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Generalized Flag Manifolds through Dynkin Diagrams

Varietà di Bandiera Generalizzate attraverso i Diagrammi di Dynkin

Tesi di Laurea Magistrale in Matematica

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Abstract

This thesis provides an overview of the connection between Dynkin diagrams and generalized flag manifolds.
After introducing the preliminary notions concerning Lie algebras and Lie groups, the focus shifts to semisimple complex Lie algebras and their root space decomposition and, consequently, abstract root systems and Dynkin diagrams.
The second part of the work focuses on homogeneous spaces and their description as coset spaces, in particular, generalized flag manifolds are presented and, using the theory introduced earlier, classified by means of *painted* Dynkin diagrams.
The effectiveness of these representations is presented through the computation of some invariants associated with generalized flag manifolds using combinatorial arguments originating from the painted Dynkin diagrams.

Abstract

Questa tesi fornisce una panoramica della connessione tra diagrammi di Dynkin e varietà di bandiera generalizzate.
Dopo aver introdotto le nozioni preliminari riguardanti le algebre e i gruppi di Lie, ci si concentra sulle algebre di Lie complesse semisemplici, la loro decomposizione in spazi di radici e, conseguentemente, sui sistemi di radici astratti e diagrammi di Dynkin.
La seconda parte dell'elaborato è dedicata agli spazi omogenei e alla loro descrizione come spazi di classi laterali, in particolare, si introducono le varietà di bandiera generalizzate e, mediante la teoria introdotta precedentemente, esse vengono classificate per mezzo dei diagrammi di Dynkin *dipinti*.
L'efficacia di tali rappresentazioni è illustrata attraverso il calcolo di alcuni invarianti associati alle varietà di bandiera generalizzate utilizzando argomenti di tipo combinatorio basati sui diagrammi di Dynkin dipinti.

Contents

1 Preliminaries **9**
- 1.1 Lie Algebras . . . 9
- 1.2 Lie Groups . . . 11
- 1.3 Projective Spaces . . . 13

2 Root Systems **15**
- 2.1 Abstract Root Systems . . . 17
- 2.2 Weyl Group . . . 18
- 2.3 Dynkin Diagrams . . . 22
- 2.4 Real Forms . . . 23
- 2.5 Parabolic Subalgebras . . . 25
- 2.6 T-roots . . . 27

3 Homogeneous Spaces **29**
- 3.1 Actions of Lie groups . . . 29
- 3.2 Flag Manifolds . . . 30
- 3.3 Adjoint Action . . . 32
- 3.4 Reductive Homogeneous Spaces . . . 33

4 Generalized Flag Manifolds **35**
- 4.1 As Coset Spaces . . . 35
- 4.2 As Adjoint Orbits . . . 36
- 4.3 Painted Dynkin Diagrams . . . 37

5 Some Invariants **43**
- 5.1 Dimension of a Generalized Flag Manifold . . . 43
- 5.2 Euler Characteristic . . . 45

Chapter 1

Preliminaries

This chapter contains the general notations and the basic definitions that will be used in this work. The material presented follows mainly [Jac62] and [Arv03]. Throughout this chapter, $\mathbb{K}$ denotes a field of characteristic zero.

1.1 Lie Algebras

Definition 1.1.1.
A $\mathbb{K}$-vector space V endowed with a bilinear map $\mu : V \times V \to V$ is called an *algebra*. A subspace W of V is

- a *subalgebra* if $\mu(W \times W) \subseteq W$.
- an *ideal* if $\mu(V \times W) \cup \mu(W \times V) \subseteq W$.

Definition 1.1.2.
An algebra $\mathfrak{g}$ over $\mathbb{K}$ is called a *Lie algebra over* $\mathbb{K}$ if its multiplication (denoted by $[\cdot, \cdot]$) satisfies the identities:

1. $[X, X] = 0$
2. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi identity)

for all $X, Y, Z \in \mathfrak{g}$.

Definition 1.1.3.
A Lie algebra $\mathfrak{g}$ is called *simple* if it is non-abelian and its ideals are trivial.

Remark 1.1.4.
Given an associative algebra $(\mathfrak{A}, \cdot)$, one can define the corresponding Lie algebra by endowing $\mathfrak{A}$ with $[X, Y] := X \cdot Y - Y \cdot X$.

Definition 1.1.5.
If $\mathfrak{a}, \mathfrak{b}$ are Lie subalgebras of a Lie algebra $\mathfrak{g}$, $[\mathfrak{a}, \mathfrak{b}]$ denotes the subalgebra of $\mathfrak{g}$ generated by all elements of the form $[A, B]$, for $A \in \mathfrak{a}$, $B \in \mathfrak{b}$.

Definition 1.1.6.
The *derived series* of a Lie algebra $\mathfrak{g}$ is given by:

$$\mathfrak{g} \supseteq \mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}] \supseteq \mathfrak{g}'' := [\mathfrak{g}', \mathfrak{g}'] \supseteq \cdots \supseteq \mathfrak{g}^{(k)} := [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}] \supseteq \cdots$$

The *central lower series* is given by:

$$\mathfrak{g} \supseteq \mathfrak{g}^2 := [\mathfrak{g}, \mathfrak{g}] \supseteq \mathfrak{g}^3 := [\mathfrak{g}^2, \mathfrak{g}] \supseteq \cdots \supseteq \mathfrak{g}^k := [\mathfrak{g}^{k-1}, \mathfrak{g}] \supseteq \cdots$$

Now denote by $\mathfrak{g}$ a finite-dimensional Lie algebra. It can always be viewed as a subalgebra of $\mathfrak{gl}_n(\mathbb{K})$ endowed with the commutator.

Definition 1.1.7.
A Lie algebra $\mathfrak{g}$ is said to be *solvable* (resp. *nilpotent*) if $\mathfrak{g}^{(k)} = 0$ (resp. $\mathfrak{g}^k = 0$) for some natural number k. Observe that nilpotent implies solvable. The maximal solvable (resp. nilpotent) ideal is called *radical* (resp. *nilradical*). A Lie algebra is called *semisimple* if its radical is zero, equivalently if it is a direct sum of its ideals which are simple algebras.

Definition 1.1.8.
Let $\mathfrak{h}$ be a subalgebra of a Lie algebra $\mathfrak{g}$. Then $N_{\mathfrak{g}}(\mathfrak{h}) = \{X \in \mathfrak{g} \,|\, [X, \mathfrak{h}] \subseteq \mathfrak{h}\}$ is a subalgebra of $\mathfrak{g}$, called the *normalizer* of $\mathfrak{h}$.
A subalgebra of a Lie algebra $\mathfrak{g}$ is defined as a:

- *Cartan subalgebra* $\mathfrak{h}$ if it is nilpotent and $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.
- *Borel subalgebra* $\mathfrak{b}$ if it is a maximal solvable subalgebra.
- *parabolic subalgebra* $\mathfrak{p}$ if it is a subalgebra which contains a Borel subalgebra.

$$\mathfrak{h} \subseteq \mathfrak{b} \subseteq \mathfrak{p}.$$

Definition 1.1.9.
The *adjoint operator* of a Lie algebra $\mathfrak{g}$ is the homomorphism of Lie algebras:

$$\begin{array}{rcl} \mathrm{ad} : \mathfrak{g} & \longrightarrow & \mathrm{End}(\mathfrak{g}) \\ X & \longmapsto & [X, \cdot] \end{array}$$

Definition 1.1.10.
The *Killing form* of $\mathfrak{g}$ is the symmetric bilinear form given by $B(X, Y) := \mathrm{tr}(\mathrm{ad}X \circ \mathrm{ad}Y)$.

Proposition 1.1.11.
The following relation holds:

$$-B(\mathrm{ad}(Y)(X), Z) = B([X, Y], Z) = B(X, [Y, Z]) = B(X, \mathrm{ad}(Y)(Z)), \qquad X, Y, Z \in \mathfrak{g}$$

Example 1.1.12.
In the table below Killing forms associated with certain complex Lie algebras are listed:

$\mathfrak{g}$				$B(X,Y)$
$\mathfrak{gl}_n(\mathbb{C})$	$=$	$M(n, \mathbb{C})$		$2n\mathrm{tr}(XY) - 2\mathrm{tr}(X)\mathrm{tr}(Y)$
$\mathfrak{sl}_n(\mathbb{C})$	$=$	$\{X \in \mathfrak{gl}_n(\mathbb{C}) \,\vert\, \mathrm{tr}(X) = 0\}$	$n \geq 2$	$2n\mathrm{tr}(XY)$
$\mathfrak{so}_n(\mathbb{C})$	$=$	$\{X \in \mathfrak{gl}_n(\mathbb{C}) \,\vert\, X = -X^T\}$	$n \geq 2$	$(n-2)\mathrm{tr}(XY)$
$\mathfrak{sp}_{2n}(\mathbb{C})$	$=$	$\{X \in \mathfrak{gl}_{2n}(\mathbb{C}) \,\vert\, JX = -X^T J\}$		$2(n+1)\mathrm{tr}(XY)$

Proposition 1.1.13.
A Lie algebra $\mathfrak{g}$ is solvable if and only if $B(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$.

Proposition 1.1.14.
The Killing form of $\mathfrak{g}$ is non-degenerate if and only if $\mathfrak{g}$ is semisimple.

Proposition 1.1.15 (Levi decomposition)**.**
Every Lie algebra can be expressed as a semidirect product of its radical $\mathfrak{r}$ and a semisimple subalgebra $\mathfrak{l}$, called *Levi factor*:

$$\mathfrak{g} = \mathfrak{l} \ltimes \mathfrak{r}.$$

1.2 Lie Groups

Definition 1.2.1.
A differentiable manifold G is called a *Lie group* if it has a group structure such that the map $(g,h) \mapsto gh^{-1}$ of $G \times G$ onto G is smooth. Analogously, a complex manifold is called a (complex) Lie group if the previous map is holomorphic.

Definition 1.2.2.
A *Lie subgroup* of a Lie group is a subgroup that is an immersed submanifold.

Definition 1.2.3.
Define the left and right multiplication as follows:

$$\begin{aligned} L : G \times G &\longrightarrow G \\ (g,h) &\longmapsto L_g(h) := gh \end{aligned} \qquad\qquad \begin{aligned} R : G \times G &\longrightarrow G \\ (g,h) &\longmapsto R_g(h) := hg \end{aligned}$$

A vector field X on a Lie group G is called *left-invariant* if $L_{g_*}X = X$ for all $g \in G$.

Proposition 1.2.4.
Let G be a Lie group, let $\mathrm{Lie}(G) = \mathfrak{g}$ be the set of all left-invariant vector fields on G. The set $\mathfrak{g}$ equipped with the commutator $[\cdot,\cdot]$ is a Lie algebra. Furthermore, $\mathfrak{g}$ can be identified with T_eG.
The map $G \mapsto \mathfrak{g}$ defines a functor, called the *Lie functor*, from the category of Lie groups to the category of Lie algebras.

Theorem 1.2.5.
Every Lie algebra of finite dimension is the Lie algebra of (at least) a group.
Every Lie subalgebra of the Lie algebra of a Lie group has exactly one connected Lie subgroup associated.

$\mathfrak{g}_1 \cong \mathfrak{g}_2$	$\Rightarrow$	G_1, G_2 locally isomorphic
H Lie subgroup of G	$\Rightarrow$	$\mathfrak{h}$ Lie subalgebra of $\mathfrak{g}$
H normal Lie subgroup of G	$\Rightarrow$	$\mathfrak{h}$ ideal of $\mathfrak{g}$

These correspondences allow to define the following properties of a connected Lie group dually by means of its Lie algebra (see [Ale96]):

	G	$\mathfrak{g}$
Solvable	$\exists\, G^{(k)} = 1$	$\exists\, \mathfrak{g}^{(k)} = 0$
Nilpotent	$\exists\, G^{k} = 1$	$\exists\, \mathfrak{g}^{k} = 0$
Commutative	$g \cdot h = h \cdot g \quad \forall g, h \in G$	$[X, Y] = 0 \quad \forall X, Y \in \mathfrak{g}$
Simple	$H \lhd G \Rightarrow H$ discrete	$\mathfrak{i} \lhd \mathfrak{g} \Rightarrow \mathfrak{i} = 0$
Semisimple	$H \trianglelefteq G$ commutative $\Rightarrow H$ discrete	$\mathfrak{i} \trianglelefteq \mathfrak{g}$ commutative $\Rightarrow \mathfrak{i} = 0$
Reductive	$\exists S$ semisimple $\lhd G : G/S$ commutative	$\exists \mathfrak{a} \leq \mathfrak{g}$ commutative, $\exists \mathfrak{b} \leq \mathfrak{g}$ semisimple: $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$

Definition 1.2.6.
The *adjoint representation* of G is the homomorphism of Lie groups:

$$\begin{aligned} \mathrm{Ad} : G &\longrightarrow \mathrm{Aut}(\mathfrak{g}) \\ g &\longmapsto (L_g \circ R_{g^{-1}})_* \end{aligned}$$

There are several important relations involving the adjoint representation:

$$d(\mathrm{Ad})_e = \mathrm{ad};$$

$$\mathrm{Ad}\Big(\exp(X)\Big)(Y) = \exp\Big(\mathrm{ad}(X)\Big)(Y), \qquad X, Y \in \mathfrak{g}.$$

Furthermore, the Killing form is invariant under the adjoint representation:

$$B(X, Y) = B\Big(\mathrm{Ad}(g)(X), \mathrm{Ad}(g)(Y)\Big), \qquad X, Y \in \mathfrak{g},\ g \in G.$$

In a matrix Lie group G, the adjoint representation has a simple expression:

$$\mathrm{Ad}(g)(X) = gXg^{-1}, \qquad\qquad g \in G, X \in \mathfrak{g}.$$

Definition 1.2.7.
Let T be a subgroup of a Lie group G.
If T is isomorphic to $(\mathbb{K}^*)^n$ for some n, it is called an *algebraic torus.* ([Hum75])
If T is isomorphic to $(S^1)^n$ for some n, it is called a *torus.*
A subgroup T of a Lie group G is a *maximal torus* if it is a torus and is not properly contained in any other torus in G.

Proposition 1.2.8.
Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then, the following hold:

- Any torus is contained in a maximal torus.
- If G is compact, then any maximal torus is a maximal connected abelian subgroup of G.
- If G is compact, T is a connected Lie subgroup whose Lie algebra $\mathfrak{t}$ is a maximal abelian subalgebra of $\mathfrak{g}$, then T is a maximal torus in G.
- If G is compact and connected, then any element is contained in some maximal torus.
- If G is compact and connected, then any two maximal tori are conjugate.

Definition 1.2.9.
The *rank* of a compact and connected Lie group is the dimension of a maximal torus.

1.3 Projective Spaces

Let V be a $\mathbb{C}$-linear space, denote by $\mathbb{P}(V)$ the space of 1-dimensional subspaces of V; define $\mathbb{P}^n := \mathbb{P}(\mathbb{C}^{n+1})$.

$$\begin{aligned} \pi : \mathbb{C}^{n+1} \smallsetminus \{0\} &\longrightarrow \mathbb{P}^n \\ (x_0, \dots, x_n) &\longmapsto \{\lambda(x_0, \dots, x_n) \,|\, \lambda \in \mathbb{C}^*\} =: (x_0 : \dots : x_n), \end{aligned}$$

where we denote by *homogeneous coordinates* the last expression.
Let $p \in \mathbb{C}[X_0, \dots, X_n]$ be a homogeneous polynomial, the zero locus in $\mathbb{P}^n$ is well defined. Given a collection S of homogeneous polynomials, denote by $\mathcal{V}(S)$ the subset of $\mathbb{P}^n$ in which all polynomials in S vanish. A set of this form, equipped with the Zariski topology, is called *projective variety*. Define $U_i := \{(x_0 : \dots : x_n) \in \mathbb{P}^n \,|\, x_i \neq 0\}$, it holds that $U_i \cong \mathbb{C}^n$ via the map $(x_0 : \dots : x_n) \mapsto (\frac{x_0}{x_i}, \dots, \frac{\hat{x_i}}{x_i}, \dots, \frac{x_n}{x_i})$.

