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# **Evolving Generalized Parton Distributions across Heavy-Quark Thresholds**

## **Evoluzione delle distribuzioni partoniche generalizzate attraverso le soglie dei quark pesanti**

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## **Abstract**

This work presents a systematic study of the one-loop matching conditions required for a consistent description of Generalized Parton Distributions across heavy-quark mass thresholds, where the number of active flavours changes. By performing perturbative calculations in the heavy-quark–gluon sector, we derive the explicit matching coefficients and show that the resulting expressions correctly reproduce the known threshold matching relations for ordinary parton distributions in the forward limit.

## **Sommario**

Questo lavoro presenta uno studio sistematico delle condizioni di matching a un loop necessarie per una descrizione consistente delle distribuzioni partoniche generalizzate attraverso le soglie di massa dei quark pesanti, in corrispondenza delle quali cambia il numero di flavour attivi. Attraverso calcoli perturbativi nel settore quark pesante–gluone, deriviamo esplicitamente i coefficienti di matching e mostriamo che le espressioni ottenute riproducono correttamente, nel limite forward, le note relazioni di matching per le distribuzioni partoniche ordinarie.



# Contents

|                                                                 |           |
|-----------------------------------------------------------------|-----------|
| <b>Introduction</b>                                             | <b>1</b>  |
| <b>1 Generalized Parton Distributions</b>                       | <b>5</b>  |
| 1.1 Hard exclusive processes . . . . .                          | 5         |
| 1.1.1 DVCS process and kinematics . . . . .                     | 7         |
| 1.1.2 Compton form factors and QCD factorisation . . . . .      | 9         |
| 1.2 Operator definition of GPDs . . . . .                       | 12        |
| 1.2.1 Partonic interpretation of GPDs . . . . .                 | 14        |
| 1.3 Properties . . . . .                                        | 17        |
| 1.3.1 Forward limit . . . . .                                   | 17        |
| 1.3.2 Symmetry properties . . . . .                             | 18        |
| 1.3.3 Sum rules and polynomiality . . . . .                     | 19        |
| 1.4 Scale evolution of GPDs . . . . .                           | 24        |
| 1.4.1 Evolution equation and evolution kernels . . . . .        | 24        |
| 1.4.2 Parton-in-parton GPDs . . . . .                           | 27        |
| 1.4.3 One-loop evolution kernels . . . . .                      | 34        |
| <b>2 GPD evolution across heavy-quark thresholds</b>            | <b>41</b> |
| 2.1 Flavour-number schemes and heavy-quark thresholds . . . . . | 41        |
| 2.2 GPD matching across a heavy-quark threshold . . . . .       | 45        |
| 2.3 Partonic framework for the matching calculation . . . . .   | 50        |
| <b>3 One-loop matching conditions for GPDs</b>                  | <b>55</b> |
| 3.1 Massive parton-in-parton GPDs at one loop . . . . .         | 55        |
| 3.1.1 Off-diagonal channels . . . . .                           | 57        |
| 3.1.2 Diagonal channels . . . . .                               | 60        |
| 3.2 One-loop matching coefficients . . . . .                    | 63        |
| 3.2.1 Off-diagonal coefficients . . . . .                       | 64        |
| 3.2.2 Diagonal coefficients . . . . .                           | 65        |
| 3.2.3 Matching-scale dependence . . . . .                       | 66        |
| 3.3 Consistency checks and forward limit . . . . .              | 66        |
| 3.3.1 UV poles and heavy-mass logarithms . . . . .              | 66        |
| 3.3.2 Forward limit . . . . .                                   | 67        |

|                                                                      |            |
|----------------------------------------------------------------------|------------|
| <b>Conclusion</b>                                                    | <b>69</b>  |
| <b>A Light-cone coordinates and high-energy kinematics</b>           | <b>71</b>  |
| A.1 Light-cone coordinates . . . . .                                 | 71         |
| A.2 Fast-moving hadrons and infinite-momentum frame . . . . .        | 72         |
| A.3 Collinear partonic kinematics . . . . .                          | 73         |
| <b>B Parton-in-parton GPDs: technical details</b>                    | <b>77</b>  |
| B.1 Gluon operator in the light-cone gauge . . . . .                 | 77         |
| B.2 Tree-level massless parton-in-parton GPDs . . . . .              | 80         |
| B.2.1 Quark-in-quark GPD . . . . .                                   | 81         |
| B.2.2 Gluon-in-gluon GPD . . . . .                                   | 82         |
| <b>C One-loop massive parton-in-parton GPDs: calculation details</b> | <b>85</b>  |
| C.1 Heavy-quark-in-gluon GPD . . . . .                               | 85         |
| C.2 Gluon-in-heavy-quark GPD . . . . .                               | 91         |
| C.3 Real heavy-quark-in-heavy-quark GPD . . . . .                    | 97         |
| <b>Bibliography</b>                                                  | <b>105</b> |

