Thus, since the product of two phases is still a phase, $k' = ck$, for some $c \in \mathcal{U}(1)$. Conversely, it is also true that

$$U'U^\dagger \in \mathcal{U}(1) \impliedby U' = cU, \text{ for } c \in \mathcal{U}(1).$$

In fact, we have

$$\begin{aligned} U' &= cU \\ U'U^\dagger &= cUU^\dagger \\ U'U^\dagger &= k'\bar{k} = c \in \mathcal{U}(1). \end{aligned}$$

From Def. 1.16, for every pair $U, U' \in \mathcal{C}_n$ that differ by a global phase, $U \sim U'$ and $[U]$ is an equivalence class. □

Example 1.5. A reasonable motivation to introduce $[U]$ as an equivalence class is the following: every pair of Clifford elements that differ by a global phase has the same action upon a Pauli string, namely

$$(cU)\,P\left(\bar{c}U^\dagger\right) = c\bar{c}UPU^\dagger = |c|^2 UPU^\dagger = UPU^\dagger \in \mathcal{P}_n.$$

Let us specify one more time that P is not necessarily equal to $UPU^\dagger$.

Finally, we are able to introduce a slightly different version of the Clifford group, defined modulo global phases.

Definition 1.26. The $n-$qubit projective Clifford group is a quotient group defined as follows:

$$\widetilde{\mathcal{C}}_n := \mathcal{C}_n / \mathcal{U}(1).$$

This is equivalent to saying that the canonical projection $\Pi_{Clifford} : \mathcal{C}_n \to \widetilde{\mathcal{C}}_n$ maps a Clifford element into the equivalence class defined in Prop. 1.22.

Proposition 1.23. *The generating set of $\mathcal{C}_n$ is $\{\mathrm{H}, \mathrm{S}, \mathrm{CZ}\}$. The generating set of $\widetilde{\mathcal{C}}_n$ is simply $\{[\mathrm{H}], [\mathrm{S}], [\mathrm{CZ}]\}$. In other words, every "new" generator is the equivalence class of the corresponding "standard" generator.*

1.5.2 The Symplectic Form of the Clifford Group

The following section is mainly based on [37]. Let $\mathcal{H}$ be a Hilbert space of dimension $d = 2^n$. If we were interested in working on qudits, then we would choose a prime number p instead of 2. Let $\mathcal{B}(\mathcal{H})$ be the algebra of the bounded linear operators acting on $\mathcal{H}$; that is, every bounded linear operator is a $2^n \times 2^n-$complex matrix. We need first a few basic notions about symplecticity.

Definition 1.27 (Symplectic Form)**.** Let V be a vector space over a field $\mathbb{F}$. The map $\omega : V \times V \longrightarrow \mathbb{F}$ is called a symplectic form if the following properties hold:

- (Bilinearity) ω is linear in each argument, separately;
- (Alternation) $\omega(v, v) = 0, \forall v \in V$;
- (Non-Degeneracy) $\omega(v, u) = 0, \forall v \in V \implies u = 0$.

Definition 1.28 (Symplectic Vector Space)**.** Let V be a vector space over the field $\mathbb{F}$, equipped with a symplectic form ω. We call V a symplectic vector space.

A useful example of a symplectic vector space is the so-called discrete phase space $\mathbb{F}_2^{2n}$.

Definition 1.29 (Galois Field)**.** Let $\mathbb{F}$ be a field. We call $\mathbb{F}$ a Galois field if it contains a finite number q of elements, and we write $\mathbb{F}_q$.

The fundamental rule for the existence of a Galois field $\mathbb{F}_q$ concerns the number of elements q. In particular, q must be the power of a prime number p, namely

$$q = p^n,$$

where n is a positive integer. The simplest Galois field is $\mathbb{F}_2 = \{0, 1\}$. However, it is forbidden to construct, for example, a Galois field with $q = 6$, since 6 is not a power of a prime number.

Definition 1.30 (Discrete Phase Space)**.** The discrete phase space is the following $2n-$dimension vector space over $\mathbb{F}_2$:

$$V(\mathbb{F}_2) := \mathbb{F}_2^{2n}.$$

An element of $\mathbb{F}_2^{2n}$ is generally written as a vector with two components:

$$v = (a, b),$$

where $a, b \in \mathbb{F}_2^n$. This means that $\mathbb{F}_2^{2n} = \mathbb{F}_2^n \oplus \mathbb{F}_2^n = \mathbb{F}_2^n \times \mathbb{F}_2^n$. For $n \geq 1$, the elements of $\mathbb{F}_2^{2n}$ are pairs of $n-$bit strings, *e.g.* $(1001 \ldots 01, 0011 \ldots 11)$. Moreover, on $\mathbb{F}_2^n$, we use the standard coordinate pairing

$$a \cdot b = \sum_{j=1}^{n} a_j b_j \in \mathbb{F}_2,$$

which depends on the chosen basis, as it is the ordinary dot product in the chosen coordinates. Furthermore, we note that $\mathbb{F}_2^{2n}$ is a symplectic vector space.

Proposition 1.24. *The vector space $\mathbb{F}_2^{2n}$ is equipped with the following symplectic form, called a commutator:*

$$[\cdot, \cdot] : \mathbb{F}_2^{2n} \times \mathbb{F}_2^{2n} \longrightarrow \mathbb{F}_2,$$

defined as

$$[v, w] := a \cdot b' - a' \cdot b,$$

where $v = (a, b)$ and $w = (a', b')$. In particular, note that $[v, w]$ is either 0, that is, v and w commute, or 1 , that is, v and w do not commute.

Example 1.6. Let us consider $v = (1, 0)$ and $w = (0, 1)$. They do not commute, since

$$[v, w] = (1)(1) - (0)(0) = 1 (\text{mod } 2).$$

On the other hand, v commutes with itself, since

$$[v, v] = (1)(0) - (1)(0) = 0.$$

The symplectic group can be constructed as follows:

Definition 1.31 (Symplectic Matrix)**.** Let M be a $2n \times 2n$-matrix with coefficients in a field $\mathbb{F}$, namely $M \in \mathbb{M}_{2n\times 2n}(\mathbb{F})$; M is called a symplectic matrix if

$$M^T \Omega M = \Omega,$$

where M^T is the transpose of M and Ω is the standard symplectic matrix.

Remark 1.3. Let us specify some properties of matrix Ω. It is typically chosen as the following block matrix:

$$\Omega = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix},$$

where $\mathbb{I}_n$ is the $n \times n-$identity matrix. Furthermore, $\Omega^{-1} = \Omega^T = -\Omega$ and $\det(\Omega) = 1$. Finally, note that $\det(M) = 1$, too.

Remark 1.4. Let M be a symplectic matrix and V be a symplectic vector space. Then, $\forall u, v \in V$, M preserves the symplectic form, that is,

$$\omega(Mu, Mv) = \omega(u, v).$$

Definition 1.32 (Symplectic Group)**.** The symplectic group $\mathcal{S}p(2n, \mathbb{F})$ is formed by all the $2n \times 2n-$symplectic matrices with coefficients in $\mathbb{F}$, namely

$$\mathcal{S}p(2n, \mathbb{F}) := \{M \in \mathbb{M}_{2n\times 2n}(\mathbb{F}) \mid M^T \Omega M = \Omega\}.$$

Example 1.7. Let us consider the symplectic group $\mathcal{S}p(2n, 2)$; that is, we choose $\mathbb{F}_2$ as a field and $\mathbb{F}_2^{2n}$ as a symplectic vector space.

It is possible to show the existence of a correspondence between quantum operations and binary geometry. A formal proof can be found in [37].

Theorem 1.5. *The projective Pauli group is isomorphic to the discrete phase space, namely*

$$\widetilde{\mathcal{P}}_n \cong \mathbb{F}_2^{2n}. \tag{1.2}$$

This is equivalent to saying that there exists an isomorphism that maps a Pauli string P – defined up to ± 1 or $\pm i$ – to a vector $v = (x, y)$ with two bit strings as components, namely

$$P \longleftrightarrow v = (x, y).$$

Proof. The proof can be found in [37]. □

Example 1.8. Let $n = 1$. In a previous example, we stated that the elements of $\mathbb{F}_2^2$ are $(0,0),(1,0),(0,1),(1,1)$. Recalling Prop. 1.2, there exists a perfect correspondence between Pauli strings and vectors with two bit strings as components. In fact, we have

$$\begin{aligned} \mathbb{I} &\longleftrightarrow (0,0), \\ X &\longleftrightarrow (1,0), \\ Z &\longleftrightarrow (0,1), \\ Y &\longleftrightarrow (1,1). \end{aligned}$$

Example 1.9. For $n \geq 1$, the same isomorphism can be found. A general element is a vector with two bit strings as components, namely $(x_1, \dots, x_n, z_1, \dots, z_n)$, each one $n-$bit long. In order to construct the corresponding Pauli string, we need to check every position. Let us consider the following vector for $n = 4$:

$$(0110, 1100).$$

The first-position bits are $x_1 = 0$ and $z_1 = 1$ – $(0,1)$ corresponds to Z; the second-position bits are 1 and 1, *i.e.*, Y; the third-position bits are 1 and 0, *i.e.*, X; Finally, we have 0 and 0, *i.e.*, $\mathbb{I}$. Thus, the corresponding Pauli string is $Z \otimes Y \otimes X \otimes \mathbb{I}$.

Finally, it is possible to find an analogous correspondence in the framework of the Clifford group. A formal proof can be found in [37].

Theorem 1.6. *The quotient of the projective Clifford group by the projective Pauli group is isomorphic to the symplectic group of degree $2n$ over $\mathbb{F}_2$, namely*

$$\widetilde{\mathcal{C}}_n / \widetilde{\mathcal{P}}_n \cong \mathcal{S}p(2n, 2).$$

Proof. The proof can be found in [37]. □

This is equivalent to saying that there exists an isomorphism that maps a Clifford unitary U – defined up to a phase and a Pauli string – to a symplectic matrix M_U with 0 and 1 entries, namely

$$U \longleftrightarrow M_U.$$

Remark 1.5. We can adapt Rem. 1.4 to the framework of the Clifford and Pauli groups. Every Clifford unitary, being a symplectic matrix, leaves the commutation relations between two Pauli strings invariant, that is,

$$\omega(M_U v, M_U v') = [M_U v, M_U v'] = \omega(v, v') = [v, v'].$$

This means that, if two Pauli strings commute before the action of a Clifford unitary, then they also commute after that. We will further formalize this result.

Chapter 2

The Quantum Resource Theory of Magic

2.1 What is Quantum Magic?

Within the framework of quantum computation, there exists a set of particular states, which are classically simulable (commonly known as *stabilizer states*). This is formally captured by the Gottesman-Knill theorem:

Theorem 2.1 (Gottesman-Knill Theorem ([1, 21]))**.** *A quantum computer performing only Clifford unitaries, measurements of Pauli strings, and Clifford unitaries conditioned on classical bits, can be perfectly simulated in polynomial time on a probabilistic classical computer, that is,*

$$\text{Stabilizerness} \implies \text{Classical simulability (Gottesman-Knill).}$$

This result implies that the set of stabilizer states cannot give any quantum advantage. Thus, it is necessary to move on towards non-stabilizer states. Introduced for the first time by S. Bravyi and A. Kitaev in [10], the expression *quantum magic* currently refers to non-stabilizerness. Theorem 2.1 can be simply reformulated in terms of magic as follows: every non-classically simulable state is a magic state.

Remark 2.1. Theorem 2.1 does not claim that magic states cannot be simulated under other paradigms — it only applies within the stabilizer setting.

An immediate consequence of Theorem 2.1 is that the notion of magic is necessary to achieve universal quantum computation.

Theorem 2.2. *The following gate set is universal:*

$$\{\mathrm{H}, \mathrm{CZ}, \mathrm{S}, \mathrm{T}\},$$

where the magic gate T *is defined as*

$$\mathrm{T} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}.$$

Proof. A proof can be found in [5]. □

Note that the minimal universal gate set is $\{\mathrm{H}, \mathrm{CZ}, \mathrm{T}\}$, since $\mathrm{S} = \sqrt{\mathrm{T}}$, but for practical purposes the S gate is usually included. Furthermore, in [10], Bravyi and Kitaev showed that universality can be achieved through *magic state injection*, that is, the consumption of a magic state by a Clifford circuit.

2.2 What is a Quantum Resource Theory?

In order to construct the Quantum Resource Theory (QRT) of magic, we need the following ingredients: (1) a set of *free states*, that is, the states we always have at our disposal; (2) a set of *free operations*, that is, the maps we are always able to use; (3) a set of *resourceful states*, that is, the non-free states – they are very important for our QRT, like fuel for a vehicle; (4) a set of resource-quantifying functions, called *monotones*. This is basically our starter pack. However, let us start from a general point of view, in order to describe later the QRT of magic.

2.3 Resource Theories: a Mathematical Perspective

While the exhaustive works in [13, 14] provide an in-depth mathematical analysis of Resource Theories (RTs), this section only aims for a general overview. In [14] Coecke et al. use the term *resource* to indicate all elements within a resource theory. Consequently, this choice of terminology leads to a distinction between *free resources* and *non-free resources*. In order to avoid ambiguity with the majority of the literature (*e.g.* [11]), we shall instead use the word *state* to indicate both *free states* and *resource states*, the latter being *non-free states*. Furthermore, the use of the term *state* is particularly natural in this framework, as our main goal is to construct a quantum resource theory. Let us first introduce a few concepts from category theory.

Definition 2.1. A category **C** consists of

- objects $A, B, C, \ldots$;

- morphisms $f, g, h, ... \in \mathbf{C}(A, B)$ for each pair of objects (A, B);
- associative composition $\circ$ of morphisms; we write $g \circ f \in \mathbf{C}(A, C)$, for any $f \in \mathbf{C}(A, B)$ and $g \in \mathbf{C}(B, C)$;
- identity morphisms $1_A \in \mathbf{C}(A, A)$ and $1_B \in \mathbf{C}(B, B)$ such that $f \circ 1_A = 1_B \circ f = f$.

The symbol $|\mathbf{C}|$ denotes the set of objects, while $\mathbf{C}(A, B)$ denotes the hom-set, *i.e.*, the set of morphisms from A to B.

Definition 2.2. A morphism $f : A \to B$ is called an isomorphism if there exists an inverse $f^{-1} : B \to A$ such that

$$f^{-1} \circ f = 1_A \qquad\qquad f \circ f^{-1} = 1_B.$$

Definition 2.3. A symmetric monoidal category is a category equipped with a symmetric monoidal tensor $\odot$, that is, an assignment for both pairs of objects and pairs of morphisms:

$$(A, B) \mapsto A \odot B$$
$$(A \xrightarrow{f} B, C \xrightarrow{g} D) \mapsto A \odot C \xrightarrow{f \odot g} B \odot D \ .$$

Let us now show the main features of a resource theory. There are two different kinds of "building blocks": states – denoted by $A, B, C, ...$ – and transformations between states – denoted by $f, g, h, ...$. A primary requirement is the parallel composition of both states and transformations. For any pair A and B, their parallel composition $A \odot B$ is also a state; analogously, for any pair $f : A \to B$ and $g : C \to D$, their parallel composition $f \odot g$ is a transformation from $A \odot C$ to $B \odot D$. Furthermore, we require the sequential composition of transformations; for any pair $h : A \to B$ and $l : B \to C$, their sequential composition $l \circ h$ is a transformation from A to C. Finally, we require the existence of a "neutral" void state I such that $A \odot I = I \odot A = A$, for any arbitrary state A. As formally illustrated by Coecke et al. in [14], it is evident that a resource theory can be understood as a specific interpretation of a symmetric monoidal category.

Definition 2.4. A Resource Theory is a symmetric monoidal category $(\mathbf{C}, \circ, \odot, I)$, where

- the objects $|\mathbf{C}|$ are the *states*;
- the morphisms in $\mathbf{C}(A, B)$ are the *transformations* of state A into state B, implementable without any cost;
- the binary operation $\circ$ represents the *sequential composition* of transformations;

- the binary operation $\odot$ represents the *parallel composition* of both states and transformations;
- the trivial object I is the *void state*.

Definition 2.5. The set of free states in $\mathbf{C}$ is defined as follows:

$$\mathrm{F}_{\mathbf{C}} := \{A \in |\mathbf{C}| \text{ such that } \mathbf{C}(I, A) \neq \varnothing\},$$

that is, there always exists at least one morphism that maps the void state I to a free state A; note also that I is free by definition. This is equivalent to saying that a valid resource theory requires a non-empty set of free states.

It is particularly natural to ask whether it is possible to convert a state into another, and under what conditions. In other words, given two states A and B, does there exist a transformation $f : A \to B$ or, equivalently, is the hom-set $\mathbf{C}(A, B)$ non-empty? In [14], the notation $A \succeq B$ means that there exists a transformation between A and B. This relation imposes a preorder structure.

Definition 2.6. Let X be an arbitrary set. The binary relation $\succeq$ is called a *preorder* if the following conditions hold:

- (Reflexivity.) For all $x \in X$, $x \succeq x$;
- (Transitivity.) For all $x, y, z \in X$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$.

The set X is called a *preordered set*.

It is possible to construct an equivalence relation $\sim$ from a preorder $\succeq$.

Definition 2.7. We say that x is equivalent to y, namely $x \sim y$, if

$$x \succeq y \qquad\qquad y \succeq x.$$

Then, the equivalence class is given by

$$[x] := \{y \in X \mid y \sim x\},$$

while the quotient set is denoted by $X/\sim$.

Definition 2.8. Let $X/\sim$ be an arbitrary quotient set. The preorder $\succeq$ is called a partial order if it is antisymmetric, *i.e.*,

$$[x] \sim [y] \implies [x] = [y],$$

for any $[x], [y] \in X/\sim$.

With a slight abuse of notation, both preorder and partial order will be denoted by the symbol $\succeq$.

Remark 2.2. Let $x, y \in X$. If $x \sim y$, that is, $x \succeq y$ and $y \succeq x$, it is not necessarily true that $x = y$. Equality holds when considering their equivalence classes, *i.e.*, by moving to the quotient set where the relation becomes a partial order.

Definition 2.9. A preorder $\succeq$ on a set X is called a *total order*, if either $x \succeq y$ or $y \succeq x$, for any $x, y \in X$. This is equivalent to saying that any two elements are always *comparable*.

Remark 2.3. Def. 2.9 can also be extended to a partial order, by merely considering the equivalence classes.

We are now able to define a theory of state convertibility.

Definition 2.10. A theory of state convertibility is a set $\mathbf{R}$ equipped with a binary operation $+$, a preorder $\succeq$, and a special element $0 \in \mathbf{R}$ such that

$$\begin{aligned} a + (b + c) &\sim (a + b) + c \\ a + b &\sim b + a \\ a + 0 &\sim 0 + a, \end{aligned}$$

for all $a, b, c \in \mathbf{R}$. We also require that

$$a \succeq b, c \succeq d \implies a + c \succeq b + d.$$

Such a theory will be denoted by $(\mathbf{R}, +, \succeq, 0)$.

The following theorem provided in [14] shows that any resource theory can be seen as a theory of state convertibility.

Theorem 2.3. *Let $(\mathbf{C}, \circ, \odot, I)$ be a resource theory. If we define $\mathbf{R} := |\mathbf{D}|$, and, for all $a, b \in |\mathbf{D}|$, we also define*

$$\begin{aligned} a + b &:= a \odot b, \\ a \succeq b &\iff \mathbf{C}(a, b) \neq \varnothing \\ 0 &:= I, \end{aligned}$$

then $(\mathbf{R}, +, \succeq, 0)$ is a theory of state convertibility.

Proof. The proof is straightforward, since $\mathbf{C}$ and $\mathbf{R}$ are assumed to be symmetric monoidal categories and we are simply re-defining their elements. □

Another natural question is how much of a resource an arbitrary state contains.

Definition 2.11. A monotone is a function $M : \mathbf{R} \to \mathbb{R}$ such that

$$a \succeq b \implies M(a) \geq M(b).$$

In the subsequent sections, we will give a slightly different definition of monotones: for practical purposes, a monotone will be taken to be a non-negative function, *i.e.*, $\mathrm{Codom}(M) = \mathbb{R}_0^+$.

Proposition 2.1. *If $a \sim b$, then $M(a) = M(b)$.*

Proof. $a \sim b$ means that both $a \succeq b$ and $b \succeq a$ hold. Consequently, both $M(a) \geq M(b)$ and $M(b) \geq M(a)$ hold, that is, $M(a) = M(b)$. □

Definition 2.12. A collection of monotones $\{M_i\}_{i \in I}$ is said to be complete, if

$$a \succeq b \iff M_i(a) \geq M_i(b), \forall i \in I.$$

This is equivalent to saying that $\{M_i\}_{i \in I}$ characterizes completely the order of a given resource theory.

Theorem 2.4. *For any resource theory* $\mathbf{R}$*, there exists a complete set of monotones.*

Proof. The original statement of the proof can be found in [14]. Let us define the index set as $I := \mathbf{R}$. For any $i \in I$, the i-th monotone $M_i : \mathbf{R} \to \mathbb{R}$ is defined as follows:

$$M_i(a) = \begin{cases} 1 & \text{if } a \succeq i \\ 0 & \text{if } a \not\succeq i \end{cases}.$$

Let $a, b \in \mathbf{R}$ such that $a \not\succeq b$. There always exists a monotone witnessing this – $M_b(b) = 1$, but $M_b(a) = 0$. Thus, $\{M_i\}_{i \in I}$ is a complete set of monotones for $\mathbf{R}$. □

For the sake of simplicity and consistency with this work, we refer the reader to [13] for formal results in SMC theory and to [14] for a comprehensive mathematical discussion on resource theories.

2.4 QRTs: a Physical Perspective

We now focus on the perspective of Physics, specifically Quantum Mechanics. An exhaustive analysis can be found in [11]. Let A and B be two quantum systems associated with two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively.

Definition 2.13. Let $\mathcal{O}$ be a map that assigns to a pair of input/output systems (A, B) a unique set of linear CPTP maps, denoted by

$$\mathcal{O}(A \to B) \equiv \mathrm{FO}^{(A\to B)}.$$

In other words, the free operations are quantum channels. In the following section, whenever the input/output systems are clear from the context, we will drop the superscript and write FO.

Let $\mathbb{C}$ be the set of complex numbers. We recall that $\mathcal{H}_{\mathbb{C}} \cong \mathcal{B}(\mathbb{C}) \cong \mathbb{C}$.

Definition 2.14. Let $\mathcal{H}_A$ be an arbitrary Hilbert space. Let $\mathcal{S}$ be the map induced by $\mathcal{O}$ that assigns to the pair $(\mathbb{C}, \mathcal{H}_A)$ a particular set of quantum channels denoted by

$$\mathcal{S}(\mathbb{C} \to A) \equiv \mathrm{FreeSt}(A).$$

In other words, the free states are a particular type of free operations. We start from the set of complex numbers $\mathbb{C}$, that is, "nothing", and we create a positive semi-definite and trace-one bounded operator of the Hilbert space $\mathcal{H}$, that is, a quantum state. As above, whenever the Hilbert space is clear from the context, we will simply write FreeSt.

Definition 2.15. Let A be a system associated with a Hilbert space $\mathcal{H}_A$. A state is called a *resourceful state* if it is not a free state, that is,

$$\mathrm{ResSt}^{(A)} := \{\rho \in \mathrm{St}(A) \mid \rho \notin \mathrm{FreeSt}(A)\}.$$

We are now able to formally construct a QRT.

Definition 2.16. The tuple $\mathbf{R} := (\mathcal{S}, \mathcal{O})$ is called a *quantum resource theory* if the following conditions hold:

- For any system A, the set $\mathcal{O}(A \to A) \equiv \mathrm{FO}^{(A\to A)}$ contains the identity map I_A;
- For any three systems A, B, C, if $\Phi \in \mathrm{FO}^{(A\to B)}$ and $\Lambda \in \mathrm{FO}^{(B\to C)}$, then $\Lambda \circ \Phi \in \mathrm{FO}^{(A\to C)}$. That is, the composition between free operations is a free operation.

It is important to justify the term *resource*. We give a sort of golden rule, that is, a theoretical way to characterize the free states.

Proposition 2.2. *For any pair of systems A and B, if $\Phi \in \mathrm{FO}^{(A\to B)}$ and $\rho \in \mathrm{FreeSt}(A)$, then $\Phi(\rho) \in \mathrm{FreeSt}(B)$. In other words, a free operation always maps a free state into another one.*

Remark 2.4. We stated that a free operation is a quantum channel. This is equivalent to saying that $\Phi : \mathrm{St}(A) \to \mathrm{St}(B)$, *i.e.*, a free operation acts on the whole set of quantum states. When ρ is free, then $\Phi(\rho)$ must also be free, according to the golden rule in Prop. 2.2. On the other hand, when ρ is not free, then $\Phi(\rho)$ may be either free or a resource, depending on the specific QRT. We will discuss this in more detail later.

It is particularly natural to introduce the set of non-free operations.

Definition 2.17. Let A and B be two input/output systems. A linear CPTP map Φ is called a *non-free operation* if

$$\Phi_{\text{resource}} := \Phi \notin \mathrm{FO}^{(A\to B)}.$$

Now, we introduce a useful structure common to many QRTs, recalling that $\mathbb{R} \cong \mathcal{H}_{\mathbb{R}} \cong \mathcal{B}(\mathbb{R})$.

Definition 2.18. Let $\mathbf{R} = (\mathcal{S}, \mathcal{O})$ be a QRT. It admits a tensor-product structure if:

- For any three systems A, B, C, if $\Phi \in \mathrm{FO}^{(A\to B)}$, then $I_C \otimes \Phi \in \mathrm{FO}^{(CA\to CB)}$ — we say that free operations must be completely free (CF);
- For any arbitrary $\sigma \in \mathrm{FreeSt}^{(B)}$, the map $\Phi_\sigma(\rho) := \rho \otimes \sigma \in \mathrm{FO}^{(A\to AB)}$;
- For any arbitrary Hilbert space $\mathcal{H}$, the set $\mathrm{FO}^{(\mathcal{H}\to\mathbb{R})}$ is not empty.

Def. 2.18 allows us to make further statements.

Proposition 2.3. *The trace map* Tr *is a free operation.*

Proof. The condition that the set $\mathbb{O}(\mathcal{H} \to \mathbb{R})$ is not empty is equivalent to saying that the trace map $\mathrm{Tr} : \mathcal{B}(\mathcal{H}) \to \mathbb{R}$ is a free operation. In fact, it is straightforward to show that Tr is a linear CPTP map, that is, a quantum channel. Furthermore, given an arbitrary quantum state σ, $\mathrm{Tr}[\sigma] = 1$ is a free state for $\mathbb{R}$. In particular, this is true for any arbitrary free state. Thus, the golden rule in Prop. 2.2 is satisfied. □

Proposition 2.4. *Let* $\Phi \in \mathrm{FO}^{(A\to B)}$ *and* $\Phi' \in \mathrm{FO}^{(A'\to B')}$ *be free maps. Then* $\Phi \otimes \Phi' \in \mathrm{FO}^{(AA'\to BB')}$.

Proof. The proof derives directly from the first condition. In fact, since $\mathbf{R}$ is a QRT with tensor-product structure, any free operation is completely free. In addition, since $\mathbf{R}$ is a QRT, the composition between any two free operations is a free operation. Thus, we have

$$\Phi \otimes \Phi' = (I_B \otimes \Phi') \circ (\Phi \otimes I_{A'}),$$

i.e., $\Phi \otimes \Phi' \in \mathrm{FO}^{(AA'\to BB')}$. □

Proposition 2.5. *The partial-trace map* $\mathrm{Tr} \otimes I$ *is a free operation.*

Proof. The proof is trivial and comes from Prop. 2.3 and Prop. 2.4. In fact, the partial-trace map is a tensor product of two free operations (Tr and I). □

Prop. 2.4 implies the following corollary:

Corollary 1. For an arbitrary joint state $\rho_{AB} \in \mathrm{FreeSt}(AB)$, the marginal states ρ_A and ρ_B are free.