Chapter 2

Root Systems

Complex semisimple Lie algebras have several convenient properties; among these, the decomposition into root spaces plays a crucial role. It is indeed possible, by means of it, to establish a correspondence with abstract root systems and, finally, with the so-called Dynkin diagrams. These correspondences lead to a complete classification of semisimple Lie algebras and of the important, as will be shown in the next chapter, class of parabolic Lie subalgebras.
In this chapter $\mathfrak{g}$ will be a complex semisimple Lie algebra (unless otherwise specified) and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. We will follow mainly [Arv03] and [Ale96].

Remark 2.0.1.
In a complex semisimple Lie algebra, Cartan subalgebras are maximal abelian subalgebras such that $\mathrm{ad}(H)$ is diagonalizable for all $H \in \mathfrak{h}$.

Definition 2.0.2.
Given an element $\alpha \in \mathfrak{h}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$,

$$\mathfrak{g}^\alpha := \{X \in \mathfrak{g} : \mathrm{ad}(H)X = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}.$$

Any element $\alpha \in \mathfrak{h}^*$ such that $\alpha \neq 0$ and $\mathfrak{g}^\alpha \neq \{0\}$ is called a *root* of $\mathfrak{g}$. In this case, the set $\mathfrak{g}^\alpha$ is called the *root space* that corresponds to the root α. The set of all roots is denoted by R and is called the *root system* of $\mathfrak{g}$ (relative to $\mathfrak{h}$).

Theorem 2.0.3 (Root space decomposition)**.**
Since the endomorphisms $\mathrm{ad}(H)$ are diagonalizable for all $H \in \mathfrak{h}$ and commute with each other, they are simultaneously diagonalizable. Hence, there is a root space decomposition of $\mathfrak{g}$ (for a given Cartan subalgebra $\mathfrak{h}$):

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}^\alpha$$

Proposition 2.0.4.
Some of the main properties of this decomposition are:

1. $R = -R$.
2. $\alpha, \beta \in R \cup \{0\}$, $B(\mathfrak{g}^\alpha, \mathfrak{g}^\beta) = 0$ if $\alpha + \beta \neq 0$.
3. $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subseteq \mathfrak{g}^{\alpha+\beta}$.

4. The subspace $[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$ of $\mathfrak{g}^0 (= \mathfrak{h})$ has dimension 1.
5. For all $\alpha \in R$, $\dim(\mathfrak{g}^\alpha) = 1$.
6. The Killing form is non-degenerate on $\mathfrak{h}$. The resulting isomorphism $\sharp_B$ between $\mathfrak{h}^*$ and $\mathfrak{h}$ associates to each root in R a so-called *root vector*.
7. Let $E := \mathrm{span}_{\mathbb{R}} R$, $\mathfrak{h}_{\mathbb{R}} := E^{\sharp_B}$ is called *real form* of $\mathfrak{h}$ and its complexification is $\mathfrak{h}$. The Killing form B is positively definite on $\mathfrak{h}_{\mathbb{R}}$, the corresponding scalar product on $\mathfrak{h}_{\mathbb{R}}$ and E will be denoted by $(\cdot,\cdot)$.

Remark 2.0.5.
As a consequence of (5), one can choose a generator E_α of each $\mathfrak{g}^\alpha$ with $\alpha \in R$. This choice can be made in such a way that $B(E_\alpha, E_{-\alpha}) = \frac{2}{(\alpha,\alpha)}$. For (6), the generator of $[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$ can be chosen as $H_\alpha := [E_\alpha, E_{-\alpha}]$.
Note that, given $H \in \mathfrak{h}$:

$$B(H_\alpha, H) = B([E_\alpha, E_{-\alpha}], H) = B(E_\alpha, [E_{-\alpha}, H]) = B(E_\alpha, \alpha(H)E_{-\alpha}) = \alpha(H)\frac{2}{(\alpha,\alpha)}$$

For $\tilde{H}_\alpha$ such that $B(\tilde{H}_\alpha, H) = \alpha(H)$,

$$\alpha(H_\alpha) = B(H_\alpha, \tilde{H}_\alpha) = \alpha(\tilde{H}_\alpha)\frac{2}{(\alpha,\alpha)} = 2$$

The subalgebra of $\mathfrak{g}$ given by $\mathbb{C}H_\alpha \oplus \mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ by the identification:

$$H_\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{-\alpha} \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Example 2.0.6 ($\mathfrak{sl}_n$)**.**
The complex Lie algebra $\mathfrak{g} = \mathfrak{sl}_n$ is simple for $n > 1$, thus it admits a root space decomposition. It is easily verified that $\mathfrak{h} = \{H = \mathrm{diag}(h_1, \ldots, h_n) \in \mathfrak{sl}_n\}$ is a Cartan subalgebra of $\mathfrak{g}$.
Define, for $i = 1, \ldots, n$,

$$\begin{aligned} \varepsilon_i : \mathfrak{h} &\longrightarrow \mathbb{C} \\ H &\longmapsto h_i. \end{aligned}$$

Observe that, given $H \in \mathfrak{h}$ and $X \in \mathfrak{g}$,

$$[H, X] = \big((h_i - h_j)x_{ij}\big) = \big((\overbrace{\varepsilon_i - \varepsilon_j}^{\varepsilon_{ij}})(H)x_{ij}\big).$$

The root system is therefore $R = \{\varepsilon_{ij} \,|\, i, j = 1, \ldots, n,\ i \neq j\}$, with root spaces $\mathfrak{g}^{\varepsilon_{ij}} = \mathbb{C}E_{ij}$ (here E_{ij} denotes the matrix with 1 in the (i,j)-entry and 0 in all other ones). Furthermore, $H_{ij} = [E_{ij}, E_{ji}] = E_{ii} - E_{jj}$: one obtains $n^2 - n$ vectors while the Cartan subalgebra $\mathfrak{h}$ is $(n-1)$-dimensional.
The following decomposition is obtained:

$$\mathfrak{sl}_n = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathbb{C}E_{ij}.$$

In matrix form:

$$\begin{pmatrix} h_1 & a_{12} & \cdots & a_{1n} \\ a_{21} & h_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1\,n} \\ a_{n1} & \cdots & a_{n\,n-1} & h_n \end{pmatrix}$$

The correspondence between $\mathfrak{h}$ and $\mathfrak{h}^*$ is given by $B(X,Y) = 2n\mathrm{tr}(XY)$ (1.1.12)

$$\begin{aligned} \mathfrak{h}^* &\longrightarrow \mathfrak{h} \\ \varepsilon_{ij} &\longmapsto \frac{1}{2n} H_{ij}. \end{aligned}$$

Therefore $(\varepsilon_{ij}, \varepsilon_{ij}) := B\left(\frac{1}{2n}H_{ij}, \frac{1}{2n}H_{ij}\right) = \frac{1}{n}$.
Observe that $(E_{\varepsilon_{ij}}, E_{\varepsilon_{ji}}) = 2n\mathrm{tr}(E_{\varepsilon_{ij}}E_{\varepsilon_{ji}}) = 2n = \frac{2}{(\varepsilon_{ij},\varepsilon_{ij})}$, as requested in 2.0.5.
Another way to approach this case is via the Lie group: the *standard torus* is defined by $T = \{t = \mathrm{diag}(t_1,\ldots,t_n) : \prod_{i=1}^n t_i = 1, t_i \in \mathbb{C}\}$.
The obvious isomorphism

$$\begin{aligned} (\mathbb{C}^*)^{n-1} &\longrightarrow T \\ (t_1,\ldots,t_{n-1}) &\longmapsto \mathrm{diag}\left(t_1,\ldots,t_{n-1}, \left(\prod_{i=1}^{n-1} t_i\right)^{-1}\right) \end{aligned}$$

shows that T is indeed an algebraic $(n-1)$-torus.
T acts on $\mathfrak{g}$ by $\mathrm{Ad}(t)(X) = tXt^{-1}$. In particular $\mathrm{Ad}(t)(E_{ij}) = tE_{ij}t^{-1} = \frac{t_i}{t_j}E_{ij}$.
Hence the map $\alpha_{ij} : T \to \mathbb{C}^*$ given by $t \mapsto \frac{t_i}{t_j}$ corresponds to a root with respect to the Cartan subalgebra $\mathfrak{h} = \mathrm{Lie}(T)$; in fact $\mathbb{C}E_{ij} = \{X \in \mathfrak{g} \mid \mathrm{Ad}(t)(X) = \alpha_{ij}(t)X$ for all $t \in T\} = \{X \in \mathfrak{g} \mid \mathrm{ad}(H)(X) = \underbrace{d(\alpha_{ij})}_{\varepsilon_{ij}}(H)X$ for all $H \in \mathfrak{h}\} = \mathfrak{g}^{\varepsilon_{ij}}$.

Theorem 2.0.7.
Put $E = \mathrm{span}_{\mathbb{R}} R$ equipped with the scalar product $(\cdot,\cdot)$. The following properties hold:

(a) For all $\alpha, \beta \in R$, $\langle \alpha \mid \beta \rangle := \frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$.

(b) For all $\alpha \in R$, $S_\alpha R = R$, where $S_\alpha : \beta \mapsto \beta - \langle \beta \mid \alpha \rangle \alpha$ is the reflection with respect to hyperplane $\alpha^\perp$ in E.

(c) If $\alpha, \lambda\alpha \in R$ for some $\lambda \in \mathbb{R}$, then $\lambda = \pm 1$.

2.1 Abstract Root Systems

A finite set R of vectors in a Euclidean space E is called an *abstract root system* if R generates E and satisfies (a), (b), (c).

Definition 2.1.1.
A set $\mathcal{B} = \{\alpha_i\}_{i=1}^\ell \subseteq R$ is called a *basis* of R if it is a basis of E and $R \subseteq -\mathrm{span}_{\mathbb{N}}\mathcal{B} \cup \mathrm{span}_{\mathbb{N}}\mathcal{B}$, where $\mathrm{span}_{\mathbb{N}}\mathcal{B} := \{\sum_{i=1}^\ell k_i\alpha_i | k_i \in \mathbb{Z}, k_i \geq 0, \alpha_i \in \mathcal{B}\}$, its elements are called

simple roots. A basis is called *irreducible* if it cannot be expressed as a nontrivial disjoint union of orthogonal sets.
The elements of $R^+ := R \cap \mathrm{span}_{\mathbb{N}}\mathcal{B}$ (resp. $R^- := R \cap (-\mathrm{span}_{\mathbb{N}}\mathcal{B})$) are called *positive roots* (resp. *negative roots*). Hence a basis defines a decomposition $R = R^+ \sqcup R^-$. A basis always exists.

Definition 2.1.2.
Two root systems are said to be isomorphic if there exists a linear isomorphism that preserves $\langle \cdot \mid \cdot \rangle$.

Example 2.1.3 ($\mathfrak{sl}_n$)**.**
As seen in 2.0.6, a root system of $\mathfrak{sl}_n$ $(n > 1)$ is $R = \{\varepsilon_{ij} \mid i, j = 1, \dots, n,\ i \neq j\}$. The standard choice of its basis is $\mathcal{B} = \{\alpha_i := \varepsilon_{i,i+1} \mid i = 1, \dots, n-1\}$ and, consequently, $R^+ = \{\varepsilon_{ij} \mid i, j = 1, \dots, n,\ i < j\}$.

Definition 2.1.4.
Given a basis $\mathcal{B} = \{\alpha_i\}_{i=1}^{\ell}$ of a root system R, $a_{ij} := \langle \alpha_i \mid \alpha_j \rangle$ defines the *Cartan matrix* of the basis $\mathcal{B}$.
It can be characterized by the following properties:

- $a_{ij} \in \mathbb{Z}$, $a_{ii} = 2$, $a_{ij} \leq 0$ for $i \neq j$.
- $a_{ij} = 0 \iff a_{ji} = 0$.
- $m_{ij} := a_{ij}a_{ji} \in \{0, 1, 2, 3\}$ for $i \neq j$.
- The matrix given by $g_{ij} := \begin{cases} 1 & \text{if } i = j \\ -\frac{1}{2}\sqrt{m_{ij}} & \text{if } i \neq j \end{cases}$ is positively defined.

Proposition 2.1.5.
Let $\alpha, \beta \in R$ be roots in an abstract root system such that $|\beta| \geq |\alpha|$ and $(\alpha, \beta) \leq 0$. Denote by θ the angle between α and β. Then all possible cases are given in the following table:

θ	$\langle \alpha \mid \beta \rangle$	$\langle \beta \mid \alpha \rangle$	$\lvert\beta\rvert^2/\lvert\alpha\rvert^2$
$\pi/2$	0	0	
$2\pi/3$	-1	-1	1
$3\pi/4$	-1	-2	2
$5\pi/6$	-1	-3	3

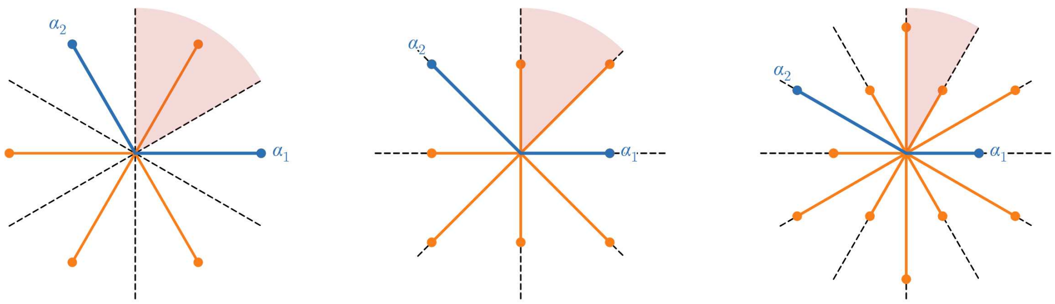
Theorem 2.1.6.
There is a one-to-one correspondence, up to isomorphism, between abstract root systems and complex semisimple Lie algebras.

2.2 Weyl Group

This section will follow mainly [BE89] and [Hal15].

Definition 2.2.1.
The subgroup $\mathfrak{W}$ of $O(E)$ defined by the reflections $S_{\alpha \in R}$ is called the *Weyl group*.
The hyperplane $\mathfrak{W}_\alpha := \{x \in E | x \perp \alpha\}$, with $\alpha \in R$, is called *wall* of α. The walls partition E into regions called *Weyl chambers* on which $\mathfrak{W}$ acts faithfully.

Given a basis $\mathcal{B}$, the positive chamber with respect to R^+ (determined by $\mathcal{B}$) is called *fundamental chamber*. Viceversa, given a chamber, there exists a unique basis for which it is fundamental. The action of $\mathfrak{W}$ on the set of Weyl chambers (and hence on the set of bases) is simply transitive.



(a) A_2 (b) B_2 (c) G_2

Walls dashed, fundamental chamber in red.
The indicated names and graphs will be clarified later.