# Introduction

One of the central goals of hadronic physics is to understand how the properties of hadrons emerge from their fundamental constituents: quarks and gluons, collectively referred to as *partons*. Quantum Chromodynamics (QCD) provides the theoretical framework describing these degrees of freedom and their interactions. Observable properties such as the hadron mass, spin, and magnetic moment are not simply inherited from the individual constituents, but arise from the way partons move, interact, and are distributed inside the hadron. Understanding this internal dynamics is therefore essential for connecting the experimentally observed properties of hadrons with the fundamental degrees of freedom of QCD.

A complete description of hadron structure requires answering several complementary questions. How is the longitudinal momentum of a fast-moving hadron shared among its partons? How are quarks and gluons distributed in the plane transverse to the hadron momentum? How are their spatial and momentum distributions correlated, and how do their spin and orbital motion contribute to the total angular momentum of the hadron? [1] Addressing these questions is particularly challenging because, at typical hadronic scales, QCD interactions are strong and the internal structure of hadrons cannot be described perturbatively. High-energy scattering processes nevertheless provide a way to probe this structure. The interaction with the external probe takes place over sufficiently short distance scales to be treated using perturbative QCD, while the non-perturbative information associated with the internal structure of the hadron is encoded in suitable partonic distributions [2]. These distributions therefore provide the link between the underlying QCD dynamics and experimentally measurable observables.

Historically, complementary aspects of hadron structure have been investigated through different classes of scattering processes. Inclusive deep-inelastic scattering led to the introduction of Parton Distribution Functions (PDFs), which describe how the longitudinal momentum of a fast-moving hadron is distributed among its quark and gluon constituents [3]. A complementary picture is provided by elastic scattering, which gives access to electromagnetic form factors encoding information on the spatial distribution of charge and magnetisation [4, 5]. These two descriptions therefore provide complementary but intrinsically partial views of hadron structure: PDFs describe the longitudinal momentum distribution of partons without direct information on their spatial localisation, whereas form factors provide spatial information without resolving how it is correlated with the longitudinal momentum carried by individual partons.

Generalized Parton Distributions (GPDs) establish a common framework in which these

complementary descriptions are unified [6–11]. They extend ordinary PDFs to non-forward kinematics, where the initial and final hadron carry different momenta, and depend simultaneously on the longitudinal momentum fraction of the active parton and on the momentum transferred to the hadron [12]. In the forward limit, GPDs reduce to ordinary PDFs, while integrating them over the parton momentum fraction gives access to elastic form factors. They therefore encode a correlation between momentum and spatial information that is absent from either description separately.

This correlation becomes particularly transparent in impact-parameter space. A Fourier transform of GPDs with respect to the transverse momentum transfer provides information on the transverse position of partons carrying a given longitudinal momentum fraction. In this sense, GPDs form the theoretical basis for a three-dimensional description of hadron structure in a mixed longitudinal-momentum and transverse impact-parameter space [13, 14]. Such a mapping is commonly referred to as *hadron tomography* and is one of the main motivations for the extensive theoretical and experimental activity devoted to GPDs.

The information encoded in GPDs is not limited to spatial imaging. Their Mellin moments are related to matrix elements of local QCD operators and, in particular, to form factors of the QCD energy–momentum tensor. Through these relations, GPDs provide access to fundamental properties of the internal dynamics of hadrons, including the partonic contributions to the total angular momentum of the hadron and information related to its mechanical structure, such as pressure and shear forces [15–18].

Experimentally, GPDs can be accessed through hard exclusive processes. A central example is Deeply Virtual Compton Scattering (DVCS), in which a highly virtual photon interacts with a hadron and a real photon is produced while the hadron remains intact [19–22]. In the deeply virtual regime, the large photon virtuality provides the hard scale required for the perturbative treatment of the interaction with the probe. In this regime, QCD factorisation allows the DVCS amplitude to be expressed in terms of perturbatively calculable coefficient functions and non-perturbative GPDs [23, 24].