Proof. Since $\rho_A = \mathrm{Tr}_B[\rho_{AB}]$ and $\rho_B = \mathrm{Tr}_A[\rho_{AB}]$, Prop. 2.4 guarantees that ρ_A and ρ_B are free states. □

Proposition 2.6. *For two arbitrary states* $\rho \in \mathrm{FreeSt}(A)$ *and* $\sigma \in \mathrm{FreeSt}(B)$*, the joint state* $\rho \otimes \sigma$ *is free.*

Proof. The proof is trivial and follows directly from the fact that appending free states is a free operation. □

Let us now focus on the question *Does there exist a transformation between two given states?* Let A be a quantum system associated with a Hilbert space $\mathcal{H}_A$ and let the set of free operations $\mathrm{FO}^{(A\to A)}$ be fixed, for example by the physical constraints. For any two arbitrary states $\rho, \sigma \in \mathrm{St}(A)$, we write

$$\rho \xrightarrow{\mathrm{FO}} \sigma$$

if there exists $\Phi \in \mathrm{FO}^{(A\to A)}$ such that $\Phi(\rho) = \sigma$. This is equivalent to saying that $\rho \succeq \sigma$, recalling the notion of preorder in Def. 2.6.

Proposition 2.7. *The set* $\mathrm{FO}^{(A\to A)}$ *imposes a preorder on the set* $\mathrm{St}(A)$*.*

Proof. The binary relation $\succeq$ is a preorder for $\mathrm{St}(A)$, since it satisfies the conditions in Def. 2.6 as follows: (1) For any state ρ, it is natural that $\rho \xrightarrow{\mathrm{FO}} \rho$, that is, reflexivity holds; (2) For any three states ρ, σ, η, if $\rho \xrightarrow{\mathrm{FO}} \sigma$ and $\sigma \xrightarrow{\mathrm{FO}} \eta$, then $\rho \xrightarrow{\mathrm{FO}} \eta$, that is, transitivity holds. □

Let us introduce some basic information-theoretic tasks, collectively known as *convertibility* – we will follow the structure of [11]. One of the most basic problems is whether there exists a free operation that perfectly transforms a state ρ into another state σ – this is commonly known as single-shot convertibility.

Definition 2.19. Let $\rho \in \mathrm{St}(A)$ and $\sigma \in \mathrm{St}(B)$ and let $\Phi \in \mathrm{FO}^{(A\to B)}$ be a linear CPTP map. It is called an exact state transformation if

$$\Phi(\rho) = \sigma.$$

As previously stated, we write $\rho \xrightarrow{\mathrm{FO}} \sigma$.

Let us introduce a central topic in the framework of QRTs, that is, the definition of resource measures, in order to answer the question *How much of a resource does an arbitrary state contain?* Let $\mathcal{H}$ be an arbitrary Hilbert space and let $\mathcal{S}(\mathcal{H})$ be the associated set of quantum states. We would like to adapt the general mathematical version of a resource-quantifying function introduced in Def. 2.11 to the quantum framework.

Definition 2.20. A function $f : \bigcup_{\mathcal{H}} \mathrm{St}(\mathcal{H}) \to \mathbb{R}_0^+$ is called a resource measure if both the following properties hold:

- $\rho \in \mathrm{FreeSt}(A) \implies f(\rho) = 0$;
- $f(\sigma) \geq f(\Phi(\sigma))$ for any $\Phi \in \mathrm{FO}^{(A \to B)}$ and $\sigma \in \mathrm{St}(A)$.

Note that the requirement of f being non-negative is natural: a state containing a negative quantity of a resource has no physical significance. Thus, throughout this paper, we will use the terms *resource measure* and *resource monotone* interchangeably. Moreover, the two defining properties justify the term *free*: every free state is, by definition, a non-resourceful state, that is, its resource content must be zero, and no free operation can increase the amount of resource in an arbitrary quantum state. Let us consider a free state ρ and a free operation Ω. According to the golden rule of QRTs, the state $\Omega(\rho)$ must be free. This is consistent with both properties, since

$$0 = f(\rho) = f(\Omega(\rho)).$$

It may be useful to introduce additional properties for resource measures. Let us focus on the converse of the first defining property.

Definition 2.21. A resource measure f is said to be faithful, if

$$f(\rho) = 0 \implies \rho \in \mathrm{FreeSt}(A).$$

Note that a resource measure f is not necessarily faithful; consequently, faithfulness must be verified for each resource measure within a given QRT.

Remark 2.5. Let us consider a complete set of monotones, namely $\{f_i\}_{i \in I}$, as previously introduced in Def. 2.12. Although faithfulness may seem a useful property, we do not require that every monotone f_i in the collection is faithful. In fact, in order to completely characterize a QRT, we only need a complete set of monotones. Furthermore, there always exists such a set, according to Prop. 2.4.

Furthermore, we may be interested in the behavior of a resource measure with respect to quantum-classical (QC) states. Let us consider a QRT $\mathbf{R} = (\mathcal{S}, \mathcal{O})$ in which physical measurements are free operations.

Definition 2.22. Let us consider an arbitrary QC state $\sigma^{QX} = \sum_i p_i \sigma_i \otimes |i\rangle\langle i|$. A non-negative function f as above is said to be convex linear on QC states if

$$f\left(\sum_i p_i \sigma_i \otimes |i\rangle\langle i|\right) = \sum_i p_i f(\sigma_i \otimes |i\rangle\langle i|).$$

Proposition 2.8. *Let f be a resource measure. If f is convex linear on QC states, then it holds that*

$$f(\rho) \geq \sum_i p_i f(\sigma_i \otimes |i\rangle\langle i|).$$

Proof. Let Φ be a measurement map, that is, a quantum channel such that $\Phi(\rho) = \sigma^{QX}$, for any quantum state ρ. Thus, we have

$$f(\rho) \geq f(\Phi(\rho)) = f(\sigma^{QX}) = \sum_i p_i f(\sigma_i \otimes |i\rangle\langle i|).$$

This concludes the proof. □

Finally, let us introduce convexity and subadditivity.

Definition 2.23. For any collection $\{\rho_i\}$ of quantum states with an associated collection $\{p_i\}$ of probability distributions, a resource measure f is said to be convex if

$$f\left(\sum_i p_i \rho_i\right) \leq \sum_i p_i f(\rho_i).$$

Definition 2.24. A resource measure f is said to be subadditive if

$$f(\rho \otimes \sigma) \leq f(\rho) + f(\sigma),$$

for all quantum states ρ, σ. When the equality holds, f is said to be additive.

2.5 The Standard Quantum Resource Theory of Magic

We are now able to introduce the *standard* version of the quantum resource theory of magic. We refer the reader to [20, 21, 55] for the original works.

2.5.1 The Stabilizer and Magic States

The stabilizer states are understood as the free states of the standard QRT of magic. Let us begin from the pure states.

Definition 2.25. Let $\mathcal{S}$ be a maximal stabilizer subgroup. The state $|\psi\rangle$ is a pure stabilizer state if

$$s|\psi\rangle = |\psi\rangle, \quad \forall s \in \mathcal{S}.$$

The set of pure stabilizer states will be denoted by PurStab.

It is fundamental to specify that $|\psi\rangle$ is a pure stabilizer state if and only if it is uniquely defined, that is, the stabilizer group $\mathcal{S}$ must be maximal. Moreover, recalling Section 1.5, we give the following operational definition of pure stabilizer states.

Definition 2.26. Any pure stabilizer state can be written as $U|0\rangle$, where $U \in \mathcal{C}_n$, and $|0\rangle$ is the n-qubit computational vacuum pure stabilizer state.

Example 2.1. In the single-qubit framework, each maximal stabilizer subgroup has order $2^n = 2$ and is generated by a single Pauli operator (chosen from $\{\pm X, \pm Y, \pm Z\}$), together with the identity $\mathbb{I}$. This yields exactly 6 pure stabilizer states, geometrically corresponding to the vertices of the stabilizer octahedron in Fig. 2.1. These states are summarized in Table 2.1.

Stabilizer State	**Computational Basis Form**	**Stabilizer Strings**
$\|0\rangle$	$\|0\rangle$	$\mathbb{I}, \mathrm{Z}$
$\|1\rangle$	$\|1\rangle$	$\mathbb{I}, -\mathrm{Z}$
$\|+\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$	$\mathbb{I}, \mathrm{X}$
$\|-\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle - \|1\rangle)$	$\mathbb{I}, -\mathrm{X}$
$\|i\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle + i\|1\rangle)$	$\mathbb{I}, \mathrm{Y}$
$\|-i\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle - i\|1\rangle)$	$\mathbb{I}, -\mathrm{Y}$

Table 2.1: The six pure stabilizer states for $n = 1$.

Example 2.2. Magic and entanglement are entirely independent resources. A suitable example is given by the four Bell states: despite being maximally entangled, they are pure stabilizer states and therefore have zero magic by construction. We refer the reader to Table 2.2.
In particular, the existence of such entangled stabilizer states shows that the Gottesman-Knill theorem's classical simulability does not rely on the absence of entanglement, but rather on the absence of magic.

Bell State	Computational Basis Form	Stabilizer Strings
$\|\psi^+\rangle$	$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$	$\mathbb{I}\otimes\mathbb{I}, X\otimes X, Z\otimes Z$
$\|\psi^-\rangle$	$\frac{1}{\sqrt{2}}(\|00\rangle - \|11\rangle)$	$\mathbb{I}\otimes\mathbb{I}, -X\otimes X, Z\otimes Z$
$\|\varphi^+\rangle$	$\frac{1}{\sqrt{2}}(\|01\rangle + \|10\rangle)$	$\mathbb{I}\otimes\mathbb{I}, X\otimes X, -Z\otimes Z$
$\|\varphi^-\rangle$	$\frac{1}{\sqrt{2}}(\|01\rangle - \|10\rangle)$	$\mathbb{I}\otimes\mathbb{I}, -X\otimes X, -Z\otimes Z$

Table 2.2: The four Bell states are entangled stabilizer states.

Moreover, let us specify the difference between a general element of the coding space $\mathcal{H}_{\mathcal{S}}$ and a pure stabilizer state, namely $|\psi\rangle$. For example, let us consider the 3-qubit bit-flip code – see [39] for further details about this code. If $n = 3$, then $|\mathcal{P}_3| = 4^4 = 256$. On one hand, a maximal stabilizer group $\mathcal{S}_{max}$ has order $|\mathcal{S}_{max}| = 2^3 = 8$ and $n = 3$ generators. On the other hand, for $k = 1$, a non-maximal stabilizer group might be given by

$$\mathcal{S} = \langle Z_1 Z_2 \mathbb{I}_3, \mathbb{I}_1 Z_2 Z_3 \rangle,$$

with $|\mathcal{S}| = 2^{n-k} = 4$. A reasonable choice for a basis is $\{|000\rangle, |111\rangle\}$, in order to express an arbitrary element of $\mathcal{H}_{\mathcal{S}}$ as $|\tau\rangle = \alpha|000\rangle + \beta|111\rangle$. In fact, we have

$$Z_1 Z_2 \mathbb{I}_3 |\tau\rangle = \alpha|000\rangle + \beta|111\rangle = |\tau\rangle,$$
$$\mathbb{I}_1 Z_2 Z_3 |\tau\rangle = \alpha|000\rangle + \beta|111\rangle = |\tau\rangle.$$

This means that $\mathcal{S} = \langle Z_1 Z_2 \mathbb{I}_3, \mathbb{I}_1 Z_2 Z_3 \rangle$ stabilizes τ, but not in a unique way. This means that τ cannot be a stabilizer state. However, it is possible to maximize $\mathcal{S} = \langle Z_1 Z_2 \mathbb{I}_3, \mathbb{I}_1 Z_2 Z_3 \rangle$. For example, if we add the generator $Z_1 \mathbb{I}_2 \mathbb{I}_3$, then the stabilizer state is $|000\rangle$. Instead, if we add $X_1 X_2 X_3$, then the stabilizer state is the so-called Greenberger-Horne-Zeilinger state (see [39]):

$$|\psi\rangle_{GHZ} := \frac{|000\rangle + |111\rangle}{\sqrt{2}}.$$

The density matrix formalism allows for a slight generalization.

Definition 2.27. Let $\mathcal{S}$ be a stabilizer group. The stabilizer state associated with $\mathcal{S}$ is

$$\rho = \frac{1}{2^n} \sum_{S \in \mathcal{S}} S.$$

Def. 2.27 only concerns uniformly distributed stabilizer states, known as *projective stabilizer states* – see for example [1, 20, 39]). The set of projective stabilizer states will be denoted by ProjStab.

Example 2.3. We provide an example in order to show that any projective stabilizer state is always associated with a specific stabilizer subgroup — see Table 2.3 for a summary.

Stabilizer Subgroup	**Associated Stabilizer State**
Maximal, $\lvert\mathcal{S}\rvert = 2^n$	Pure
Trivial, $\lvert\mathcal{S}\rvert = 2^0 = 1$	Maximally Mixed
Neither Maximal Nor Trivial, $\lvert\mathcal{S}\rvert = 2^{n-k} \mid k < n$	Mixed

Table 2.3: The hierarchy of the projective stabilizer states.

In the two-qubit framework, since $n = 2$ and $k = 0, 1, 2$, there are three different possible stabilizer subgroups. First, let us consider the maximal one, *i.e.*, $\mathcal{S} = \{\mathbb{I}^{\otimes 2}, X_1, X_2, X_1X_2\}$ with $\lvert\mathcal{S}\rvert = 2^{2-0} = 4$. The stabilizer state associated with $\mathcal{S}_{\text{maximal}}$ is given by

$$\rho_{\text{pure}} = \frac{1}{4}\left(\mathbb{I}\otimes\mathbb{I} + \mathbb{I}\otimes X + X\otimes\mathbb{I} + X\otimes X\right) = \lvert{+}{+}\rangle\langle{+}{+}\rvert\,.$$

The trivial stabilizer subgroup is $\mathcal{S}_{\text{trivial}} = \{\mathbb{I}^{\otimes 2}\}$ with $\lvert\mathcal{S}\rvert = 2^{2-2} = 1$. Then, the stabilizer state associated with $\mathcal{S}_{\text{trivial}}$ is maximally mixed, namely

$$\rho_{\text{max.mixed}} = \frac{\mathbb{I}\otimes\mathbb{I}}{4}.$$

The last option is covered by a stabilizer subgroup satisfying $\lvert\mathcal{S}\rvert = 2^{2-1} = 2$, *e.g.*, $\mathcal{S} = \{\mathbb{I}^{\otimes 2}, X_1X_2\}$. The associated stabilizer state is

$$\rho_{\text{mixed}} = \frac{1}{4}\left(\mathbb{I}\otimes\mathbb{I} + X\otimes X\right).$$

Now, we provide a novel form for stabilizer states through geometrical considerations. Within the general framework introduced in [55] by Veitch et al., the full set of stabilizer states can be understood as the convex hull of the set of pure stabilizer states, namely

$$\mathsf{Stab} = \mathsf{Conv}(\mathsf{PurStab}) := \Big\{\rho = \sum_{i=1}^{n}\lambda_i\rho_i^{(s)} \mid \rho_i^{(s)} \in \mathsf{PurStab},\ \lambda_i \geq 0, \sum_{i=1}^{n}\lambda_i = 1\Big\}.$$

Lemma 2.5. *Let* ProjStab *and* PurStab *be the sets of projective stabilizer states and pure stabilizer states, respectively. Then, we have*

$$\mathsf{ProjStab} \subseteq \mathsf{Conv}(\mathsf{PurStab}).$$

Proof. From Def. 2.27, it is evident that an arbitrary projective stabilizer state is a convex combination of pure stabilizer states with equal coefficients. □

Corollary 2. As an immediate consequence, it follows that

$$\mathsf{Conv}(\mathsf{ProjStab}) \subseteq \mathsf{Conv}(\mathsf{PurStab}).$$

Proof. It suffices to note that $\mathrm{Conv}(\mathrm{Conv}(X)) = \mathrm{Conv}(X)$, where X is an arbitrary finite set, and adapt this to Lemma 2.5:

$$\begin{aligned} \mathsf{ProjStab} &\subseteq \mathsf{Conv}(\mathsf{PurStab}) \\ \mathsf{Conv}(\mathsf{ProjStab}) &\subseteq \mathsf{Conv}(\mathsf{Conv}(\mathsf{PurStab})) \\ \mathsf{Conv}(\mathsf{ProjStab}) &\subseteq \mathsf{Conv}(\mathsf{PurStab}), \end{aligned}$$

that is, any arbitrary convex combination of projective stabilizer states is a convex combination of pure stabilizer states. □

Lemma 2.6. *Let* ProjStab *and* PurStab *be the sets of projective stabilizer states and pure stabilizer states, respectively. Then, we have*

$$\mathsf{Conv}(\mathsf{ProjStab}) = \mathsf{Conv}(\mathsf{PurStab}).$$

Proof. From Corollary 2, it follows that $\mathsf{Conv}(\mathsf{ProjStab}) \subseteq \mathsf{Conv}(\mathsf{PurStab})$.
The converse is trivial. From Def. 2.27, it suffices to consider a maximal stabilizer subgroup to obtain a pure stabilizer state, that is, $\mathsf{PurStab} \subseteq \mathsf{ProjStab}$. Then, it holds that $\mathsf{Conv}(\mathsf{PurStab}) \subseteq \mathsf{Conv}(\mathsf{ProjStab})$. The claim follows. □

Let us define the following set:

Definition 2.28. The union of all stabilizer subgroups up to phases is expressed as

$$\mathcal{A}_n := \bigcup_{\mathcal{S}_{max}} \mathcal{S}_{max},$$

where n is the number of qubits.

Remark 2.6. Note that it would be redundant to consider all the stabilizer subgroups – any non-maximal stabilizer subgroup is contained in a maximal one, thus it suffices to consider the union of maximal stabilizer subgroups. Thus, we have

$$\mathcal{A}_n = \bigcup_{\mathcal{S}_{max}} \mathcal{S}_{max} = \bigcup_{\mathcal{S}} \mathcal{S}.$$

Lemma 2.7. *The union of all maximal stabilizer subgroups up to phases is equal to the projective Pauli group, namely*

$$\mathcal{A}_n = \widetilde{\mathcal{P}}_n.$$

Proof. The proof is given for maximal stabilizer subgroups, according to Rem. 2.6. It is evident by construction that $\mathcal{A}_n \subseteq \widetilde{\mathcal{P}}_n$. Indeed, a basic property of unions guarantees that

$$\mathcal{S}_{max} \subset \widetilde{\mathcal{P}}_n \implies \bigcup_{\mathcal{S}_{max}} \mathcal{S}_{max} = \mathcal{A}_n \subseteq \widetilde{\mathcal{P}}_n.$$

Conversely, let us consider an arbitrary $\tilde{P} \in \widetilde{\mathcal{P}}_n$. Note first that, if $\tilde{P} = \mathbb{I}^{\otimes n}$, then $\tilde{P} \in \mathcal{S}_{max}$ by construction, that is, $\tilde{P} \in \mathcal{A}_n$. This is true also for the other elements of $\widetilde{\mathcal{P}}_n$, since it is always possible to find a maximal stabilizer subgroup that contains a given Pauli string. Hence, $\widetilde{\mathcal{P}}_n \subseteq \mathcal{A}_n$ and the equality holds. □

We are now able to formalize the following result:

Theorem 2.8. *An arbitrary stabilizer state can be expressed as*

$$\rho = \frac{1}{2^n} \sum_{A \in \mathcal{A}_n} \phi_A A,$$

where $\mathcal{A}_n$ is defined up to phases, and $\phi_A := \mathrm{Tr}[\rho A]$.

Proof. The proof is a consequence of the fact that $\mathsf{Stab} = \mathrm{Conv}(\mathsf{PurStab}) = \mathrm{Conv}(\mathsf{ProjStab})$, that is, any stabilizer state can be expressed as a convex combination of projective stabilizer states. Let $\{p_i\}$ be a set of elements such that $p_i \geq 0$ and $\sum_i p_i = 1$. Let us consider $A \in \mathcal{A}_n$ and $\{\rho_i^{proj}\} \subseteq \mathsf{ProjStab}$. Then:

$$\phi_A := \mathrm{Tr}[\rho A] = \mathrm{Tr}\left[\sum_i p_i \rho_i^{(proj)} A\right] = \sum_i p_i \,\mathrm{Tr}\left[\rho_i^{(proj)} A\right].$$

Let us calculate the trace separately as follows:

$$\mathrm{Tr}\left[\rho_i^{(proj)} A\right] = \mathrm{Tr}\left[\frac{1}{2^n} \sum_{S \in \mathcal{S}_i} SA\right] = \frac{1}{2^n} \sum_{S \in \mathcal{S}_i} \mathrm{Tr}[SA].$$

If $S = \pm A$, then $\mathrm{Tr}\left[\pm A^2\right] = \pm \mathrm{Tr}\left[\mathbb{I}^{\otimes n}\right] = \pm 2^n$, since any squared stabilizer string is the identity; otherwise, the trace is null. Thus, we have

$$\mathrm{Tr}\left[\rho_i^{(proj)} A\right] = \frac{1}{2^n} \begin{cases} \pm 2^n & \text{if } S = \pm A \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \pm 1 & \text{if } S = \pm A \\ 0 & \text{otherwise} \end{cases}, \forall i.$$

Hence, the coefficients ϕ_A are a convex average of $-1, 0, +1$, that is, they assume values within $[-1; +1]$. Furthermore, we have

$$\rho = \sum_i p_i \rho_i^{(proj)} = \sum_i p_i \left(\frac{1}{2^n} \sum_{S \in \mathcal{S}_i} s \right) =$$
$$= \frac{1}{2^n} \sum_{A \in \mathcal{A}_n} \left(\sum_i p_i \operatorname{Tr}\left[\rho_i^{(proj)} A \right] \right) A = \frac{1}{2^n} \sum_{A \in \mathcal{A}_n} \phi_A A.$$

Note that the coefficients ϕ_A account for the phases previously ruled out in the definition of $\mathcal{A}_n$. □

Finally, we are able to formally define the set of free states.

Definition 2.29. The free states of the QRT of magic are the stabilizer states, namely

$$\mathsf{Stab} := \{ \rho = \frac{1}{2^n} \sum_{A \in \mathcal{A}_n} \phi_A A \}.$$

The definition of magic states follows immediately:

$$\mathsf{Magic} := \{ \rho \notin \mathsf{Stab} \}.$$

Theorem 2.8 can be justified by geometric considerations. In particular, the convex hull $\mathrm{Conv}(X)$ of a finite set X is always a polytope, *i.e.,* a geometric solid with flat sides, called *facets*, and vertices, called *extreme points*. In this sense, a polytope can be seen as an n-dimensional generalization of a polygon.
Consequently, $\mathsf{Stab} = \mathrm{Conv}(\mathsf{PurStab})$ is a polytope with pure stabilizer states as vertices and mixed stabilizer states within the polytope. A graphic example for 1-qubit scenario is shown in Fig. 2.1.
Furthermore, in [34], Leone et al. provided an explicit expression fully consistent with Theorem 2.8, representing a special case thereof:

$$\rho_L = \frac{1}{2^n} \left(\mathbb{I}^{\otimes n} + \sum_{P \in \mathcal{G}} \phi_P P \right), \tag{2.1}$$

where $\mathcal{G}$ is an abelian subgroup of $\widetilde{\mathcal{P}}_n \setminus \{[\mathbb{I}^{\otimes n}]\}$, while $\phi_P := \operatorname{Tr}[P\rho_L]$. This is equivalent to saying that Leone et al. consider only a subset of Stab, that is, $\mathsf{ProjStab}$. Then, the only possible values of ϕ_P are $-1, 0, +1$. Let us formalize this statement as follows:

Proposition 2.9. *The expression given in Eq. 2.1 is equivalent to the one given in Def. 2.27.*

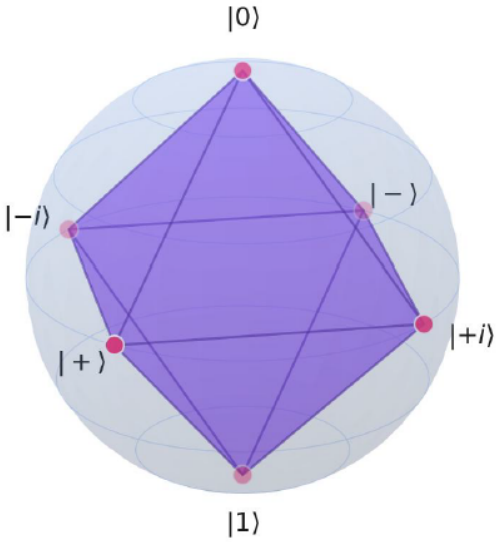


Figure 2.1: The 1-qubit polytope is a octahedron within the Bloch sphere.

Proof. Let $\mathcal{S}$ be a stabilizer subgroup as in Def. 1.22; for practical purposes, let us define $\widetilde{\mathcal{P}}$ as $\mathcal{S}$ up to phases. Let us consider $P \in \widetilde{\mathcal{S}}$ and $\rho \in \mathsf{ProjStab}$. Then, we have

$$\phi_P := \mathrm{Tr}[\rho P] = \mathrm{Tr}\left[\frac{1}{2^n}\sum_{S\in\mathcal{S}} SP\right] = \frac{1}{2^n}\sum_{S\in\mathcal{S}} \mathrm{Tr}[SP].$$

As previously stated, if $S = \pm P$, then $\mathrm{Tr}\left[\pm P^2\right] = \pm\,\mathrm{Tr}\left[\mathbb{I}^{\otimes n}\right] = \pm 2^n$; otherwise, it is null. Thus, we have

$$\phi_P = \frac{1}{2^n}\begin{cases}\pm 2^n & \text{if } S = \pm P \\ 0 & \text{otherwise}\end{cases} = \begin{cases}\pm 1 & \text{if } S = \pm P \\ 0 & \text{otherwise}\end{cases}.$$

Hence,

$$\rho = \frac{1}{2^n}\sum_{S\in\mathcal{S}} S = \frac{1}{2^n}\sum_{P\in\widetilde{\mathcal{P}}} \phi_P P = \frac{1}{2^n}\left(\mathbb{I}^{\otimes n} + \sum_{P\in\mathcal{G}} \phi_P P\right) = \rho_L,$$

where $\mathcal{G} := \widetilde{\mathcal{P}} \setminus \{[\mathbb{I}^{\otimes n}]\} \subseteq \widetilde{\mathcal{P}}_n$. Finally, note that the coefficients ϕ_P account for the phases ± 1 previously ruled out in the definition of $\widetilde{\mathcal{P}}$. □

2.5.2 The Stabilizer Operations

In the following section, we adhere to the choice made by Veitch et al. in [55] concerning the *free operations*, which will be called hereafter the *stabilizer operations*. The set of such maps will be denoted by SO.

Definition 2.30. Let $\mathcal{H}_A$, $\mathcal{H}_B$ be two Hilbert spaces. The map $\Lambda : \mathrm{St}(A) \to \mathrm{St}(B)$ is called a stabilizer operation if it is one of the following operations or a composition thereof:

- Applying a Clifford unitary U, that is, $\rho \mapsto U\rho U^\dagger$;
- Appending a stabilizer state, that is, $\rho \mapsto \rho \otimes \chi$, where $\chi \in \mathsf{Stab}$;
- Discarding a (sub)system, that is, $\rho \mapsto \mathrm{Tr}_{\mathcal{H}}[\rho]$;
- Measuring in the computational basis, that is,

$$\rho \mapsto (\mathbb{I} \otimes |i\rangle\langle i|)\rho(\mathbb{I} \otimes |i\rangle\langle i|)/\operatorname{Tr}[\rho \mathbb{I} \otimes |i\rangle\langle i|];$$

- The above operations conditioned on measurement outcomes.

Every map which is not a stabilizer operation is called *magic operation*.

Example 2.4 ([10])**.** An illustrative example of a non-Clifford operation is the T gate, defined as

$$\mathrm{T} := \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}.$$

Let us consider the action of T upon the pure stabilizer state $|+\rangle$, namely

$$\mathrm{T}|+\rangle = \mathrm{T}\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) = \frac{\mathrm{T}|0\rangle + \mathrm{T}|1\rangle}{\sqrt{2}} = \frac{|0\rangle + e^{i\frac{\pi}{4}}|1\rangle}{\sqrt{2}},$$

which is usually referred to as *T-type magic state*.

Let us provide a few fundamental properties of the maps in SO.

Proposition 2.10. *The standard QRT of magic admits a tensor-product structure.*

Proof. By construction, the stabilizer operations satisfy Def. 2.18. □

Let us demonstrate that the stabilizer operations follow the important properties shown in Section 2.4.