Example 2.2.2 ($\mathfrak{sl}_n$)**.**
The Weyl group $\mathfrak{W}$ acts on $\mathfrak{sl}_n$ as follows:

$$
\begin{array}{rrcll}
S_{\alpha_k}: & R & \longrightarrow & R & \\
& \varepsilon_{ij} & \longmapsto & \varepsilon_{ij} - \overbrace{\langle \varepsilon_{ij} \mid \alpha_k \rangle}^{\delta_{ik} - \delta_{i\,k+1} - \delta_{jk} + \delta_{j\,k+1}} \alpha_k & \\
& \pm\alpha_k & \longmapsto & \mp\alpha_k & \\
& \varepsilon_{kj} & \longmapsto & \varepsilon_{k+1\,j} & j \neq k+1 \\
& \varepsilon_{k+1\,j} & \longmapsto & \varepsilon_{k\,j} & j \neq k \\
& \varepsilon_{ik} & \longmapsto & \varepsilon_{i\,k+1} & i \neq k+1 \\
& \varepsilon_{i\,k+1} & \longmapsto & \varepsilon_{ik} & i \neq k
\end{array}
$$

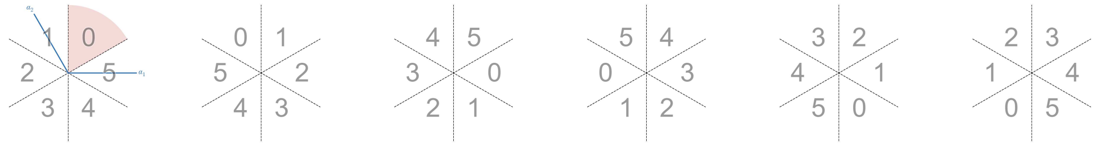
and it leaves the other roots unchanged.
Observe that it simply consists in the permutation of the indices $(k\ k+1)$: $\mathfrak{W} \cong S_n$.

Definition 2.2.3.
Fixed a basis $\mathcal{B}$, given $w \in \mathfrak{W}$, $\ell(w) := \min\{n \in \mathbb{N} \,|\, w = S_{\alpha_1} \circ \cdots \circ S_{\alpha_n},\ \alpha_i \in \mathcal{B}\}$ is called the *length* of w; such a decomposition is called a *reduced expression*. There is a unique element of maximal length ([BE89]).

Definition 2.2.4 (Bruhat ordering)**.**
Let $w_1, w_2 \in \mathfrak{W}$ such that $\ell(w_2) = \ell(w_1) + 1$ and there exists $\alpha \in R$ with $w_2 = S_\alpha \circ w_1$; denoted by $w_1 \to w_2$. The induced partial ordering relation $\leq$ on $\mathfrak{W}$ is called *Bruhat ordering*.

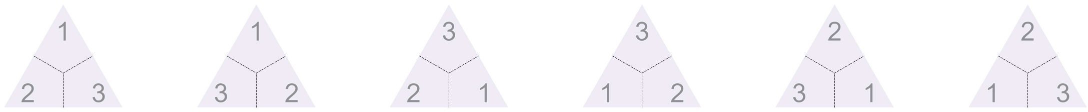
Example 2.2.5 ($\mathfrak{sl}_3$)**.**
Denote by $\langle i_1 \cdots i_n \rangle$ the element of the Weyl group $S_{\alpha_{i_1}} \circ \cdots \circ S_{\alpha_{i_n}}$.

Note that $\mathfrak{W}$ is isomorphic to S_3:



(a) $\langle\rangle$ (b) $\langle 1\rangle$ (c) $\langle 2\rangle$ (d) $\langle 12\rangle$ (e) $\langle 21\rangle$ (f) $\langle 121\rangle$

Action of $\mathfrak{W}$ on the numbered Weyl chambers of $\mathfrak{sl}_3$.



(a) () (b) (23) (c) (13) (d) (123) (e) (132) (f) (12)

Action of S_3 on numbered adjacent Weyl chambers.

$$\begin{tikzcd}
& \langle 1\rangle \arrow[r] \arrow[ddr] & \langle 21\rangle \arrow[dr] & \\
\langle\rangle \arrow[ur] \arrow[dr] & & & \langle 121\rangle \\
& \langle 2\rangle \arrow[r] \arrow[uur] & \langle 12\rangle \arrow[ur] &
\end{tikzcd}$$

Bruhat ordering on W.

Definition 2.2.6 (Weyl Group W)**.**
Given a connected Lie group G and a torus T in it, the quotient

$$W(G,T) \coloneqq N_G(T)/C_G(T)$$

is called *Weyl group of G relative to T*. Since all maximal tori are conjugate, all their Weyl groups are isomorphic; such a group is called the *Weyl group of G* and it is denoted by $W(G)$.

Theorem 2.2.7.
Let G be a connected complex semisimple Lie group, then $W(G)$ is isomorphic to the Weyl group $\mathfrak{W}$ associated with $\mathfrak{g} \coloneqq \mathrm{Lie}(G)$.

Proof. It can be proved that $C_G(T) = T$, with T a maximal algebraic torus, in a semisimple Lie group (see [Hal15]). Denote by $\mathfrak{h} \coloneqq \mathrm{Lie}(T)$ the Cartan subalgebra associated with T. It can be proved ([Hal15], §11) that the following map defines a group isomorphism.

$$\begin{aligned} \Phi : N_G(T)/T &\longrightarrow \mathfrak{W} \\ [g] &\longmapsto \mathrm{Ad}(g)_{|\mathfrak{h}_{\mathbb{R}}} \end{aligned}$$

We will focus on the proof of the fact that $\mathfrak{W}$ is in fact contained in $\mathrm{Im}(\Phi)$ and on finding an explicit map g from $\mathfrak{W}$ to $N_G(T)$ such that $\Phi \circ g = \mathbb{1}_{\mathfrak{W}}$.

- Let $g \in N_G(T)$, $t \in T$, one obtains $\mathrm{Ad}(gt)(H) = g\, \underbrace{tHt^{-1}}_{\mathrm{Ad}(t)(H)}\, g^{-1} = gHg^{-1} = \mathrm{Ad}(g)(H)$. Therefore Φ is well-defined.

- Since $g \in N_G(T)$, $\mathrm{Ad}(g)(\mathfrak{h}) \subseteq \mathfrak{h}$. Given $\alpha \in R$, it suffices to show that $\mathrm{Ad}(g)(H_\alpha) = H_\beta$ for some $\beta \in R$. Let $H \in \mathfrak{h}$ (here H_α denotes the vector in $\mathfrak{h}$ corresponding to $\alpha \in \mathfrak{h}^*$ by the Killing form):

$$\begin{aligned} B(\mathrm{Ad}(g)(H_\alpha), H) &= B(H_\alpha, \mathrm{Ad}(g^{-1})(H)) = \alpha(\mathrm{Ad}(g^{-1})(H)) = \\ &= \underbrace{(\alpha \circ \mathrm{Ad}(g^{-1}))}_{=:\beta\in\mathfrak{h}^*}(H) = \beta(H). \end{aligned}$$

Thus $\mathrm{Ad}(g)(H_\alpha) = H_\beta$, it remains to show that $\beta \in R$. Let $H \in \mathfrak{h}$:

$$\begin{aligned} [H, \mathrm{Ad}(g)(E_\alpha)] &= [\mathrm{Ad}(g)(\overbrace{\mathrm{Ad}(g^{-1}(H))}^{=:\tilde{H}}), \mathrm{Ad}(g)(E_\alpha)] = \mathrm{Ad}(g)\left([\tilde{H}, E_\alpha]\right) = \\ &= \mathrm{Ad}(g)(\alpha(\tilde{H})E_\alpha) = \alpha(\tilde{H})(\mathrm{Ad}(g)(E_\alpha)) = \beta(H)(\mathrm{Ad}(g)(E_\alpha)). \end{aligned}$$

Thus $\mathrm{Ad}(g)(E_\alpha) \in \mathfrak{g}^\beta \smallsetminus 0$.

- Using the notation in Remark 2.0.5, let $\alpha \in R$, then $g_\alpha := \exp\left(\frac{\pi}{2}(E_\alpha - E_{-\alpha})\right) \in N_G(T)$ and $\mathrm{Ad}(g_\alpha) = S_\alpha \in \mathfrak{W}$:

$$\mathrm{Ad}(g_\alpha)(H) = \mathrm{Ad}(\exp(\frac{\pi}{2}(E_\alpha - E_{-\alpha})))(H) = \exp(\frac{\pi}{2}\mathrm{ad}(E_\alpha - E_{-\alpha}))(H)$$

Thus, if $\alpha \perp \beta$,

$$\mathrm{Ad}(g_\alpha)(H_\beta) = H_\beta + \frac{\pi}{2}\underbrace{[E_\alpha - E_{-\alpha}, H_\beta]}_{-\alpha(H_\beta)(E_\alpha - E_{-\alpha})} + \cancel{\cdots} = H_\beta.$$

Observe that:

$$\begin{aligned} &[E_\alpha - E_{-\alpha}, iE_\alpha + iE_{-\alpha} + H_\alpha] = 2iH_\alpha - \overbrace{\alpha(H_\alpha)}^{2}(E_\alpha + E_{-\alpha}) = 2i(iE_\alpha + iE_{-\alpha} + H_\alpha); \\ &[E_\alpha - E_{-\alpha}, iE_\alpha + iE_{-\alpha} - H_\alpha] = -2i(iE_\alpha + iE_{-\alpha} - H_\alpha). \end{aligned}$$

Therefore, $\Phi([g_\alpha])(H_\alpha) = -H_\alpha$:

$$\begin{aligned} &\exp\left(\mathrm{ad}(\frac{\pi}{2}(E_\alpha - E_{-\alpha}))\right)(H_\alpha) = \\ &= \frac{1}{2}\exp\left(\mathrm{ad}(\frac{\pi}{2}(E_\alpha - E_{-\alpha}))\right)\Big((iE_\alpha + iE_{-\alpha} + H_\alpha) - (iE_\alpha + iE_{-\alpha} - H_\alpha)\Big) = \\ &= e^{i\pi}\frac{1}{2}\Big((iE_\alpha + iE_{-\alpha} + H_\alpha) - (iE_\alpha + iE_{-\alpha} - H_\alpha)\Big) = -H_\alpha. \end{aligned}$$

One can finally conclude that $\Phi([g_\alpha]) = S_\alpha$.

□

Proposition 2.2.8 (Bruhat Decomposition)**.**
Let G be a connected complex reductive algebraic group. Given a Borel subgroup B and a maximal torus T associated to it, note that the double coset BwB is well-defined for all $w \in W(G,T)$. The following decomposition holds:

$$G = \bigsqcup_{w \in W(G,T)} BwB.$$

Example 2.2.9 (SL_3)**.**
Consider $G = \mathrm{SL}_3$ with the standard torus $T = \{\mathrm{diag}(t_1, \dots, t_3) \mid \prod_{i=1}^{3} t_i = 1\}$ and the Borel subgroup of upper-triangular matrices B.

- $N_G(T) = \{$ generalized permutation matrices $\in \mathrm{SL}_3\}$
- $N_G(T)/T = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\} \cong S_3.$
- Computing BwB for each $w \in W(G)$, one obtains:

$$G = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \right\} \sqcup \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} : a_{21} \neq 0 \right\} \sqcup \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} \neq 0 & a_{33} \end{pmatrix} : a_{32} \neq 0 \right\} \sqcup$$
$$\sqcup \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} : a_{31} \neq 0, \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \neq 0 \right\} \sqcup \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} : a_{31} \neq 0, \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = 0 \right\} \sqcup$$
$$\sqcup \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} : a_{21}, a_{32} \neq 0 \right\},$$
it is implicit that every matrix has determinant equal to 1.

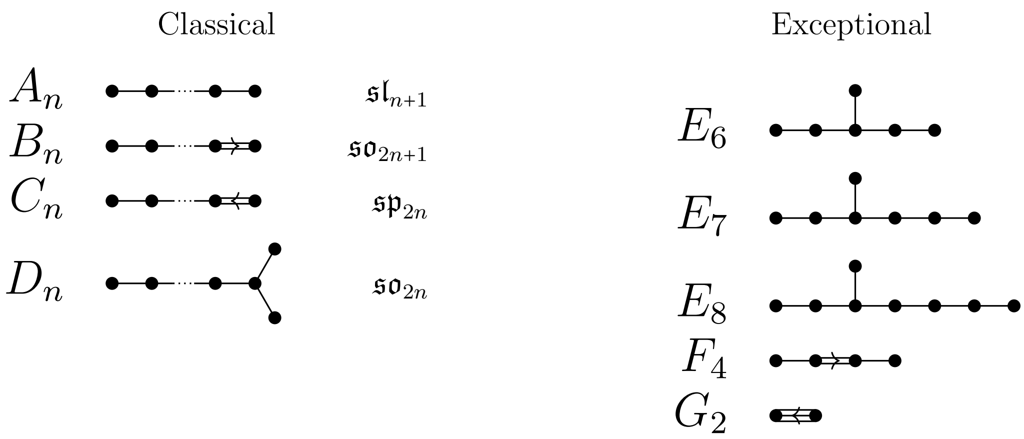
2.3 Dynkin Diagrams

Definition 2.3.1.
Given a basis $\mathcal{B} = \{\alpha_i\}_{i=1}^{\ell}$ of a root system R, the *Dynkin diagram* $\Delta(\mathcal{B})$ is the graph that has ℓ vertices $\{v_i\}_{i=1}^{\ell}$. Two vertices v_i and v_j are joined by $m_{ij} := a_{ij}a_{ji}$ edges. If $|\alpha_i| > |\alpha_j|$ and $(\alpha_i, \alpha_j) \neq 0$, then the edge is oriented from v_i to v_j. The bijection $\alpha_i \mapsto v_i$ is called an *equipment* of the Dynkin diagram Δ.

Theorem 2.3.2.
The Dynkin diagram determines, up to root system isomorphism, the root system.

Remark 2.3.3.
A basis is irreducible if and only if its Dynkin diagram is connected, moreover the direct orthogonal sum of root systems is a root system. Therefore only irreducible bases are considered in this section.

Theorem 2.3.4.
The only possible Dynkin diagrams are the following:



2.4 Real Forms

A real Lie algebra $\mathfrak{g}$ can be extended to a complex Lie algebra $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C}$, called the *complexification* of $\mathfrak{g}$. Viceversa, $\mathfrak{g}$ is called the *real form* of $\mathfrak{g}^{\mathbb{C}}$.
Any complex semisimple Lie algebra admits a unique (up to a conjugation by an automorphism) *compact* real form, i.e. a real form on which the Killing form is negative definite.

Definition 2.4.1.
Given a complex semisimple Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathbb{C}E_\alpha$, define the antiholomorphic involutive automorphism σ on $\mathfrak{g}$ such that

$$\begin{array}{rcll} \sigma : \mathfrak{g} & \longrightarrow & \mathfrak{g} & \\ E_\alpha & \longmapsto & -E_{-\alpha} & \text{for } \alpha \in R \\ h & \longmapsto & -h & \text{for } h \in \mathfrak{h}_{\mathbb{R}} \end{array}$$

The compact real form is the set of fixed points of σ:

$$\mathfrak{g}^\sigma = i\mathfrak{h}_{\mathbb{R}} \oplus \bigoplus_{\alpha \in R^+} \operatorname{span}_{\mathbb{R}}(E_\alpha - E_{-\alpha}, i(E_\alpha + E_{-\alpha})).$$

Example 2.4.2 ($\mathfrak{sl}_3$)**.**
Using the decomposition seen in 2.0.6:

$$(\mathfrak{sl}_3)^\sigma = \left\{ \begin{pmatrix} ih_1 & a + i\alpha & b + i\beta \\ -a + i\alpha & ih_2 & c + i\gamma \\ -b + i\beta & -c + i\gamma & ih_3 \end{pmatrix} : h_1 + h_2 + h_3 = 0 \right\} = \{A \in \mathfrak{sl}_3 \,|\, A = -A^\dagger\} = \mathfrak{su}(3)$$

Proposition 2.4.3.
A description of classical root systems can be provided using the following notation (from [Ale96]):

$$\operatorname{Aut}(V, b) := \{A \in \mathfrak{gl}(V) \,|\, b(A\cdot, \cdot) + b(\cdot, A\cdot) = 0\}$$

where b is a non degenerate bilinear form in the linear space V. Denote the dual space of V as V^* and the dual basis of $\{e_i\}$ as $\{e_i^*\}$, $x \vee y := x \otimes y + y \otimes x$, $x \wedge y := x \otimes y - y \otimes x$.