The extraction of GPDs from experimental data nevertheless requires more than a factorisation theorem. GPDs are not scale-independent quantities: they are defined as matrix elements of renormalised non-local light-cone operators and consequently depend on the factorisation scale. Their scale dependence is governed by perturbative evolution equations, whose kernels follow from the ultraviolet renormalisation of the underlying operators [12, 25]. In phenomenological applications, GPDs defined at an initial scale must therefore be evolved to the factorisation scale relevant for the process under consideration. The same evolution is also required to consistently compare measurements performed at different hard scales. A reliable treatment of GPD evolution is therefore an essential ingredient for precision phenomenological studies, playing a role analogous to PDF evolution in ordinary PDF phenomenology.

The treatment of scale evolution becomes more challenging when the factorisation scale crosses the mass threshold of a heavy quark. While the light quarks can usually be treated as effectively massless, heavier flavours introduce additional mass scales that cannot be neglected. Below a heavy-quark threshold, the corresponding quark is not included among the active flavours of the evolution, whereas above the threshold it becomes part of the par-

tonic description [26]. Crossing a heavy-quark threshold therefore changes the number of active flavours used in the evolution equations. The distributions defined below and above the threshold correspond to different numbers of active flavours and must be consistently related to one another. This relation is provided by the *matching conditions*, which constitute the central subject of this thesis. Matching conditions ensure that GPD evolution can be consistently continued across a heavy-quark mass threshold. Together with the fixed-flavour evolution within each region, they provide the necessary tools to describe the scale dependence of GPDs over ranges extending across one or more heavy-quark masses.

For ordinary PDFs, heavy-quark threshold matching is a standard component of variable-flavour-number schemes. The perturbative relations between distributions with different numbers of active flavours have been studied extensively and are an essential ingredient of modern PDF evolution [27, 28]. A similar matching procedure applies to GPDs, but the non-forward nature of these distributions introduces additional kinematic dependencies. The corresponding matching coefficients must retain the information associated with the longitudinal momentum transfer and must reduce to the established PDF results in the forward limit.

The purpose of this thesis is to extend the heavy-quark matching framework from ordinary PDFs to GPDs and to investigate the corresponding matching conditions across heavy-quark thresholds at one-loop accuracy. At this order, the heavy-quark effects relevant for the matching arise from contributions involving the heavy quarks and gluons, which are therefore the focus of this work. The calculation is formulated in terms of parton-in-parton GPDs. Physical GPDs are hadronic matrix elements containing genuinely non-perturbative information. However, the perturbative quantities governing their scale evolution and their matching across heavy-quark thresholds are universal and independent of the specific hadronic state. They can therefore be determined by replacing the hadron with an external quark or gluon state [25]. The corresponding parton-in-parton GPDs are perturbatively calculable. Their ultraviolet singularities determine the evolution kernels, while, in the presence of a finite heavy-quark mass, the massive parton-in-parton calculation contains finite contributions that determine the matching coefficients across the heavy-quark threshold.

The thesis is organised as follows. Chapter 1 introduces the theoretical framework of GPDs. Starting from hard exclusive processes and DVCS, we define quark and gluon GPDs and discuss their partonic interpretation, support properties, symmetry relations, and connection to ordinary PDFs and form factors. We then introduce their factorisation-scale dependence and the corresponding evolution equations. The chapter concludes with the calculation of massless one-loop parton-in-parton GPDs, whose ultraviolet structure reproduces the known leading-order evolution kernels and provides a first validation of the perturbative framework used throughout this work.

Chapter 2 introduces the heavy-quark threshold problem and develops the formalism of the matching conditions. We first describe GPD evolution with a fixed number of active quark flavours and then explain how the descriptions below and above a heavy-quark threshold are related. The matching kernel is combined with the corresponding evolution operators to obtain a consistent description of evolution across the threshold. Finally, the

problem is formulated at the partonic level, establishing the explicit connection between the matching coefficients and massive parton-in-parton GPDs.

Chapter 3 presents the main perturbative calculation of this thesis. Massive parton-in-parton GPDs are evaluated in the heavy-quark–gluon sector and used to determine the finite terms entering the matching conditions. The resulting expressions are finally checked against the known structure of GPD evolution and, in the forward limit, against the established heavy-flavour matching results for ordinary parton distributions.