Proposition 2.11. *Any stabilizer operation satisfies the golden rule of QRTs expressed in Prop. 2.2, namely*

$$\rho \in \mathsf{Stab}, \Lambda \in \mathsf{SO} \implies \Lambda(\rho) \in \mathsf{Stab}.$$

Proof. Let us focus first on the most relevant type of stabilizer operations, *i.e.*, the Clifford unitaries. Let $\rho \in \mathsf{Stab}$, that is, $\rho = \frac{1}{2^n}\sum_{A\in\mathcal{A}_n} \phi_A A$, where A is a Pauli string (up to phases) and $\phi_A \in [-1,+1]$. Given an arbitrary $U \in \mathcal{C}_n$, it follows that

$$\begin{aligned} U\rho U^\dagger = U\left(\frac{1}{2^n}\sum_{A\in\mathcal{A}_n}\phi_A A\right)U^\dagger = \\ = \frac{1}{2^n}\sum_{A\in\mathcal{A}_n}\phi_A UAU^\dagger = \frac{1}{2^n}\sum_{A'\in\mathcal{A}_n}\phi_{A'}A' = \rho', \end{aligned}$$

where A' is a Pauli string (up to phases), ρ' is a stabilizer state, and $\phi_{A'}$ is the product of ϕ_A and the phase generated by the adjoint action of U. Now, let us focus on appending a stabilizer state and discarding a subsystem. In particular, we have

$$\begin{aligned} \rho\otimes\sigma = \frac{1}{2^n}\sum_{A\in\mathcal{A}_n}\phi_A A \otimes \frac{1}{2^n}\sum_{A'\in\mathcal{A}_n}\phi_{A'}A' = \\ = \frac{1}{2^{2n}}\sum_{B\in\mathcal{A}_{2n}}\phi_B B, \end{aligned}$$

where $B := A\otimes A'$ is a $2n$-qubit Pauli string (up to phases) and $\phi_B := \phi_A\phi_{A'} \in [-1,+1]$. Moreover, let us consider an orthonormal basis $\{|\psi\rangle\}$ for $\mathcal{H}_B$. Then, we have

$$\begin{aligned} \mathrm{Tr}_B[\rho\otimes\sigma] = \sum_\psi (\mathbb{I}\otimes\langle\psi|)(\rho\otimes\sigma)(\mathbb{I}\otimes|\psi\rangle) = \\ = \sum_\psi (\mathbb{I}\otimes\langle\psi|)\left(\frac{1}{2^{2n}}\sum_{A\in\mathcal{A}_n}\phi_A A\otimes\sum_{A'\in\mathcal{A}_n}\phi_{A'}A'\right)(\mathbb{I}\otimes|\psi\rangle) = \\ = \frac{1}{2^{2n}}\sum_{A\in\mathcal{A}_n}\phi_A A\otimes\sum_{A'\in\mathcal{A}_n}\phi_{A'}\sum_\psi\langle\psi|A'|\psi\rangle = \frac{1}{2^n}\sum_{A\in\mathcal{A}_n}\phi_A A = \rho, \end{aligned}$$

where $\sum_\psi \langle\psi|A'|\psi\rangle = \mathrm{Tr}[A'] = 2^n$ if $A' = \mathbb{I}$ (otherwise it is zero), and $\phi_{\mathbb{I}} = +1$. Finally, this can be straightforwardly extended to measuring in the computational basis. For example, we have

$$\rho \in \mathsf{Stab} \mapsto (\mathbb{I}\otimes|i\rangle\langle i|)\rho(\mathbb{I}\otimes|i\rangle\langle i|)/\,\mathrm{Tr}[\rho\mathbb{I}\otimes|i\rangle\langle i|] = \rho_i\otimes|i\rangle\langle i|\,,$$

where ρ conditioned on the outcome i, namely ρ_i, is a stabilizer state, and $|i\rangle\langle i|$ is a (pure) stabilizer state. This means that $\rho_i\otimes|i\rangle\langle i|$ is also a stabilizer state and therefore measuring in the computation basis satisfies the golden rule. □

Proposition 2.12. *The stabilizer operations satisfy the identity and composition properties introduced in Def. 2.16.*

Proof. Identity property trivially holds, since the identity map $\mathbb{I}$ is a special case of Clifford unitaries, namely

$$\mathbb{I}(\rho) = \mathbb{I}\rho\mathbb{I} = \rho,$$

for any quantum state ρ. Furthermore, also composition property holds. Since any quantum state ρ can be decomposed in the Pauli basis, we only focus on the action upon a single Pauli string P. Let U, V be two Clifford unitaries. Then, we have

$$(UV)(P)(UV)^\dagger = (UV)P(V^\dagger U^\dagger) = U(VPV^\dagger)U^\dagger = UP'U^\dagger = P''$$

where $P' = VPV^\dagger$ and $P'' = UP'U^\dagger$ are Pauli strings. Finally, analogous arguments apply to the remaining stabilizer operations. □

Remark 2.7. An immediate consequence is that any composition of stabilizer operations satisfies the golden rule of QRTs.

Furthermore, let us consider $U \in \mathcal{C}_n$, a pure state ρ, and a non-pure state σ. Then, we have that $U\rho U^\dagger$ is a pure state and $U\sigma U^\dagger$ is a non-pure state — in other words, U preserves the purity of a given state. This can be proven as follows. The rank of a matrix is invariant under multiplication by an invertible matrix, namely $\mathrm{rk}[AB] = \mathrm{rk}[A]$ if B is invertible, and similarly $\mathrm{rk}[AB] = \mathrm{rk}[B]$ if A is invertible. Since the Clifford unitaries U and $U^\dagger$ are both invertible, then the following expression holds for an arbitrary quantum state η:

$$\mathrm{rk}[\eta U^\dagger] = \mathrm{rk}[\eta].$$

Appending U does not change the rank, as previously stated, namely

$$\mathrm{rk}[\eta] = \mathrm{rk}[\eta U^\dagger] = \mathrm{rk}[U\eta U^\dagger],$$

that is, the rank of a density matrix is invariant under the action of a Clifford unitary. Since $\mathrm{rk}[\rho] = 1$ for pure quantum states and $\mathrm{rk}[\sigma] > 1$ for non-pure quantum states, then any Clifford unitary maps pure states into pure states and non-pure states into non-pure states, respectively.

Remark 2.8. More generally, the previous property holds for any unitary operator.

Remark 2.9. It is possible to manipulate the rank of a given quantum state as follows:

$$\mathrm{rk}(\rho \otimes \sigma) = \mathrm{rk}(\rho)\mathrm{rk}(\sigma),$$
$$\mathrm{rk}\left(\mathrm{Tr}_B[\rho_{AB}]\right) \leq \mathrm{rk}(\rho_{AB}).$$

This means that appending non-pure states increases the rank (by multiplying it) and discarding subsystems decreases (or preserves) it.

2.5.3 The Magic Preorder

In the following subsection, the QRT of magic will be equipped with an ordering structure. In particular, we recall Def. 2.6 in order to construct a *magic preorder*.

Proposition 2.13. *The magic preorder is a relation between two quantum states* $\succeq$ *which satisfies the following properties:*

- $\rho \succeq \rho$*, for any* $\rho \in \mathrm{St}$*,*
- $\rho \succeq \sigma$ *and* $\sigma \succeq \chi \implies \rho \succeq \chi$*, for any* $\rho, \sigma, \chi \in \mathrm{St}$*.*

Proof. Reflexivity holds, since $\rho \succeq \rho$, that is, there exists $\Lambda \in \mathsf{SO}$ such that $\Lambda(\rho) = \rho$, *i.e.*, $\Lambda = \mathrm{I}$. Also transitivity holds. Let us consider $\rho, \sigma, \chi \in \mathrm{St}$ such that $\rho \succeq \sigma$ and $\sigma \succeq \chi$, that is, there exist $\Lambda, \Gamma \in \mathsf{SO}$ such that

$$\Lambda(\rho) = \sigma \text{ and } \Gamma(\sigma) = \chi,$$

respectively. Then, there exists $\Omega := \Lambda \circ \Gamma \in \mathsf{SO}$ such that $\Omega(\rho) = \Gamma(\Lambda(\rho)) = \Gamma(\sigma) = \chi$. □

Let us formalize the fact that the magic preorder is not total.

Lemma 2.9. *The magic preorder* $\succeq$ *is not total.*

Proof. Let us suppose that the magic preorder is total, that is, for any pair of quantum states (ρ, σ), either $\rho \succeq \sigma$ or $\sigma \succeq \rho$. Then, let us consider $\chi \in \mathsf{Stab}$ and $\eta \in \mathsf{Magic}$ such that $\chi \succeq \eta$. This means that there exists $\Lambda \in \mathsf{SO}$ such that $\Lambda(\chi) = \eta$. This contradicts the golden rule of QRTs in Prop. 2.2 – a stabilizer operation is forbidden to produce a magic state starting from a stabilizer one. □

2.5.4 The Monotones of Magic

In order to quantify how magic an arbitrary state is, the notion of magic monotone naturally arises – we recall Section 2.2 and in particular Def. 2.20.

Definition 2.31. A function $M : \bigcup_{\mathcal{H}} \mathcal{S}(\mathcal{H}) \to \mathbb{R}_0^+$ is called a *magic monotone*, if the following conditions are satisfied:

- $\rho \in \mathsf{Stab} \implies M(\rho) = 0$;
- $M(\sigma) \geq M(\Lambda(\sigma)), \forall \Lambda \in \mathsf{SO}$ and $\forall \sigma \in \mathcal{S}(\mathcal{H})$.

In other words, every stabilizer state contains no magic, and every stabilizer operation is forbidden to increase the magic content of an arbitrary quantum state.

Although we do not require faithfulness as a defining property of monotones, we recall that many authors do impose it, leading to

$$M(\rho) = 0 \iff \rho \in \mathsf{Stab}.$$

As previously shown in Section 2.3, in order to fully characterize an arbitrary QRT, a complete set of monotones is needed. By virtue of Theorem 2.4, there always exists a complete set of monotones for any QRT, in particular for the QRT of magic. However, the explicit construction of such a set is a formidable task. As a manifestation of this difficulty, Datta et al. provided two useful no-go theorems in [16].

Theorem 2.10 (Datta-Ganardi-Kundra-Streltsov I)**.** *For any non-trivial resource theory with pure free states, there does not exist a finite complete set of monotones that are both faithful and continuous.*

Proof. We omit the proof, but it can be found in [16]. □

Theorem 2.11 (Datta-Ganardi-Kundra-Streltsov II)**.** *Any resource theory has a single complete monotone if and only if the theory is totally ordered.*

Proof. The proof can be found in [16]. □

It is evident that these results can be adapted to the framework of the QRT of magic.

Proposition 2.14. *The following conditions hold for the standard QRT of magic:*

- *There does not exist a complete and finite set of magic monotones that are both faithful and continuous;*
- *There does not exist a single magic monotone which is complete.*

Proof. The first condition is simply a specialization of Theorem 2.10. The second condition follows from Lemma 2.9 and Theorem 2.11. □

In other words, in order to find a complete set of magic monotones, we shall renounce faithfulness, continuity, finiteness, or a combination thereof, but it is not possible to have a single complete monotone.

2.5.5 An Overview of α-Stabilizer Rényi Entropies

While early pioneering works introduced the foundational magic monotones, such as the relative entropy of magic in [55], and the robustness of magic in [26], recent efforts have focused on the α-stabilizer Rényi entropies (SREs) – see [23, 33, 34].

Let us consider a n-qubit Pauli string up to phases, namely $P \in \widetilde{\mathcal{P}}_n$, and a pure quantum state, namely $|\psi\rangle$. Let us consider the normalized squared expectation value of P with respect to $|\psi\rangle$, namely

$$\Xi_P(|\psi\rangle) := \frac{1}{d} \langle\psi| P |\psi\rangle^2,$$

where $d = 2^n$ is the dimension of the Hilbert space $\mathcal{H}$. Notably, since $\Xi_P(|\psi\rangle) \geq 0$ and $\sum_{P\in\widetilde{\mathcal{P}}_n} \Xi_P = \mathrm{Tr}[|\psi\rangle\langle\psi|] = 1$, the set $\{\Xi_P(|\psi\rangle)\}_{P\in\widetilde{\mathcal{P}}_n}$ is a probability distribution, where $\Xi_P(|\psi\rangle)$ can be understood as the probability of finding P in the representation of the pure state $|\psi\rangle$.

Definition 2.32. The SRE associated with a pure state $|\psi\rangle$ is expressed as follows:

$$M_\alpha(|\psi\rangle) := \frac{1}{1-\alpha} \log\left(\sum_{P\in\widetilde{\mathcal{P}}_n} |\Xi_P(|\psi\rangle)|^\alpha \right) - \log(d) =$$
$$= \frac{1}{1-\alpha} \log\left(\sum_{P\in\widetilde{\mathcal{P}}_n} \frac{1}{d^\alpha} |\langle\psi| P |\psi\rangle|^{2\alpha} \right) - \log(d),$$

where $\alpha \in \mathbb{R} \setminus \{0\}$.

The shift $-\log(d)$ guarantees the nullity of $M_\alpha(|\psi\rangle)$ when $|\psi\rangle \in \mathsf{Stab}$. In particular, when $|\psi\rangle \in \mathsf{Stab}$, the expectation value $\langle\psi| P |\psi\rangle = \pm 1$ only for d mutually commuting Pauli strings. Thus

$$M_\alpha(|\psi\rangle) = \frac{1}{1-\alpha} \log\left(\sum_{P\in\widetilde{\mathcal{P}}_n} \frac{1}{d^\alpha} |\langle\psi| P |\psi\rangle|^{2\alpha} \right) - \log(d) =$$
$$= \frac{1}{1-\alpha} \log\left(d^{-\alpha+1}\right) - \log(d) =$$
$$= \log(d) - \log(d) = 0.$$

Furthermore, let $\Xi(|\psi\rangle)$ be the probability vector with d^2 entries labeled by $\Xi_P(|\psi\rangle)$. Its ℓ_α norm is defined as

$$\|\Xi(|\psi\rangle)\|_\alpha := \left(\sum_{P\in\widetilde{\mathcal{P}}_n} |\Xi_P(|\psi\rangle)|^\alpha \right)^{1/\alpha}.$$

Then, M_α can be rewritten in terms of the ℓ_α norm of $\Xi(|\psi\rangle)$, namely

$$M_\alpha(|\psi\rangle) = \frac{1}{1-\alpha} \log\left(\|\Xi(|\psi\rangle)\|_\alpha^\alpha \right) - \log(d) =$$
$$= \frac{\alpha}{1-\alpha} \log\left(\|\Xi(|\psi\rangle)\|_\alpha \right) - \log(d).$$

Now, we introduce some fundamental properties of the SREs.

Proposition 2.15. *The SREs are valid magic monotones only for $\alpha \geq 2$.*

Proof. An exhaustive proof can be found in [23]. □

Proposition 2.16. *Given $\alpha \geq 2$, the SREs satisfy the following properties:*

- $M_\alpha(|\psi\rangle) = 0 \iff |\psi\rangle \in \mathsf{Stab}$*, i.e., they are faithful;*
- $M_\alpha(U|\psi\rangle) = M_\alpha(|\psi\rangle)$ *with* $U \in \mathcal{C}_n$*, i.e., they are invariant under Clifford unitaries;*
- $M_\alpha(|\psi\rangle \otimes |\phi\rangle) = M_\alpha(|\psi\rangle) + M_\alpha(|\phi\rangle)$*, i.e., they are additive;*
- *They are upper bounded by* $M_\alpha(|\psi\rangle) \leq \log(d) = n\log(2)$.

Proof. A proof can be found in the Supplemental Material of [34]. □

The set $\{M_\alpha\}_{\alpha\geq 2}$ does not violate Prop. 2.14, being an infinite family of faithful and continuous monotones, and hence it might appear to be a good candidate for completeness. We show, however, that this is not the case, at least within the single-shot convertibility regime.

Definition 2.33. The *stabilizer purity* associated with $|\psi\rangle$ is given by

$$\Pi_\alpha(|\psi\rangle) := d^{\alpha-1} \sum_{P\in\widetilde{\mathcal{P}}_n} |\Xi_P(|\psi\rangle)|^\alpha = \frac{1}{d} \sum_{P\in\widetilde{\mathcal{P}}_n} |\langle\psi| P |\psi\rangle|^{2\alpha},$$

that is, the stabilizer purity is the α-moment of the probability distribution $\{\Xi_P(|\psi\rangle)\}$, with rescaling factor $d^{\alpha-1}$.

Proposition 2.17. *The SREs can be rewritten as follows:*

$$M_\alpha(|\psi\rangle) = \frac{1}{1-\alpha}\log\left(\Pi_\alpha(|\psi\rangle)\right)$$

Proof. Recalling Def. 2.33, we have that

$$\begin{aligned} M_\alpha(|\psi\rangle) &= \frac{1}{1-\alpha}\log\left(\sum_{P\in\widetilde{\mathcal{P}}_n} \frac{1}{d^\alpha}|\langle\psi| P |\psi\rangle|^{2\alpha}\right) - \log(d) = \\ &= \frac{1}{1-\alpha}\left[\log\left(\sum_{P\in\widetilde{\mathcal{P}}_n} \frac{1}{d^\alpha}|\langle\psi| P |\psi\rangle|^{2\alpha}\right) + \log\left(d^{\alpha-1}\right)\right] = \\ &= \frac{1}{1-\alpha}\log\left(\sum_{P\in\widetilde{\mathcal{P}}_n} \frac{1}{d}|\langle\psi| P |\psi\rangle|^{2\alpha}\right) = \frac{1}{1-\alpha}\log\left(\Pi_\alpha(|\psi\rangle)\right). \end{aligned}$$

The claim holds. □

Proposition 2.18. *The set $\{M_\alpha\}_{\alpha\geq 2}$ is not a complete set of magic monotones in the single-shot convertibility regime.*

Proof. Since the α-SREs are faithful and every two stabilizer states can always be mapped into each other by a Clifford unitary, any counterexample to completeness must involve magic states. Let us consider the following pair of pure magic states:

$$|\psi\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + e^{i\frac{\pi}{4}}|11\rangle\right),$$
$$|\phi\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + e^{-i\frac{\pi}{4}}|11\rangle\right).$$

As a concrete counterexample, let us compute the 2-stabilizer Rényi entropy for both $|\psi\rangle$ and $|\phi\rangle$. We can split the 2-qubit projective Pauli group into two different subsets, namely

$$\mathcal{P}^+ := \{\mathbb{I}\otimes\mathbb{I}, \mathbb{I}\otimes X, X\otimes\mathbb{I}, X\otimes X, X\otimes Z, Z\otimes X, Y\otimes Y, \mathbb{I}\otimes Z, Z\otimes\mathbb{I}, Z\otimes Z\},$$
$$\mathcal{P}^- := \{\mathbb{I}\otimes Y, Y\otimes\mathbb{I}, X\otimes Y, Y\otimes X, Y\otimes Z, Z\otimes Y\}.$$

Note that

$$\langle\psi|P|\psi\rangle = \begin{cases} +\langle\phi|P|\phi\rangle & \text{if } P\in\mathcal{P}^+ \\ -\langle\phi|P|\phi\rangle & \text{if } P\in\mathcal{P}^- \end{cases}.$$

Then, the 2-stabilizer purities of $|\psi\rangle$ and $|\phi\rangle$ are equal, that is,

$$\begin{aligned} P_2(|\psi\rangle) &= \frac{1}{d}\sum_{P\in\mathcal{P}_2} |\langle\psi|P|\psi\rangle|^4 = \frac{1}{d}\left(\sum_{P\in\mathcal{P}^+} |\langle\psi|P|\psi\rangle|^4 + \sum_{P\in\mathcal{P}^-} |\langle\psi|P|\psi\rangle|^4\right) = \\ &= \frac{1}{d}\left(\sum_{P\in\mathcal{P}^+} |\langle\phi|P|\phi\rangle|^4 + \sum_{P\in\mathcal{P}^-} |-\langle\phi|P|\phi\rangle|^4\right) = \\ &= \frac{1}{d}\left(\sum_{P\in\mathcal{P}^+} |\langle\phi|P|\phi\rangle|^4 + \sum_{P\in\mathcal{P}^-} |\langle\phi|P|\phi\rangle|^4\right) = \\ &= \frac{1}{d}\sum_{P\in\mathcal{P}_2} |\langle\phi|P|\phi\rangle|^4 = P_2(|\phi\rangle). \end{aligned}$$

This means that

$$M_2(|\psi\rangle) = M_2(|\phi\rangle) \approx 0.5737,$$

and this equivalence holds also for the other Rényi indexes. However, since Clifford unitaries preserve the Pauli group up to ± 1 phases, the phase factors $e^{\pm i\frac{\pi}{4}}$ cannot be interconverted by a Clifford unitary. This means that both $|\psi\rangle \not\succeq |\phi\rangle$ and $|\phi\rangle \not\succeq |\psi\rangle$, that is, the magic content of $|\psi\rangle$ must be different from the one of $|\phi\rangle$. Thus, the α-SREs cannot distinguish all pairs of magic states with different magic content, and therefore completeness is not satisfied. □

2.6 Generalization of the QRT of Magic

We now introduce two tools that will simplify the treatment of qubit magic within the framework of QRTs and also provide a mathematical generalization of such a theory: an enlargement of the operational set from Stabilizer Operations (SO) to *Completely Stabilizer-Preserving Operations* (CSPO), and a subsequent quotienting following from the notion of magic partial order.

2.6.1 The Completely Stabilizer-Preserving Operations

The most natural approach to enlarging the set of free operations is to consider the maximal such set, that is, by considering all the quantum operations that map arbitrary stabilizer states to other stabilizer states. In the framework of qubit magic, such maps are represented by the *Stabilizer-Preserving Operations* (SPO) — we refer the reader to the foundational work in [2].

Definition 2.34. Let Λ be a quantum channel, that is, a linear CPTP map. It is called a *Stabilizer-Preserving Operation* (SPO) if the following condition holds:

$$\Lambda : \mathsf{Stab}_n \to \mathsf{Stab}_m,$$

where the integers n, m denoting the number of qubits will be omitted when the situation is clear.

An interesting thing to note is that the set SPO is not closed under tensor product, that is, stabilizer-preserving maps may fail to be free operations when applied to subsystems — this result can be achieved from the general treatment in [11]. In order to avoid this feature, which would be strongly invalidating for the QRT setting, completely stabilizer-preserving operations have been studied (see [24, 43, 44]).

Definition 2.35. Let Λ be a quantum channel and let $k \in \mathbb{Z}_0^+$. It is called a k-stabilizer-preserving operation, if

$$\Lambda \otimes \mathrm{id}_k : \mathsf{Stab}_{n+k} \to \mathsf{Stab}_{m+k},$$

where id_k denotes the identity map associated with a suitable ancilla.

Definition 2.36. Let Λ be a quantum channel. It is called a completely stabilizer-preserving operation if it is a k-stabilizer-preserving operation for every $k \in \mathbb{Z}_0^+$, namely

$$\Lambda \otimes \mathrm{id}_k : \mathsf{Stab}_{n+k} \to \mathsf{Stab}_{m+k}, \forall k.$$

It may be useful to express $\Lambda \in \mathsf{CSPO}$ via the Choi isomorphism. Let $\mathcal{H}_A \cong \left(\mathbb{C}^2\right)^{\otimes n}$ and $\mathcal{H}_B \simeq \left(\mathbb{C}^2\right)^{\otimes m}$ be two Hilbert spaces, with $d_A = 2^n$ and $d_B = 2^m$ respectively, and let $\mathcal{H}_{A'} \simeq \mathcal{H}_A$ be an auxiliary Hilbert space associated with the ancillary system A'. Moreover, let $|\varphi\rangle = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |i\rangle_A \otimes |i\rangle_{A'}$ be the normalized maximally entangled state with respect to the computational basis. Then, the Choi representation of Λ is given by

$$J_\Lambda = \frac{1}{2^n} \Lambda \otimes \mathrm{id}_{A'} \left(|\varphi\rangle\langle\varphi|\right) \in \mathcal{B}\left(\mathcal{H}_B \otimes \mathcal{H}_{A'}\right),$$

where $\mathrm{Tr}\left[J_\Lambda\right] = 1$, $J_\Lambda \geq 0$, and $\mathrm{Tr}_B[J_\Lambda] = \frac{\mathrm{id}_{A'}}{2^n}$. Furthermore, the question of whether SO is strictly included in CSPO is extensively addressed in [24], establishing a sharp separation.

Theorem 2.12. *For a number of qubit $n \geq 2$, it holds that*

$$\mathsf{SO} \subsetneq \mathsf{CSPO}.$$

Moreover, the two sets coincide in the single-qubit framework, namely

$$\mathsf{SO}_1 = \mathsf{CSPO}_1.$$

Proof. A rigorous and exhaustive proof, generalized to the qudit setting, can be found in [24]. □

2.6.2 The Magic Quotient

We are now able to introduce the second simplifying tool, *i.e.*, the *magic quotient*. First, it is natural to extend the notion of magic preorder to the one of magic partial order.

Definition 2.37. Let us consider two quantum states, namely ρ and σ. They are said to be equivalent, namely $\rho \sim \sigma$, if

$$\rho \succeq \sigma \text{ and } \sigma \succeq \rho.$$

Definition 2.38. The quotient set $\mathrm{St}/\sim$ is the set of equivalence classes of the form

$$[\rho] := \{\sigma \in \mathrm{St} \mid \sigma \sim \rho\}.$$

Proposition 2.19. *It is possible to construct a magic partial order by equipping the magic preorder with antisymmetry property.*

Proof. The magic partial order is well-defined. Let us consider ρ, ρ' such that $\rho \sim \rho'$, that is, both ρ and ρ' are elements of the same equivalence class. This is also equivalent to saying that

$$\exists \Gamma \in \mathsf{CSPO} \mid \Gamma(\rho) = \rho',$$
$$\exists \Gamma' \in \mathsf{CSPO} \mid \Gamma'(\rho') = \rho.$$

Let us suppose that $\rho \succeq \sigma$, that is,

$$\exists \Lambda \in \mathsf{CSPO} \mid \Lambda(\rho) = \sigma,$$

where σ is not in the same equivalence class of ρ. We can construct the following map:

$$\Omega := \Lambda \circ \Gamma' \in \mathsf{CSPO}.$$

Thus, Ω acting upon ρ' produces σ:

$$\Omega(\rho') = \Lambda(\Gamma'(\rho')) = \Lambda(\rho) = \sigma,$$

that is, $\rho' \succeq \sigma$. This is true for any element of the equivalence class of ρ. The same argument applies also for a pair of output states, namely σ, σ', in the same equivalence class. Finally, the antisymmetry property is guaranteed by the structure of the quotient set $\mathrm{St}/\sim$, as previously introduced in Def. 2.8. □

We are now able to extend the notion of stabilizer set and magic set introduced within the standard QRT of magic.

Definition 2.39. The set of equivalence classes of quantum states $\mathrm{St}/\sim$ can be divided into a stabilizer set and a magic set, defined respectively as follows:

$$\mathsf{Stab}/\sim := \{[\rho] \mid \rho \in \mathsf{Stab}\},$$
$$\mathsf{Magic}/\sim := \{[\rho] \mid \rho \in \mathsf{Magic}\} = (\mathrm{St}/\sim) \backslash (\mathsf{Stab}/\sim).$$

Theorem 2.13. *The stabilizer states are equivalent, that is, the entire set* $\mathsf{Stab}/\sim$ *collapses to a singleton, namely*

$$\mathsf{Stab}/\sim = \{[\mathsf{Stab}]\}.$$

Proof. Let us consider two stabilizer states, namely ρ and ρ'. In order to prove the fact that $\rho \sim \rho'$, it suffices to adapt the operational definition of a stabilizer state provided in Def. 2.26 to the framework of SPO. In particular, ρ can be expressed as

$$\rho = \Lambda\left(|0\rangle\langle 0|\right)^{\otimes n},$$

for a suitable $\Lambda \in \mathsf{CSPO}$ (such as a Clifford unitary). It suffices to transform ρ into $(|0\rangle\langle 0|)^{\otimes n}$ via a measurement operation (which is a valid stabilizer-preserving operation), and then consider the action of an appropriate $\Lambda' \in \mathrm{CSPO}$ such that

$$\Lambda' (|0\rangle\langle 0|)^{\otimes n} = \rho',$$

that is, $\rho \mapsto \rho'$ via the composition of stabilizer-preserving operations, which is again a valid stabilizer-preserving operation. The converse is trivial. □

Theorem 2.14. *No magic state is equivalent to a stabilizer state, namely*

$$[\rho] \neq [\mathsf{Stab}],$$

with $[\rho] \in \mathsf{Magic}/\sim$.