A_n: Denote by $\{e_i\}_{i=1}^{n+1}$ the standard basis of $\mathbb{C}^{n+1}$. There is an identification between $\mathfrak{gl}_{n+1}$ and $\mathbb{C}^{n+1} \otimes (\mathbb{C}^{n+1})^*$.

B_n: Denote by $\{e_i\}_{i=-n}^{n}$ the standard basis of $\mathbb{C}^{2n+1}$. Define the symmetric bilinear form $g := e_0^* \otimes e_0^* + \sum_{i=1}^{n} e_i^* \vee e_{-i}^*$, it determines an identification with the dual space of $\mathbb{C}^{2n+1}$ and B_n can be expressed as $\mathrm{Aut}(\mathbb{C}^{2n+1}, g)$. A Cartan subalgebra can be described as $\mathfrak{h} = \{\sum_{i=1}^{n} x_i e_i \wedge e_{-i}\}$, define $\varepsilon_i : \mathfrak{h} \to \mathbb{C}$ such that $\varepsilon_i(h) = x_i$.

C_n: Denote by $\{e_i, e_{-i}\}_{i=1}^{n}$ the standard basis of $\mathbb{C}^{2n}$. Define the skew-symmetric bilinear form $\omega := \sum_{i=1}^{n} e_i^* \wedge e_{-i}^*$, it determines an identification with the dual space of $\mathbb{C}^{2n}$ and C_n can be expressed as $\mathrm{Aut}(\mathbb{C}^{2n}, \omega)$. A Cartan subalgebra can be described as $\mathfrak{h} = \{\sum_{i=1}^{n} x_i e_i \vee e_{-i}\}$, define $\varepsilon_i : \mathfrak{h} \to \mathbb{C}$ such that $\varepsilon_i(h) = x_i$.

D_n: Denote by $\{e_i, e_{-i}\}_{i=1}^{n}$ the standard basis of $\mathbb{C}^{2n}$. Define the symmetric bilinear form $g := \sum_{i=1}^{n} e_i^* \vee e_{-i}^*$, it determines an identification with the dual space of $\mathbb{C}^{2n}$ and D_n can be expressed as $\mathrm{Aut}(\mathbb{C}^{2n}, g)$. A Cartan subalgebra can be described as $\mathfrak{h} = \{\sum_{i=1}^{n} x_i e_i \wedge e_{-i}\}$, define $\varepsilon_i : \mathfrak{h} \to \mathbb{C}$ such that $\varepsilon_i(h) = x_i$.

Type	$\mathfrak{g}$	$\dim \mathfrak{g}$	$\mathfrak{h}$	Root vectors	Root system R	$\mathcal{B}$	$\mathfrak{W}$
A_n	$\mathfrak{sl}_{n+1}(\mathbb{C})$	$n(n+2)$	$\sum_{i=1}^{n+1} x_i e_i \otimes e_i^*$ $\sum_{i=1}^{n+1} x_i = 0$	$e_i \otimes e_j^*$ $i \neq j$	$\varepsilon_i - \varepsilon_j$	$\alpha_i = \varepsilon_i - \varepsilon_{i+1}$	S_{n+1}
B_n	$\mathfrak{so}_{2n+1}(\mathbb{C})$	$n(2n+1)$	$\sum_{i=1}^{n} x_i e_i \wedge e_{-i}$	$e_i \wedge e_{-j}$ $e_i \wedge e_j$ $e_{-i} \wedge e_{-j}$ $e_{\pm i} \wedge e_0$	$\varepsilon_i - \varepsilon_j$ $\varepsilon_i + \varepsilon_j$ $-\varepsilon_i - \varepsilon_j$ $\pm\varepsilon_i$	$\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $i = 1, \dots, n-1$ $\alpha_n = \varepsilon_n$	$S_n \ltimes \mathbb{Z}_2^n$
C_n	$\mathfrak{sp}_{2n}(\mathbb{C})$	$n(2n+1)$	$\sum_{i=1}^{n} x_i e_i \vee e_{-i}$	$e_i \vee e_{-j}$ $e_i \vee e_j$ $e_{-i} \vee e_{-j}$	$\varepsilon_i - \varepsilon_j$ $\varepsilon_i + \varepsilon_j$ $-\varepsilon_i - \varepsilon_j$	$\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $i = 1, \dots, n-1$ $\alpha_n = 2\varepsilon_n$	$S_n \ltimes \mathbb{Z}_2^n$
D_n	$\mathfrak{so}_{2n}(\mathbb{C})$	$n(2n-1)$	$\sum_{i=1}^{n} x_i e_i \wedge e_{-i}$	$e_i \wedge e_{-j}$ $e_i \wedge e_j$ $e_{-i} \wedge e_{-j}$	$\varepsilon_i - \varepsilon_j$ $\varepsilon_i + \varepsilon_j$ $-\varepsilon_i - \varepsilon_j$	$\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $i = 1, \dots, n-1$ $\alpha_n = \varepsilon_{n-1} + \varepsilon_n$	$S_n \ltimes \mathbb{Z}_2^{n-1}$

In all these root systems the action of the Weyl group is given by permutations of indices and arbitrary inversions of signs (an even number thereof in D_n, arbitrary in B_n, C_n).

2.5 Parabolic Subalgebras

Given the semisimple Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}^\alpha$, one can generate a subset of it by choosing $R_0 \subseteq R$:

$$\mathfrak{g}(R_0) := \underbrace{\langle \{H_\alpha\}_{\alpha \in R_0 \cap -R_0} \rangle}_{\mathfrak{h}_{R_0}} \oplus \bigoplus_{\alpha \in R_0} \mathfrak{g}^\alpha.$$

For $\mathfrak{g}(R_0)$ to be a subalgebra, it is necessary and sufficient that, for all $\alpha, \beta \in R_0$, $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subseteq \mathfrak{g}(R_0)$; for (3) in Proposition 2.0.4, i.e., if $\alpha + \beta \in R$, then $\alpha + \beta \in R_0$. A subset with this characteristic is called *closed*.
A closed subset of R can be provided via a subset $\mathcal{B}_0$ of $\mathcal{B}$, define $[\mathcal{B}_0]^\pm := \langle \mathcal{B}_0 \rangle \cap R^\pm$: the set $R_{\mathcal{B}_0} := [\mathcal{B}_0]^- \sqcup R^+$ is closed.

Proof. Let $\mathcal{B} = \{\alpha_i\}_{i=1}^n \subset R$. For simplicity, define $\mathcal{B}_0 = \{\alpha_i\}_{i=1}^m$, $m \leq n$. Observe that both $[\mathcal{B}_0]$ and R^+ are closed, thus it is sufficient to verify if, given $\alpha \in [\mathcal{B}_0]^-$ and $\beta \in R^+$ such that $\alpha + \beta \in R^-$, $\alpha + \beta \in R_{\mathcal{B}_0}$. $\alpha = \sum_{i=1}^m (-a_i)\alpha_i$, $\beta = \sum_{i=1}^n (b_i)\alpha_i$, with $a_i, b_i \in \mathbb{Z} \setminus \mathbb{Z}^-$.

$$\alpha + \beta = \sum_{i=1}^m \underbrace{(b_i - a_i)}_{\leq 0} \alpha_i + \sum_{i=m+1}^n \underbrace{b_i}_{\leq 0} \alpha_i$$

Therefore $b_i = 0$ for $i = m+1, \dots, n$ and one obtains that $\alpha + \beta \in [\mathcal{B}_0] \subset R_{\mathcal{B}_0}$. □

Denote with $\mathfrak{z}_{\mathcal{B}_0}$ the orthogonal complement in $\mathfrak{h}$ to the subalgebra $\mathfrak{h}_{[\mathcal{B}_0]} = \mathfrak{h}_{R_{\mathcal{B}_0}}$,

$$\mathfrak{p}_{\mathcal{B}_0} := \mathfrak{z}_{\mathcal{B}_0} \oplus \mathfrak{g}(R_{\mathcal{B}_0}).$$

Note that, by including $\mathfrak{z}_{\mathcal{B}_0}$, the Cartan subalgebra $\mathfrak{h}$ is always contained and $\mathfrak{p}_{\mathcal{B}_0}$ is itself a subalgebra, hence a parabolic one: $\mathfrak{z}$ is trivially closed by bracket and $\mathfrak{g}(R_{\mathcal{B}_0})$ is closed by construction, let $\tilde{H} \in \mathfrak{z}_{\mathcal{B}_0}$ and $H + X \in \mathfrak{h}_{R_0} \oplus \bigoplus_{\alpha \in R\mathcal{B}_0} \mathfrak{g}^\alpha = \mathfrak{g}(R_{\mathcal{B}_0})$, with $X = \sum_{\alpha \in R_{\mathcal{B}_0}} x_\alpha E_\alpha$,

$$[\tilde{H}, H + X] = \underbrace{[\tilde{H}, H]}_{0} + [\tilde{H}, X] = \sum_{\alpha \in R_{\mathcal{B}_0}} x_\alpha \alpha(\tilde{H}) E_\alpha \in \bigoplus_{\alpha \in R_{\mathcal{B}_0}} \subseteq \mathfrak{p}_{\mathcal{B}_0}.$$

Remark 2.5.1.
It is possible to express $\mathfrak{p}_{\mathcal{B}_0}$ as a vector space in several ways: $\mathfrak{p}_{\mathcal{B}_0}$ is a direct sum of a maximal reductive subalgebra (that is, a direct sum of an abelian and a semisimple subalgebra) and the nilradical.

$$\begin{aligned} \mathfrak{p}_{\mathcal{B}_0} &= \underbrace{\mathfrak{h} \oplus \bigoplus_{\alpha \in [\mathcal{B}_0]} \mathfrak{g}^\alpha}_{\substack{\mathfrak{k}_{\mathcal{B}_0} \\ \text{maximal reductive}}} \oplus \underbrace{\bigoplus_{\alpha \in R^+ \setminus [\mathcal{B}_0]} \mathfrak{g}^\alpha}_{\substack{\mathfrak{n}_{\mathcal{B}_0} \\ \text{nilradical}}} = \\ &= \underbrace{\mathfrak{z}_{\mathcal{B}_0}}_{\text{abelian}} \oplus \underbrace{\mathfrak{g}([\mathcal{B}_0])}_{\text{semisimple}} \oplus \mathfrak{n}_{\mathcal{B}_0}. \end{aligned}$$

Observe that, in general, $\mathfrak{k}_{\mathcal{B}_0}$ is not an ideal. This decomposition is a version, for parabolic Lie algebras, of the Levi decomposition (see Proposition 1.1.15), here the Levi factor is given by $\mathfrak{g}([\mathcal{B}_0])$.
Using this notation, the entire Lie algebra $\mathfrak{g}$ can be decomposed (depending on $\mathcal{B}_0$) as follows:

$$\mathfrak{g} = \mathfrak{k}_{\mathcal{B}_0} \oplus \overbrace{\underbrace{\mathfrak{g}(R^- \setminus [\mathcal{B}_0])}_{n^-_{\mathcal{B}_0}} \oplus \underbrace{\mathfrak{g}(R^+ \setminus [\mathcal{B}_0])}_{n^+_{\mathcal{B}_0}}}^{\mathfrak{m}_{\mathcal{B}_0}}. \qquad \text{(generalized Gauss decomposition)}$$

In particular, for $\mathcal{B}_0 = \varnothing$, one obtains:

$$\mathfrak{g} = \underbrace{\mathfrak{g}(R^-)}_{\mathfrak{n}^-} \oplus \mathfrak{h} \oplus \underbrace{\mathfrak{g}(R^+)}_{\mathfrak{n}^+}. \qquad \text{(Gauss decomposition)}$$

Example 2.5.2.
Fix $\mathfrak{g} = \mathfrak{sl}_4$ with the standard equipment (2.1.3).

- $\mathcal{B}_0 = \varnothing \rightsquigarrow R_{\mathcal{B}_0} = R^+ = \{\varepsilon_{ij} \,|\, i < j\}$.

$$\mathfrak{p} = \mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{i<j} \mathbb{C}E_{ij} = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix} \in \mathfrak{sl}_4 \right\}.$$

  Note that the Gauss decomposition for $\mathfrak{g} = \mathfrak{sl}_n$ is therefore a decomposition in strictly lower triangular ($\mathfrak{n}^-$), diagonal ($\mathfrak{h}$) and strictly upper triangular ($\mathfrak{n}^+$) matrices.

- $\mathcal{B}_0 = \{\alpha_1, \alpha_2\} \rightsquigarrow [\mathcal{B}_0] = \{\pm\varepsilon_{12}, \pm\varepsilon_{23}, \pm\varepsilon_{13}\} \rightsquigarrow R_{\mathcal{B}_0} = \{\varepsilon_{21}, \varepsilon_{32}, \varepsilon_{31}\} \cup \{\varepsilon_{ij} \,|\, i < j\}$;
  $\mathfrak{h}_{[\mathcal{B}_0]} = \langle\{\pm H_{12}, \pm H_{23}, \pm H_{13}\}\rangle = \{\mathrm{diag}(h_1, h_2, h_3, 0) \in \mathfrak{sl}_4\}$.
  Let $Z = \mathrm{diag}(z_1, z_2, z_3, z_4) \in \mathfrak{z}_{\mathcal{B}_0} \subseteq \mathfrak{h}$, a basis for $\mathfrak{h}_{[\mathcal{B}_0]}$ is given, for example, by $\{H_{12}, H_{13}\}$. As seen in 1.1.12, $B(X, Y) = 8\mathrm{tr}(XY)$, the orthogonality condition can therefore be satisfied by imposing

$$\begin{cases} B(H_{12}, Z) = 8(z_1 - z_2) = 0 \\ B(H_{13}, Z) = 8(z_1 - z_3) = 0 \end{cases}$$

  i.e. $z_1 = z_2 = z_3$. It follows that $\mathfrak{z}_{\mathcal{B}_0} = \{\mathrm{diag}(k, k, k, -3k) \,|\, k \in \mathbb{C}\}$.

$$\begin{aligned} \mathfrak{p} &= \mathfrak{h} \oplus \bigoplus_{\alpha \in [\mathcal{B}_0]^-} \mathfrak{g}^\alpha \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}^\alpha = \\ &= \mathfrak{h} \oplus \mathbb{C}E_{21} \oplus \mathbb{C}E_{32} \oplus \mathbb{C}E_{31} \oplus \bigoplus_{i<j} \mathbb{C}E_{ij} = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix} \in \mathfrak{sl}_4 \right\} = \end{aligned}$$

$$= \mathfrak{z}_{\mathcal{B}_0} \oplus \mathfrak{h}_{[\mathcal{B}_0]} \oplus \bigoplus_{\alpha \in [\mathcal{B}_0]} \mathfrak{g}^\alpha \oplus \bigoplus_{\alpha \in R^+ \setminus [B_0]} \mathfrak{g}^\alpha = \left\{ \begin{pmatrix} h_1 + k & a_{12} & a_{13} & a_{14} \\ a_{21} & h_2 + k & a_{23} & a_{24} \\ a_{31} & a_{32} & h_3 + k & a_{34} \\ 0 & 0 & 0 & -3k \end{pmatrix} \in \mathfrak{sl}_4,\ k \in \mathbb{C} \right\}.$$

Proposition 2.5.3.
Any parabolic subalgebra is conjugated to a subalgebra of the form $\mathfrak{p}_{\mathcal{B}_0}$, for a subset $\mathcal{B}_0$ of $\mathcal{B}$.