The appendices collect technical material supporting the derivations presented in the main text. Appendix A introduces the light-cone conventions used throughout this work and discusses their connection with high-energy and collinear partonic kinematics. Appendix B provides technical details on the parton-in-parton GPD formalism, including the reduction of the gluon operator in light-cone gauge and the explicit derivation of the tree-level quark-in-quark and gluon-in-gluon GPDs. Appendix C contains the detailed calculation of the real massive one-loop contributions entering the heavy-quark matching coefficients.

Finally, the conclusions summarise the main results of this work, discuss their consistency with GPD evolution and the forward limit, and outline the remaining steps towards a complete treatment of heavy-quark thresholds in GPD evolution.

# Chapter 1

## Generalized Parton Distributions

This chapter introduces the theoretical framework underpinning the description of Generalized Parton Distributions (GPDs). We first present the hard exclusive processes that provide access to GPDs, with particular emphasis on Deeply Virtual Compton Scattering. We then introduce the operator definition of quark and gluon GPDs, before discussing their partonic interpretation and main properties. Finally, we address the scale dependence and evolution of GPDs, establishing the notation and conventions employed throughout the following chapters.

### 1.1 Hard exclusive processes

Lepton–hadron scattering provides one of the main experimental tools for investigating the internal structure of hadrons. An incoming lepton  $\ell$  with momentum  $k$  interacts with a hadron of momentum  $p$  through the exchange of a virtual photon. Denoting the momentum of the scattered lepton by  $k'$ , the four-momentum of the virtual photon is

$$q = k - k'. \quad (1.1)$$

In the scattering kinematics considered here, the exchanged photon is spacelike ( $q^2 < 0$ ), and its virtuality is conventionally defined as

$$Q^2 \equiv -q^2 > 0. \quad (1.2)$$

The photon virtuality  $Q^2$  determines the characteristic spatial scale probed in the interaction: in natural units, one may associate with the exchanged virtual photon a characteristic wavelength, or resolution scale,

$$\lambda \sim \frac{1}{Q}. \quad (1.3)$$

Thus, increasing  $Q^2$  allows progressively shorter-distance structures inside the hadron to be resolved. This qualitative behaviour is illustrated in Fig. 1.1 for the specific case of electron–proton scattering.

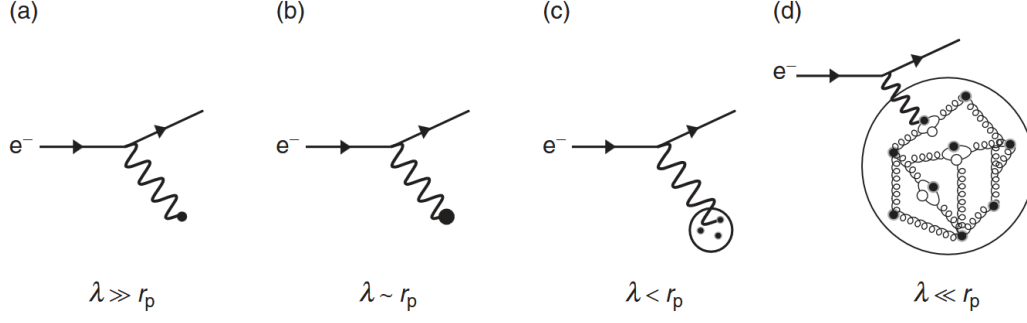


**Figure 1.1:** Schematic representation of proton resolution at different wavelength scales  $\lambda$  of the exchanged virtual photon. (a) At low resolution ( $\lambda \gg r_p$ ), the proton acts as a point-like particle. (b) As  $\lambda \sim r_p$ , the photon probes the overall charge and magnetic distribution (form factors). (c) At higher resolutions ( $\lambda < r_p$ ), the probe resolves individual valence quarks. (d) At very high resolutions ( $\lambda \ll r_p$ ), the photon resolves the complex partonic sea of quarks, antiquarks, and gluons. Reproduced from Ref. [29].

When the photon wavelength is much larger than the characteristic size of the proton, its internal structure cannot be resolved, and the proton behaves approximately as a point-like object. As the wavelength decreases, the interaction first probes the global electromagnetic structure of the proton and eventually resolves its quark and gluon constituents. The same qualitative resolution argument applies more generally to electromagnetic lepton–hadron scattering.