Proof. Let us consider $\rho \in \mathsf{Magic}$ and $\sigma \in \mathsf{Stab}$. We are always able to map ρ into $(|0\rangle\langle 0|)^{\otimes n}$ via a suitable measurement, and then to achieve σ via a suitable $\Lambda \in \mathsf{CSPO}$ such that

$$\Lambda(|0\rangle\langle 0|)^{\otimes n} = \sigma.$$

On the contrary, since the golden rule of QRTs holds, it is forbidden for a stabilizer operation to produce a magic state, that is,

$$\rho \nsim \sigma.$$

This argument immediately leads to

$$[\rho] \neq [\mathsf{Stab}],$$

for any $[\rho] \in \mathsf{Magic}/\sim$. □

Proposition 2.20. *The following conditions hold within the framework of equivalence classes:*

- *There does not exist a complete and finite set of magic monotones that are both faithful and continuous;*
- *There does not exist a single magic monotone which is complete.*

Proof. The first condition can be proven analogously to Prop. 2.14. The second condition relies on the fact that the magic partial order is not total. In fact, an analogous argument of Lemma 2.9 can be easily extended to the magic partial order. □

We have shown that Prop. 2.14 holds, essentially unchanged, when passing to the setting of the magic quotient, thus yielding another no-go theorem. Therefore, quotienting does not resolve the obstruction represented by Prop. 2.20 — its role is merely to "lighten" the sets of states involved, potentially easing the study of magic monotones and their completeness. It could be interesting, as a direction for future work, to investigate whether an analogous quotienting procedure could be applied to the operations of CSPO as well.

Chapter 3

Introduction to Gaussian Formalism

Before analyzing the QRT of fermionic non-Gaussianity, some relevant mathematical and physical tools are needed. In this chapter, while elementary concepts adhere to standard conventions, more specific topics employ a custom notation; nevertheless, the original works will always be cited.

3.1 Mathematical Preliminaries

In order to properly introduce the fermionic framework, let us begin from a few mathematical basics, following in particular Appendix A of [52].

Definition 3.1. Let $\mathbb{K}$ be a field and V be a vector space over $\mathbb{K}$ equipped with a binary operation $\star : V \times V \to V$. Suppose that $\star$ is also bilinear, that is,

- (Right distributivity) $(x + y) \star z = x \star z + y \star z$;
- (Left distributivity) $x \star (y + z) = x \star y + x \star z$;
- (Compatibility with scalars) $(ax) \star (by) = (ab)(x \star y)$.

with $a, b \in \mathbb{K}$ and $x, y, z \in V$. Then, V is called a $\mathbb{K}$-algebra. Hereafter, we will write algebras in calligraphic style, such as $\mathscr{V}$.

Definition 3.2. A $\mathbb{K}$-algebra $\mathscr{A}$ with $\star : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ is said to be

- Associative, if $\star$ is associative;
- Commutative, if $\star$ is commutative;
- Unital, if there exists a trivial element, that is, I such that $I \star x = x \star I = x$, for all $x \in \mathscr{A}$.

Let us introduce the notion of graded algebra.

Definition 3.3 ($\mathbb{Z}_2$-Graded Vector Space)**.** A $\mathbb{Z}_2$-graded $\mathbb{K}$-vector space V is a pair

$$V = (V^0, V^1) := V^0 \cup V^1,$$

where V^p is a vector space over the field $\mathbb{K}$, with $p \in \mathbb{Z}_2$, *i.e.*, $p = 0, 1$.

Remark 3.1. The sum is defined only for elements of V that belong to the same vector space, that is,

$$O_1, O_2 \in V^p \implies O_1 + O_2 \in V^p, \text{ for } p \in \mathbb{Z}_2.$$

Definition 3.4. A unital $\mathbb{Z}_2$-graded $\mathbb{K}$-algebra A is a $\mathbb{Z}_2$-graded $\mathbb{K}$-vector space

$$A = (A^0, A^1) := A^0 \cup A^1$$

endowed with an associative distributive product such that

$$O_1 \in A^p, O_2 \in A^q \implies O_1 O_2 \in A^{p \oplus q},$$

where $p \oplus q$ means $p + q$ modulo 2; $I_A \in A^0$ is the trivial element. As previously stated, $\mathbb{Z}_2$-graded algebras will be denoted by a calligraphic symbol, such as $\mathscr{A}$.

Remark 3.2. In Def. 3.4, the $\mathbb{Z}_2$-graded algebra $\mathscr{A}$ is assumed to be both associative and unital for the sake of simplicity. It would be unnatural not to include those properties in the definition.

Remark 3.3. The dimension of a $\mathbb{Z}_2$-graded algebra $\mathscr{A}$ is straightforwardly expressed as follows:

$$\dim(\mathscr{A}) = \dim(\mathscr{A}^0) + \dim(\mathscr{A}^1).$$

Remark 3.4. When the context is clear, the trivial element will be simply denoted by I; similarly, $O \in \mathscr{A}$ will stand for $O \in \mathscr{A}^p$.

Definition 3.5. Let $\mathscr{A}$ and $\mathscr{B}$ be two $\mathbb{Z}_2$-graded algebras; $\mathscr{B}$ is called a $\mathbb{Z}_2$-graded subalgebra of $\mathscr{A}$ if

$$\mathscr{B}^p = \mathscr{B} \cap \mathscr{A}^p.$$

We write $\mathscr{B} \subseteq \mathscr{A}$.

Definition 3.6. Let $\mathscr{A}$ be a $\mathbb{Z}_2$-graded algebra. Let $g : \mathscr{A} \to \mathbb{Z}_2$ be the parity map defined as follows:

$$O \in \mathscr{A}^p \mapsto g(O) = p \in \mathbb{Z}_2.$$

When $g(O) = 0$, then O is called an even operator; on the contrary, when $g(O) = 1$, then O is called an odd operator.

Definition 3.7. Let $\eta : \mathscr{A} \to \mathscr{B}$ be a map between two $\mathbb{Z}_2$-graded algebras. If $g(\eta(O)) = g(O)$ for any $O \in \mathscr{A}$, then η is called an even map; on the contrary, if $g(\eta(O)) = g(O) \oplus 1$ for any $O \in \mathscr{A}$, then η is called an odd map.

We are now able to generalize the notions of commutator and anticommutator.

Definition 3.8. Let $\mathscr{A}$ be a $\mathbb{Z}_2$-graded algebra and let $O_1, O_2 \in \mathscr{A}$. The graded commutator is defined as follows:

$$\{[O_1, O_2]\} := O_1 O_2 - (-1)^{g(O_1)g(O_2)} O_2 O_1.$$

Proposition 3.1. *The graded commutator in Def. 3.8 corresponds to the anticommutator if O_1, O_2 are both odd operators, and to the commutator otherwise.*

Proof. O_1 and O_2 are odd if $g(O_1) = 1 = g(O_2)$. Thus

$$\{[O_1, O_2]\} = O_1 O_2 - (-1)^1 O_2 O_1 = O_1 O_2 + O_2 O_1 = \{O_1, O_2\}.$$

Otherwise, the product $g(O_1)g(O_2)$ is always zero, that is

$$\{[O_1, O_2]\} = O_1 O_2 - (-1)^0 O_2 O_1 = O_1 O_2 - O_2 O_1 = [O_1, O_2].$$

This concludes the proof. □

Let us now introduce the concepts of graded morphism and graded tensor product.

Definition 3.9. Let $\mathscr{A}$ and $\mathscr{B}$ be two $\mathbb{Z}_2$-graded algebras. A map $M : \mathscr{A} \longrightarrow \mathscr{B}$ is called a graded morphism if

$$M(k_1 O_1 + k_2 O_2 O_3) = k_1 M(O_1) + k_2 M(O_2) M(O_3),$$

for any $k_1, k_2 \in \mathbb{K}$ and $O_1, O_2, O_3 \in \mathscr{A}$, with $M(\mathscr{A}^p) \subseteq \mathscr{B}^p$.

Definition 3.10. Let $\mathscr{A}$ and $\mathscr{B}$ be two $\mathbb{Z}_2$-graded algebras. The graded tensor product algebra $\mathscr{A} \boxtimes \mathscr{B}$ is the $\mathbb{Z}_2$-graded algebra given by

$$\mathscr{A} \boxtimes \mathscr{B} = \left((\mathscr{A} \boxtimes \mathscr{B})^0, (\mathscr{A} \boxtimes \mathscr{B})^1\right), \text{ with}$$
$$(\mathscr{A} \boxtimes \mathscr{B})^r := \bigoplus_{\substack{p,q=0,1| \\ p \oplus q = r}} \mathscr{A}^p \otimes \mathscr{B}^q,$$

where $\otimes$ indicates the standard tensor product of $\mathbb{K}$-vector spaces.

Remark 3.5. For any $A \in \mathscr{A}$ and any $B \in \mathscr{B}$, an element of the graded tensor product algebra $\mathscr{A} \boxtimes \mathscr{B}$ will be simply denoted by $A \boxtimes B$.

Remark 3.6. Let $A_1, A_2 \in \mathscr{A}$ and $B_1, B_2 \in \mathscr{B}$. Then, we have

$$(A_1 \boxtimes B_1)(A_2 \boxtimes B_2) = (-1)^{g(B_1)g(A_2)}(A_1A_2 \boxtimes B_1B_2).$$

Proposition 3.2. *The graded tensor product satisfies the following braiding relation:*

$$\{[(A \boxtimes I_B), (I_A \boxtimes B)]\} = 0.$$

Proof. Def. 3.8 implies that

$$\{[(A \boxtimes I_B), (I_A \boxtimes B)]\} = (A \boxtimes I_B)(I_A \boxtimes B) - (-1)^{g(A\boxtimes I_B)(I_A\boxtimes B)}(I_A \boxtimes B)(A \boxtimes I_B).$$

Furthermore, from Remark 3.6, we have that

$$\begin{aligned}(A \boxtimes I_B)(I_A \boxtimes B) &= (-1)^{g(I_B)g(I_A)}(AI_A \boxtimes I_BB) = A \boxtimes B \text{ and}\\ (I_A \boxtimes B)(A \boxtimes I_B) &= (-1)^{g(B)g(A)}(A \boxtimes B).\end{aligned}$$

Thus,

$$\{[(A \boxtimes I_B), (I_A \boxtimes B)]\} = A \boxtimes B - (-1)^{g(A\boxtimes I_B)(I_A\boxtimes B)}(-1)^{g(B)g(A)}A \boxtimes B.$$

The two multiplicative factors might assume the following values:

$$(-1)^{g(A\boxtimes I_B)(I_A\boxtimes B)} = \begin{cases}+1 & \text{if } A \boxtimes I_B \text{ and/or } I_A \boxtimes B \text{ are even}\\ -1 & \text{if both } A \boxtimes I_B \text{ and } I_A \boxtimes B \text{ are odd}\end{cases}$$

$$(-1)^{g(B)g(A)} = \begin{cases}+1 & \text{if } A \text{ and/or } B \text{ are even}\\ -1 & \text{if both } A \text{ and } B \text{ are odd}\end{cases}$$

Since the trivial element is by definition an even operator, then $g(A \boxtimes I_B) = g(A)$ and $g(I_A \boxtimes B) = g(B)$; that is, both multiplicative factors are either $+1$ or -1. Thus,

$$\{[(A \boxtimes I_B), (I_A \boxtimes B)]\} = A \boxtimes B - A \boxtimes B = 0.$$

This concludes the proof. □

Remark 3.7. Since $g(I) = 0$ by definition, the following properties hold:

$$\begin{aligned}g(A_1A_2) &= g(A_1) \oplus g(A_2), \text{ for any pair } A_1, A_2 \in \mathscr{A};\\ g(A \boxtimes B) &= g(A) \oplus g(B), \text{ for any } A \in \mathscr{A} \text{ and } B \in \mathscr{B}.\end{aligned}$$

In order to construct a $\mathbb{Z}_2$-graded C^*-algebra, let us introduce some fundamental notions.

Definition 3.11. Let V be a vector space over a field $\mathbb{K}$; let $\|\cdot\| : V \to \mathbb{R}_0^+$ be a norm, satisfying the following conditions:

- $\|v\| = 0 \iff v = 0$;
- $\|kv\| = k\|v\|$, for any $k \in \mathbb{K}$ and any $v \in V$;
- $\|v + w\| \leq \|v\| + \|w\|$.

the pair $(V, \|\cdot\|)$ is called a normed space.

The norm $\|\cdot\|$ induces naturally a metric given by

$$d(v, w) := \|v - w\| = \|w - v\|,$$

for any $v, w \in V$. This means that the pair (V, d) is also a metric space.

Definition 3.12. Let V be a normed space and consequently a metric space. Let $\{x_j\}_{j=1}^N$ a sequence of elements in V. It is called a Cauchy sequence if, for every $\varepsilon > 0$, there exists a positive integer M such that

$$d(x_m, x_n) < \varepsilon,$$

for all positive integers $m, n > M$.

When every Cauchy sequence within a normed space satisfies a precise condition, then we are able to construct a Banach space.

Definition 3.13. A normed space V is called a Banach space, if it is complete with respect to the induced norm, that is, every Cauchy sequence in V converges in V.

According to Def. 3.1, we are able to construct a Banach algebra as a Banach space equipped with a particular binary map.

Definition 3.14. Let $\mathbb{C}$ be the complex field and V be a Banach space over $\mathbb{C}$ equipped with a bilinear binary map $\star : V \times V \to V$. Then, V is called a Banach algebra. Also Banach algebras will be written in calligraphic style, such as $\mathscr{V}$.

Definition 3.15. Let $\mathscr{A}$ be a Banach algebra over $\mathbb{C}$ equipped with an involution map $*$ satisfying the following conditions:

- $(a^*)^* = a, \forall a \in \mathscr{A}$;
- $(a + b)^* = a^* + b^*, \forall a, b \in \mathscr{A}$;
- $(\lambda a)^* = \bar{\lambda} a^*, \forall \lambda \in \mathbb{C}$ and $\forall a \in \mathscr{A}$;

- $(ab)^* = b^*a^*, \forall a, b \in \mathscr{A}$.

Moreover, it is natural to define $\mathscr{A}$ both as an associative and unital algebra.

We are now able to construct a $\mathbb{Z}_2$-graded C^*-algebra by equipping a $\mathbb{Z}_2$-graded algebra with the proper structures of norm and involution.

Definition 3.16. A C^*-algebra is a Banach $*$-algebra $\mathscr{A}$ satisfying an extra condition, that is,

$$\|A^*A\| = \|A\|\|A^*\| = \|A\|^2.$$

Definition 3.17. Let $\mathscr{A}$ and $\mathscr{B}$ be two C^*-algebras. A map $M : \mathscr{A} \to \mathscr{B}$ is called a $*$-morphism if

$$M(c_1O_1 + c_2O_2^\dagger O_3) = c_1M(O_1) + c_2M(O_2)^\dagger M(O_3),$$

for any $c_1, c_2 \in \mathbb{C}$ and $O_1, O_2, O_3 \in \mathscr{A}$.

Definition 3.18. Let $\mathscr{A}$ be both a $\mathbb{Z}_2$-graded algebra and a C^*-algebra. Then, it is called a $\mathbb{Z}_2$-graded C^*-algebra.

Remark 3.8. As stated in [52], if $\mathscr{A}$ and $\mathscr{B}$ are both $\mathbb{Z}_2$-graded C^*-algebras with involution $\dagger$, then $\mathscr{A} \boxtimes \mathscr{B}$ is also a $\mathbb{Z}_2$-graded C^*-algebra.

Remark 3.9. We adhere to the choice of norm in [52], that is,

$$\|A \boxtimes I\| = \|A\|_A \text{ and } \|I \boxtimes B\| = \|B\|_B, \text{ for any } A \in \mathscr{A} \text{ and for any } B \in \mathscr{B}.$$

Furthermore, note that

$$(A \boxtimes B)^\dagger = (I_A \boxtimes B)^\dagger (A \boxtimes I_B)^\dagger = (-1)^{g(B)g(A)} \left(A^\dagger \boxtimes B^\dagger\right).$$

3.1.1 The Local Fermionic Modes

For the following subsection, we refer the reader to [9, 15].

Definition 3.19. A local fermionic mode (LFM) is a site that may be either empty or occupied by a spinless fermionic particle.

Let us consider a system composed of N LFMs. The associated fermionic algebra $\mathscr{F}(N)$ is generated by the fermionic operators

$$\{b_i, b_i^\dagger \mid i = 1, \dots, N\}.$$

They satisfy the canonical anticommutation relations (CAR):

$$\begin{aligned} \{b_i; b_j\} &= 0 \\ \{b_i; b_j^\dagger\} &= \delta_{ij} I. \end{aligned} \tag{3.1}$$

Moreover, b_i and $b_i^\dagger$ act respectively as the annihilation operator and the creator operator for $b_i^\dagger b_i$. If $|\lambda\rangle$ is an eigenvector of $b_i^\dagger b_i$ with eigenvalue 1(0), then $b_i |\lambda\rangle (b_i^\dagger |\lambda\rangle)$ is an eigenvector with eigenvalue 0(1), and $b_i^\dagger |\lambda\rangle = 0 (b_i |\lambda\rangle = 0)$.

Definition 3.20. The state $|\Omega\rangle$ is called a vacuum state, if it is a simultaneous eigenvector of $b_i^\dagger b_i$ with eigenvalue 0 for all i. This is equivalent to saying that all the LFMs are empty.

Definition 3.21. Raising the vacuum state $|\Omega\rangle$ in all possible ways yields the Fock basis, that is, an orthonormal basis composed of 2^N vectors in the occupation number representation, namely

$$\{|n_i\rangle\}_{i=1}^N,$$

where $|n_1, \dots, n_N\rangle_F := (b_1^\dagger)^{n_1} \dots (b_N^\dagger)^{n_N} |\Omega\rangle$, and $n_i = 0, 1$ is the occupation number of the i-th site.

Definition 3.22. The antisymmetric Fock space is the completion of the linear span of the Fock basis with respect to the inner product norm, namely

$$\mathcal{F}_N := \overline{\text{span}\left(\{|n_1, \dots, n_N\rangle_F \mid n_i \in \{0, 1\} \forall i\}\right)}.$$

By construction, $\mathcal{F}_N$ is a well-defined Hilbert space.

It is possible to outline a relation between fermionic and qubits frameworks – we refer the reader to [9, 15] for detailed proofs. In particular, the antisymmetric Fock space and the Hilbert space of N qubits are isomorphic, namely

$$\mathcal{F}_N \cong \left(\mathbb{C}^2\right)^{\otimes N}.$$

Let us consider a joint eigenvector of the qubit operator Z_j with $j = 1, \dots, N$, namely $|n_1, \dots, n_N\rangle_Q$ and a permutation of N elements, namely π, acting on the Fock basis. Then, there exists a unitary U_π such that

$$|n_1, \dots, n_N\rangle_F^\pi := (-1)^{\text{sign}(\pi)} |n_1, \dots, n_N\rangle_F \xrightarrow{U_\pi} |n_1, \dots, n_N\rangle_Q .$$

For a given permutation π, the unitary U_π induces the *Jordan-Wigner transformation* (JWT), that is, a $*$-algebra isomorphism between the CAR fermionic algebra and the algebra of Pauli operators. We refer the curious reader to the original work of Jordan and Wigner ([27]).

Example 3.1. The 2-LFM operator $b_i^\dagger b_j$ is mapped by JWT into

$$J_\pi(b_i^\dagger b_j) = S^+_{\pi(i)} \prod_{\pi(i)<k<\pi(j)} Z_k S^-_{\pi(j)},$$

where J_π indicates the JWT representation, and $S_k^\pm := \frac{X_k \pm iY_k}{2}$.

3.1.2 The Majorana Operators

As previously stated, we follow [52]. Let us consider a lattice $\mathbb{G}$ of cells, each containing a finite number of LFMs. Every cell is associated with a $\mathbb{Z}_2$-graded C^*-algebra over $\mathbb{C}$, that is,

$$x \in \mathbb{G} \longleftrightarrow \mathcal{A}_x = (\mathcal{A}_x^0, \mathcal{A}_x^1).$$

Definition 3.23. Let $x \in \mathbb{G}$. The algebra $\mathcal{A}_x$ associated with x can be seen as the CAR algebra generated by the following d LFMs:

$$\{b_{x,k}, b_{x,k}^\dagger \mid k = 1, \dots, d\}.$$

When $\Lambda \subset \mathbb{G}$ is a finite subset, the algebra of Λ is simply denoted by $\mathcal{A}(\Lambda) = \bigotimes_{x\in\Lambda} \mathcal{A}_x$, where $\otimes$ is a suitable tensor product. Furthermore, recalling Def. 3.6, we call $O \in \mathcal{A}_x^p$ an even operator if $p = 0$, otherwise an odd operator if $p = 1$.

Remark 3.10. It is evident that $\{b_{x,j}, b_{y,k}\} = 0$ implies and is implied by $\{b_{x,j}^\dagger, b_{y,k}^\dagger\} = 0$, for any pair of cells x, y.

Hereafter, when we do not need to specify the cell label x, we will simply drop it, yielding b_k and $b_k^\dagger$ for the annihilation and creation operators, with $k = 1, \dots, d$ as the fermionic index.

Proposition 3.3. *For any fermionic index $k = 1, \dots, d$, it holds that*

$$(b_k)^2 = 0 \text{ and } \left(b_k^\dagger\right)^2 = 0.$$

Proof. The proof is a consequence of the canonical anticommutation relations, that is,

$$0 = \{b_k, b_k\} = 2b_k^2.$$

Thus, $(b_k)^2 = 0$ for any fermionic mode k. It follows straightforwardly that $\left(b_k^\dagger\right)^2 = 0$. □

Definition 3.24. It is convenient to define the following new operators, referred to as *Majorana operators*:

$$X_k := b_k^\dagger + b_k \in A^1;$$
$$Y_k := i\left(b_k^\dagger - b_k\right) \in A^1.$$

Remark 3.11. Many authors – see for example [9] – identify $X_k = \gamma_{2k}$ and $Y_k = \gamma_{2k+1}$, splitting the index set in two subsets, even and odd.

Definition 3.25. For practical purposes, it is convenient to introduce the following operator:

$$Z_k := b_k b_k^\dagger - b_k^\dagger b_k \in A^0.$$

Proposition 3.4. *The operator in Def. 3.25 can be expressed as follows:*

$$Z_k = (-1)^{n_k},$$

where $n_k := b_k^\dagger b_k$ *is the fermionic number operator.*

Proof. Since $b_k b_k^\dagger = 1 - b_k^\dagger b_k$, we have

$$Z_k = 1 - 2b_k^\dagger b_k = 1 - 2n_k = (-1)^{n_k},$$

where $1 - 2n_k = (-1)^{n_k}$ as $n_k = 0, 1$. □

Remark 3.12. We will refer to $Z_k = (-1)^{n_k}$ as the *local fermionic parity operator*.

Proposition 3.5. *The local fermionic parity operator can be also written in terms of the Majorana operators, namely*

$$Z_k = -iX_k Y_k.$$

Proof. From Def. 3.24, it is possible to write the following expression:

$$\begin{aligned} X_k Y_k &= \left(b_k^\dagger + b_k\right) i \left(b_k^\dagger - b_k\right) = i\left[\left(b_k^\dagger\right)^2 - b_k^\dagger b_k + b_k b_k^\dagger - (b_k)^2\right] = \\ &= i\left(b_k b_k^\dagger - b_k^\dagger b_k\right), \end{aligned}$$

where the squared terms vanish according to Prop. 3.3. Consequently, we have

$$-iX_k Y_k = \left(b_k b_k^\dagger - b_k^\dagger b_k\right) = Z_k,$$

for any fermionic mode k. □

Let us introduce some fundamental properties of the Majorana operators.

Proposition 3.6. *The Majorana operators are self-adjoint, namely*

$$\gamma_k = \gamma_k^\dagger,$$

where $\gamma_k = X_k, Y_k$.

Proof. The proof is straightforward. □

Proposition 3.7. *The Majorana operators satisfy the following CAR:*

$$\{\gamma_i;\gamma_j\} = 2\delta_{ij}\mathbb{I}.$$

Proof. The proof follows directly from Def. 3.24. □

Corollary 3. For any fixed fermionic index k, the Majorana operators are involutory, that is, square to the identity.

Proof. It suffices to adapt the Majorana CAR for $i = j$. □

Remark 3.13. The latter result holds also for Z_k. Nevertheless, Z_k is not a Majorana operator, since it does not satisfy the Majorana CAR for different fermionic index $i \neq j$.

Proposition 3.8. *It holds that*

$$X_k Y_k = -\mathbb{I}.$$

Proof. Since $\{X_k;Y_k\} = 0$ from the Majorana CAR, we have that

$$\begin{aligned} X_k Y_k &= -Y_k X_k \\ (X_k Y_k)^2 &= -Y_k X_k X_k Y_k = -\mathbb{I}. \end{aligned}$$

□

Finally, it is possible to express the Majorana operator via the Jordan-Wigner transformation, as follows:

$$\begin{aligned} X_k &= \left(\prod_{j<k} Z_j^{(Pauli)}\right) X_k^{(Pauli)}, \\ Y_k &= \left(\prod_{j<k} Z_j^{(Pauli)}\right) Y_k^{(Pauli)}, \end{aligned}$$

where the label $(Pauli)$ denotes a Pauli operator.

3.2 The Mathematical Structure of Fermionic Gaussianity

In order to construct the QRT of fermionic non-Gaussianity, we now introduce the mathematical structure of fermionic Gaussianity. This section is primarily based on [22, 31] for the mathematical content and on [28] for the physical one. When needed, the notation has been adapted for the sake of consistency and clarity.

3.2.1 The Majorana Complex Clifford Algebra

The following section presents a summary description of the algebra generated by the Majorana operators. For further details and formal results, we refer the reader to [31].

Definition 3.26. Let V be a vector space equipped with a non-degenerate symmetric bilinear form $\delta : V \times V \to \mathbb{R}$.

Remark 3.14. Let us specify the defining properties of $\delta : V \times V \to \mathbb{R}$:

- (Bilinearity) Linearity with respect to both arguments;
- (Symmetry) $\delta(u, v) = \delta(v, u)$, for any $u, v \in V$;
- (Non-degeneracy) If $\delta(u, v) = 0 \; \forall v \in V$, then $u = 0$.

Definition 3.27. A Clifford algebra $\mathscr{C}\ell(V, \delta)$ is an associative algebra over $\mathbb{R}$ equipped with a linear map $l : V \to \mathscr{C}\ell(V, \delta)$ such that

$$l^2(v) = \delta(v, v)\mathbb{I},$$

and moreover, given any other unital associative algebra $\mathscr{A}$ over $\mathbb{R}$ equipped with a linear map $j : V \to \mathscr{A}$ such that $m^2(v) = \delta(v, v)\mathbb{I}$, there exists a unique algebra homomorphism $f : \mathscr{C}\ell(V, \delta) \to \mathscr{A}$ such that

$$f \circ l = m.$$

We are now able to adapt Def. 3.27 to the Majorana operators framework.

Definition 3.28. The vector space V can be identified as the real span of $2n$ Majorana operators, namely

$$V \equiv G_{2n} := \mathrm{span}_{\mathbb{R}}(\gamma_1, \dots, \gamma_{2n}) \cong \mathbb{R}^{2n}.$$

The elements of G_{2n} are given by

$$v = \sum_i^{2n} v^i \gamma_i,$$

where $v^i \in \mathbb{R}$ represent the coefficients associated with the Majorana operators γ_i.

Remark 3.15. Since $\{\gamma_i\}$ form trivially an orthonormal basis of G_{2n}, it holds that $\delta(\gamma_i, \gamma_j) = \delta_{ij}$, and $l(\gamma_i) = \gamma_i$. This also means that

$$l^2(v) = l(v)l(v) = vv = v^2 = \delta(v, v)\mathbb{I},$$

for any $v \in G_{2n}$.