Definition 2.5.4.
A *painted Dynkin diagram* is a Dynkin diagram Δ together with a subset of its vertices Δ_0 (represented graphically by white vertices).

2.6 T-roots

Let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}^\alpha$ be a complex semisimple Lie algebra, let $\mathcal{B}_0 \subseteq \mathcal{B}$. Denote by $\mathfrak{t} := \mathfrak{z}_{\mathcal{B}_0} \cap \mathfrak{h}_{\mathbb{R}}$.

Definition 2.6.1.
Given a root $\alpha \in R \setminus [\mathcal{B}_0]$, its restriction $\tilde{\alpha} := \alpha_{|\mathfrak{t}}$ is called a *T-root*. The set of T-roots will be denoted by R_T. An element $t \in \mathfrak{t}$ is called *regular* if $\tilde{\alpha}(t) \neq 0$ for all $\tilde{\alpha} \in R_T$. The set of regular elements will be denoted by $\mathfrak{t}_{\text{reg}}$, its connected components are called *T-chambers*.

Definition 2.6.2.
The subgroup of the Weyl group defined by $\mathfrak{W}^{\mathcal{B}_0} := \{w \in \mathfrak{W} | w[\mathcal{B}_0] = [\mathcal{B}_0]\}$ is called a *T-Weyl group*.

Example 2.6.3.
Fix $\mathfrak{g} = \mathfrak{sl}_3$ with the standard equipment and consider $\mathfrak{b} = \mathfrak{p}_\varnothing$.
So that $\mathfrak{t} = \mathfrak{z}_\varnothing \cap \mathfrak{h}_{\mathbb{R}} = \mathfrak{h}_{\mathbb{R}}$.
Let $H \in \mathfrak{t}$, $\tilde{\varepsilon}_{ij}(H) = h_i - h_j = 0 \iff h_i = h_j$.
T-chambers are therefore given by: $\{h_{\tau(1)} < h_{\tau(2)} < h_{\tau(3)}\}_{\tau \in S_3}$.

Chapter 3

Homogeneous Spaces

The combination of differential and group structures characteristic of Lie groups has an immediate application to actions on manifolds. Remarkably, the latter can be described entirely in terms of a quotient of the given Lie group and, consequently, through a quotient of the associated Lie algebra. This chapter will follow mainly [Arv03].

3.1 Actions of Lie groups

The first part of this section will follow [Lee13] and [Var74].

Definition 3.1.1.
A (complex) manifold endowed with a transitive smooth (holomorphic) action by a (complex) Lie group G is called a *homogeneous G-space*.

Theorem 3.1.2.
Let G be a Lie group and let K be a closed subgroup of G. The left coset space G/K is a topological manifold of dimension equal to $\dim G - \dim K$, and has a unique structure such that the quotient map $\pi : G \to G/K$ is a smooth (holomorphic) submersion.
The left action of G on G/K given by $g_1 \cdot [g_2] = \tau_{g_1}([g_2]) := [g_1 g_2]$ turns G/K into a homogeneous G-space.

Theorem 3.1.3.

Let G be a Lie group, M a homogeneous G-space.
Fixed a point p in M, the *isotropy group* $G_p := \{g \in G : g \cdot p = p\}$ is a closed subgroup of G, and the map

$$\begin{aligned}\Phi : G/G_p &\to M\\ [g] &\mapsto g \cdot p\end{aligned}$$

is an equivariant diffeomorphism (biholomorphism).

$$\begin{tikzcd} G/G_p \arrow[r, "\Phi"] \arrow[d, "\tau_g"'] & M \arrow[d, "\tau_g"] \\ G/G_p \arrow[r, "\Phi"'] & M \end{tikzcd}$$

Proposition 3.1.4.
The map

$$d\pi_e : \mathfrak{g} \longrightarrow T_{eK}(G/K)$$

defines a canonical isomorphism

$$\mathfrak{g}/\mathfrak{k} \cong T_{eK}(G/K).$$

Proof. From Theorem 3.1.2, we know that π is a submersion, hence it is sufficient to show that $\mathfrak{k} = \ker(d\pi_e)$:

$$d\pi_e(X) = \frac{d}{dt}_{|0} \exp(tX)K = 0 \iff \exp(\mathbb{R}X) \in K \iff X \in \mathfrak{k}.$$

□

Definition 3.1.5.
The *isotropy representation* of the homogeneous space G/K is the homomorphism

$$\begin{aligned} \mathrm{Ad}^{G/K} : K &\to \mathrm{Aut}(T_{eK}(G/K)) \\ k &\mapsto d(\tau_k)_{eK}. \end{aligned}$$

3.2 Flag Manifolds

A useful type of manifolds to apply actions of groups is the one given by flag manifolds.

Definition 3.2.1.
Let V be a n-dimensional $\mathbb{K}$-vector space and $\mathbf{p} = (p_1, \dots, p_r)$ an r-tuple of natural numbers such that $0 < p_1 < \dots < p_r < n$. A *flag of type* $\mathbf{p}$ is a system $f = (V_1, \dots, V_r)$ of subspaces of V such that $V_i \subseteq V_{i+1}$ and $\dim V_i = p_i$. Define $\mathcal{F}_{\mathbf{p}}(V)$ the set of all $\mathbf{p}$-flags of V.
A flag of type $(1, \dots, n-1)$ is called a *full flag.*
The set of (p)-flags, with $0 < p < n$, is called a *Grassmannian of type* p and is denoted by $\mathrm{Gr}_{\mathbb{K}}(p, n)$. Whenever not specified, assume $\mathbb{K} = \mathbb{C}$.

Definition 3.2.2 (Plücker Embedding)**.**
Given a k-subspace $W = \langle v_1, \dots, v_k \rangle$ of V, the set $\bigwedge^k W$ can be viewed as a 1-subspace of $\bigwedge^k V$. Denote with $\{e_i\}_{i=1}^n$ a basis of V, let $A \in M_{k\times n}(\mathbb{C})$ be the matrix with v_i (with respect to the chosen basis) as its i-th row and define the *Plücker coordinates* ([EH16]) $p_{i_1,\dots,i_k}$ as the determinant of the minor made from the columns $i_1, \dots, i_k$.

Observe that

$$v_1 \wedge \cdots \wedge v_k = \sum_{i_1 < \cdots < i_k} \underbrace{p_{i_1,\ldots,i_k}}_{\text{Plücker coordinates}} e_{i_1} \wedge \cdots \wedge e_{i_k}.$$

The choice of A is not unique, in fact a matrix $A' \in M_{k\times n}(\mathbb{C})$ corresponds to the subspace W if and only if there exists a matrix $\Omega \in \mathrm{GL}_k(\mathbb{C})$ such that $A' = \Omega A$. The effect on the Plücker coordinates is $p'_{i_1,\ldots,i_k} = \det(\Omega) p_{i_1,\ldots,i_k}$.
The Grassmannian can be embedded in $\mathbb{P}(\wedge^k V)$ as follows:

$$\begin{array}{rcll} \mathrm{Gr}(k,n) & \longrightarrow & \mathbb{P}(\wedge^k V) \cong \mathbb{P}^{\binom{n}{k}-1} & \textit{Plücker embedding} \\ \langle v_1,\ldots,v_k\rangle & \longmapsto & [v_1 \wedge \cdots \wedge v_k] = (p_{1,2,\ldots,k-1,k} \overset{\text{lex}}{:\cdots:} p_{n-k+1,\ldots,n}). & \end{array}$$

This embedding gives to $\mathrm{Gr}(k,n)$ the structure of a projective variety ([Hum75]).

Remark 3.2.3.
Note that a flag can be viewed as

$$\mathcal{F}_{\mathbf{p}} = \{(V_1,\ldots,V_r) \in \prod_{i=1}^{r} \mathrm{Gr}(p_i,n) \,|\, V_1 \subseteq \cdots \subseteq V_r\}.$$

It can be proven that flags are projective varieties.

Example 3.2.4.
Consider an element of $\mathbb{C}^n$ as a $n \times 1$-matrix, thus a flag manifold can be seen as collection of systems of sets of column matrices. The group $G = \mathrm{SL}_n(\mathbb{C})$ acts by left multiplication on $M = \mathcal{F}_{\mathbf{p}}(\mathbb{C}^n)$:

$$\begin{array}{c} G \times M \longrightarrow M \\ (A, f = (V_1 \subset \cdots \subset V_r)) \longmapsto Af := (AV_1 \subset \cdots \subset AV_r), \end{array}$$

where $AV_j := \{Av \,|\, v \in V_j\}$: observe that, being A invertible, $\dim(AV_i) = \dim(V_i) = p_i$. Furthermore, this action is transitive: let $f_1 = (V_1 \subset \cdots \subset V_r), f_2 = (W_1 \subset \cdots \subset W_r) \in M$, there are two bases $\{v_i\}_{i=1}^n$, $\{w_i\}_{i=1}^n$ of $\mathbb{C}^n$ such that $\langle v_i\rangle_{i=1}^{p_j} = V_j$ and $\langle w_i\rangle_{i=1}^{p_j} = W_j$, the matrix A such that $Av_i = w_i$ is invertible, thus, let $c^n = det(A)^{-1}$, $\tilde{A} := cA \in \mathrm{SL_n} = G$ and $\tilde{A} f_1 = f_2$.

Let $\{e_i\}_{i=1}^n$ be the standard basis of $\mathbb{C}^n$, the *standard* $\mathbf{p}$*-flag*, $\mathbf{p} = (p_1,\ldots,p_r)$ is defined as $f_0 = (\mathrm{span}(e_1,\ldots,e_{p_i}))_{i=1}^r$. The stabilizer can be computed directly: let $A \in G$,

$$A \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_{p_i} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_{p_i} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \iff A = \begin{pmatrix} a_{11} & \cdots & a_{1\,p_i} & a_{1\,p_i+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{p_i\,1} & \cdots & a_{p_i\,p_i} & a_{p_i\,p_i+1} & \cdots & a_{p_i\,n} \\ 0 & \cdots & 0 & a_{p_i+1\,p_i+1} & \cdots & a_{p_i+1\,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & a_{n\,p_i+1} & \cdots & a_{n\,n} \end{pmatrix}.$$

The stabilizer G_{f_0} can be therefore expressed as

$$\bigcap_{i=1}^{r}\{A \in G \,|\, a_{ij} = 0 \text{ if } i > p_i, j < p_i + 1\},$$

that is the set of all upper block-triangular matrices in $\mathrm{SL}_n(\mathbb{C})$ of the following form:

$$P = G_{f_0} = \left\{ \begin{pmatrix} A_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & A_{r+1} \end{pmatrix} \in \mathrm{SL}_n(\mathbb{C}) \mid A_i \in \mathrm{GL}_{p_i - p_{i-1}}(\mathbb{C}),\ i = 1, \ldots, r+1 \right\}$$

with $p_0 := 0$ and $p_{r+1} := n$.
Similarly, one may proceed with any subgroup of SL_n. For example, considering the compact Lie group $U(n)$, the stabilizer becomes $P \cap U(n)$:

$$(U(n))_{f_0} = \left\{ \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_{r+1} \end{pmatrix} \in \mathrm{GL}_n(\mathbb{C}) \mid A_i \in \mathrm{U}(p_i - p_{i-1}),\ i = 1, \ldots, r+1 \right\}.$$

Therefore $\mathcal{F}_{\mathbf{p}}(\mathbb{C}^n) \cong \mathrm{SL}_n/P \cong U(n)/(U(p_1)\times U(p_2 - p_1)\times\cdots\times U(p_{r+1}-p_r))$. As a particular case, full flags can be described as $U(n)/(U(1))^n = U(n)/T^n$.The appearance is not exclusive to full flags: it should be observed that, in general, $U(n_1)\times\cdots\times U(n_r)$ is the centralizer of the torus $\{\mathrm{diag}(\underbrace{e^{i\theta_1}, \ldots, e^{i\theta_1}}_{n_1 \text{ times}}, \cdots, \underbrace{e^{i\theta_r}, \ldots, e^{i\theta_r}}_{n_r \text{ times}}),\quad \theta_i \in \mathbb{R},\ i = 1, \ldots, r\}$.

3.3 Adjoint Action

Proposition 3.3.1.
Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. The adjoint representation defines a smooth action of G on $\mathfrak{g}$:

$$\begin{aligned} \mathrm{Ad} : G \times \mathfrak{g} &\longrightarrow \mathfrak{g} \\ (g, X) &\longmapsto g \cdot X := \mathrm{Ad}(g)(X). \end{aligned}$$

In general, this action is not transitive: denote by $\mathfrak{g}_X := \mathrm{Ad}(G)(X)$ the orbit of $X \in \mathfrak{g}$. One obtains, using the fact that the action is transitive on an orbit, that

$$G/G_X \cong \mathfrak{g}_X.$$

Example 3.3.2 ($U(3)$)**.**
The unitary group $U(3)$ corresponds to the real Lie algebra $\mathfrak{u}(3) = \{X \in M(3,\mathbb{C}) \,|\, X = -X^\dagger\}$.

- Fix $X = 0 \in \mathfrak{u}(3)$: $\mathfrak{g}_0 = \{0\}$, $G_0 = G$.

- Fix $X = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{u}(3)$, $G_X = \left\{ \begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix} \mid \theta_i \in \mathbb{R} \right\} \cong U(1)^3$

Proposition 3.3.3.
Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Given $X \in \mathfrak{g}$, the isotropy subgroup G_X coincides with the centralizer of the torus $T_X := \overline{\exp(\mathbb{R}X)}$. Furthermore, the Lie algebra of G_X is the kernel of the adjoint endomorphism $\operatorname{ad}(X) = [X, \cdot]$.

Proof.
We want to show that $\operatorname{Ad}(g)(X) = gXg^{-1} = X \iff g \in C_G(T_X) = C_G(\exp(\mathbb{R}X))$ (by continuity of the product in G). Using that $g\exp(tX)g^{-1} = \exp(gtXg^{-1})$:

$$X = gXg^{-1} \iff \frac{d}{dt}_{|0} \exp(tX) = \frac{d}{dt}_{|0} \exp(gtXg^{-1}).$$

Let $Y \in \operatorname{Lie}(G_X)$, we have $\operatorname{Ad}(\exp(tY))(X) = X$ for all t: $0 = \frac{d}{dt}_{|0} \operatorname{Ad}(\exp(tY))(X) = \operatorname{ad}(Y)(X) = -\operatorname{ad}(X)(Y)$, that is $Y \in \ker(\operatorname{ad}(X))$. Viceversa, let $Y \in \ker(\operatorname{ad}(X))$, we have $\operatorname{Ad}(\exp(tY))(X) = \exp(\operatorname{ad}(tY))(X) = X$: $\exp(\mathbb{R}Y) \subseteq G_X$, hence $Y \in \operatorname{Lie}(G_X)$. □

3.4 Reductive Homogeneous Spaces

This section will follow [Arv03] (Chapters 4, 7).

Definition 3.4.1.
A homogeneous space G/K is called *reductive* if there exists an $\operatorname{Ad}(K)$-invariant subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Observe that $\mathfrak{m} \cong \mathfrak{g}/\mathfrak{k}$, hence (from 3.1.4) $\mathfrak{m} \cong T_{eK}(G/K)$.