The information that can be extracted from lepton–hadron scattering depends not only on the spatial resolution set by  $Q^2$ , but also on which particles are identified in the final state. In the following, we focus on a nucleon target, denoted by  $N$ . In inclusive deep-inelastic scattering, the scattered lepton is detected, whereas the hadronic final state, collectively denoted by  $X$ , is not resolved:

$$\ell(k) + N(p) \longrightarrow \ell(k') + X.$$

The cross section is therefore summed over all hadronic final states compatible with the measured lepton kinematics. In the deep-inelastic regime, the large photon virtuality probes the nucleon at sufficiently short distances to resolve its partonic constituents. The corresponding observables give access to Parton Distribution Functions (PDFs), which describe how the longitudinal momentum of a fast-moving nucleon is distributed among its quarks and gluons. A different situation occurs in elastic scattering,

$$\ell(k) + N(p) \longrightarrow \ell(k') + N(p'),$$

where the nucleon remains intact and receives a finite momentum transfer. This process therefore involves non-forward kinematics: the initial and final nucleon states carry different four-momenta,  $p' \neq p$ . Elastic-scattering measurements determine form factors, which encode the response of the nucleon to the momentum transfer and provide information on

global properties of the nucleon, such as its spatial distributions of charge and magnetisation. Unlike PDFs, however, form factors do not resolve how the longitudinal momentum of the nucleon is shared among its partons.

Hard exclusive processes combine the partonic resolution characteristic of deep-inelastic scattering with the non-forward kinematics of elastic scattering: a highly virtual photon probes the partonic structure of the incoming nucleon, while the nucleon receives a finite momentum transfer and remains intact in the final state. This class of processes gives access to GPDs, which unify the information contained in PDFs and form factors within a common framework [12, 15, 19, 20, 30]. The precise relations between these quantities are discussed in Sec. 1.3. Beyond these connections, GPDs encode correlations between the longitudinal momentum of partons and the transverse spatial structure of the nucleon, thereby revealing 3D features of nucleon structure that cannot be accessed through PDFs or form factors alone [13, 14, 31–33].

Among hard exclusive processes, Deeply Virtual Compton Scattering (DVCS) plays a central role: it is often regarded as the *golden channel* for the extraction of GPDs, due to its relatively clean theoretical description and the extensive experimental data set available for phenomenological analyses.

### 1.1.1 DVCS process and kinematics

Experimentally, DVCS is accessed through exclusive photon leptonproduction,

$$\ell(k) + N(p) \longrightarrow \ell(k') + \gamma(q') + N(p'), \quad (1.4)$$

in which a lepton scatters off a nucleon through the exchange of a virtual photon, producing an outgoing real photon and a recoiling nucleon. The underlying photon–nucleon subprocess is

$$\gamma^*(q) + N(p) \longrightarrow \gamma(q') + N(p'), \quad (1.5)$$

where the incoming virtual photon carries momentum  $q = k - k'$  and virtuality  $Q^2 \equiv -q^2 > 0$ , as introduced above. A schematic representation of the DVCS contribution to exclusive photon leptonproduction is shown in Fig. 1.2.

In addition to the photon virtuality, which sets the hard scale of the process, two further Lorentz invariants are conventionally used: the Bjorken variable

$$x_B \equiv \frac{Q^2}{2p \cdot q}, \quad (1.6)$$

and the squared momentum transfer to the nucleon,

$$t \equiv (p' - p)^2. \quad (1.7)$$

The deeply virtual regime corresponds to the Bjorken limit,

$$Q^2 \rightarrow \infty, \quad x_B \text{ fixed}, \quad t \text{ fixed}, \quad (1.8)$$

in which the photon virtuality becomes asymptotically large while  $x_B$  and  $t$  remain fixed.

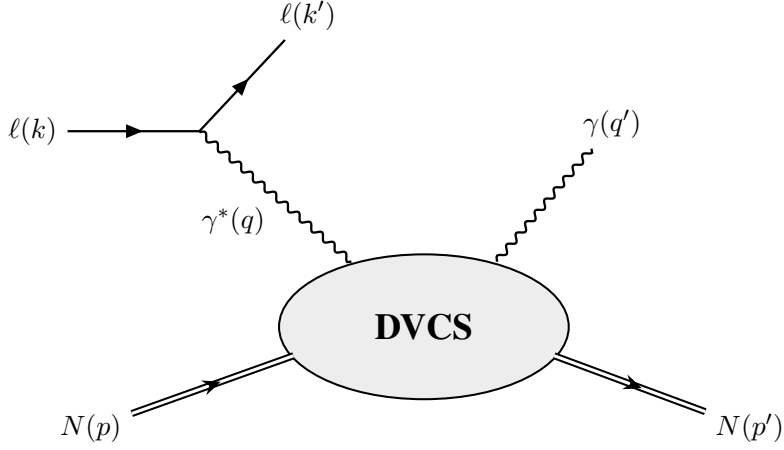


**Figure 1.2:** DVCS contribution to exclusive photon lepton production.

Following the standard convention of Ref. [6], we define the average nucleon momentum  $P$  and the momentum-transfer  $\Delta$  as