Proposition 3.9. *The non-degenerate symmetric bilinear form $\delta : G_{2n} \times G_{2n} \to \mathbb{R}$ is the standard Euclidean inner product.*

Proof. Let us consider $u, v \in G_{2n}$, where $u = \sum_i^{2n} u^i \gamma_i$ and $v = \sum_j^{2n} v^j \gamma_j$ – it trivially follows that $u + v \in G_{2n}$. Moreover, bilinearity and symmetry of δ guarantee that

$$\delta(u+v, u+v) = \delta(u,u) + 2\delta(u,v) + \delta(v,v),$$

that is,

$$\delta(u,v) = \frac{1}{2}\left[\delta(u+v, u+v) - \delta(u,u) - \delta(v,v)\right].$$

Since $\delta(\gamma_i, \gamma_j) = \delta_{ij}$, it follows that

$$\delta(u,u) = \delta\left(\sum_i^{2n} u^i \gamma_i, \sum_j^{2n} u^j \gamma_j\right) = \sum_i^{2n}\sum_j^{2n} u^i u^j \delta(\gamma_i, \gamma_j) = \sum_i^{2n} (u^i)^2,$$

and similarly

$$\delta(v,v) = \sum_i^{2n} (v^i)^2,$$

$$\delta(u+v, u+v) = \sum_i^{2n} \left[(u^i)^2 + 2u^i v^+ (v^i)^2\right].$$

Then, we have

$$\delta(u,v) = \sum_i^{2n} u^i v^i,$$

that is, the form δ is the standard Euclidean inner product. □

We are now able to construct the Majorana Clifford algebra.

Definition 3.29. The Majorana Clifford algebra $\mathcal{C}\ell(G_{2n}, \delta)$ is a Clifford algebra over $\mathbb{R}$ where δ denotes the standard Euclidean inner product.

Remark 3.16. For practical purposes, the real Majorana Clifford algebra generated by $2n$ Majorana operators will be denoted by $\mathcal{C}\ell_{2n}(\mathbb{R})$.

Proposition 3.10. *The Majorana CAR naturally arise from the structure of the Majorana Clifford algebra.*

Proof. For the Majorana Clifford algebra $\mathscr{C}\ell(G_{2n}, \delta)$, it holds that

$$v^2 = \delta(v, v)\mathbb{I},$$

for any $v \in G_{2n}$. This must be true also for $v = \gamma_i + \gamma_j$, yielding

$$(\gamma_i + \gamma_j)^2 = \delta(\gamma_i + \gamma_j, \gamma_i + \gamma_j)\mathbb{I}.$$

On one hand, Prop. 3.9 implies that

$$\delta(\gamma_i + \gamma_j, \gamma_i + \gamma_j) = \delta(\gamma_i, \gamma_i) + 2\delta(\gamma_i, \gamma_j) + \delta(\gamma_j, \gamma_j) = 2 + 2\delta_{ij},$$

that is,

$$(\gamma_i + \gamma_j)^2 = (2 + 2\delta_{ij})\mathbb{I}.$$

On the other hand, we have that

$$(\gamma_i + \gamma_j)^2 = \gamma_i^2 + \gamma_i\gamma_j + \gamma_j\gamma_i + \gamma_j^2 = 2\mathbb{I} + \{\gamma_i; \gamma_j\}.$$

Thus, the CAR hold:

$$\{\gamma_i; \gamma_j\} = 2\delta_{ij}.$$

□

In order to properly work within the fermionic framework, the complexification of the Majorana real Clifford algebra is needed, namely

$$\mathscr{C}\ell_{2n}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathscr{C}\ell_{2n}(\mathbb{C}).$$

Hereafter, in order to justify the employment of $\mathbb{Z}_2$-graded algebras, we present some fundamental results shown in [52]. Recalling Def. 3.4, the Majorana complex Clifford algebra is also a $\mathbb{Z}_2$-graded $\mathbb{C}$-algebra, namely

$$\mathscr{C}\ell_{2n}(\mathbb{C}) = \mathscr{C}\ell^0_n(\mathbb{C}) \sqcup \mathscr{C}\ell^1_n(\mathbb{C}),$$

where both the even subalgebra and the odd subalgebra has dimension equal to 2^{n-1}. Furthermore, if an even number $2n$ of Majorana generators are considered, the following isomorphism holds:

$$\mathscr{C}\ell_{2n}(\mathbb{C}) \cong \mathfrak{m}(\mathbb{C}^{2^{2n-1}|2^{2n-1}}),$$

where $\mathfrak{m}(\mathbb{C}^{p|q}) := \mathfrak{m}(\mathbb{C}^{p|q})^0 \sqcup \mathfrak{m}(\mathbb{C}^{p|q})^1$ is the matrix algebra, which is a $\mathbb{Z}_2$-graded $\mathbb{C}$-algebra. Finally, for the sake of completeness, let us stress how the $\mathbb{Z}_2$-grading highlights the superselection structure – for further details, we refer the reader to [52].

Definition 3.30. Let $\mathcal{S}$ be a $\mathbb{Z}_2$-graded $\mathbb{C}^*$-algebra. A state is a linear map $\omega : \mathcal{S} \to \mathbb{C}$ such that $\omega(S^\dagger S) \geq 0$ for any $S \in \mathcal{S}$, and $\omega(\mathbb{I}_{\mathcal{S}}) = 1$.

Definition 3.31. Let $\mathcal{S}$ be a $\mathbb{Z}_2$-graded $\mathbb{C}^*$-algebra. A superselected state is a state such that $\omega(\mathcal{S}^p) \subseteq \mathbb{C}^{p\oplus 1}$.

Remark 3.17. Both $\mathcal{C}\ell_{2n}(\mathbb{C})$ and $\mathfrak{m}(\mathbb{C}^{p|q})$ are $\mathbb{C}^*$-algebras. In fact, they can be endowed with the operator norm $\|S\| := \sup_\omega \sqrt{\omega(S^\dagger S)}$, where $S \in \mathcal{S}$ and ω is a superselected state.

Remark 3.18 ([52]). The formulation of $\mathbb{Z}_2$-grading based on disjoint unions can be justified by the fact that the $\mathbb{Z}_2$-grading is able to distinguish superselection sectors corresponding to even (bosonic) and odd (fermionic) elements, which cannot be coherently superposed.

3.2.2 The Majorana Lie Algebra

From the structure of the Majorana complex Clifford algebra, it is possible to construct the so-called *Majorana Lie algebra*. We recall Def. 3.1, but Lie algebras will be denoted by Fraktur symbols, *e.g.*, $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$, and not by calligraphic symbols as is done for generic algebras.

Definition 3.32. A Lie algebra $\mathfrak{l}$ is an algebra over a field $\mathbb{K}$ equipped with a binary operation $[\ ,\ ] : \mathfrak{l} \times \mathfrak{l} \to \mathfrak{l}$ called the Lie bracket, satisfying the following conditions:

- (Bilinearity) $[ax + by, z] = a[x, z] + b[y, z]$ and $[z, ax + by] = a[z, x] + b[z, y]$, for any $a, b \in \mathbb{K}$ and $x, y, z \in \mathfrak{l}$;
- (Alternating property) $[x, x] = 0$, for any $x \in \mathfrak{l}$;
- (Jacobi identity) $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$, for any $x, y, z \in \mathfrak{l}$.

Proposition 3.11. *Given a Lie algebra $\mathfrak{l}$, the anticommutativity property holds, namely*

$$[x, y] = -[y, x],$$

for any $x, y \in \mathfrak{l}$.

Proof. Let us consider the following Lie bracket: $[x + y, x + y]$. It can be expressed both via the alternating property and via bilinearity:

$$\begin{aligned} 0 = [x + y, x + y] &= [x, x] + [x, y] + [y, x] + [y, y] = \\ &= [x, y] + [y, x], \end{aligned}$$

that is, $[x, y] = -[y, x]$. □

Let $\{\gamma_i\}_{i=1}^{2n}$ be the set of $2n$ Majorana operators, that is, the generators of the Majorana complex Clifford algebra.

Definition 3.33. A quadratic Majorana operator is expressed as the linear combination of products of two Majorana operators, namely

$$\gamma_{\blacksquare} = \sum_{i,j=1}^{2n} c_{ij}\gamma_i\gamma_j.$$

Proposition 3.12. *The matrix C with coefficients c_{ij} in Def. 3.33 is self-adjoint.*

Proof. Any quadratic Majorana operator is self-adjoint by definition, namely

$$\gamma_{\blacksquare} = \gamma_{\blacksquare}^{\dagger}$$
$$\sum_{i,j=1}^{2n} c_{ij}\gamma_i\gamma_j = \sum_{i,j=1}^{2n} c_{ij}^*\gamma_j^\dagger\gamma_i^\dagger.$$

Let us swap indexes i, j in the second term, yielding

$$\sum_{i,j=1}^{2n} c_{ij}\gamma_i\gamma_j = \sum_{i,j=1}^{2n} c_{ji}^*\gamma_i^\dagger\gamma_j^\dagger$$
$$\sum_{i,j=1}^{2n} c_{ij}\gamma_i\gamma_j = \sum_{i,j=1}^{2n} c_{ji}^*\gamma_i\gamma_j.$$

Then, we have

$$c_{ij} = c_{ji}^*,$$

that is,

$$C = C^\dagger.$$

□

Proposition 3.13. *The matrix C is purely imaginary.*

Proof. When $i \neq j$, the Majorana CAR implies that

$$\gamma_i\gamma_j = -\gamma_j\gamma_i.$$

Given indexes $i, j = 1, \ldots, 2N$, let us express a quadratic Majorana operator as follows:

$$\gamma_{\blacksquare} = \sum_{i=j} c_{ii}\gamma_i^2 + \sum_{i\neq j} c_{ij}\gamma_i\gamma_j.$$

Since $\gamma_i^2 = \mathbb{I}$, the first term is simply a negligible shift factor, namely

$$\sum_{i=j} c_{ii}\gamma_i^2 = \sum_{i=j} c_{ii}\mathbb{I} = \mathrm{Tr}\{A\mathbb{I}\}.$$

Moreover, let us decompose matrix C into the sum of its symmetric and skew-symmetric parts, that is,

$$C = S + W,$$

where $S := \frac{C+C^{\mathrm{T}}}{2}$ is the symmetric part and $W := \frac{C-C^{\mathrm{T}}}{2}$ is the skew-symmetric part. Then, we have

$$\sum_{i\neq j} c_{ij}\gamma_i\gamma_j = \sum_{i\neq j} s_{ij}\gamma_i\gamma_j + \sum_{i\neq j} w_{ij}\gamma_i\gamma_j.$$

Note that

$$\sum_{i\neq j} s_{ij}\gamma_i\gamma_j = \sum_{j\neq i} s_{ji}\gamma_j\gamma_i = -\sum_{i\neq j} s_{ji}\gamma_i\gamma_j,$$

that is, $s_{ij} = -s_{ji}$. Since $s_{ij} = s_{ji}$ due to symmetry, it follows that $s_{ji} = -s_{ji}$, *i.e.*, the matrix S vanishes, and the matrix C is skew-symmetric. Hence, C is purely imaginary. □

For practical purposes, let us express the matrix C as

$$C = \frac{i}{2}A,$$

where A is a real matrix, and the factor $\frac{1}{2}$ avoids double counting over the indexes i, j. Consequently, any quadratic Majorana operator can be written as a real linear combination of products of two Majorana operators, namely

$$\gamma_{\blacksquare} = \frac{i}{2}\sum_{i,j=1}^{2N} a_{ij}\gamma_i\gamma_j = i\sum_{i<j} a_{ij}\gamma_i\gamma_j.$$

The mathematical structure underlying quadratic Majorana operators is called the Majorana Lie algebra.

Definition 3.34. The Majorana Lie algebra is a Lie algebra spanned by the products of two Majorana operators, namely

$$\mathfrak{m}_{2n}(\mathbb{R}) := \langle \gamma_i\gamma_j \rangle,$$

where $i, j = 1, \ldots, 2n$ with $i < j$.

Definition 3.35. The vector space of real skew-symmetric matrices, equipped with the standard matrix commutator [;], is the following algebra:

$$\mathfrak{so}_{2n}(\mathbb{R}) := \{A \in \mathbb{M}_{2n\times 2n}(\mathbb{R}) \mid A^{\mathrm{T}} = -A\}.$$

In particular, $\mathfrak{so}_{2n}(\mathbb{R})$ is a Lie algebra, where the Lie bracket is the standard matrix commutator.

We can formalize the relationship between these two structures through the following property:

Proposition 3.14. *The Majorana Lie algebra is isomorphic to the Lie algebra of real skew-symmetric matrices, namely*

$$\mathfrak{m}_{2n}(\mathbb{R}) \cong \mathfrak{so}_{2n}(\mathbb{R}).$$

3.2.3 The Majorana Lie Group

We refer the reader to [38] for a general introduction to topology, and to [32] for a rigorous discussion on smooth manifolds.

Definition 3.36. A Lie group $\mathcal{L}$ is both a group and a smooth manifold, in which the group operations of multiplication and inversion, that is,

$$\begin{aligned} *:\ \mathcal{L}\times\mathcal{L} &\to \mathcal{L} \\ (a,b) &\mapsto a*b \end{aligned} \quad \text{and} \quad \begin{aligned} \mathrm{inv}:\ \mathcal{L} &\to \mathcal{L} \\ a &\mapsto a^{-1} \end{aligned},$$

are smooth.

A Lie group can be constructed from a Lie algebra by means of the so-called *exponential map* – this procedure is called *exponentiation*.

Definition 3.37. Let $\mathfrak{l}$ be a Lie algebra and $\mathcal{L}$ be a Lie group. For any $l \in \mathfrak{l}$, let $\phi_l : \mathbb{R} \to \mathcal{L}$ be a smooth Lie group homomorphism such that $\phi_l(0) = e$, where e is the identity of $\mathcal{L}$, and $\dot{\phi}_l(0) = L$. The exponential map $\exp : \mathfrak{l} \to \mathcal{L}$ is defined as follows:

$$\exp(l) := \phi_l(1).$$

For practical purposes, let us specialize Def. 3.37 to the framework of the Lie algebra of real skew-symmetric matrices.

Definition 3.38. The spin group, denoted by $\mathrm{Spin}(2n)$, is a Lie group defined as follows:

$$\mathrm{Spin}(2n) := \{s \in \mathscr{C}\ell_{2n}(\mathbb{R}) \mid s = v_1 v_2 \dots v_{2k}, v_i \in \mathbb{R}^{2n}, |v_i| = 1\}.$$

Proposition 3.15. *The exponential map relates the Lie algebra of real skew-symmetric matrices to the spin group, namely*

$$\exp_{\mathsf{Spin}} : \mathfrak{so}_{2n}(\mathbb{R}) \to \mathsf{Spin}(2n).$$

Proof. The proof can be found in [31]. □

Let us now introduce the Lie group of special orthogonal matrices, denoted by $\mathcal{SO}_{2n}(\mathbb{R})$.

Definition 3.39. The special orthogonal group is a Lie group defined as follows:

$$\mathcal{SO}_{2n}(\mathbb{R}) := \{O \in \mathbb{M}_{2n\times 2n}(\mathbb{R}) \mid O^{\mathrm{T}}O = \mathbb{I}, \det(O) = 1\}.$$

Proposition 3.16. *The exponential map relates the Lie algebra of real skew-symmetric matrices to the Lie group of special orthogonal matrices, namely*

$$\exp_{so} : \mathfrak{so}_{2n}(\mathbb{R}) \to \mathcal{SO}_{2n}(\mathbb{R}),$$

Proof. Let $A \in \mathfrak{so}_{2n}(\mathbb{R})$ and let us denote the action of the exponential map exp upon A as follows:

$$O := e^{A}.$$

The matrix O is orthogonal:

$$O^{\mathrm{T}} = \left(e^{A}\right)^{\mathrm{T}} = e^{A^{\mathrm{T}}} = e^{-A} = \left(e^{A}\right)^{-1} = O^{-1},$$

that is,

$$O^{\mathrm{T}}O = \mathbb{I}.$$

Finally, the matrix O is also special orthogonal:

$$\det(O) = \det\left(e^{A}\right) = e^{\mathrm{Tr}\{A\}} = e^{0} = 1.$$

Then, $O \in \mathcal{SO}_{2n}(\mathbb{R})$. □

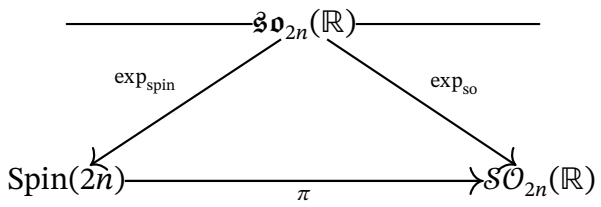
Proposition 3.17. *The spin group* $\mathsf{Spin}(2n)$ *is the (two-sheeted) universal covering space of the Lie group* $\mathcal{SO}_{2n}(\mathbb{R})$*. The two Lie groups are linked through the following universal covering map:*

$$\pi : \mathsf{Spin}(2n) \to \mathcal{SO}_{2n}(\mathbb{R}),$$

with $\mathsf{Ker}(\pi) = \{\pm 1\}$.

Proof. The proof can be found in [22]. □

Theorem 3.1. *The following diagram commutes:*



Equivalently,

$$\pi \circ \exp_{\text{spin}} = \exp_{\text{so}} .$$

Proof. An exhaustive and rigorous proof can be found in [31]. □

We are now able to focus on the physical framework. In particular, we recall Def. 3.33 and Prop. 3.14.

Definition 3.40. Let $\gamma_{\blacksquare}$ be a quadratic Majorana operator. A fermionic Gaussian unitary is defined as follows:

$$U(A) := e^{-i\gamma_{\blacksquare}} = \exp\left(\sum_{i<j} a_{ij}\gamma_i\gamma_j\right) = \exp\left(\frac{1}{2}\sum_{i\neq j} a_{ij}\gamma_i\gamma_j\right),$$

where the matrix A with coefficients a_{ij} is real and skew-symmetric. The set of fermionic Gaussian unitaries will be called the *Majorana Lie group* and will be denoted by FGU.

Remark 3.19. The operator $U(A)$ is unitary by construction, since any quadratic Majorana operator is self-adjoint.

Let us introduce the following first result concerning the fermionic Gaussian unitaries:

Proposition 3.18. *For any $U(A) \in$ FGU and any Majorana operator γ_j, it holds that*

$$U^\dagger(A)\gamma_j U(A) = \sum_{k=1}^{2n} o_{jk}\gamma_k,$$

where the matrix O with coefficients o_{jk} is such that

$$O = e^{2A} \in \mathcal{SO}_{2n}(\mathbb{R}).$$

Proof. Given $p, q = 1, \dots, 2n$ such that $p \neq q$, let us define the following one-parameter family:

$$U(t) := \exp\left(\frac{t}{2}\sum_{p,q=1}^{2n} a_{pq}\gamma_p\gamma_q\right).$$

The adjoint action can be expressed via the following function:

$$\chi_j(t) := U^\dagger(t)\gamma_j U(t),$$

such that $\chi_j(0) = \gamma_j$. The derivative with respect to the parameter t is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\chi_j(t) = U^\dagger(t)\left[-\frac{1}{2}\sum_{p,q=1}^{2n} a_{pq}\gamma_p\gamma_q;\gamma_j\right]U(t). \tag{3.2}$$

The Majorana CAR guarantee that

$$[\gamma_p\gamma_q;\gamma_j] = 2\delta_{qj}\gamma_p - 2\delta_{pj}\gamma_q. \tag{3.3}$$

Substituting Eq. 3.3 in Eq. 3.2 and employing skew-symmetry of the matrix A, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\chi_j(t) = 2\sum_k a_{kj}\chi_k(t). \tag{3.4}$$

Since $\chi_j(0) = \gamma_j$, the solution of Eq. 3.4 is

$$\chi_j(t) = \sum_k \left(e^{2tA}\right)_{kj}\gamma_k,$$

that is, for $t = 1$,

$$\chi_j(1) = U^\dagger(A)\gamma_j U(A) = \sum_k \left(e^{2A}\right)_{kj}\gamma_k,$$

that is, $O = e^{2A}$. Finally, $O \in \mathcal{SO}_{2n}(\mathbb{R})$ from Prop. 3.16. □

Remark 3.20. The factor 2 in the expression for $O = e^{2A}$ is a direct consequence of the choices made concerning the definition of the Majorana operators. Some authors, for example in [49], define the Majorana operators with a factor $\frac{1}{\sqrt{2}}$ in order to yield $O = e^A$.

Definition 3.41. The action of $U(A) \in \mathsf{FGU}$ produces a rotated Majorana operator, expressed as

$$\widetilde{\gamma}_j := U^\dagger(A)\gamma_j U(A) = \sum_{k=1}^{2n} o_{jk}\gamma_k.$$

Proposition 3.19. *Any rotated Majorana operator is a Majorana operator.*

Proof. It follows straightforwardly that

$$\widetilde{\gamma}_j = \widetilde{\gamma}_j^\dagger$$

and

$$\{\widetilde{\gamma}_i;\widetilde{\gamma}_j\} = 2\delta_{ij}\mathbb{I}.$$

□

Note that Prop. 3.18 induces a well-defined group homomorphism, namely

$$\phi : \mathsf{FGU} \to \mathcal{SO}_{2n}(\mathbb{R})$$

where

$$U(A) \mapsto \phi(U(A)) = e^{2A}.$$

Since both $U(A)$ and $-U(A)$ yield the same adjoint action, we have that

$$\mathsf{Ker}(\phi) = \{\pm 1\}.$$

The following theorem, whose proof is given in [22], makes this fact precise.

Theorem 3.2. *Any fermionic Gaussian unitary defined modulo phases can be seen as a special orthogonal matrix with real coefficients, namely*

$$\mathsf{FGU}/\{\pm 1\} \cong \mathcal{SO}_{2n}(\mathbb{R}).$$

Remark 3.21. Furthermore, it also holds that

$$\mathsf{FGU} \cong \mathsf{Spin}(2n).$$

Thus, the adjoint action ϕ of Prop. 3.18 and the universal covering map π of Prop. 3.17 are the same entity, namely

$$\phi \equiv \pi.$$

Remark 3.22. Although $\mathrm{Spin}(2n)$ plays an important role within the mathematical framework of fermionic Gaussianity, hereafter we will focus only on $\mathcal{SO}_{2n}(\mathbb{R}) \cong \mathsf{FGU}/\{\pm 1\}$, that is, fermionic Gaussian unitaries as in Def. 3.40 up to phases.

Chapter 4

The Quantum Resource Theory of Fermionic Non-Gaussianity

In the following chapter, we manage to extend the notion of quantum magic to the framework of fermionic non-Gaussianity, in order to construct an appropriate quantum resource theory.

4.1 What is Fermionic Non-Gaussianity?

Interest towards fermionic Gaussianity has arisen since the foundational works by E. Knill ([29]), S. Bravyi ([8]), and R. Jozsa and A. Miyake ([28]) in the framework of Fermionic Linear Optics (FLO). Like stabilizerness in the qubit framework, fermionic Gaussianity also plays an important role with respect to fermionic computation. According to the simulability theorem for free-fermion systems (see [8, 29, 51, 54]), any circuit acting only on nearest-neighbor qubits and composed of fermionic Gaussian unitaries is efficiently simulable by a deterministic classical computer, that is,

$$\text{Fermionic Gaussianity} \implies \text{Classical Simulability (Gauss)}.$$

This automatically means that

$$\text{Non-Classical Simulability (Gauss)} \implies \text{Fermionic Non-Gaussianity},$$

that is, every non-classically simulable fermionic state (in the Gaussian sense) must be a fermionic non-Gaussian state.

4.1.1 Universal Gate Set for Fermionic Computation

A universal gate set for fermionic computation was originally provided in [9], namely

$$\left\{ \begin{array}{l} \mathrm{CT} = \exp\left(i\frac{\pi}{4} b_0^\dagger b_0\right) \\ \mathrm{CZ} = \exp\left(i\pi b_0^\dagger b_0 b_1^\dagger b_1\right) \\ \mathrm{G} = \exp\left(-i\frac{\pi}{4}\left(b_0 - b_0^\dagger\right)\left(b_1 + b_1^\dagger\right)\right) \end{array} \right\}, \tag{4.1}$$

where CT and CZ are the fermionic counterparts of the qubit gates controlled-T and controlled-Z, respectively, and G is related to the fermionic version of the Hadamard gate through the relation $\mathrm{H} = \left(\mathrm{CS}^\dagger\right)\mathrm{G}\left(\mathrm{CS}^\dagger\right)$. Furthermore, it is possible to express Eq. 4.1 through the Majorana operators – despite the results being shown in [9], here we provide an independent step-by-step derivation.

Theorem 4.1. *The universal gate set (up to phases) for fermionic computation is given by*

$$\left\{ \begin{array}{l} \mathrm{CT} = \exp\left(\frac{\pi}{8} X_0 Y_0\right) \\ \mathrm{F4} = \exp\left(i\frac{\pi}{4} X_0 Y_0 X_1 Y_1\right) \end{array} \right\}.$$

Hereafter, we refer to the gate denoted by F4 *as the Fermionic-Four gate.*

Proof. The following relations can be immediately derived by Def. 3.24:

$$\begin{aligned} b_k &= \frac{1}{2}\left(X_k + iY_k\right), \\ b_k^\dagger &= \frac{1}{2}\left(X_k - iY_k\right), \end{aligned}$$

where $k = 0, 1$ in our framework. Then, let us compute the following term:

$$\begin{aligned} b_0^\dagger b_0 &= \frac{1}{4}\left(X_0 - iY_0\right)\left(X_0 + iY_0\right) = \frac{1}{4}\left(2\mathbb{I} + iX_0Y_0 - iY_0X_0\right) = \\ &= \frac{1}{4}\left(2\mathbb{I} + 2iX_0Y_0\right) = \frac{1}{2}\left(\mathbb{I} + iX_0Y_0\right). \end{aligned}$$

This means that

$$\begin{aligned} \mathrm{CT} &= \exp\left(i\frac{\pi}{4} b_0^\dagger b_0\right) = \exp\left(i\frac{\pi}{4}\frac{1}{2}(\mathbb{I} + iX_0Y_0)\right) = \\ &= \exp\left(i\frac{\pi}{8}(\mathbb{I} + iX_0Y_0)\right) \propto \exp\left(\frac{\pi}{8} X_0Y_0\right). \end{aligned}$$

Moreover, since

$$\left(b_0 - b_0^\dagger\right)\left(b_1 + b_1^\dagger\right) = iY_0X_1,$$

we have that

$$G = \exp\left(-i\frac{\pi}{4}\left(b_0 - b_0^\dagger\right)\left(b_1 + b_1^\dagger\right)\right) = \exp\left(\frac{\pi}{4}Y_0X_1\right).$$

Finally, let us bring CZ to an explicit Majorana form. Since

$$\begin{aligned} b_0^\dagger b_0 b_1^\dagger b_1 &= \frac{1}{16}(X_0 - iY_0)(X_0 + iY_0)(X_1 - iY_1)(X_1 + iY_1) = \\ &= \frac{1}{16}(2\mathbb{I} + 2iX_0Y_0)(2\mathbb{I} + 2iX_1Y_1) = \\ &= \frac{1}{4}(\mathbb{I} + iX_0Y_0 + iX_1Y_1 - X_0Y_0X_1Y_1), \end{aligned}$$

it follows that

$$\begin{aligned} \mathrm{CZ} &= \exp\left(i\pi b_0^\dagger b_0 b_1^\dagger b_1\right) = \exp\left(i\frac{\pi}{4}(\mathbb{I} + iX_0Y_0 + iX_1Y_1 - X_0Y_0X_1Y_1)\right) = \\ &= \exp\left(-i\frac{\pi}{4}\right)\exp\left(\frac{\pi}{4}X_0Y_0\right)\exp\left(\frac{\pi}{4}X_1Y_1\right)\exp\left(i\frac{\pi}{4}X_0Y_0X_1Y_1\right) \propto \\ &\propto \exp\left(\frac{\pi}{4}X_0Y_0\right)\exp\left(\frac{\pi}{4}X_1Y_1\right)\exp\left(i\frac{\pi}{4}X_0Y_0X_1Y_1\right), \end{aligned}$$

where we define the Four-Fermionic gate as $\mathrm{F4} := \exp\left(i\frac{\pi}{4}X_0Y_0X_1Y_1\right)$. Since $\exp(\phi X_0Y_0)$ for an arbitrary parameter ϕ can be generated by CT, both G and the two-Majorana factors in CZ are redundant. Thus, up to phases, the universal gate set for fermionic computation reduces to the pair

$$\left\{\begin{array}{l} \mathrm{CT} = \exp\left(\frac{\pi}{8}X_0Y_0\right) \\ \mathrm{F4} = \exp\left(i\frac{\pi}{4}X_0Y_0X_1Y_1\right) \end{array}\right\}. \tag{4.2}$$

□

4.2 Basics of Algebra of Skew-Symmetric Forms

In order to better analyze the following section about the fermionic Gaussian states, we need a few fundamental notions about skew-symmetric matrices – we refer the reader to [25] for further details.