Proposition 3.4.2.
A homogeneous space G/K, with G a compact Lie group, is reductive.

Proposition 3.4.3.
Let G/K be a reductive homogeneous space and Ad^G (resp. Ad^K) the adjoint representation of G (resp. K). Let $k \in K$, $X \in \mathfrak{k}$ and $Y \in \mathfrak{m}$. Then

$$\operatorname{Ad}^G(k)(X + Y) = \operatorname{Ad}^K(k)X + \operatorname{Ad}^{G/K}(k)Y.$$

Proof. (from [Arv03])
Since $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, it suffices to consider:

$Y = 0$: Trivial.

$X = 0$: We want to prove that $\operatorname{Ad}^G(k)(Y) = d(\tau_k)_{eK}(Y)$, that is, the following diagram commutes:

$$\begin{tikzcd} \mathfrak{m} \arrow[r, "\operatorname{Ad}(k)"] \arrow[d, "d(\pi)_e"] & \mathfrak{m} \arrow[d, "d(\pi)_e"] \\ T_{eK}(G/K) \arrow[r, "d(\tau_k)_{eK}"] & T_{eK}(G/K) \end{tikzcd}$$

$$(d(\pi)_e \circ \mathrm{Ad}(k))(Y) = \frac{d}{dt}_{|0} \exp\Big(t\mathrm{Ad}(k)(Y)\Big)K = \frac{d}{dt}_{|0} \exp\Big(\mathrm{Ad}(k)(tY)\Big)K = \frac{d}{dt}_{|0}\Big(k\exp(tY)k^{-1}\Big)K = \frac{d}{dt}_{|0}\tau_k\Big(\exp(tY)K\Big) = (d(\tau_k)_{eK} \circ d(\pi)_e)(Y).$$

□

Definition 3.4.4.
A G-invariant almost complex structure on $M = G/K$ is a linear map J on $T_{eK}M \cong \mathfrak{m}$ such that $J^2 = -\mathbb{1}$ and, for all $k \in K$, the following diagram commutes:

$$\begin{tikzcd} \mathfrak{m} \arrow[r, "\mathrm{Ad}^{G/K}(k)"] \arrow[d, "J"'] & \mathfrak{m} \arrow[d, "J"] \\ \mathfrak{m} \arrow[r, "\mathrm{Ad}^{G/K}(k)"'] & \mathfrak{m} \end{tikzcd}$$

Define the eigenspaces of $J^{\mathbb{C}}$ in $\mathfrak{m}^{\mathbb{C}}$ with respect to eigenvalues $\pm i$ as $\mathfrak{m}^{\pm}$, giving the decomposition

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^+ \oplus \mathfrak{m}^-.$$

Chapter 4

Generalized Flag Manifolds

An important class of manifolds is given by the generalization of flag varieties: it can be obtained, from a real Lie group, by focusing on the presence of a torus, or, from a complex Lie group, by parabolic subgroups (as observed in 3.2.4). Furthermore, they can be obtained as orbits of the adjoint representation, defining a correspondence between these orbits and regular elements modulo the T-Weyl group. Of particular interest for this work is the fact that it is possible to provide a complete description via painted Dynkin diagrams. This chapter will follow mainly [Ale96], [Ale98] and [Arv03].

4.1 As Coset Spaces

Definition 4.1.1.
A homogeneous space of the form $G/C_G(T)$, where G is a compact semisimple Lie group and T is a torus, is called a *generalized flag manifold*.

Proposition 4.1.2.
There is a correspondence between generalized flag manifolds $G/C_G(T)$ and the homogeneous space $\tilde{G}/P$, where $\tilde{G}$ is a semisimple complex Lie group and P a parabolic subgroup.

Proof. (from [Ale96])
Let G/K, with $K = C_G(T)$, be a generalized flag manifold, there exists a maximal torus $\tilde{T}$ such that $T \subseteq \tilde{T}$, hence $\tilde{T} = C_G(\tilde{T}) \subseteq C_G(T) = K$. It follows that $\mathfrak{h}^{\mathbb{C}} \subseteq \mathfrak{k}^{\mathbb{C}}$, where $\mathfrak{h} := \mathrm{Lie}(\tilde{T})$, a Cartan subalgebra in $\mathfrak{g}^{\mathbb{C}}$. Thus we can decompose $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}^{\alpha}$ and there exists a subset $R_K \subseteq R$ such that $\mathfrak{k}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in R_K} \mathfrak{g}^{\alpha}$. The choice of a basis (see 4.1.5) $\mathcal{B}$ determines a unique parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}^{\mathbb{C}}$ associated with R_K: we get that G/K is diffeomorphic to $G^{\mathbb{C}}/P$.
Viceversa, let G be a connected complex semisimple Lie group ($\mathrm{Lie}(G) = \mathfrak{g}$) and P a parabolic subgroup ($\mathrm{Lie}(P) = \mathfrak{p} \subseteq \mathfrak{g}$). The compact Lie group G^{σ}, associated with $\mathfrak{g}^{\sigma}$, acts transitively on G/P, thus $G/P \cong G^{\sigma}/G^{\sigma}_{eP} = G^{\sigma}/G^{\sigma} \cap P$. It can be proved that G^{σ}_{eP} is connected. Hence, it is determined by the Lie algebra $\mathfrak{g}^{\sigma} \cap \mathfrak{p} = \mathfrak{p}^{\sigma}$,with

$\mathfrak{p} := \mathrm{Lie}(P)$. Using the notation in 2.5, one obtains:

$$\mathfrak{p}^\sigma = \overbrace{\mathfrak{z}^\sigma \oplus \mathfrak{g}([\mathcal{B}_0])^\sigma}^{\mathfrak{k}^\sigma} \oplus \underbrace{\mathfrak{n}^\sigma}_{0} = \mathfrak{k}^\sigma.$$

Being $\mathfrak{z}^\sigma$ abelian in a compact Lie algebra, its Lie subgroup $T \leq G^\sigma$ is an abelian and connected subgroup of a compact Lie group, since it is also closed, it is a torus. Observe that $\mathfrak{p}^\sigma = \mathfrak{k}^\sigma = C_{\mathfrak{g}^\sigma}(\mathfrak{z}^\sigma)$. Being the centralizer of a torus in a connected compact semisimple Lie group connected, this implies $G^\sigma_{eP} = C_{G^\sigma}(T)$. □

As a consequence, the following definition is equivalent to the latter.

Definition 4.1.3.
A homogeneous space of the form G/P, where G is a complex semisimple Lie group and P is a parabolic subgroup, is called a *generalized flag manifold.*

Remark 4.1.4.
Let $M = G/K$, with $K = C_G(T)$, be a generalized flag manifold. Being G compact, from 3.4.2, its Lie algebra has a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and $T_{eK}M \cong \mathfrak{m}$. Let $\mathfrak{h} := \mathrm{Lie}(\tilde{T})$, with $\tilde{T}$ a maximal torus containing T (as seen in the proof above), $\mathfrak{g}^\mathbb{C} = \mathfrak{h}^\mathbb{C} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}^\alpha$ and $\mathfrak{k}^\mathbb{C} = \mathfrak{h}^\mathbb{C} \oplus \bigoplus_{\alpha \in R_K} \mathfrak{g}^\alpha$, from this one obtains:

$$(T_{eK}M)^\mathbb{C} \cong \mathfrak{m}^\mathbb{C} = \bigoplus_{\alpha \in R \setminus R_K} \mathfrak{g}^\alpha.$$

Remark 4.1.5.
Given a generalized flag manifold $M = G/K$, $K = C_G(T)$, a basis $\mathcal{B}$ of R (determined by $\mathfrak{h} := \mathrm{Lie}(\tilde{T})$) such that $\mathcal{B}_0 := \mathcal{B} \cap R_K$ is a basis of R_K determines a G-invariant complex structure on M defined through the parabolic subalgebra $\mathfrak{p}_{\mathcal{B}_0} = \mathfrak{h}^\mathbb{C} \oplus \bigoplus_{\alpha \in [\mathcal{B}_0] \cup R^+}$ (where R^+ is determined by the basis $\mathcal{B}$):

$$\mathfrak{m}^+ := d\pi_e^\mathbb{C}(\mathfrak{p}_{\mathcal{B}_0}).$$

4.2 As Adjoint Orbits

Definition 4.2.1.
Given a generalized flag manifold $M = G/P \cong G^\sigma/K^\sigma$, an element $X \in \mathfrak{g}^\sigma$ is said to *define a realization* of M if $G^\sigma_X = K^\sigma$. In this case there is a canonical identification

$$\begin{aligned} G^\sigma/K^\sigma &\cong \mathrm{Ad}_{G^\sigma}(X) \\ eK^\sigma &\leftrightarrow X \end{aligned}$$

Proposition 4.2.2.
An element $X \in \mathfrak{g}^\sigma$ defines a realization of G^σ/K^σ if and only if $iX \in \mathfrak{t}_{\mathrm{reg}}$.
Furthermore, if $X_1, X_2 \in \mathfrak{g}^\sigma$ define a realization and are in the same orbit of G^σ, then they are in the same orbit of the T-Weyl group.

Therefore, there is an identification between $\mathfrak{t}_{\text{reg}}/\mathfrak{W}^P$ and the set of adjoint orbits isomorphic to M.

Proof. (from [Ale96])
If $X \in \mathfrak{g}^\sigma$ has stabilizer K^σ, then $\mathfrak{k}^\sigma = C_{\mathfrak{g}^\sigma}(X)$. Hence, $X \in Z(\mathfrak{k}^\sigma) = \mathfrak{z}^\sigma = i\mathfrak{h}_{\mathbb{R}} \cap \mathfrak{z}$, as seen in 2.4.1. It holds that $[\mathcal{B}_0] = \{\alpha \in R | \alpha(X) = 0\}$:

- $\boxed{\subseteq}$ Let $\alpha \in [\mathcal{B}_0]$, observe that $\alpha(X)(E_\alpha) = [X, \overset{\in \mathfrak{k} = C_{\mathfrak{g}}(X)}{E_\alpha}] = 0$. It follows that $\alpha(X) = 0$.

- $\boxed{\supseteq}$ Let $\alpha \in R$ such that $\alpha(X) = 0$, we have $0 = \alpha(X)(E_\alpha) = [X, E_\alpha]$, therefore $E_\alpha \in C_{\mathfrak{g}}(X) = \mathfrak{k}$. It follows that $\alpha \in [\mathcal{B}_0]$.

One concludes that $R_T = R \smallsetminus [\mathcal{B}_0] = \{\alpha \in R | \alpha(X) \neq 0\}$ and thus, considering that $iX \in \mathfrak{h}_{\mathbb{R}} \cap \mathfrak{z} = \mathfrak{t}$, $iX \in \mathfrak{t}_{\text{reg}}$. Analogously the viceversa.

Let $X_1, X_2 \in \mathfrak{g}^\sigma$ in the same orbit, such that $G^\sigma_{X_1} = G^\sigma_{X_2}$. Then $\mathrm{Ad}(g)(X_1) = gX_1g^{-1} = X_2$ for some $g \in G^\sigma$, this implies $K^\sigma = G^\sigma_{X_1} = g^{-1}G^\sigma_{X_2}g = g^{-1}K^\sigma g$, one obtains $g \in N_{G^\sigma}(K^\sigma)$. Hence $\mathrm{Ad}(g) \in \mathrm{Aut}(\mathfrak{k}^\sigma)$, that implies that the center $\mathfrak{z}^\sigma$ is fixed. Let $\mathfrak{h}_1^\sigma = \mathfrak{z}^\sigma \oplus \tilde{\mathfrak{h}}_1$ be a Cartan subalgebra and $\mathfrak{h}_2^\sigma := \mathrm{Ad}(g)(\mathfrak{h}_1) = \mathfrak{z}^\sigma \oplus \tilde{\mathfrak{h}}_2$. Being $\mathfrak{k}^\sigma = \mathfrak{z}^\sigma \oplus (\mathfrak{k}^\sigma)'$ with $(\mathfrak{k}^\sigma)'$ semisimple and $\tilde{\mathfrak{h}}_1$ and $\tilde{\mathfrak{h}}_2$ maximal abelian subalgebras in $(\mathfrak{k}^\sigma)'$, they are Cartan subalgebras in $(\mathfrak{k}^\sigma)'$. Cartan subalgebras are conjugated, $\tilde{\mathfrak{h}}_1 = k\tilde{\mathfrak{h}}_2k^{-1}$for some $k \in K^\sigma$, thus $\mathrm{Ad}(kg)$ fixes $\tilde{\mathfrak{h}}_1$ and, consequently,$\mathfrak{h}_1^\sigma$: $kg \in N_{G^\sigma}(T_1)$, with T_1 the maximal torus associated to $\tilde{\mathfrak{h}}_1$, i.e. $\mathrm{Ad}(kg) \in \mathfrak{W}$. Furthermore, since $\mathrm{Ad}(kg)$ fixes $\mathfrak{k}^\sigma$, $\mathrm{Ad}(kg) \in \mathfrak{W}^P$. This, combined with the fact that $\mathrm{Ad}(kg)(X_1) = X_2$, implies that X_1 and X_2 are in the same orbit of the T-Weyl group.

□

4.3 Painted Dynkin Diagrams

Remark 4.3.1.
Given a Dynkin diagram Δ, denote the associated simple Lie algebra by $\mathfrak{g}$ and fix an equipment with a basis $\mathcal{B}$ of the root system R.
A painted Dynkin diagram (Δ, Δ_0) represents a generalized flag manifold:

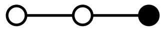
- The equipment associates to Δ_0 a subset of $\mathcal{B}$, denoted by $\mathcal{B}_0$.
- Following (2.5), $\mathcal{B}_0$ determines the parabolic subalgebra $\mathfrak{p} := \mathfrak{p}_{\mathcal{B}_0}$.
- The Lie algebra $\mathfrak{g}$ and its subalgebra $\mathfrak{p}$ uniquely determine a simply connected Lie group G and a parabolic subgroup P, respectively.
- The desired generalized flag manifold is G/P.

Alternatively, it is sufficient to consider the group G^σ associated with the compact form $\mathfrak{g}^\sigma$ and its subgroup K^σ, associated with $\mathfrak{p}^\sigma = \mathfrak{k}^\sigma_{\mathcal{B}_0} \oplus \mathfrak{g}(R^+ \smallsetminus [\mathcal{B}_0])^\sigma = \mathfrak{k}^\sigma_{\mathcal{B}_0}$. The sought manifold is G^σ/K^σ.
Note that the subgraph obtained by removing the black vertices represents the semisimple part of $\mathfrak{p}$, that is $\mathfrak{g}([B_0]) = \mathfrak{k}'$. The remaining summand $\mathfrak{z}_{\mathcal{B}_0}$ has a compact form isomorphic to $\mathfrak{u}(1)^{\#\text{ black vertices}}$.

Example 4.3.2.
The examples shown in 2.5.2 can therefore be represented as follows:

- $\mathcal{B}_0 = \varnothing$ 

  The corresponding manifold is SL_4/B: a full flag (as seen in 3.2.4).

- $\mathcal{B}_0 = \{\alpha_1, \alpha_2\}$ 

  The corresponding manifold is $\mathrm{SL}_4/P_{\{\alpha_1,\alpha_2\}} \cong \mathrm{Gr}(3,4)$ (as seen in 3.2.4).

Proposition 4.3.3.
Fixing an equipment of the Dynkin diagram, every possible painting determines non-G-isomorphic manifolds.