$$P \equiv \frac{p + p'}{2}, \quad \Delta \equiv p' - p. \quad (1.9)$$

The initial and final nucleon momenta can therefore be written as

$$p = P - \frac{\Delta}{2}, \quad p' = P + \frac{\Delta}{2}, \quad (1.10)$$

while momentum conservation gives

$$\Delta = p' - p = q - q', \quad t = \Delta^2. \quad (1.11)$$

An additional quantity relevant to the non-forward kinematics is the skewness variable  $\xi$ , which characterizes the longitudinal momentum asymmetry between the initial and final nucleon states. To define it in a Lorentz-invariant way, we introduce the average photon momentum

$$K \equiv \frac{q + q'}{2}. \quad (1.12)$$

The skewness variable is then defined as

$$\xi \equiv -\frac{\Delta \cdot K}{2P \cdot K}. \quad (1.13)$$

Using  $q^2 = -Q^2$ ,  $q'^2 = 0$ , together with  $\Delta = q - q'$  from Eq. (1.11), this definition can be expressed in terms of the DVCS invariants introduced above as

$$\xi = \frac{x_B}{2 - x_B + x_B t / Q^2}. \quad (1.14)$$

In the Bjorken limit, where  $t/Q^2$  is power suppressed, this reduces to

$$\xi = \frac{x_B}{2 - x_B} + \mathcal{O}\left(\frac{t}{Q^2}\right). \quad (1.15)$$

In this regime, the nucleon carries a large momentum along a preferred longitudinal direction, so that light-cone components provide a natural parametrisation of longitudinal momentum fractions. A brief discussion of light-cone coordinates and their high-energy interpretation is given in App. A. We therefore introduce an auxiliary light-like vector  $n$ , with  $n^2 = 0$ , specifying the light-cone direction along which longitudinal momenta are projected. This allows us to introduce the light-cone skewness, defined as

$$\xi_{\text{LC}} \equiv -\frac{(p' - p) \cdot n}{(p + p') \cdot n} = -\frac{\Delta \cdot n}{2P \cdot n}. \quad (1.16)$$

At leading power,  $\xi_{\text{LC}}$  coincides with the invariant skewness introduced above. We therefore denote both quantities by  $\xi$  in the following. With this identification, combining Eq. (1.16) with Eq. (1.10), the longitudinal projections of the initial and final nucleon momenta are

$$p \cdot n = (1 + \xi)P \cdot n, \quad p' \cdot n = (1 - \xi)P \cdot n, \quad (1.17)$$

so that the initial and final nucleons carry longitudinal momentum fractions  $1 + \xi$  and  $1 - \xi$ , respectively, with respect to the average momentum  $P$ .

The relation in Eq. (1.15) connects the experimentally accessible Bjorken variable  $x_B$  to the skewness variable  $\xi$  entering the non-forward partonic kinematics. Together with  $t$  and  $Q^2$ , the skewness  $\xi$  completes a convenient set of kinematic variables for the description of the DVCS amplitude developed in the following subsection.

### 1.1.2 Compton form factors and QCD factorisation

The exclusive photon leptonproduction process introduced in Eq. (1.4) does not receive contributions from DVCS alone. The same initial and final states can also be connected by the Bethe–Heitler (BH) process. In DVCS, the real photon is emitted from the hadronic subprocess, whereas in the BH process it is radiated from either the incoming or the outgoing lepton, as shown in Fig. 1.3. Therefore, the total leptonproduction amplitude is

$$\mathcal{T} = \mathcal{T}_{\text{BH}} + \mathcal{T}_{\text{DVCS}}, \quad (1.18)$$

and the squared amplitude entering the cross section can be written as

$$|\mathcal{T}|^2 = |\mathcal{T}_{\text{BH}}|^2 + |\mathcal{T}_{\text{DVCS}}|^2 + \mathcal{I}, \quad (1.19)$$

where

$$\mathcal{I} = 2 \operatorname{Re} [\mathcal{T}_{\text{BH}} \mathcal{T}_{\text{DVCS}}^*] \quad (1.20)$$

denotes the BH–DVCS interference term.