Definition 4.1. An $m \times m$ matrix A is skew-symmetric if

$$A^{\mathrm{T}} = -A.$$

Remark 4.1. It is straightforward to show that any real skew-symmetric matrix admits pure imaginary eigenvalues of the form $\pm i\lambda$, where $\lambda \in \mathbb{R}_0^+$ – note that 0 might be a valid eigenvalue.

Lemma 4.2. *Let A be a skew-symmetric matrix. Then, A is normal, that is,*

$$AA^{\mathrm{T}} = A^{\mathrm{T}}A.$$

Proof. The fact that $A^{\mathrm{T}} = -A$ implies both

$$A^{\mathrm{T}}A = -AA = -A^2 = AA^{\mathrm{T}}.$$

This means that A is normal. □

Lemma 4.3. *Let $A \in \mathbb{M}_{m\times m}(\mathbb{C})$. Then, A is unitarily diagonalizable if and only if A is a normal matrix.*

Proof. Since this is a standard result within the matrix framework, we omit the proof, which can be found in [25]. □

The following theorem guarantees a useful decomposition for any real skew-symmetric matrix A.

Theorem 4.4. *Let A be an $m \times m$ real skew-symmetric matrix. Then, A can be brought into a real block diagonal form via a special orthogonal matrix O, namely*

$$A = O\Sigma O^{\mathrm{T}},$$

where $\Sigma = \bigoplus_{j=1}^{m/2} \left(\begin{smallmatrix} 0 & \lambda_j \\ -\lambda_j & 0 \end{smallmatrix}\right)$ is a real block diagonal matrix, with $\lambda_j \geq 0$.

Proof. For the sake of simplicity, we only provide a sketch of the proof. Since A is normal from Lemma 4.2, then Lemma 4.3 applies to A as follows:

$$A = UDU^{\dagger},$$

where D is a diagonal matrix such that the eigenvalues $\pm i\lambda_j$ of A is situated on the diagonal of D, and U is a unitary matrix. It is possible to adapt this result to the real framework, yielding

$$A = O\Sigma O^{\mathrm{T}}.$$

□

Let us introduce another useful result.

Lemma 4.5. *Let A be a $m \times m$ skew-symmetric matrix. Then*

$$\det(A) = (-1)^m \det(A)$$

Proof. Since A is skew-symmetric, it holds that

$$\det(A) = \det\left(A^{\mathrm{T}}\right) = \det(-A) = (-1)^m \det(A).$$

□

Corollary 4. If A is a $m \times m$ skew-symmetric matrix with m odd, *i.e.*, $m = 2n+1$, then

$$\det(A) = 0.$$

Proof. Since $\det(A) = (-1)^m \det(A)$ and m is odd, we have that

$$\det(A) = -\det(A),$$

that is, $\det(A) = 0$. □

Definition 4.2. Let A be a $m \times m$ skew-symmetric matrix. The Pfaffian of A is defined as follows:

$$\mathrm{Pf}(A) := \begin{cases} \frac{1}{2^n n!} \sum_{\sigma \in \mathcal{S}_{2n}} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1),\sigma(2i)} & \text{if } m = 2n \\ 0 & \text{if } m \text{ is odd} \end{cases},$$

where $\sigma \in \mathcal{S}_{2n}$ is a permutation, and $\mathrm{sgn}(\sigma)$ is the sign of σ.

Hereafter, since the Pfaffian of an odd skew-symmetric matrix is zero, only the even case will be analyzed.

Proposition 4.1. *Let A be a $2n \times 2n$ skew-symmetric matrix. The Pfaffian of A can be expressed in the following form:*

$$\mathrm{Pf}(A) = \sum_{\mu} \mathrm{sgn}(\mu) \prod_{(i,j)\in\mu} a_{ij},$$

where μ is a perfect matching, i.e., a partition of the index set $\{1, \ldots, 2n\}$ into n disjoint pairs of the form (i, j) with $i < j$.

Proof. We refer the reader to the foundational work in [47] for a rigorous proof. □

Proposition 4.2. *Let A be a $2n \times 2n$ block diagonal skew-symmetric matrix, namely*

$$A = \bigoplus_{i=1}^{n} A_i.$$

The Pfaffian of A is the product of the Pfaffians of each diagonal block, that is,

$$\mathrm{Pf}(A) = \prod_{i=1}^{n} \mathrm{Pf}(A_i)$$

Proof. The proof follows straightforwardly from the Def. 4.2 and Prop. 4.1. □

Theorem 4.6. *If A is a $2n \times 2n$ skew-symmetric matrix, then*

$$\det(A) = \mathrm{Pf}^2(A).$$

Proof. Since Corollary 4 holds, only matrices with $m = 2n$ will be analyzed. Recalling Theorem 4.4, the matrix A can be decomposed as $A = O\Sigma O^{\mathrm{T}}$. According to Binet's formula, we have that

$$\det(A) = \det\left(O\Sigma O^{\mathrm{T}}\right) = \det(O)\det(\Sigma)\det\left(O^{\mathrm{T}}\right) = \det(\Sigma).$$

where $\det(O) = \det\left(O^{\mathrm{T}}\right) = 1$ since O is a special orthogonal matrix. Moreover, the determinant of each block of Σ is equal to λ_j^2, for every index $j = 1, \dots, n$. Then, the total determinant is given by the following product:

$$\det(\Sigma) = \lambda_1^2 \lambda_2^2 \dots \lambda_{2n}^2 = (\lambda_1 \lambda_2 \dots \lambda_{2n})^2 .$$

This means that

$$\det(A) = (\lambda_1 \lambda_2 \dots \lambda_n)^2 = \mathrm{Pf}^2(A),$$

where the Pfaffian Pf(A) is equal to $(\lambda_1 \lambda_2 \dots \lambda_n)$ by construction. □

Remark 4.2. Theorem 4.6 trivially holds also within the odd framework. In fact, given an odd skew-symmetric matrix A, Def. 4.2 guarantees that $\mathrm{Pf}(A) = 0$, and Corollary 4 states that $\det(A) = 0$.

4.3 The Resource Theory of Fermionic Non-Gaussianity

We are now able to introduce the QRT of fermionic non-Gaussianity, which is composed of: (1) fermionic Gaussian states as free states;0 (2) fermionic Gaussian operations as free operations, with the fermionic Gaussian unitaries as a special case thereof; (3) the fermionic non-Gaussian states as resource states; (4) a set of fermionic non-Gaussian monotones.

4.3.1 Fermionic Gaussian and Fermionic Non-Gaussian States

The notion of fermionic Gaussianity lies in the definition of fermionic Gaussian states. Although the following section is primarily based on [8, 49], our notation may differ for internal consistency and clarity.

Definition 4.3. A fermionic Gaussian state is defined as

$$\rho = \frac{e^{-\gamma_{\blacksquare}}}{K} = \frac{1}{K}\exp\left(-\frac{i}{2}\sum_{i,j=1}^{2n} a_{ij}\gamma_i\gamma_j\right),$$

where $K := \mathrm{Tr}\left[e^{-\gamma_{\blacksquare}}\right]$ is a normalization factor, $\gamma_{\blacksquare}$ is a quadratic Majorana operator, and A is a skew-symmetric matrix with real coefficients a_{ij}. The set of fermionic Gaussian states will be denoted by Gauss.

Remark 4.3. Throughout this chapter, we restrict to even states, that is, states commuting with the fermionic parity operator. In the parity-preserving framework, odd expectation values vanish and consequently any fermionic Gaussian state is determined by the Majorana two-point correlation matrix. This will be discussed in detail in Subsections 4.3.2 and 4.3.3.

Proposition 4.3. *The matrix A can be decomposed as follows:*

$$A = OA_DO^{\mathrm{T}},$$

where $O \in \mathcal{SO}_{2n}(\mathbb{R})$ can be understood as a fermionic Gaussian unitary (modulo phases), and $A_D = \bigoplus_{k=1}^{n}\left(\begin{smallmatrix} 0 & \alpha_k \\ -\alpha_k & 0 \end{smallmatrix}\right)$ is a real block diagonal matrix, with $\alpha_k \in \mathbb{R}$ for every index k.

Proof. Since A is a real skew-symmetric matrix, Theorem 4.4 holds. □

We are now able to introduce a crucial element in the framework of fermionic Gaussianity, that is, the *correlation matrix*.

Definition 4.4. The correlation matrix Γ is defined as follows:

$$\Gamma_{ij} := \frac{i}{2}\mathrm{Tr}\left[\rho[\gamma_i;\gamma_j]\right].$$

Proposition 4.4. *The correlation matrix Γ is a skew-symmetric matrix with real coefficients and vanishing diagonal.*

Proof. Let us recall the Majorana CAR, that is,

$$\{\gamma_i;\gamma_j\} = 2\delta_{ij}\mathbb{I}.$$

First, we prove that the diagonal of Γ vanishes by reformulating Γ itself as follows:

$$\begin{aligned}\Gamma_{ij} &= \frac{i}{2}\mathrm{Tr}\left[\rho(\gamma_i\gamma_j - \gamma_j\gamma_i)\right] = \frac{i}{2}\mathrm{Tr}\left[2\gamma_i\gamma_j - 2\delta_{ij}\mathbb{I}\right] = \\ &= i\,\mathrm{Tr}\left[\rho\gamma_i\gamma_j\right] - i\delta_{ij} = \begin{cases} i\,\mathrm{Tr}[\rho] - i\delta_{ii} = i - i = 0 & \text{if } i = j \\ i\,\mathrm{Tr}\left[\rho\gamma_i\gamma_j\right] - i\delta_{ij} = i\,\mathrm{Tr}\left[\rho\gamma_i\gamma_j\right] & \text{if } i \neq j \end{cases}.\end{aligned}$$

Moreover, skew-symmetry of Γ is trivial when $i = j$, since $\Gamma_{ii} = 0 = -\Gamma_{ii}$. When $i \neq j$, we have that

$$\Gamma_{ij} = i \operatorname{Tr}\big[\rho\gamma_i\gamma_j\big] = -i \operatorname{Tr}\big[\rho\gamma_j\gamma_i\big] = -\Gamma_{ji},$$

that is, Γ is a skew-symmetric matrix. Finally, reality of Γ is trivial when $i = j$, since $\Gamma_{ii} = 0 \in \mathbb{R}$. When $i \neq j$, we have that

$$\begin{aligned}\Gamma^*_{ij} &= -i \operatorname{Tr}\left[(\rho\gamma_i\gamma_j)^\dagger\right] = -i \operatorname{Tr}\left[\gamma_j^\dagger\gamma_i^\dagger\rho^\dagger\right] = \\ &= -i \operatorname{Tr}\left[\gamma_j\gamma_i\rho\right] = -i \operatorname{Tr}\left[\rho\gamma_j\gamma_i\right] = i \operatorname{Tr}\left[\rho\gamma_i\gamma_j\right] = \Gamma_{ij},\end{aligned}$$

that is, Γ is a real matrix. □

Proposition 4.5. *The correlation matrix Γ can be decomposed as follows:*

$$\Gamma = O\Gamma_D O^{\mathrm{T}},$$

where $O \in \mathcal{SO}_{2n}(\mathbb{R})$ can be understood as a fermionic Gaussian unitary (modulo phases), and $\Gamma_D = \bigoplus_{k=1}^{n} \left(\begin{smallmatrix} 0 & \eta_k \\ -\eta_k & 0 \end{smallmatrix}\right)$ is a real block diagonal matrix, with $\eta_k \geq 0$ for every index k.

Proof. Since Γ is a real skew-symmetric matrix, Theorem 4.4 holds. □

Let us prove that any fermionic Gaussian state is a valid quantum state through the following property:

Proposition 4.6. *The density matrix ρ in Def. 4.3 represents a valid quantum state, that is, ρ is positive semi-definite, self-adjoint, and trace-one.*

Proof. First, let us compute the trace of ρ as follows:

$$\operatorname{Tr}[\rho] = \operatorname{Tr}\left[\frac{e^{-\gamma_\blacksquare}}{K}\right] = \frac{\operatorname{Tr}\left[e^{-\gamma_\blacksquare}\right]}{\operatorname{Tr}\left[e^{-\gamma_\blacksquare}\right]} = 1.$$

Furthermore, self-adjointness of ρ is a direct consequence of the self-adjointness of $\gamma_\blacksquare$. In fact, we have that:

$$\rho^\dagger = \frac{1}{K^*}\left(\exp(-\gamma_\blacksquare)\right)^\dagger = \frac{1}{K^*}\exp\left(-\gamma_\blacksquare^\dagger\right) = \frac{1}{K}\exp(-\gamma_\blacksquare) = \rho,$$

where $K \in \mathbb{R}$, since $K^* = \operatorname{Tr}\left[\exp(-\gamma_\blacksquare)\right]^* = \operatorname{Tr}\left[\exp(-\gamma_\blacksquare)^\dagger\right] = \operatorname{Tr}\left[\exp\left(-\gamma_\blacksquare^\dagger\right)\right] = \operatorname{Tr}\left[\exp(-\gamma_\blacksquare)\right] = K$. Finally, ρ is also positive semi-definite. Since $\gamma_\blacksquare$ is self-adjoint, the following expression holds:

$$\gamma_\blacksquare \left|k\right\rangle = \lambda_k \left|k\right\rangle.$$

Then, we have that

$$\rho\,|k\rangle = \frac{1}{K}\exp\left(-\gamma_{\blacksquare}\right)|k\rangle = \frac{1}{K}\exp(-\lambda_k)\,|k\rangle = \nu_k\,|k\rangle\,,$$

where $\nu_k := \frac{\exp(-\lambda_k)}{K} > 0$, since $\exp(-\lambda_k) > 0$ and $K = \mathrm{Tr}[\exp(-\lambda_k)] > 0$. If we consider $|\psi\rangle = \sum_k c_k\,|k\rangle$, where $c_k = \langle k|\psi|k|\psi\rangle$, then we have that

$$\langle\psi|\,\rho\,|\psi\rangle = \sum_{k,k'} c^*_{k'} c_k\,\langle k'|\,\rho\,|k\rangle = \sum_{k,k'} c^*_{k'} c_k \nu_k\,\langle k'|k|k'|k\rangle = \sum_k |c_k|^2 \nu_k \geq 0,$$

that is, $\rho \geq 0$. □

4.3.2 Wick's Theorem

An explicit form of the non-diagonal elements of the correlation matrix Γ has been given in the proof of Prop. 4.4, namely

$$\Gamma_{ij} = i\,\mathrm{Tr}\left[\rho\gamma_i\gamma_j\right],$$

for $i \neq j$. This means that the correlation matrix Γ encodes the two-point correlations between the Majorana operators γ_i and γ_j. Wick's theorem ensures that higher-order correlations are completely determined by the two-point correlations, that is, by the matrix Γ.

Theorem 4.7. *Let us consider $\rho \in$ Gauss and a set of Majorana operators $\{\gamma_i\}_{i=1}^{2n}$. Then*

$$\mathrm{Tr}\left[\rho\gamma_{q_1}\gamma_{q_2}\ldots\gamma_{q_p}\right] = \mathrm{Pf}\left(\Gamma_{|q_1,q_2,\ldots,q_p}\right),$$

where $1 \leq q_1 < q_2 < \cdots < q_p \leq 2n$, and $\Gamma_{|q_1,q_2,\ldots,q_p}$ is the restriction of the correlation matrix Γ to the monomial of Majorana operators $\{\gamma_j\}_{j=q_1}^{q_p}$.

Proof. An exhaustive proof can be found in the Appendix G in [36], but some examples are provided in [8, 49]. □

Corollary 5. Let $\{\gamma_i\}_{i=1}^{2n}$ a set of Majorana operators. The expectation value of a fermionic Gaussian state ρ with respect to a monomial of even Majorana operators reduces to the product of two-point correlations, namely

$$\mathrm{Tr}[\rho\gamma_{q_1}\ldots\gamma_{q_{2l}}] = \sum_{\mu}\mathrm{sgn}(\mu)\prod_{(i,j)\in\mu} i\,\mathrm{Tr}\left[\rho\gamma_{q_i}\gamma_{q_j}\right],$$

where μ is a perfect matching for $\{q_1,\ldots,q_{2l}\}$, and $1 \leq q_1 < q_2 < \cdots < q_{2l} \leq 2n$.

Proof. Let us express the Pfaffian of the reduced correlation matrix $\Gamma_{|q_1,\dots,q_{2l}}$ through Prop. 4.1:

$$\mathrm{Pf}\left(\Gamma_{|q_1\dots,q_{2l}}\right) = \sum_{\mu} \mathrm{sgn}(\mu) \prod_{(i,j)\in\mu} \Gamma_{q_i q_j}.$$

Since $\Gamma_{q_i q_j} = i\,\mathrm{Tr}\left[\rho\gamma_{q_i}\gamma_{q_j}\right]$ and by virtue of Theorem 4.7, it holds that

$$\mathrm{Tr}\left[\rho\gamma_{q_1}\dots\gamma_{q_{2l}}\right] = \sum_{\mu} \mathrm{sgn}(\mu) \prod_{(i,j)\in\mu} i\,\mathrm{Tr}\left[\rho\gamma_{q_i}\gamma_{q_j}\right].$$

□

Corollary 6. Let us consider a set of Majorana operators $\{\gamma_i\}_{i=1}^n$. The expectation value of a fermionic Gaussian state ρ with respect to a monomial of odd Majorana operators is zero, namely

$$\mathrm{Tr}\left[\rho\gamma_{q_1}\dots\gamma_{q_{2l+1}}\right] = 0,$$

where $1 \le q_1 < q_2 < q_{2l+1} < 2n$.

Proof. First, note that the reduced correlation matrix $\Gamma_{|q_1,\dots,q_{2l+1}}$ is an odd skew-symmetric matrix. By construction (see Def. 4.2), we have that

$$\mathrm{Pf}\left(\Gamma_{|q_1,\dots,q_{2l+1}}\right) = 0.$$

By virtue of Theorem 4.7, it holds that

$$\mathrm{Tr}\left[\rho\gamma_{q_1}\dots\gamma_{q_{2l+1}}\right] = \mathrm{Pf}\left(\Gamma_{|q_1,\dots,q_{2l+1}}\right) = 0.$$

□

Remark 4.4. Corollary 6 corresponds to the superselection rule for fermions.

4.3.3 Representations of Fermionic Gaussian States

It is natural to link the correlation matrix Γ to the matrix A, in order to yield a new expression for any fermionic Gaussian state ρ based only on the correlation matrix. First, let us recall Prop. 4.3 and Prop. 4.5, where the k-th block of Γ_D and A_D is respectively given by

$$(\Gamma_D)_k = \begin{pmatrix} 0 & \eta_k \\ -\eta_k & 0 \end{pmatrix} \text{ and } (A_D)_k = \begin{pmatrix} 0 & \alpha_k \\ -\alpha_k & 0 \end{pmatrix}.$$

The following representation for ρ is provided in [8] and will be called *Bravyi form*, but the step-by-step derivation is independently developed in this thesis work.

Theorem 4.8. *Any* $\rho \in \mathsf{Gauss}$ *can be expressed as follows:*

$$\rho = \frac{1}{2^n} \prod_{k=1}^{n} (\mathbb{I} + i\eta_k X_k Y_k),$$

where $\eta_k = -\tanh(\alpha_k)$*, for every index* $k = 1, \dots, n$*.*

Proof. Without loss of generality, we can evaluate the relation between α_k and η_k directly in the block-diagonal basis where $A = A_D = \bigoplus_{k=1}^{n} \left(\begin{smallmatrix} 0 & \alpha_k \\ -\alpha_k & 0 \end{smallmatrix}\right)$, and $\Gamma = \Gamma_D = \bigoplus_{k=1}^{n} \left(\begin{smallmatrix} 0 & \eta_k \\ -\eta_k & 0 \end{smallmatrix}\right)$. This means that ρ will be expressed through the rotated Majorana operators introduced in Def. 3.41 as follows:

$$\rho = \frac{1}{K} \exp\left(-i \sum_{m=1}^{n} \alpha_m X_m Y_m\right) = \frac{1}{K} \prod_{m=1}^{n} \exp\left(-i\alpha_m X_m Y_m\right) = \frac{1}{K} \prod_{m=1}^{n} E_m, \tag{4.3}$$

where the second equality holds as the elements of different blocks commute, and $E_m := \exp(-i\alpha_m X_m Y_m)$ for practical purposes. Let us expand the exponential E_m in Eq. 4.3 into a Taylor series, recalling that $(X_m Y_m)^2 = -\mathbb{I}$:

$$\begin{aligned}
E_m = \exp(-i\alpha_m X_m Y_m) &= \sum_{n=0}^{+\infty} \frac{(-i\alpha_m)^n}{n!} (X_m Y_m)^n = \\
&= \sum_{k=0}^{+\infty} \frac{(-i\alpha_m)^{2k}}{(2k)!} (X_m Y_m)^{2k} + \sum_{k=0}^{+\infty} \frac{(-i\alpha_m)^{2k+1}}{(2k+1)!} (X_m Y_m)^{2k+1} = \\
&= \sum_{k=0}^{+\infty} \frac{(-i\alpha_m)^{2k}}{(2k)!} (-1)^k \mathbb{I} + \sum_{k=0}^{+\infty} \frac{(-i\alpha_m)^{2k+1}}{(2k+1)!} (-1)^k X_m Y_m = \\
&= \sum_{k=0}^{+\infty} \frac{(-1)^k (-i)^{2k} \alpha_m^{2k}}{(2k)!} \mathbb{I} - i \sum_{k=0}^{+\infty} \frac{(-1)^k (-i)^{2k} \alpha_m^{2k+1}}{(2k+1)!} X_m Y_m = \\
&= \sum_{k=0}^{+\infty} \frac{(-1)^k (-1)^k \alpha_m^{2k}}{(2k)!} \mathbb{I} - i \sum_{k=0}^{+\infty} \frac{(-1)^k (-1)^k \alpha_m^{2k+1}}{(2k+1)!} X_m Y_m = \\
&= \sum_{k=0}^{+\infty} \frac{\alpha_m^{2k}}{(2k)!} \mathbb{I} - i \sum_{k=0}^{+\infty} \frac{\alpha_m^{2k+1}}{(2k+1)!} X_m Y_m = \cosh(\alpha_m)\mathbb{I} - i \sinh(\alpha_m) X_m Y_m.
\end{aligned} \tag{4.4}$$

In addition, let us compute the value of K:

$$K = \mathrm{Tr}\left[\prod_{m=1}^{n} \exp\left(-i\alpha_m X_m Y_m\right)\right] = \mathrm{Tr}\left[\prod_{m=1}^{n} \left(\cosh(\alpha_m)\mathbb{I} - i\sinh(\alpha_m) X_m Y_m\right)\right] =$$
$$= \prod_{m=1}^{n} \left(\mathrm{Tr}\left[\cosh(\alpha_m)\mathbb{I}\right] - i\,\mathrm{Tr}\left[\sinh(\alpha_m) X_m Y_m\right]\right) = \prod_{m=1}^{n} 2\cosh(\alpha_m) =$$
$$= 2^n \prod_{m=1}^{n} \cosh(\alpha_m), \tag{4.5}$$

where $\mathrm{Tr}[X_m Y_m] = 0$ for every index m. Let us substitute Eq. 4.4 and Eq. 4.5 into Eq. 4.3, yielding

$$\rho = \frac{1}{2^n} \prod_{m=1}^{n} \frac{1}{\cosh(\alpha_m)} \left\{\prod_{m=1}^{n} \left(\cosh(\alpha_m)\mathbb{I} - i\sinh(\alpha_m) X_m Y_m\right)\right\} =$$
$$= \frac{1}{2^n} \prod_{m=1}^{n} \left(\mathbb{I} - i\tanh(\alpha_m) X_m Y_m\right). \tag{4.6}$$

Furthermore, the coefficient η_k it the k-th diagonal block of $\Gamma = \Gamma_D$ is given by

$$\eta_k = \frac{i}{2}\,\mathrm{Tr}\left[\rho(X_k Y_k - Y_k X_k)\right] = i\,\mathrm{Tr}\left[\rho X_k Y_k\right]. \tag{4.7}$$

In order to find an explicit expression for η_k, let us substitute Eq. 4.6 into Eq. 4.7, yielding

$$\eta_k = i\,\mathrm{Tr}\left[\frac{1}{2^n} \prod_{m=1}^{n} \left(\mathbb{I} - i\tanh(\alpha_m) X_m Y_m\right) X_k Y_k\right] =$$
$$= \frac{i}{2^n}\,\mathrm{Tr}\left[\left(\mathbb{I}_k - i\tanh(\alpha_k) X_k Y_k\right) \prod_{m\neq k} \left(\mathbb{I}_m - i\tanh(\alpha_m) X_m Y_m\right) X_k Y_k\right] =$$
$$= \frac{i}{2^n}\,\mathrm{Tr}\left[\left(X_k Y_k + i\tanh(\alpha_k)\mathbb{I}_k\right) \prod_{m\neq k} \left(\mathbb{I}_m - i\tanh(\alpha_m) X_m Y_m\right)\right] =$$
$$= \frac{i}{2^n}\,\mathrm{Tr}\left[X_k Y_k \prod_{m\neq k} \left(\mathbb{I}_m - i\tanh(\alpha_m) X_m Y_m\right) + i\tanh(\alpha_k)\mathbb{I}_k \prod_{m\neq k} \left(\mathbb{I}_m - i\tanh(\alpha_m) X_m Y_m\right)\right]. \tag{4.8}$$

Since the trace operator is linear, we can split the trace in Eq. 4.8 into two terms. The first one vanishes as $\mathrm{Tr}[X_k Y_k] = 0$, that is,

$$\mathrm{Tr}\left[X_k Y_k \prod_{m\neq k} \left(\mathbb{I}_m - i\tanh(\alpha_m) X_m Y_m\right)\right] = \prod_{m\neq k} \left(\mathbb{I}_m - i\tanh(\alpha_m) X_m Y_m\right) \mathrm{Tr}[X_k Y_k] = 0.$$

The other one can be expressed as follows:

$$
\begin{aligned}
& i \tanh(\alpha_k) \operatorname{Tr}\left[\mathbb{I}_k \prod_{m \neq k} (\mathbb{I}_m - i \tanh(\alpha_m) X_m Y_m) \right] = \\
& = i \tanh(\alpha_k) \operatorname{Tr}[\mathbb{I}_k] \prod_{m \neq k} \operatorname{Tr}\left[(\mathbb{I}_m - i \tanh(\alpha_m) X_m Y_m) \right] = \\
& = 2i \tanh(\alpha_k) \prod_{m \neq k} (\operatorname{Tr}[\mathbb{I}_m] - i \tanh(\alpha_m) \operatorname{Tr}[X_m Y_m]) = \\
& = 2i \tanh(\alpha_k) \prod_{m \neq k} \operatorname{Tr}[\mathbb{I}_m] = 2i 2^{n-1} \tanh(\alpha_k) = i 2^n \tanh(\alpha_k). \qquad (4.9)
\end{aligned}
$$

Substituting Eq. 4.9 into Eq. 4.8, we obtain

$$\eta_k = \frac{i^2}{2^n} 2^n \tanh(\alpha_k) = -\tanh(\alpha_k), \qquad (4.10)$$

for every index $k = 1, \dots, n$. Thus, ρ can be written in the Bravyi form. □

Remark 4.5. Since $\eta_k = -\tanh(\alpha_k)$ and $\alpha_k \in \mathbb{R}$ for every index $k = 1, \dots, n$, it holds that $\eta_k \in (-1, +1)$. Hereafter, for practical purposes, we extend the range of acceptable values of η_k to $[-1; +1]$, that is, we include the limits of η when $\alpha_k \to \pm\infty$.