Proposition 4.3.4.
Fix a Cartan subalgebra $\mathfrak{h}$ contained in two parabolic subalgebras $\mathfrak{p}_1$ and $\mathfrak{p}_2$, respectively generated by $\mathcal{B}_0^1 \subseteq \mathcal{B}_1$ and $\mathcal{B}_0^2 \subseteq \mathcal{B}_2$. The subalgebras $\mathfrak{p}_1$, $\mathfrak{p}_2$ are conjugated if and only if there is an element $w \in \mathfrak{W}$ such that $w(R_{\mathcal{B}_0^1}) = R_{\mathcal{B}_0^2}$.

Proof.
If $w \in \mathfrak{W}$ exists, consider $g \in N_G(T)$ associated to it, we have:

$$\mathfrak{p}_2 = \mathfrak{h} \oplus \bigoplus_{\alpha \in R_{\mathcal{B}_0^2}} \mathfrak{g}^\alpha = \mathrm{Ad}(g)(\mathfrak{h}) \oplus \bigoplus_{\alpha \in R_{\mathcal{B}_0^1}} \mathrm{Ad}(g)(\mathfrak{g}^\alpha) = \mathrm{Ad}(g)(\mathfrak{p}_1).$$

Viceversa, let $\mathrm{Ad}(g)(\mathfrak{p}_1) = \mathfrak{p}_2$ for some $g \in G$. Both P_1 and P_2 contain the maximal torus T and its image $gTg^{-1} =: T' \subseteq P_2$. Two maximal tori in a parabolic group are conjugated: there exists $k \in P_2$ such that $T = kT'k^{-1}$. One obtains $T = kgT(kg)^{-1}$, thus $kg \in N_G(T)$, and $kgP_1(kg)^{-1} = P_2$. Consider $w \in \mathfrak{W}$ associated to kg.

$$\mathfrak{h} \oplus \bigoplus_{\alpha \in R_{\mathcal{B}_0^2}} \mathfrak{g}^\alpha = \mathfrak{p}_2 = \mathrm{Ad}(kg)(\mathfrak{p}_1) = \mathfrak{h} \oplus \bigoplus_{\alpha \in R_{\mathcal{B}_0^1}} \mathrm{Ad}(kg)(\mathfrak{g}^\alpha) = \mathfrak{h} \oplus \bigoplus_{\alpha \in R_{\mathcal{B}_0^1}} \mathfrak{g}^{w(\alpha)}$$

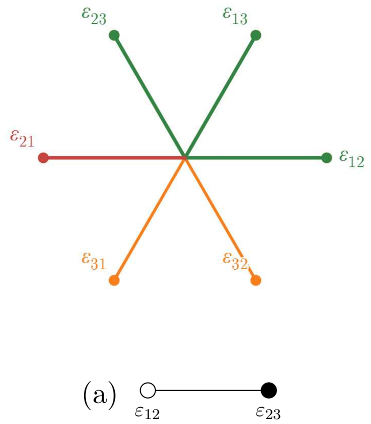
$\square$

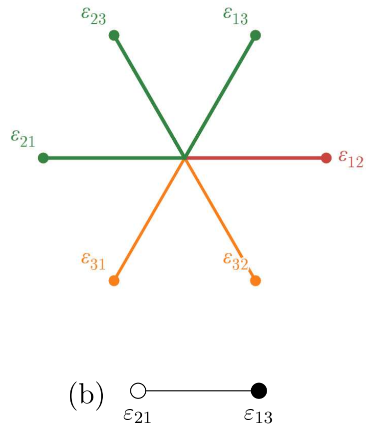
Example 4.3.5 (A_2)**.**
Observe that, although it is possible to recover both $[\mathcal{B}_0]$ and $R^+ \setminus [\mathcal{B}_0]$ having $R_{\mathcal{B}_0}$, it is not possible, in general, to obtain $\mathcal{B}_0$ itself.
Consider, for $\mathfrak{g} = \mathfrak{sl}_3$, the following parabolic subalgebra:

$$\mathfrak{p} = \left\{ \begin{pmatrix} h_1 & a_{12} & a_{13} \\ a_{21} & h_2 & a_{23} \\ 0 & 0 & h_3 \end{pmatrix} \in \mathfrak{sl}_3 \right\}$$

It corresponds to the following subset of the root system:

$$\begin{pmatrix} h_1 & a_{12} & a_{13} \\ a_{21} & h_2 & a_{23} \\ 0 & 0 & h_3 \end{pmatrix} \qquad \begin{pmatrix} h_1 & a_{12} & a_{13} \\ a_{21} & h_2 & a_{23} \\ 0 & 0 & h_3 \end{pmatrix}$$





R^+ in green, $[\mathcal{B}_0]^-$ in red, $R^- \smallsetminus [\mathcal{B}_0]$ in orange.

One can transform the first basis in the second by the element of $\mathfrak{W}$ given by $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = S_{\varepsilon_{12}}$. As seen in Theorem 2.2.7, an element $g \in N_G(T)$ associated to it can be obtained as $\exp\left(\frac{\pi}{2}(E_{12} - E_{21})\right) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Corollary 4.3.6.
Fixed a painted Dynkin diagram (Δ, Δ_0), two equipments $\varepsilon_1 : \mathcal{B}_1 \to V(\Delta)$ and $\varepsilon_2 : \mathcal{B}_2 \to V(\Delta)$ determine G-isomorphic manifolds if and only if there exists an element $w \in \mathfrak{W}$ such that $(\varepsilon_2 \circ w)^{-1}(\Delta_0) = \varepsilon_1^{-1}(\Delta_0)$.

Remark 4.3.7.
Given two bases $\mathcal{B}_1$, $\mathcal{B}_2$, an element $\tilde{w} \in \mathfrak{W}$ such that $\tilde{w}(\mathcal{B}_1) = \mathcal{B}_2$ always exists (see Definition 2.2.1), thus, given two equipments $\varepsilon_1 : \mathcal{B}_1 \to V(\Delta)$ and $\varepsilon_2 : \mathcal{B}_2 \to V(\Delta)$, define $\tilde{\varepsilon}_2 := \varepsilon_2 \circ \tilde{w} : \mathcal{B}_1 \to V(\Delta)$. Equipments from the same basis correspond to automorphisms of Δ: using Corollary 4.3.6, fixed a painted Dynkin diagram and a basis, two equipments determine G-isomorphic flag manifolds if and only if they are associated with an automorphism of Δ that preserves the set Δ_0.
Non-trivial automorphisms of this kind are only possible if Δ is A_n, D_n or E_6.

At this point, the search for G-isomorphic generalized flag manifolds appears too restrictive: we will consider also automorphisms of G not given by conjugation.

Definition 4.3.8.
Two generalized flag manifolds G/K_1 and G/K_2 are called *equivalent* if there exists an automorphism φ of G such that $\varphi(K_1) = K_2$.

Theorem 4.3.9.
Let G be a compact, connected, simply connected semisimple Lie group, and let K_1 and K_2 be centralizers of tori. Then the following are equivalent:

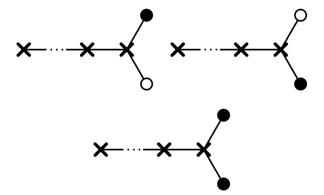
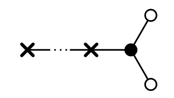
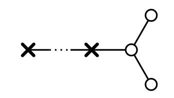
- There exists an automorphism of G mapping K_1 onto K_2.
- There exists an isomorphism of the corresponding root systems mapping the

root subsystem of K_1 onto the root subsystem of K_2.

After identifying the maximal tori, the equivalence is described by the action of $\mathrm{Aut}(R)$. The Weyl group corresponds to inner automorphisms, while Dynkin diagram automorphisms corresponds to outer automorphisms.

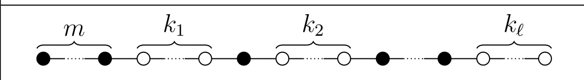
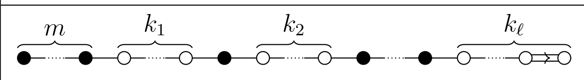
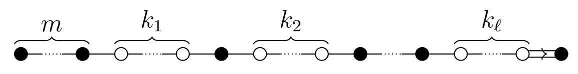
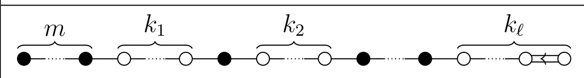
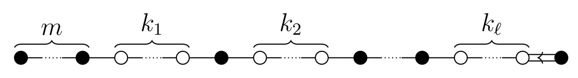
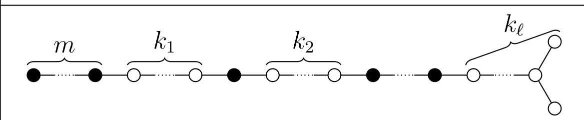
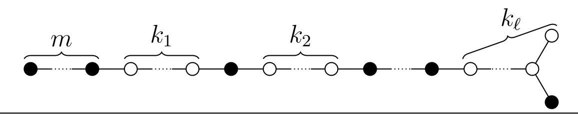
Observe that changing the equipment on a painted Dynkin diagram defines $\mathrm{Aut}(G)$-isomorphic manifolds: it is sufficient to consider unlabeled painted Dynkin diagrams.

Definition 4.3.10.
Two painted Dynkin diagrams (Δ, Δ_1) and (Δ, Δ_2) are called *equivalent* if their subgraphs (associated with Δ_i) coincide and, if $\Delta = D_\ell$, for $\ell \geq 5$, when they are in the same class:

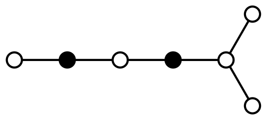
B	**WWB**	**WWW**
		

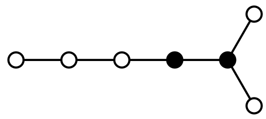
Theorem 4.3.11.
There is a one-to-one correspondence, up to equivalence, between generalized flag manifolds of the form $G/C_G(T)$, with G a compact, connected, simply connected semisimple Lie group, and painted Dynkin diagrams ([Ale96]).

Theorem 4.3.12.
Up to equivalence, generalized flag manifolds of classical Lie groups are classified as follows (the standard forms are indicated, the groups are not necessarily simply connected) [Arv03] pag. 100:

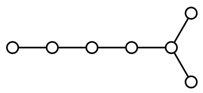
	Dynkin diagram type	Generalized Flag Manifold
A_n		$\dfrac{\mathrm{SU}(n+1)}{S(\mathrm{U}(k_1+1)\times\cdots\times\mathrm{U}(k_\ell+1)\times U(1)^m)}$
B_n		$\dfrac{\mathrm{SO}(2n+1)}{\mathrm{U}(k_1+1)\times\cdots\times\mathrm{U}(k_{\ell-1}+1)\times\mathrm{SO}(2k_\ell+1)\times U(1)^m}$
B_n		$\dfrac{\mathrm{SO}(2n+1)}{\mathrm{U}(k_1+1)\times\cdots\times\mathrm{U}(k_\ell+1)\times U(1)^m}$
C_n		$\dfrac{\mathrm{Sp}(2n)}{\mathrm{U}(k_1+1)\times\cdots\times\mathrm{U}(k_{\ell-1}+1)\times\mathrm{Sp}(2k_\ell)\times U(1)^m}$
C_n		$\dfrac{\mathrm{Sp}(2n)}{\mathrm{SU}(k_1+1)\times\cdots\times\mathrm{SU}(k_\ell+1)\times U(1)^m}$
D_n		$\dfrac{\mathrm{SO}(2n)}{\mathrm{SU}(k_1+1)\times\cdots\times\mathrm{SU}(k_{\ell-1}+1)\times\mathrm{SO}(2k_\ell)\times U(1)^m}$
D_n		$\dfrac{\mathrm{SO}(2n)}{\mathrm{SU}(k_1+1)\times\cdots\times\mathrm{SU}(k_\ell+1)\times U(1)^m}$

Remark 4.3.13.
Observe that the form used in the previous table does not, in general, describe a unique class (with respect to equivalence) of generalized flag manifolds.
Consider the following painted diagrams:

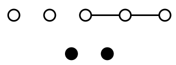




They can both be decomposed as follows:



$\mathfrak{g} = D_7 = \mathfrak{so}_{14}(\mathbb{C}) \overset{\text{compact form}}{\rightsquigarrow} \mathfrak{g}^\sigma = \mathfrak{so}(14)$



$\mathfrak{k}' = A_1 \oplus A_1 \oplus A_3 = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_4(\mathbb{C}) \overset{\text{compact form}}{\rightsquigarrow} (\mathfrak{k}')^\sigma = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(4)$

$\mathfrak{z} = \mathbb{C} \oplus \mathbb{C} \qquad \mathfrak{z}^\sigma = \mathfrak{u}(1) \oplus \mathfrak{u}(1)$

Nevertheless the two diagrams correspond to non-equivalent manifolds.

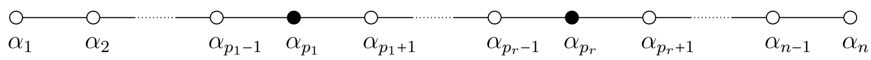
Chapter 5

Some Invariants

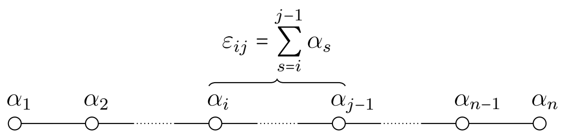
The representation via painted Dynkin diagrams allows, using combinatorial tools, for the explicit computations of some invariants associated with generalized flag manifolds. The notation in this chapter will follow [Bou08].

5.1 Dimension of a Generalized Flag Manifold

The complex dimension of a generalized flag manifold is equal to $\dim(\mathfrak{g}(R^- \smallsetminus [\mathcal{B}_0]^-)) = \dim(\mathfrak{g}(R^+ \smallsetminus [\mathcal{B}_0]^+))$, which is the positive roots not expressible as linear combinations of $\mathcal{B}_0$.
A flag manifold $\mathrm{SL_{n+1}}/P$ with $\mathrm{Lie}(P) = \mathfrak{p}$ associated with $\mathcal{B} \smallsetminus \{\alpha_{p_i}\}_{i=1}^r$ is depicted as follows:



The set of positive roots $R^+ = \{\varepsilon_{ij} \,|\, i < j\}$ can be described in terms of intervals in the diagram:

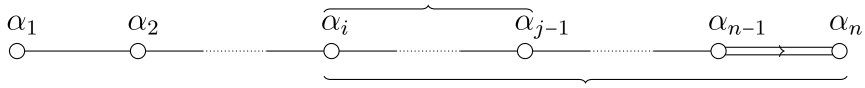


The dimension of $\mathcal{F}_{\tilde{\mathbf{p}}}(\mathbb{C}^{n+1})$ ($\tilde{\mathbf{p}}$ excluded indices) is given by the number of these intervals that cross at least one black vertex: the computation can be carried out by summing over the nodes in which the left endpoint of the bracket can be placed. Explicitly: $\sum_{s=1}^r (p_s - p_{s-1})(n - p_s + 1)$, where $p_0 := 0$.