Lemma 4.9. *The normalizer factor K can be expressed in terms of the correlation matrix Γ as*

$$\frac{1}{K} = \frac{\det\left(\mathbb{I} + \Gamma^2\right)^{\frac{1}{4}}}{2^n}.$$

Proof. The normalizer factor K is shown to depend only on the coefficient η_k in Eq. 4.5, where $\eta_k = -\tanh(\alpha_k)$. Since the trigonometric identity $\cosh(x) = \frac{1}{\sqrt{1-\tanh^2(x)}}$ holds, we have that

$$K = 2^n \prod_{k=1}^{n} \cosh(\alpha_k) = 2^n \prod_{k=1}^{n} \frac{1}{\sqrt{1-\eta_k^2}},$$

that is,

$$\frac{1}{K} = \frac{1}{2^n} \prod_{k=1}^{n} \sqrt{1-\eta_k^2}. \qquad (4.11)$$

Now, let us construct the auxiliary matrix $\mathbb{I} + \Gamma^2$. First, since $\Gamma = O\Gamma_D O^{\mathrm{T}}$, we have

$$\Gamma^2 = O\Gamma_D O^{\mathrm{T}} O\Gamma_D O^{\mathrm{T}} = O\Gamma_D^2 O^{\mathrm{T}}.$$

It follows immediately that

$$\mathbb{I} + \Gamma^2 = \mathbb{I} + O\Gamma_D^2 O^{\mathrm{T}} = O\mathbb{I}O^{\mathrm{T}} + O\Gamma_D^2 O^{\mathrm{T}} = O(\mathbb{I} + \Gamma_D^2)O^{\mathrm{T}},$$

where $\Gamma^2 = \Gamma_D^2 = \bigoplus_{k=1}^{n} (-\eta_k^2 \mathbb{I}_{2\times 2})$ and consequently $\mathbb{I}+\Gamma_D^2 = \bigoplus_{k=1}^{n} \left(\begin{smallmatrix} 1-\eta_k^2 & 0 \\ 0 & 1-\eta_k^2 \end{smallmatrix}\right)$, with $1-\eta_k^2 \geq 0$ since $\eta_k \in [-1;+1]$. Moreover, since $\det(O) = \det\left(O^{\mathrm{T}}\right) = 1$, the determinant of the auxiliary matrix is given by

$$\det\left(\mathbb{I} + \Gamma^2\right) = \det\left(\mathbb{I} + \Gamma_D^2\right) = \prod_{k=1}^{n} (1 - \eta_k^2)^2.$$

Thus, Eq. 4.11 becomes

$$\frac{1}{K} = \frac{1}{2^n}\sqrt{\sqrt{\det(\mathbb{I} + \Gamma^2)}} = \frac{\det\left(\mathbb{I} + \Gamma^2\right)^{\frac{1}{4}}}{2^n}.$$

□

Lemma 4.10. *The correlation matrix* Γ *can be written in terms of the matrix* A*, that is,*

$$\Gamma = -\tan(A). \tag{4.12}$$

Proof. As previously done, let us consider A and Γ expressed in the block-diagonal basis. Since each block of both A and Γ is a real skew-symmetric matrix, the eigenvalues of the k-th block are $a_k^{(\pm)} = \pm i\alpha_k$ and $g_k^{(\pm)} = \pm i\eta_k$, respectively. Then, the global diagonalization of the matrix A simply yields

$$A = A_D = \bigoplus_{k=1}^{n} P_k \begin{pmatrix} i\alpha_k & 0 \\ 0 & -i\alpha_k \end{pmatrix} P_k^{-1},$$

where $P_k := \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right)$ is the k-th change-of-basis matrix. Similarly, given the dependence $\eta_k = -\tanh(\alpha_k)$, the correlation matrix can be written as

$$\Gamma = \Gamma_D = \bigoplus_{k=1}^{n} P_k \begin{pmatrix} i\eta_k & 0 \\ 0 & -i\eta_k \end{pmatrix} P_k^{-1} = \bigoplus_{k=1}^{n} P_k \begin{pmatrix} -i\tanh(\alpha_k) & 0 \\ 0 & i\tanh(\alpha_k) \end{pmatrix} P_k^{-1},$$

In order to find a relation between A and Γ, we work at the single-block level by employing the spectral decomposition. Since $\eta_k = -\tanh(\alpha_k)$, $(A_D)_k$ and $(\Gamma_D)_k$ must be functions of each other and consequently share the same spectral projectors $\Pi_k^{(\pm)}$. Then, their spectral decompositions are respectively

$$(A_D)_k = a_k^{(+)}\Pi_k^{(+)} + a_k^{(-)}\Pi_k^{(-)} = i\alpha_k\Pi_k^{(+)} - i\alpha_k\Pi_k^{(-)}, \tag{4.13}$$

and

$$(\Gamma_D)_k = -i\tanh(\alpha_k)\Pi_k^{(+)} + i\tanh(\alpha_k)\Pi_k^{(-)}, \tag{4.14}$$

where $\Pi_k^{(\pm)}$ are the shared projectors of the k-th block onto the $\pm$ eigenspaces. In order to relate Eq. 4.14 to Eq. 4.13, we evaluate $\tan(A)$ as follows:

$$\begin{aligned}\tan((A_D)_k) &= \tan(i\alpha_k)\Pi^{(+)} + \tan(-i\alpha_k)\Pi^{(-)} = \\ &= i\tanh(\alpha_k)\Pi^{(+)} - i\tanh(\alpha_k)\Pi^{(-)},\end{aligned}$$

where tan and tanh are odd functions related by $\tan(ix) = i\tanh(x)$. Then, Eq. 4.14 becomes

$$(\Gamma_D)_k = -\tan\left((A_D)_k\right).$$

Since every function f acts upon each block independently from the others, we can conclude that

$$\Gamma = -\tan(A).$$

□

Remark 4.6. It follows straightforwardly that

$$A = -\arctan(\Gamma).$$

It is possible to yield a second representation for a fermionic Gaussian state, which is equivalent to the Bravyi form.

Theorem 4.11. *Any* $\rho \in \mathsf{Gauss}$ *can be written in terms of the correlation matrix* Γ *as follows:*

$$\rho = \frac{\det\left(\mathbb{I} + \Gamma^2\right)^{\frac{1}{4}}}{2^n} \exp\left(\frac{i}{2}\sum_{i,j=1}^{2n}(-\arctan(\Gamma))_{ij}\gamma_i\gamma_j\right).$$

Proof. Let us consider the Bravyi form of ρ introduced in Theorem 4.8. By applying Lemma 4.9 and Lemma 4.12, it is straightforward to achieve the Γ-form. □

Remark 4.7. Theorem 4.11 can be interpreted in the following way. Given an arbitrary fermionic Gaussian state ρ, it is completely determined by its correlation matrix Γ. More precisely, Γ uniquely identifies ρ, namely

$$\Gamma \longleftrightarrow \rho.$$

It is possible to derive new conditions on Γ concerning the physicality of ρ by considering Rem. 4.5. The following condition is to be understood completely equivalent to the ones in Prop. 4.6.

Theorem 4.12. *The correlation matrix* Γ *must satisfy the following condition in order to yield a valid quantum state:*

$$-\Gamma^2 \leq \mathbb{I}.$$

Proof. It suffices to recall Rem.4.5, that is, $\eta_k \in [-1, +1]$. Let us consider a fermionic Gaussian state, namely ρ, expressed in the Γ-form. First, ρ is positive semi-definite only when $\eta \in [-1, +1]$, which corresponds to

$$\mathbb{I} + \Gamma^2 \geq 0,$$

or equivalently

$$-\Gamma^2 \leq \mathbb{I}.$$

In fact, the eigenvalues of the k-th block of $\mathbb{I} + \Gamma^2$, namely $1 - \eta_k^2$, are non-negative only for $\eta \in [-1; +1]$. Furthermore, ρ has unit trace. From Lemma 4.9, we recall that

$$K = 2^n \prod_{k=1}^{n} \cosh(\alpha_k) = \frac{2^n}{\det\left(\mathbb{I} + \Gamma^2\right)^{\frac{1}{4}}},$$

that is,

$$\frac{\det\left(\mathbb{I} + \Gamma^2\right)^{\frac{1}{4}}}{2^n} = \frac{1}{2^n \prod_{k=1}^{n} \cosh(\alpha_k)}.$$

By expressing ρ through the rotated Majorana operators and expanding the exponential into a Taylor series, it follows straightforwardly that

$$\mathrm{Tr}\left[\exp\left(\frac{i}{2}\sum_{i,j=1}^{2n}\left(-\arctan(\Gamma)\right)_{ij}\gamma_i\gamma_j\right)\right] = 2^n \prod_{k=1}^{n}\cosh(\alpha_k).$$

This means that $\mathrm{Tr}[\rho] = 1$ only when Γ is a real skew-symmetric matrix with $\eta_k \in [-1; +1]$ – note that this condition was already established in Prop. 4.4 and Rem. 4.5. Finally, let us analyze the self-adjointness of ρ:

$$\begin{aligned}\rho^\dagger &= \frac{\left(\det\left(\mathbb{I} + \Gamma^2\right)^{\frac{1}{4}}\right)^*}{2^n}\exp\left(-\frac{i}{2}\sum_{i,j=1}^{2n}\left(-\arctan(\Gamma)\right)^*_{ji}\gamma_j^\dagger\gamma_i^\dagger\right) = \\ &= \frac{\left(\det\left(\mathbb{I} + \Gamma^2\right)^{\frac{1}{4}}\right.}{2^n}\exp\left(\frac{i}{2}\sum_{i,j=1}^{2n}\left(-\arctan(\Gamma)\right)_{ij}\gamma_i\gamma_j\right) = \rho,\end{aligned}$$

where $-\arctan(\Gamma) = A$ is a real skew-symmetric matrix by construction, and $\det\left(\mathbb{I} + \Gamma^2\right) \geq 0$ since $\mathbb{I} + \Gamma^2 \geq 0$ as previously shown. □

Corollary 7. An equivalent condition is

$$\Gamma\Gamma^\dagger \leq \mathbb{I}.$$

Proof. Since Γ is a real skew-symmetric matrix, it is also skew-Hermitian, that is,

$$\Gamma^\dagger = \left(\Gamma^{\mathrm{T}}\right)^* = (-\Gamma)^* = -\Gamma.$$

Then, by virtue of Theorem 4.12, we have that

$$\Gamma\Gamma^\dagger = -\Gamma^2 \leq \mathbb{I},$$

that is,

$$\Gamma\Gamma^\dagger \leq \mathbb{I}.$$

□

Corollary 8. Another equivalence condition is

$$-\mathbb{I} \leq i\Gamma \leq +\mathbb{I},$$

where the matrix $i\Gamma$ is called the *physical correlation matrix.*

Proof. It suffices to note that $-\Gamma^2 = (i\Gamma)^2$. Then, Theorem 4.12 guarantees that

$$-\Gamma^2 \leq \mathbb{I}$$
$$(i\Gamma)^2 \leq \mathbb{I},$$

that is,

$$-\mathbb{I} \leq i\Gamma \leq +\mathbb{I}.$$

□

Finally, let us introduce a standard characterization of pure fermionic Gaussian states.

Proposition 4.7. *Let $\rho \in$ Gauss be a pure state. Then, its correlation matrix Γ is such that*

$$\Gamma^2 = -\mathbb{I},$$

that is, $\eta_k = \pm 1$ for every index k.

Proof. A rigorous proof can be found in [8, 49]. □

Corollary 9. It follows immediately that Γ is an orthogonal matrix for any pure state $\rho \in$ Gauss.

Proof. By virtue of Corollary 7, we have that $-\Gamma^2 = \Gamma\Gamma^{\mathrm{T}} \leq \mathbb{I}$. If $\rho \in$ Gauss is a pure state, it follows from Prop. 4.7 that $\Gamma\Gamma^{\mathrm{T}} = \mathbb{I}$, that is, Γ is an orthogonal matrix. □

4.3.4 Fermionic Gaussian Operations

Now, we are able to introduce the free operations of the QRT of fermionic non-Gaussianity, *i.e.*, the fermionic Gaussian operations. Their set will be denoted by FGO. We refer the reader to Section 2 of the recent work in [50].

Definition 4.5. Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be two Hilbert spaces. The map $\Lambda : \mathrm{St}(A) \to \mathrm{St}(B)$ is called a fermionic Gaussian operation if it is one of the following operations or a composition thereof:

- Applying a fermionic Gaussian unitary;
- Appending a fermionic Gaussian state;
- Discarding a subsystem, that is, taking a partial trace;
- Performing an occupation-number measurement;
- The above operations conditioned on measurement outcomes.

Let us analyze the action of a fermionic Gaussian unitary upon a fermionic Gaussian state.

Proposition 4.8. *Let $U \in$ FGU and let $\rho \in$ Gauss, that is, ρ is completely specified by its correlation matrix Γ. Then, the following relation holds:*

$$\Gamma \mapsto \Gamma' = O\Gamma O^{\mathrm{T}},$$

where $O \in \mathcal{SO}_{2n}(\mathbb{R})$, and Γ' is the correlation matrix of $\rho' = U\rho U^\dagger \in$ Gauss.

Proof. The correlation matrix Γ' can be expressed as follows:

$$\begin{aligned}\Gamma'_{jk} &= i\,\mathrm{Tr}\left[\rho'\gamma_j\gamma_k\right] - i\delta_{jk} = i\,\mathrm{Tr}\left[U\rho U^\dagger\gamma_j\gamma_k\right] - i\delta_{jk} = \\ &= i\,\mathrm{Tr}\left[\rho U^\dagger\gamma_j U U^\dagger\gamma_k U\right] - i\delta_{jk}. \end{aligned} \tag{4.15}$$

Let us recall Prop. 3.18, that is,

$$U^\dagger\gamma_j U = \sum_{k=1}^{2n} o_{jk}\gamma_k,$$

where the matrix O with coefficients o_{jk} is a real special orthogonal matrix, *i.e.*, $O \in \mathcal{SO}_{2n}(\mathbb{R})$. Then, Eq. 4.15 can be written as

$$\begin{aligned}\Gamma'_{jk} &= i\,\mathrm{Tr}\left[\rho\sum_l o_{jl}\gamma_l\sum_m o_{km}\gamma_m\right] - i\sum_{l,m} o_{jl}o_{km}\delta_{lm} = \\ &= \sum_{l,m} o_{jl}o_{km}\left(i\,\mathrm{Tr}\left[\rho\gamma_l\gamma_m\right] - i\delta_{lm}\right) = \sum_{l,m} o_{jl}o_{km}\Gamma_{lm} = \\ &= \sum_{l,m} o_{jl}\Gamma_{lm}o_{km},\end{aligned}$$

that is, $\Gamma' = O\Gamma O^{\mathrm{T}}$. □

Remark 4.8. Since $O = e^{2A}$, where A is a real skew-symmetric matrix such that $A = \arctan(\Gamma)$, it also holds that

$$\Gamma' = e^{2A}\Gamma e^{-2A},$$

where $O^{\mathrm{T}} = \left(e^{2A}\right)^{\mathrm{T}} = e^{-2A}$.

Proposition 4.9. *Let $\Xi_\sigma \in \mathsf{FGO}$ be the map that performs an appending of a fermionic Gaussian state σ, that is, $\Xi_\sigma(\rho) = \rho \otimes \sigma \in \mathsf{Gauss}$, with $\rho \in \mathsf{Gauss}$. Then, the following relation holds:*

$$\Gamma_{\rho\otimes\sigma} = \Gamma_\rho \oplus \Gamma_\sigma.$$

Proof. The proof is straightforward. Since the tensor-product state $\rho \otimes \sigma$ presents no correlations, its correlation matrix is simply the direct sum of the correlation matrices of ρ and σ, namely

$$\Gamma_{\rho\otimes\sigma} = \Gamma_\rho \oplus \Gamma_\sigma = \begin{pmatrix} \Gamma_\rho & 0 \\ 0 & \Gamma_\sigma \end{pmatrix}.$$

□

Proposition 4.10. *Let us consider the partial trace map, associated with the possibility to discarding a subsystem. Given a fermionic Gaussian state ρ_{AB}, its partial trace over the subsystem B is given by $\mathrm{Tr}_B[\rho_{AB}] = \rho_A \in \mathsf{Gauss}$. Then, the following relation holds:*

$$\Gamma_A = \mathrm{Tr}_B[\Gamma_{AB}].$$

Proof. For the sake of simplicity, we omit the proof, but it can be found in [49]. □

Proposition 4.11. *Let us consider the map $\Phi^{(j)}$ that performs a measurement of the occupation number n_j for a single fermionic mode j. Then, the correlation matrix Γ of the input state $\rho \in \mathsf{Gauss}$ is mapped into*

$$\Gamma'_{RR} = \Gamma_{RR} - \Gamma_{Rj}\left(\Gamma_{jj} - \lambda\mathbb{I}\right)^{-1}\Gamma_{jR},$$

where $R := \{1, \ldots, n\} \setminus \{j\}$, and $\lambda = \pm 1$ are the eigenvalues of the j-th parity operator, namely $Z_j = -iX_jY_j$.

Proof. For the sake of simplicity, we omit the proof, but it can be found in [8]. □

Let us recall the universal gate set for fermionic computation expressed in Eq. 4.2. In light of the arguments presented so far concerning the fermionic Gaussian operations, it is evident that the F4 gate can be identified as the fermionic non-Gaussian gate, while the CT gate represents a valid fermionic Gaussian unitary.

4.3.5 A Novel Monotone for Fermionic Non-Gaussianity

Recent efforts have stimulated interest in quantifying non-Gaussianity, providing monotones such as the fermionic antiflatness ([46]) and the bridge degree ([50]). In this work, we introduce a novel monotone called the *entropy of non-Gaussianity*. It is worth noting that an analogous monotone exists in the bosonic setting, introduced in [19]. First, we introduce some fundamental tools from quantum information theory — for a detailed discussion, we refer the reader to [39].

Definition 4.6. Let ρ be an arbitrary quantum state. The von Neumann entropy of ρ is defined as

$$\mathrm{S}(\rho) := -\operatorname{Tr}\left[\rho \log(\rho)\right].$$

Definition 4.7. Let us consider two quantum states, namely ρ and σ. Their quantum relative entropy is defined as follows:

$$\mathrm{S}(\rho \parallel \sigma) := \operatorname{Tr}\left[\rho \left(\log(\rho) - \log(\sigma)\right)\right].$$

By convention, $\mathrm{S}(\rho \parallel \sigma) = +\infty$ if $\mathrm{Supp}(\rho) \nsubseteq \mathrm{Supp}(\sigma)$.

Remark 4.9. Let L be a linear operator acting on a Hilbert space $\mathcal{H}_L$. The support of L is defined as the orthogonal complement of its kernel, namely

$$\mathrm{Supp}(L) := \left(\mathrm{Ker}(L)\right)^{\perp},$$

where $\mathrm{Ker}(L) := \{|\psi\rangle \in \mathcal{H}_L \mid L|\psi\rangle = 0\}$.

Proposition 4.12 (Klein's Inequality)**.** *The quantum relative entropy is a non-negative function, namely*

$$\mathrm{S}(\rho \parallel \sigma) \geq 0,$$

for any pair of quantum state ρ, σ.

Proof. A rigorous proof can be found in [39]. □

Lemma 4.13 (Data-Processing Inequality)**.** *Let ρ, σ be two quantum states associated with two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$, respectively. Let $\Lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}')$ be a quantum channel, that is, a linear CPTP map. Then*

$$\mathrm{S}\left(\rho \parallel \sigma\right) \geq \mathrm{S}\left(\Lambda(\rho) \parallel \Lambda(\sigma)\right).$$

Proof. We refer the reader to [39]. □

Lemma 4.14. *Let ρ, σ be two quantum states of the same Hilbert space $\mathcal{H}$, and let U be a unitary upon $\mathcal{H}$. Then, the quantum relative entropy is invariant under the action of U, namely*

$$\mathrm{S}\left(U\rho U^{\dagger} \parallel U\sigma U^{\dagger}\right) = \mathrm{S}\left(\rho \parallel \sigma\right).$$

Proof. A proof can be found in [39]. □

Lemma 4.15. *The quantum relative entropy is additive under tensor products of quantum states, namely*

$$\mathrm{S}(\rho_A \otimes \rho_B \parallel \sigma_A \otimes \sigma_B) = \mathrm{S}(\rho_A \parallel \sigma_A) + \mathrm{S}(\rho_B \parallel \sigma_B),$$

where ρ_A, σ_A is associated with the system A, while ρ_B, σ_B with the system B.

Proof. A proof can be found in [39]. □

Now, let us introduce a central object in our analysis, *i.e.*, the *Gaussianization map*.

Definition 4.8. Let ρ be a quantum (even) state. Its Gaussianization is defined as

$$\mathcal{G}(\rho) := \rho_{\Gamma_\rho},$$

where ρ_{Γ_ρ} is the unique fermionic Gaussian state associated with the correlation matrix Γ_ρ, namely

$$\Gamma_{\mathcal{G}(\rho)} = \Gamma_\rho.$$

Remark 4.10. The map $\mathcal{G}$ is not a quantum channel, since it is nonlinear. For two fermionic modes, $|00\rangle$ and $|11\rangle$ are fermionic Gaussian states, but

$$\rho = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right)$$

has vanishing correlation matrix. Thus, $\mathcal{G}(\rho) = \frac{1}{4}$, but

$$\frac{1}{2}\mathcal{G}(|00\rangle\langle 00|) + \frac{1}{2}\mathcal{G}(|11\rangle\langle 11|) \neq \frac{1}{4}.$$

Proposition 4.13. *The Gaussianization map satisfies what follows:*

$$\mathcal{G}(\rho) = \rho \iff \rho \in \mathsf{Gauss}.$$

Proof. Let us assume that $\mathcal{G}(\rho) = \rho$. Since the Gaussianization map $\mathcal{G}$ maps a quantum state into its Gaussian analogue, if $\mathcal{G}: \rho \mapsto \rho$, then ρ must be a fermionic Gaussian state. Conversely, let $\rho \in \mathsf{Gauss}$. This means that ρ is the unique Gaussian state associated with the correlation matrix Γ_ρ, namely $\rho \equiv \rho_{\Gamma_\rho}$. Hence, $\mathcal{G}(\rho) = \rho_{\Gamma_\rho} = \rho$. □

Lemma 4.16. *Let $U \in \mathsf{FGU}$ and let ρ be a quantum state. Then*

$$U\mathcal{G}(\rho)U^\dagger = \mathcal{G}\left(U\rho U^\dagger\right).$$

Proof. Let us consider the mapping $\rho \to U\rho U^\dagger$, with $U \in \mathsf{FGU}$ — note that the output state is Gaussian. On one hand, by virtue of Prop. 4.8, the correlation matrix of the output state is given by

$$\Gamma_{U\rho U^\dagger} = O\Gamma_\rho O^{\mathrm{T}},$$

with $O \in \mathcal{SO}_{2n}(\mathbb{R})$. On the other hand, let us consider the mapping $\mathcal{G}(\rho) \to U\mathcal{G}(\rho)U^\dagger$, where the output state is still Gaussian. Its correlation matrix follows one more time from Prop. 4.8:

$$\Gamma_{U\mathcal{G}(\rho)U^\dagger} = O\Gamma_{\mathcal{G}(\rho)}O^{\mathrm{T}} = O\Gamma_\rho O^{\mathrm{T}},$$

where the last equality holds by definition of the Gaussianization map $\mathcal{G}$. By uniqueness of the Gaussian state associated with a fixed correlation matrix, it holds that

$$U\mathcal{G}(\rho)U^\dagger = U\rho U^\dagger = \mathcal{G}\left(U\rho U^\dagger\right),$$

where the last equality is guaranteed by Prop. 4.13. □

Proposition 4.14. *The Gaussianization map $\mathcal{G}$ is idempotent, that is,*

$$\mathcal{G}^2 = \mathcal{G}.$$

Proof. Let us consider a quantum state ρ. By definition, its Gaussianization is $\mathcal{G}(\rho) = \rho_{\Gamma_\rho}$. On one hand, applying the Gaussianization map gives

$$\mathcal{G}\left(\mathcal{G}(\rho)\right) = \mathcal{G}\circ\mathcal{G}(\rho) = \mathcal{G}^2(\rho).$$

On the other hand, employing Prop. 4.13, we have

$$\mathcal{G}\left(\mathcal{G}(\rho)\right) = \mathcal{G}\left(\rho_{\Gamma_\rho}\right) = \rho_{\Gamma_\rho} = \mathcal{G}(\rho).$$

Thus, $\mathcal{G}^2 = \mathcal{G}$. □

Theorem 4.17. *Let ρ be an arbitrary fermionic state. Let $\mathcal{W}\left(\Gamma_\rho\right) := \{\sigma$ quantum state $\mid \Gamma_\sigma = \Gamma_\rho\}$ be the set of states that share the correlation matrix with ρ. Then, the Gaussianization of ρ maximizes the von Neumann entropy of such states, namely*

$$\mathcal{G}(\rho) = \underset{\sigma\in\mathcal{W}(\Gamma_\rho)}{\arg\max}\, \mathrm{S}(\sigma).$$

Proof. Let us set $\tau := \mathcal{G}(\rho)$, and let $\sigma \in \mathcal{W}\left(\Gamma_\rho\right)$. Since $\Gamma_\rho = \Gamma_\sigma$, the Gaussianization of σ coincides with the one of ρ, namely $\mathcal{G}(\sigma) = \tau$. The quantum relative entropy between σ and τ can be expressed as

$$\begin{aligned}\mathrm{S}\left(\sigma \parallel \tau\right) &= \mathrm{Tr}\left[\sigma\log(\sigma)\right] - \mathrm{Tr}\left[\sigma\log(\tau)\right] = \\ &= -\mathrm{S}(\sigma) - \mathrm{Tr}\left[\sigma\log(\tau)\right].\end{aligned}$$

Since τ is a fermionic Gaussian state by construction, it can be written as

$$\tau = \frac{e^{-Q}}{K}, \qquad Q = \frac{i}{4}\sum_{i,j} h_{ij}\gamma_j\gamma_k,$$

with Q quadratic in the Majorana operators. Therefore

$$\log(\tau) = \log\left(\frac{e^{-Q}}{K}\right) = -Q - \log(K),$$

that is,

$$Q = -\log(\tau) - \log(K).$$

Moreover, since σ and τ share the same correlation matrix, we have that

$$\begin{aligned}
\mathrm{Tr}\left[\sigma Q\right] &= \mathrm{Tr}\left[\tau Q\right] \\
\mathrm{Tr}\left[\sigma\left(-\log(\tau) - \log(K)\right)\right] &= \mathrm{Tr}\left[\tau\left(-\log(\tau) - \log(K)\right)\right] \\
\mathrm{Tr}\left[\sigma\log(\tau) + \sigma\log(K)\right] &= \mathrm{Tr}\left[\tau\log(\tau) + \tau\log(K)\right] \\
\mathrm{Tr}\left[\sigma\log(\tau)\right] + \log(K)\,\mathrm{Tr}[\sigma] &= \mathrm{Tr}\left[\tau\log(\tau)\right] + \log(K)\,\mathrm{Tr}[\tau]. \qquad (4.16)
\end{aligned}$$

Since $\mathrm{Tr}[\sigma] = \mathrm{Tr}[\tau] = 1$, Eq. 4.16 becomes

$$\mathrm{Tr}[\sigma\log(\tau)] = \mathrm{Tr}[\tau\log(\tau)] = -\mathrm{S}(\tau).$$

Thus, we have that

$$\mathrm{S}\left(\sigma \parallel \tau\right) = \mathrm{S}\left(\tau\right) - \mathrm{S}(\sigma).$$

By virtue of the Klein's inequality in Prop. 4.12, we have that

$$\mathrm{S}(\sigma) \leq \mathrm{S}(\tau),$$

where the equality holds if and only if $\sigma = \tau$. Thus, $\tau = \mathcal{G}(\rho)$ is the unique maximum-entropy state with the prescribed correlation matrix, and the claim holds. □

We are now able to define a natural measure of non-Gaussianity based on the Gaussianization map: intuitively, it quantifies how distinguishable a state is from its own Gaussian counterpart.

Definition 4.9. The entropy of non-Gaussianity of ρ is defined as the quantum relative entropy between ρ and its Gaussianization $\mathcal{G}(\rho)$, namely

$$\mathfrak{G}(\rho) := \mathrm{S}\left(\rho \parallel \mathcal{G}(\rho)\right).$$

The following results emphasize the physical and informational meaning of the entropy of non-Gaussianity, making it also a relevant tool in the framework of quantum information theory.

Theorem 4.18. *For every quantum state ρ, the entropy of non-Gaussianity can be seen as the von Neumann entropic difference between the Gaussianization of ρ and ρ itself, namely*

$$\mathfrak{G}(\rho) = \mathrm{S}\left(\mathcal{G}(\rho)\right) - \mathrm{S}(\rho).$$

Proof. Let us set $\tau := \mathcal{G}(\rho)$. Then, the entropy of non-Gaussianity can be expressed as

$$\begin{aligned}\mathfrak{G}(\rho) = \mathrm{S}\left(\rho \parallel \tau\right) = \mathrm{Tr}\left[\rho \log(\rho)\right] - \mathrm{Tr}\left[\rho \log(\tau)\right] = \\ = -\mathrm{S}(\rho) - \mathrm{Tr}\left[\rho \log(\tau)\right].\end{aligned}$$

By virtue the same entropy formula applied in the proof of Theorem 4.17, it holds that

$$\mathrm{Tr}\left[\rho \log(\tau)\right] = \mathrm{Tr}\left[\tau \log(\tau)\right] = -\mathrm{S}(\tau).$$

Thus, recalling that $\tau = \mathcal{G}(\rho)$, we have that

$$\mathfrak{G}(\rho) = \mathrm{S}\left(\mathcal{G}(\rho)\right) - \mathrm{S}(\rho).$$

The claim holds. □

Corollary 10. The entropy of non-Gaussianity satisfies the following:

$$\mathfrak{G}(\rho) = \max_{\sigma \in \mathcal{W}\left(\Gamma_\rho\right)} \mathrm{S}(\sigma) - \mathrm{S}(\rho).$$

Proof. The proof directly follows from Theorem 4.17 and Theorem 4.18:

$$\begin{aligned}\mathfrak{G}(\rho) = \mathrm{S}\left(\mathcal{G}(\rho)\right) - \mathrm{S}(\rho) = \mathrm{S}\left(\underset{\sigma \in \mathcal{W}\left(\Gamma_\rho\right)}{\arg\max}\, \mathrm{S}(\sigma)\right) - \mathrm{S}(\rho) = \\ = \max_{\sigma \in \mathcal{W}\left(\Gamma_\rho\right)} \mathrm{S}(\sigma) - \mathrm{S}(\rho).\end{aligned}$$

The claim holds. □

Theorem 4.19. *For every fermionic state ρ, the entropy of non-Gaussianity can be seen as the minimum of the quantum relative entropy between ρ and any Gaussian state σ, namely*

$$\mathfrak{G}(\rho) = \min_{\sigma \in \mathrm{Gauss}} \mathrm{S}\left(\rho \parallel \sigma\right).$$

The minimum is attained at $\sigma = \mathcal{G}(\rho)$.

Proof. Let us set $\mathcal{G}(\rho) = \tau$, and let $\sigma \in \mathsf{Gauss}$. Then

$$\log(\sigma) = -Q - \log(K),$$

with Q quadratic in the Majorana operators. Since ρ and τ have the same correlation matrix, it holds that

$$\mathrm{Tr}\left[\rho \log(\sigma)\right] = \mathrm{Tr}\left[\tau \log(\sigma)\right].$$

Therefore

$$\begin{aligned} \mathrm{S}\left(\rho \parallel \sigma\right) &= -\mathrm{S}(\rho) - \mathrm{Tr}\left[\rho \log(\sigma)\right] = \\ &= -\mathrm{S}(\rho) - \mathrm{Tr}\left[\tau \log(\sigma)\right] = \\ &= \mathrm{S}(\tau) - \mathrm{S}(\rho) + \mathrm{S}\left(\tau \parallel \sigma\right), \end{aligned}$$

where the last equality holds since

$$\mathrm{S}\left(\tau \parallel \sigma\right) = -\mathrm{S}(\tau) - \mathrm{Tr}\left[\tau \log(\sigma)\right].$$

Thus, by virtue of the Klein's inequality in Prop. 4.12, we have that

$$\mathrm{S}\left(\rho \parallel \sigma\right) = \mathfrak{G}(\rho) + \mathrm{S}\left(\tau \parallel \sigma\right) \geq \mathfrak{G}(\rho),$$

where the equality holds if and only if $\sigma = \tau$. In other words, the Gaussian minimizer is precisely $\tau = \mathcal{G}(\rho)$. This proves the claim. □

Now, let us show that the entropy of non-Gaussianity is a valid monotone for the QRT of fermionic non-Gaussianity. In doing so, we will also prove non-negativity and faithfulness, despite these not being required by Def. 2.11. Moreover, we will particularly focus on linear CPTP maps which characterize its monotonicity.

Theorem 4.20. *The entropy of non-Gaussianity is a non-negative map, namely*

$$\mathfrak{G}(\rho) \geq 0.$$

Moreover, it is faithful, that is,

$$\mathfrak{G}(\rho) = 0 \iff \rho \in \mathsf{Gauss}.$$

Proof. The non-negativity follows directly from Prop. 4.12. Furthermore, from Prop. 4.13, we have that

$$\rho \in \mathsf{Gauss} \iff \mathcal{G}(\rho) = \rho \iff \mathfrak{G}(\rho) = 0,$$

since $\mathrm{S}(\rho \parallel \rho) = 0$ by construction. □

Proof. A rigorous proof can be found in [39]. □

Theorem 4.21. *Let Φ be a parity-preserving CPTP map such that it commutes with the Gaussianization map, namely*

$$\mathcal{G}(\Phi(\rho)) = \Phi(\mathcal{G}(\rho)),$$

for every quantum state ρ. Then, the entropy of non-Gaussianity is non-increasing under the action of Φ, namely

$$\mathfrak{G}(\rho) \geq \mathfrak{G}(\Phi(\rho)).$$

Proof. Since Φ and $\mathcal{G}$ commute by assumption, it holds that

$$\begin{aligned}\mathfrak{G}(\Phi(\rho)) &= \mathrm{S}(\Phi(\rho) \parallel \mathcal{G}(\Phi(\rho))) = \\ &= \mathrm{S}(\Phi(\rho) \parallel \Phi(\mathcal{G}(\rho))).\end{aligned}$$

By virtue of Lemma 4.13, we have that

$$\begin{aligned}\mathfrak{G}(\Phi(\rho)) &= \mathrm{S}(\Phi(\rho) \parallel \Phi(\mathcal{G}(\rho))) \leq \\ &\leq \mathrm{S}(\rho \parallel \mathcal{G}(\rho)) = \mathfrak{G}(\rho).\end{aligned}$$

The claim holds. □

Notably, the CPTP maps introduced in Theorem 4.21 preserve the fermionic Gaussian structure and here are termed *Fermionic-Gaussian-Preserving Operations* (FGPO). In order to guarantee closure under tensor products, as done in the stabilizer setting, we naturally introduce the *Completely Fermionic-Gaussian-Preserving Operations* (CFGPO). We leave the question of whether a full separation exists between the set of Fermionic Gaussian Operations (FGO) and CFGPO for future investigation, mirroring the result proven for the stabilizer framework by Heimendahl et al. in [24].

Theorem 4.22. *The entropy of non-Gaussianity is invariant under the action of fermionic Gaussian unitaries, namely*

$$\mathfrak{G}\left(U\rho U^\dagger\right) = \mathfrak{G}(\rho),$$

where $U \in$ FGU and ρ is a quantum state.

Proof. By virtue of Lemma 4.14 and Lemma 4.16, we have that

$$\begin{aligned}\mathfrak{G}\left(U\rho U^\dagger\right) &= \mathrm{S}\left(U\rho U^\dagger \parallel \mathcal{G}\left(U\rho U^\dagger\right)\right) = \\ &= \mathrm{S}\left(U\rho U^\dagger \parallel U\mathcal{G}(\rho)U^\dagger\right) = \\ &= \mathrm{S}(\rho \parallel \mathcal{G}(\rho)) = \mathfrak{G}(\rho).\end{aligned}$$

The claim holds. □

We are now able to focus on an additional property, that is, additivity with respect to the tensor product of states, which is particularly desirable in the framework of QRTs. Notably, unlike other monotones proposed in the literature, the entropy of non-Gaussianity satisfies additivity, which further supports its role as a suitable measure of non-Gaussianity.

Theorem 4.23. *Let us consider a quantum state of the form $\rho_{AB} = \rho_A \otimes \rho_B$. Then, the entropy of non-Gaussianity is additive, namely*

$$\mathfrak{G}(\rho_A \otimes \rho_B) = \mathfrak{G}(\rho_A) + \mathfrak{G}(\rho_B).$$

Consequently

$$\mathfrak{G}\left(\rho^{\otimes n}\right) = n\mathfrak{G}\left(\rho\right).$$

Proof. By virtue of Prop. 4.9, the correlation matrix of $\rho_A \otimes \rho_B$ is given by

$$\Gamma_{\rho_A \otimes \rho_B} = \Gamma_{\rho_A} \oplus \Gamma_{\rho_B}.$$

The Gaussian state associated with a block-diagonal correlation matrix is the tensor product of the Gaussian states associated with each block. Therefore

$$\mathcal{G}\left(\rho_A \otimes \rho_B\right) = \mathcal{G}(\rho_A) \otimes \mathcal{G}(\rho_B).$$

By virtue of Lemma 4.15, we have that

$$\begin{aligned}\mathfrak{G}\left(\rho_A \otimes \rho_B\right) &= \mathrm{S}((\rho_A \otimes \rho_B \parallel \mathcal{G}\left(\rho_A \otimes \rho_B\right)) = \\ &= \mathrm{S}((\rho_A \otimes \rho_B \parallel \mathcal{G}(\rho_A) \otimes \mathcal{G}(\rho_B)) = \\ &= \mathrm{S}\left(\rho_A \parallel \mathcal{G}(\rho_A)\right) + \mathrm{S}\left(\rho_A \parallel \mathcal{G}(\rho_B)\right) = \\ &= \mathfrak{G}(\rho_A) + \mathfrak{G}(\rho_B).\end{aligned}$$

Iteration gives the formula for $\rho^{\otimes n}$. The statement is verified. □

4.4 There is also Fermionic Magic

In the following chapter, we manage to extend the notion of quantum magic to the framework of fermionic non-stabilizerness. Throughout this section, we will use the term *fermagic* to indicate the property of fermionic magic, that is, fermionic non-stabilizerness.

4.4.1 Mathematical Preliminaries

In the following subsection, we will try to mirror the Pauli and Clifford structures already presented in the stabilizer case, in order to naturally extend it to the fermionic framework. In particular, we refer the reader to [4, 35].

Definition 4.10. Given n fermionic modes, a $2n$-Majorana string is a product of Majorana operators, namely

$$M = \prod_{k=1}^{n} \gamma_k,$$

where $\gamma_k \in \{\mathbb{I}_k, X_k, Y_k, Z_k\}$.

Definition 4.11. Given n fermionic modes, the $2n$-Majorana group is formed by Majorana strings multiplied by phase factors $(\pm 1, \pm i)$, namely

$$\mathcal{M}_{2n} := \{i^\alpha M \mid \alpha \in \mathbb{Z}_4\}.$$

Since the Majorana CAR hold (see Prop. 3.7), the Majorana strings do not behave identically as the Pauli strings.

Proposition 4.15. *Let M be a $2n$-Majorana string. Let $\Lambda(M)$ be the length of the string M, defined as the number of X_k and Y_k in M (recalling that $Z_k = -iX_kY_k$), namely $\Lambda(M) = \#(X_k, Y_k \mid k = 1, \dots, n)$. Then, it holds that M is Hermitian and involutory, that is,*

$$M^\dagger = M \qquad M^2 = \mathbb{I},$$

when $\lfloor \Lambda(M)/2 \rfloor$ is even; on the contrary, M is anti-Hermitian and skew-involutory, that is,

$$M^\dagger = -M \qquad M^2 = -\mathbb{I},$$

when $\lfloor \Lambda(M)/2 \rfloor$ is odd.

Proof. A rigorous proof can be found in [35]. □

Example 4.1. Let us consider the Majorana string $M = X_1 Y_2 \prod_{k=3}^{n} \mathbb{I}_k$. Since $\Lambda(M) = 2$, it is straightforward to compute $\lfloor \Lambda(M)/2 \rfloor = 1$. Then, M can be shown to be both anti-Hermitian and skew-involutory, that is,

$$M^\dagger = \left(\prod_{k=3}^{n} \mathbb{I}_k\right)^\dagger Y_2^\dagger X_1^\dagger = \left(\prod_{k=3}^{n} \mathbb{I}_k\right) Y_2 X_1 =$$
$$= -X_1 Y_2 \prod_{k=3}^{n} \mathbb{I}_k = -M$$

and

$$M^2 = \left(X_1 Y_2 \prod_{k=3}^{n} \mathbb{I}_k\right)^2 = \left(X_1 Y_2 \prod_{k=3}^{n} \mathbb{I}_k\right)\left(X_1 Y_2 \prod_{k=3}^{n} \mathbb{I}_k\right) =$$
$$= X_1 Y_2 X_1 Y_2 \prod_{k=3}^{n} \mathbb{I}_k = -X_1^2 Y_2^2 \prod_{k=3}^{n} \mathbb{I}_k = -\mathbb{I}_1 \mathbb{I}_2 \prod_{k=3}^{n} \mathbb{I}_k =$$
$$= -\prod_{k=1}^{n} \mathbb{I}_k = -\mathbb{I}.$$

Now, we are able to introduce the Majorana Clifford group in analogy with the action of the Clifford group upon the Pauli group.

Definition 4.12. Given n fermionic modes, the Majorana Clifford group is the normalizer of the Majorana group, namely

$$\widehat{\mathcal{C}}_{2n} := \{U \in \mathcal{U}(2^n) \mid U^\dagger \mathcal{M}_{2n} U = \mathcal{M}_{2n}\}.$$

In other words, the Majorana Clifford group is formed by the unitaries that map Majorana strings into Majorana strings, namely

$$\widehat{\mathcal{C}}_{2n} := \{U \in \mathcal{U}(2^n) \mid U^\dagger M U = \phi_U M'\},$$

where $\phi_U \in \{\pm 1, \pm i\}$, and $M, M' \in \mathcal{M}_{2n}$. Moreover, $U \in \widehat{\mathcal{C}}_{2n}$ will be referred to as the *Majorana Clifford unitary*.

Proposition 4.16. *The generating set of the Majorana Clifford group is given by* $\{e^{i\theta}\mathbb{I}, e^{i\frac{\pi}{4}X_k}, e^{\frac{\pi}{4}Y_k X_k}, e^{\frac{\pi}{4}Y_k X_k X_p}\}$*, where* $k, p \in \{1, \dots, n\}$ *with* $k \neq p$*, that is,*

$$\widehat{\mathcal{C}}_{2n} = \langle e^{i\theta}\mathbb{I}, e^{i\frac{\pi}{4}X_k}, e^{\frac{\pi}{4}Y_k X_k}, e^{\frac{\pi}{4}Y_k X_k X_p}\rangle.$$

Proof. An exhaustive proof can be found in [35]. □

Proposition 4.17. *It holds that*

$$\mathcal{P}_n \cong \mathcal{M}_{2n} \text{ and } \mathcal{C}_n \cong \widehat{\mathcal{C}}_{2n},$$

where $\mathcal{P}_n$ *and* $\mathcal{C}_n$ *are the n-qubit Pauli group and the n-qubit Clifford group, respectively.*

Proof. The proof can be found in [35]. □

Due to the superselection rule, it is not possible to generate a coherent superposition of states with even and odd fermionic numbers. This means that the only physically realizable Majorana strings are the so-called *parity-preserving Majorana strings*, that is, the ones commuting with the parity-preserving operator, namely

$$[M;\Pi] = 0,$$

where $\Pi := (-i)^n \prod_{k=1}^{n} Y_k X_k = \prod_{k=1}^{n} Z_k$. The set of the parity-preserving Majorana strings will be denoted by $\mathcal{M}_{2n}^{\Pi}$.

Proposition 4.18. *Any parity-preserving Majorana string is even-length.*

Proof. The proof is straightforward. Since the parity-preserving operator Π contains exactly one X_k and one Y_k for each fermionic mode $k \in \{1, \dots, n\}$, a Majorana string M with length $\Lambda(M)$ anticommutes with a number $\Lambda(M)$ factors of Π. Because each anticommutation contributes a sign of -1, the overall sign will be $(-1)^{\Lambda(M)}$, which is equal to $+1$ if and only if $\Lambda(M)$ is even. □

Now, we are able to introduce the *parity-preserving Majorana Clifford group*, which will be denoted by $\widehat{\mathcal{C}}_{2n}^{\Pi}$.

Definition 4.13. Given n fermionic modes, the parity-preserving Majorana Clifford group is the normalizer of the parity-preserving Majorana group, namely

$$\widehat{\mathcal{C}}_{2n}^{\Pi} := \{U \in \mathcal{U}(2^n) \mid U^\dagger \mathcal{M}_{2n}^{\Pi} U = \mathcal{M}_{2n}^{\Pi}\}.$$

In other words, the parity-preserving Majorana Clifford group is formed by the unitaries that map even-length Majorana strings into even-length Majorana strings, namely

$$\widehat{\mathcal{C}}_{2n}^{\Pi} := \{U \in \mathcal{U}(2^n) \mid U^\dagger M U = \phi_U M'\},$$

where $\phi_U \in \{\pm 1\}$, and $M, M' \in \mathcal{M}_{2n}^{\Pi}$. Moreover, $U \in \widehat{\mathcal{C}}_{2n}^{\Pi}$ will be referred to as *Majorana Clifford unitary*.

Proposition 4.19. *The generating set of the parity-preserving Majorana Clifford group is simply given by the so-called* BRAID4 *gate, namely*

$$\widehat{\mathcal{C}}_{2n}^{\Pi} = \langle e^{i\frac{\pi}{4}\gamma_a\gamma_b\gamma_c\gamma_d} \rangle,$$

where $\gamma_k \in \{X_k, Y_k, \}$, with $k = a, b, c, d$.

Proof. We refer the reader to [4]. □

Let us introduce the fermionic stabilizer subgroup, referring to [4].

Definition 4.14. A fermionic stabilizer subgroup $\mathcal{S}_{2n}$ is an abelian subgroup of the parity-preserving Majorana group $\mathcal{M}_{2n}^{\Pi}$, that is generated by $m \leq n$ independent even-length Majorana strings and does not contain the operator $-\mathbb{I}$. When $\mathcal{S}_{2n}$ has $m = n$ generators, it is called a maximal subgroup.

The notion of pure fermionic stabilizer state follows naturally.

Definition 4.15. Let $\mathcal{S}_{2n}$ be a maximal fermionic stabilizer subgroup. The state $|\psi\rangle$ is a pure fermionic stabilizer state if

$$s\,|\psi\rangle = |\psi\rangle\,, \forall s \in \mathcal{S}_{2n}.$$

By complement, a fermionic pure state that is non-stabilizer is called a *fermagic state*.

Analogously within the stabilizer setting, it is fundamental to specify that $|\psi\rangle$ is a pure fermionic stabilizer state if and only if it is uniquely defined, that is, the stabilizer subgroup $\mathcal{S}_{2n}$ must be maximal.

Definition 4.16. Any pure fermionic stabilizer state can be written as $U\,|0\rangle$, where $U \in \widehat{\mathcal{C}}_{2n}^{\Pi}$ and $|0\rangle$ is the n-mode vacuum pure fermionic stabilizer state.

4.4.2 What is Fermagic?

Interest towards fermionic stabilizerness has arisen since the foundational work of S. Bravyi, B. M. Terhal, and B. Leemhuis in [7]. Since we aim at replicating the stabilizer theory within the fermionic framework, and given that the aforementioned isomorphisms in Prop. 4.17 hold, the Gottesman-Knill theorem directly applies to this setting as well.

Theorem 4.24. *([4]) A quantum computer performing only Clifford unitaries, measurements of Pauli strings, and Clifford unitaries conditioned on classical bits, can be perfectly simulated in polynomial time on a probabilistic classical computer, that is,*

$$\text{Fermionic Stabilizerness} \implies \text{Classical simulability}.$$

Proof. We refer the reader to [4]. □

It is possible to reformulate Theorem 4.24 in terms of fermagic, namely

$$\text{Non-Classical Simulability} \implies \text{Fermagic}.$$

In other words, every fermionic state that is non-classically simulable (in the sense of fermionic stabilizerness) is a fermagic state. Let us recall the universal gate set for fermionic quantum computation expressed in Eq. 4.2. Notably, in the fermagic framework, the CT and the F4 gates swap their roles compared to the fermionic Gaussian

case. In particular, F4 is a BRAID4 gate by construction, and therefore is a generator of $\widehat{\mathcal{C}}_{2n}^{\Pi}$; hence, it can be seen as a free operation, in the sense that it preserves the fermionic stabilizerness. On the contrary, the CT gate represents a fermagic-generating gate.

Conclusions and Future Perspectives

In this thesis work, starting from the question of what makes a quantum computer more powerful than a classical one, we have seen that the Gottesman-Knill theorem ([20, 21]) singles out magic as the resource necessary for quantum computational advantage ([10]). In particular, the universal gate set for quantum computation has been shown to be {H, CNOT, S, T} ([5]), where the T gate is magic-generating. We have formalized the notion of magic within the language of QRTs ([11, 13, 14]), based on: (1) stabilizer states (Stab) as the free states; (2) Stabilizer Operations (SO) as the free operations; (3) magic states as the resourceful states; (4) a set of magic-quantifying functions, called monotones. A proper monotone M must satisfy the following conditions:

- $\rho \in \mathsf{Stab} \implies M(\rho) = 0$, that is, a monotone returns zero for a stabilizer input state;
- For every quantum state σ and $\Phi \in \mathsf{SO}$, $M(\sigma) \geq M(\Phi(\sigma))$, that is, no stabilizer operation is allowed to generate magic.

Moreover, we addressed the completeness problem concerning magic monotones ([16]), showing how the standard QRT can be extended through a larger class of free maps ([2, 24]), called the Completely Stabilizer-Preserving Operations (CSPO), and a suitable quotient structure. The main results are the following:

- The family of α-Stabilizer Rényi Entropies (SREs), despite being an infinite set of faithful and continuous monotones ([23, 33, 34]), fails to be complete in the single-shot convertibility regime. We constructed an explicit counterexample, based on the following pair of two-qubit magic states:

$$|\psi^{\pm}\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + e^{\pm i\pi/4}\,|11\rangle\right).$$

  Their SREs are computed to be equal (with a value of 0.5737), but no stabilizer operation can convert one state into the other one, implying that their magic content must be different.

- In order to generalize the QRT of magic, we enlarged the free operations from SO to CSPO, *i.e.*, all the maps that preserve the stabilizer structure and are closed under tensor products, namely $\Phi : \mathsf{Stab} \to \mathsf{Stab}$. We addressed the strict separation between SO and CSPO for a number of qubits $n \geq 2$, namely

$$\mathsf{SO} \subsetneq \mathsf{CSPO}.$$

- A second generalizing tool is represented by the magic partial order and the associated quotient set. In particular, we proved that all stabilizer states collapse to a single equivalence class, namely [Stab], while no magic state can ever be equivalent to a stabilizer state. However, we showed that even within the framework of equivalence classes it is not possible to evade the following no-go theorem: no finite, continuous, and faithful set of monotones is able to fully characterize the theory.

In the second part of the thesis, we turned to fermionic systems, where two distinct notions of non-classical simulability coexist — the fermionic stabilizer paradigm ([4, 35]) and the Gaussian paradigm ([8, 28, 29, 51, 54]). In particular, we analyzed the behavior of the universal gate set for fermionic computation ([9]), *i.e.*, {CT, F4}, in both frameworks. We also built an independent QRT for fermionic non-Gaussianity, based on: (1) fermionic Gaussian states (Gauss) as the free states; (2) fermionic Gaussian operations (FGO) as the free maps; (3) fermionic non-Gaussian states as the resourceful states; (4) a novel monotone, called the entropy of non-Gaussianity. Every fermionic Gaussian state assumes the form

$$\rho \propto \exp\left(\frac{i}{2}\sum_{i,j=1}^{2n}\left(-\arctan(\Gamma)\right)_{ij}\gamma_i\gamma_j\right),$$

where Γ is the correlation matrix and γ_k is the Majorana operator of the k-th fermionic mode. By Wick's theorem ([49]), every fermionic Gaussian state is completely and uniquely specified by its correlation matrix.
The key findings are the following:

- The entropy of non-Gaussianity is defined as the quantum relative entropy between a fermionic state and its Gaussianization, namely

$$\mathfrak{G}(\rho) := \mathrm{S}\left(\rho \parallel \mathcal{G}(\rho)\right),$$

  where $\mathcal{G}$ denotes the Gaussianization map, that is, $\mathcal{G}(\rho) \in \mathsf{Gauss}$. This novel monotone is shown to be: (1) a non-negative function, that is, $\mathfrak{G}(\rho) \geq 0$;
  (2) faithful, that is, $\mathfrak{G}(\rho) = 0 \iff \rho \in \mathsf{Gauss}$; (3) additive under tensor products of states, that is, $\mathfrak{G}(\rho^{\otimes n}) = n\mathfrak{G}(\rho)$; (4) invariant under Gaussian unitaries, that is, $\mathfrak{G}\left(U\rho U^\dagger\right) = \mathfrak{G}(\rho)$.

- However, the entropy of non-Gaussianity is much more than a new resource-quantifying function. Indeed, it carries a precise informational meaning: it can be understood as the difference, in terms of von Neumann entropies, between the Gaussianization of a fermionic state and the state itself, namely

$$\mathfrak{G}(\rho) = \mathrm{S}\left(\mathcal{G}(\rho)\right) - \mathrm{S}(\rho).$$

  In addition, it also offers an entropic-geometric interpretation: it quantifies the minimum distance, in terms of quantum relative entropy, between a fermionic state and an arbitrary Gaussian state, namely

$$\mathfrak{G}(\rho) = \min_{\sigma \in \mathrm{Gauss}} \mathrm{S}\left(\rho \parallel \sigma\right).$$

- The stabilizer paradigm in the fermionic setting is completely equivalent to the one in the qubit setting, yielding the fermionic version of the Gottesman-Knill theorem. In particular, the CT gate is a fermionic-magic-generating gate, in the same way that the T gate is a qubit-magic-generating gate.

- Within the Gaussian setting, the CT and F4 gates swap their roles: the latter represents a non-Gaussian-generating gate, while the former turns out to be Gaussian-preserving.

The results obtained in this thesis leave several questions open, both within the qubit and the fermionic settings, and naturally suggest a number of directions for future research.
While cellular automata were not treated in this thesis, they represent a natural playground for the study of magic and its fermionic counterparts. A Quantum Cellular Automaton (QCA) is an infinite lattice of sites occupied by qubits evolving under local rules that generate complex global behavior ([17, 41]). In particular, the following might be pursued:

- It would be valuable to understand under which conditions a QCA reduces to a stabilizer or magic circuit, and how the magic content behaves under different QCA evolution scenarios; this might help provide a classification, or a hierarchy, of QCAs (*e.g.*, [18, 42]).

- The production of magic states, known as distillation ([6, 40]), is the process of converting many copies of weakly-magic states into fewer, highly-magic ones. The question is whether a suitably structured QCA can act as a magic factory, that is, a lattice-based architecture capable of distilling a huge number of magic states. This aims at connecting the resource-theoretic perspective developed here to the operational task of magic state distillation.

The fermionic side of the theory raises an analogous set of questions. Fermionic Cellular Automata (FCAs) — the fermionic counterpart of QCAs ([52]) — admit exotic architectures, such as the Majorana shift automaton ([53]), that cannot be reproduced by simple fermionic quantum circuits.

- Extending the analysis of fermionic magic and fermionic non-Gaussianity to these exotic FCAs should help clarify how far the parallel between the qubit and fermionic pictures can be pushed, and where it breaks down. In doing so, other differences between the stabilizer and Gaussian paradigms might emerge.

Furthermore, this thesis has established a no-go theorem concerning the completeness of magic monotones, valid even after quotienting by the magic partial order.

- A natural continuation is to carry out the analogous analysis for the QRT of fermionic non-Gaussianity, where the ordering relation between states has not yet been fully characterized. Doing so would simplify the underlying QRT structure, aiding the search for a set of monotones that is not only complete but also computable in polynomial time.

- Beyond the single-copy convertibility regime treated here, the extension of the analysis within the many-copy and asymptotic regimes naturally arises. This might have useful implications within the quantum computation framework.

Finally, the strict separation between SO and CSPO (for $n \geq 2$ qubits) has only been proven within the qubit setting (see [24]). An open question is whether an analogous relation holds in the fermionic Gaussian framework, namely whether the set FGO is strictly contained in a suitably defined class of Completely Fermionic-Gaussian-Preserving Operations (CFGPO). Establishing such a separation would enrich and help complete the QRT of fermionic non-Gaussianity, both on the operational and on the structural side.

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