Analogously, one can analyze the other classical Lie algebras:

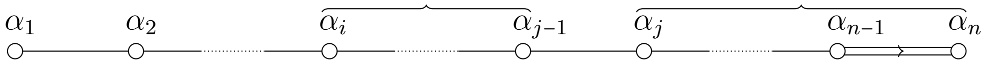
SO_{2n+1}/P: Here $R^+ = \{\varepsilon_i - \varepsilon_j \,|\, i < j\} \cup \{\varepsilon_i \,|\, i = 1, \dots, n\} \cup \{\varepsilon_i + \varepsilon_j \,|\, i < j\}$

$$\varepsilon_i - \varepsilon_j = \sum_{s=i}^{j-1} \alpha_s$$



$$\varepsilon_i = \sum_{s=i}^{n} \alpha_s$$

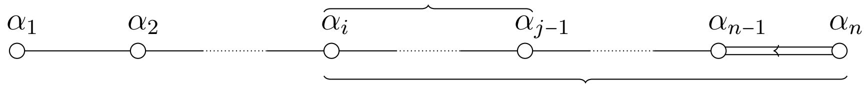
$$\varepsilon_i + \varepsilon_j = \sum_{s=i}^{j-1} \alpha_s + 2\sum_{s=j}^{n} \alpha_s$$



The number of positive roots of the second and third type depends on the number $n - p_r$ of consecutive white vertices at the end of the diagram. There are p_r positive roots of the second type, the number of subdivisions of these ones gives the number of positive roots of the third type. $\dim(\mathrm{SO}_{2n+1}/P) = \sum_{s=1}^{r}(p_s - p_{s-1})(n - p_s) + p_r + \sum_{s=1}^{p_r}(n - s)$.

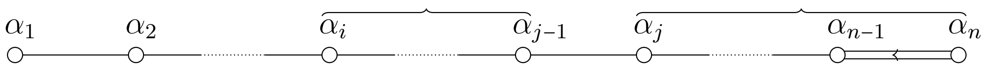
Sp_{2n}/P: Here $R^+ = \{\varepsilon_i - \varepsilon_j \,|\, i < j\} \cup \{2\varepsilon_i \,|\, i = 1, \ldots, n\} \cup \{\varepsilon_i + \varepsilon_j \,|\, i < j\}$

$$\varepsilon_i - \varepsilon_j = \sum_{s=i}^{j-1} \alpha_s$$



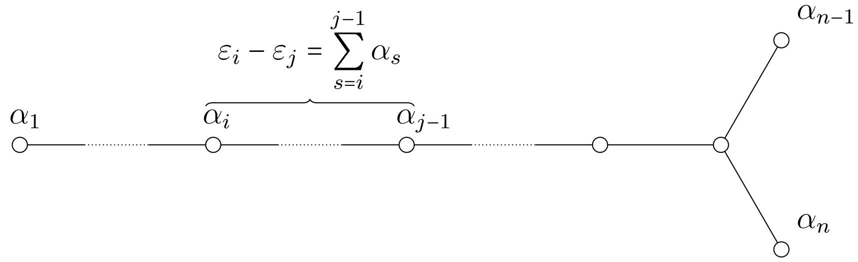
$$2\varepsilon_i = 2\sum_{s=i}^{n-1} \alpha_s + \alpha_n$$

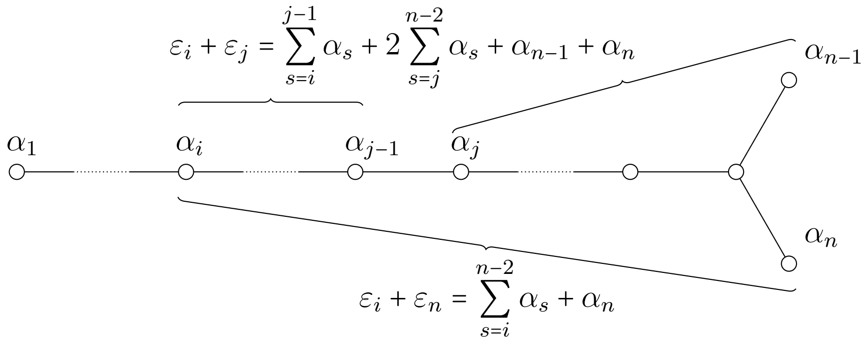
$$\varepsilon_i + \varepsilon_j = \sum_{s=i}^{j-1} \alpha_s + 2\sum_{s=j}^{n-1} \alpha_s + \alpha_n$$



It is clear that the computation is the same as the previous case.
$\dim(\mathrm{Sp}_{2n}/P) = \sum_{s=1}^{r}(p_s - p_{s-1})(n - p_s) + p_r + \sum_{s=1}^{p_r}(n - s)$.

SO_{2n}/P: Here $R^+ = \{\varepsilon_i - \varepsilon_j \,|\, i < j\} \cup \{\varepsilon_i + \varepsilon_j \,|\, i < j\}$





In this case the term depending on a single index is missing and $\varepsilon_i + \varepsilon_n$ excludes α_{n-1}.

$$\dim(\mathrm{SO}_{2n}/P) = \textstyle\sum_{s=1}^{r}(p_s - p_{s-1})(n - p_s) + \begin{cases} \frac{n(n-1)}{2} & \text{if } p_r = n; \\ \frac{(n-1)(n-2)}{2} + p_{r-1} & \text{if } p_r = n-1; \\ \sum_{s=1}^{p_r}(n-s) & \text{if } p_r < n-1. \end{cases}$$

5.2 Euler Characteristic

This section will follow [Bri04].
A flag in $\mathcal{F}_{(p_1,\ldots,p_r)}(\mathbb{C}^n)$ consisting of coordinate subspaces is called a *coordinate flag.*
A full coordinate flag in $X := \mathcal{F}_{(1,\ldots,n-1)}(\mathbb{C}^n) = \mathrm{SL}_n/B$ can be associated with a permutation in $S_n = W(T)$, with T the standard torus in SL_n:

$$\langle e_{i_1}\rangle \subset \langle e_{i_1}, e_{i_2}\rangle \subset \cdots \subset \langle e_{i_1}, e_{i_2}, \ldots, e_{i_n}\rangle$$

$$\updownarrow$$

$$S_n \ni \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix} = (a_{j\,i_j}) \in W(T).$$

Therefore, each full coordinate flag can be expressed as $f_w := wf_0$, with f_0 the standard full flag and $w \in W(T)$.
The set in X defined as $C_w := Bf_w$, with $w \in W(T)$, is called a *Schubert cell.* It can

be proved that $Bf_w = Uf_w$, where U is the set of unipotent matrices in SL_n:

$$U = \left\{ \begin{pmatrix} 1 & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & 1 \end{pmatrix} \in \mathrm{SL}_n \right\}.$$

Observe that the stabilizer $U_w := U_{f_w}$ of f_w in U is given by unipotent matrices in wUw^{-1}:

$wUw^{-1} \cap U \subseteq U_w$: $wUw^{-1}f_w = wUw^{-1}wf_0 = wUf_0 = wf_0 = f_w$.

$U_w \subseteq wUw^{-1} \cap U$: Let $g \in U_w$, we have that $w^{-1}gwf_0 = f_0$ and therefore $w^{-1}gw \in U$.

The set U_w can be therefore expressed as:

$$U_w = U \cap wUw^{-1} = \{(a_{ij}) \in U \,|\, a_{ij} = 0 \text{ if } w^{-1}(i) > w^{-1}(j)\}.$$

The "complementary" set to it is given by:

$$U^w := U \cap wU^-w^{-1} = \{(a_{ij}) \in U \,|\, a_{ij} = 0 \text{ if } w^{-1}(i) < w^{-1}(j)\},$$

where $U^- := U^T$.

Proposition 5.2.1.
The map $U^w \times U_w \to U$ defined by matrix multiplication is an isomorphism of varieties.

Theorem 5.2.2.
A full flag manifold can be expressed as disjoint union of Schubert cells:

$$X = \bigsqcup_{w \in W(T)} C_w.$$

As seen in Definition 3.2.2, flag manifolds are projective varieties, hence it is possible to consider the Zariski topology on them. Define $X_w := \overline{C_w}$ (closure in the Zariski topology), with $w \in W(T)$, as a *Schubert variety.*

Theorem 5.2.3.
A Schubert variety can be expressed as a union of Schubert cells:

$$X_w = \bigcup_{v \in W(T),\, v \leq w} C_v.$$

Here $(W(T), \leq)$ is the Bruhat ordering defined in 2.2.4.

Theorem 5.2.4.
Given $w \in W(T)$, the Schubert cell C_w is isomorphic to $\mathbb{C}^{\ell(w)}$, with ℓ the length defined in 2.2.3.

Proof. Using Proposition 5.2.1. The map

$$\begin{aligned} U^w &\longrightarrow C_w = U f_w \\ g &\longmapsto g f_w \end{aligned}$$

defines an isomorphism. Indeed, given $g \in U$, $g = g^w g_w$ for $g^w \in U^w$ and $g_w \in U_w$, but U_w stabilizes f_w: $g f_w = g^w f_w$. Viceversa, $g_1 f_w = g_2 f_w$ only if $g_1^{-1} g_2 \in U_w$, therefore $g_1^{-1} g_2 \in U^w \cap U_w = \{\mathbb{1}\}$.
Hence $\mathbb{C}^m \cong U^w \cong C_w$, where m is the number of pairs $(i,j) \in \{1,\dots,n\}^2$ such that $w^{-1}(i) > w^{-1}(j)$ and $i < j$; this number is equal to the number of pairs such that $w(i) > w(j)$ and $i < j$: the number of inversions of the permutation w, that is exactly the definition of length with respect to the standard basis for the root system of $\mathrm{SL_n}$. □

These results can be extended to all generalized flag manifolds G/P: Schubert cells and varieties are now parametrized by the coset space $W^P := W/W_P$, with W_P the Weyl group of (the semisimple part of) P:

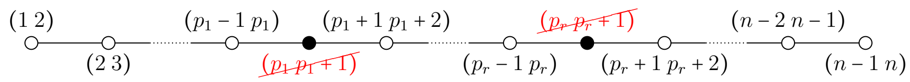
$$C_{[w]} := BwP/P \subseteq G/P, \qquad\qquad [w] \in W^P.$$

Theorem 5.2.5.
Let G/P be a generalized flag manifold, the following decomposition holds:

$$G/P = \bigsqcup_{[w]\in\mathfrak{W}^P} C_{[w]} = \bigsqcup_{[w]\in\mathfrak{W}^P} \mathbb{C}^{\min\{\ell(w)|w\in[w]\}}. \tag{5.2.1}$$

Therefore the Euler characteristic χ is given by $|W^P|$.

Example 5.2.6 (SL_n)**.**
For $G = \mathrm{SL}_n$, the Weyl group W_P associated to P, given by $\mathcal{B} \smallsetminus \{\alpha_{p_i}\}_{i=1}^r$, can be visualized by labelling each vertex with the permutation induced on the set $\{\varepsilon_i\}_{i=1}^n$:



Therefore, we have $W \cong S_n$, $W_P \cong S_{p_1} \times S_{p_2-p_1} \times \cdots \times S_{n-p_r}$. (2)

The search for a minimal representative in a coset of W^P can be done by choosing an arbitrary element of the fixed coset and writing it as a permutation "separated" according to $\mathcal{F}_{\tilde{\mathbf{p}}}$:

$$\left(\begin{array}{ccc|ccc|ccc} 1 & \cdots & p_1 & p_1+1 & \cdots & p_r & p_r+1 & \cdots & n \\ w(1) & \cdots & w(p_1) & w(p_1+1) & \cdots & w(p_r) & w(p_r+1) & \cdots & w(n) \end{array}\right).$$

Permutations inside the obtained blocks always produce an element of the same coset and the minimal representative is the one of the following form:

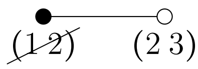
$$\left(\begin{array}{ccccc|ccccc|ccccc} 1 & & \cdots & & p_1 & p_1+1 & & \cdots & & p_r & p_r+1 & & \cdots & & n \\ w(1) & < & \cdots & < & w(p_1) & w(p_1+1) & < & \cdots & < & w(p_r) & w(p_r+1) & < & \cdots & < & w(n) \end{array}\right).$$

Therefore, one obtains $\chi = |W^P| = \frac{|W_G|}{|W_P|}$: that is, as seen in (2),

$$\frac{n!}{\prod_{s=0}^{r}(p_{s+1}-p_s)!} = \binom{n}{p_1, p_2 - p_1, \ldots, n - p_r}.$$

Example 5.2.7.
Consider the following generalized flag manifolds:

Gr$(1,3)$:



We have $W_P = \{(), (23)\}$.

$$W^P = \{()W_P = \{(), (23)\}, (12)W_P = \{(12), (123)\}, (13)W_P = \{(13), (132)\}\}.$$

Following the method above, the minimal representatives with respect to the length are:

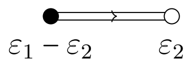
$$\left(\begin{array}{c|cc} 1 & 2 & 3 \\ 1 & 2 & 3 \end{array}\right) = () \rightsquigarrow \ell = 0 \qquad \left(\begin{array}{c|cc} 1 & 2 & 3 \\ 2 & 1 & 3 \end{array}\right) = (12) \rightsquigarrow \ell = 1 \qquad \left(\begin{array}{c|cc} 1 & 2 & 3 \\ 3 & 1 & 2 \end{array}\right) = (132) \rightsquigarrow \ell = 2$$

The decomposition is therefore

$$\mathrm{Gr}(1,3) = \mathbb{C}^0 \sqcup \mathbb{C}^1 \sqcup \mathbb{C}^2,$$

with an Euler characteristic $\chi = 3$.

Q_3: The generalized flag manifold SO_5/P_{α_2} can be identified with the quadric threefold Q_3.

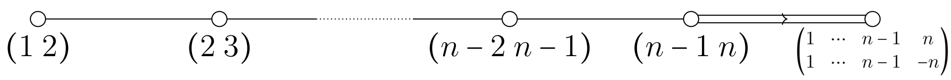


In B_n, denote by $\begin{pmatrix} 1 & 2 & \cdots & n \\ \pm\sigma(1) & \pm\sigma(2) & \cdots & \pm\sigma(n) \end{pmatrix}$ an element of the Weyl group, where $\sigma \in S_n$, acting on R as follows:

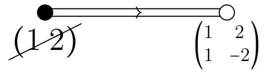
$$\begin{pmatrix} 1 & 2 & \cdots & n \\ s_1\sigma(1) & s_2\sigma(2) & \cdots & s_n\sigma(n) \end{pmatrix} \left(\pm_1 \varepsilon_i \pm_2 \varepsilon_j\right) = \pm_1 s_i \varepsilon_{\sigma(i)} \pm_2 s_j \varepsilon_{\sigma(j)},$$

where $s_i \in \{\pm 1\}$.

In general, one can write



In this case:



We have $W_P = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix} \right\}$. Using the notation in 2.2.5, one can write:

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \langle \rangle, \quad \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \langle 1 \rangle, \quad \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix} = \langle 2 \rangle, \quad \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} = \langle 21 \rangle,$$
$$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} = \langle 12 \rangle, \quad \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix} = \langle 121 \rangle, \quad \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix} = \langle 212 \rangle, \quad \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix} = \langle 1212 \rangle.$$

We have that the set W^P has as elements:

- $\langle \rangle W_P = \{\langle \rangle, \langle 2 \rangle\} \rightsquigarrow \ell = 0$;
- $\langle 1 \rangle W_P = \{\langle 1 \rangle, \langle 12 \rangle\} \rightsquigarrow \ell = 1$;
- $\langle 21 \rangle W_P = \{\langle 21 \rangle, \langle 212 \rangle\} \rightsquigarrow \ell = 2$;
- $\langle 121 \rangle W_P = \{\langle 121 \rangle, \langle 1212 \rangle\} \rightsquigarrow \ell = 3$.

Finally, the obtained decomposition is

$$Q_3 = \mathbb{C}^0 \sqcup \mathbb{C}^1 \sqcup \mathbb{C}^2 \sqcup \mathbb{C}^3,$$

with an Euler characteristic $\chi = 4$.

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