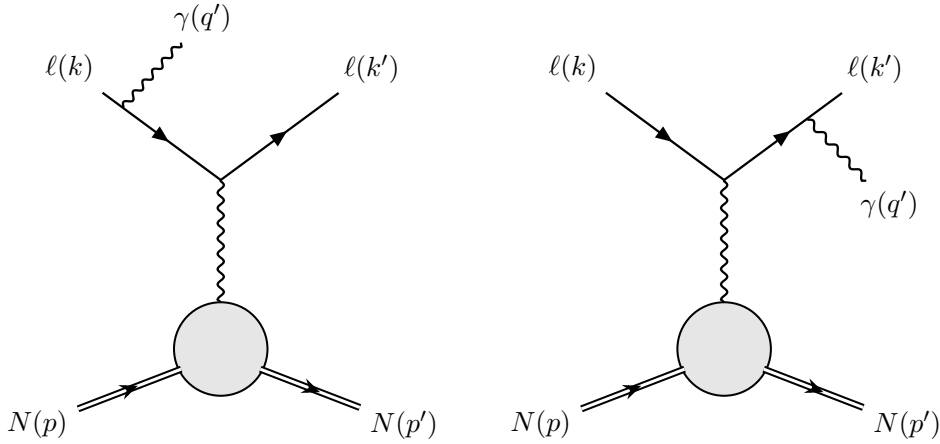


**Figure 1.3:** Bethe–Heitler contribution to exclusive photon lepton production.

The BH amplitude is calculable in QED in terms of the well-known nucleon electromagnetic form factors. By contrast, the DVCS amplitude is parametrised in terms of Compton Form Factors (CFFs) [22]. These quantities cannot be measured directly, but can be constrained through fits to exclusive photon lepton production cross sections. Since the DVCS amplitude depends linearly on the CFFs, the pure DVCS contribution in Eq. (1.19) is quadratic in the CFFs, whereas the interference term in Eq. (1.20) depends linearly on them. For the kinematic region currently accessible at facilities such as Jefferson Lab (JLab), the BH process typically dominates over the pure DVCS process. As a result, through the interference term, the BH amplitude acts as a known reference, enhancing the experimental sensitivity to the DVCS amplitude and thus to the CFFs.

In the Bjorken limit, contributions to the DVCS amplitude are organised in inverse powers of the hard scale  $Q$  and conventionally classified according to twist<sup>1</sup>. The twist-two sector, corresponding to leading twist, provides the leading-power contribution, whereas higher-twist terms are suppressed by additional powers of  $1/Q$ . Throughout this work, we restrict the analysis to the leading-twist sector. This approximation has both a theoretical and an experimental basis.

On the theoretical side, at leading power the DVCS amplitude admits a factorised description, in which short- and long-distance dynamics are separated [35]. Collins and Freund proved that this factorisation holds to all orders in perturbative QCD, up to corrections suppressed by inverse powers of the hard scale  $Q$  [23]. Such power-suppressed corrections include not only genuine higher-twist contributions, but also finite- $Q$  kinematic effects associated with the nucleon mass and momentum transfer, such as terms proportional to  $M^2/Q^2$  and  $|t|/Q^2$ . On the experimental side, scaling tests performed by the JLab Hall A Collaboration provided evidence for twist-two dominance at moderate values of  $Q^2$  [36]. This

<sup>1</sup>We use the working definition of twist of Ref. [34], in which twist classifies contributions according to their power suppression in  $M/Q$ , where  $M$  denotes a typical hadronic mass scale. A twist- $t$  contribution starts at order  $(M/Q)^{t-2}$ , up to logarithmic corrections. This should be distinguished from the geometric twist of a local operator, defined as the difference between its mass dimension and Lorentz spin.

indicates that the leading-twist factorised description can already be applicable at experimentally accessible hard scales.

The short-distance dynamics of DVCS is associated with the partonic subprocess, which constitutes the hard part and is perturbatively calculable, whereas the long-distance dynamics is associated with the non-perturbative structure of the nucleon, encoded in GPDs, and constitutes the soft part. The hard-soft separation established by the factorisation theorem is illustrated by the handbag diagram in Fig. 1.4 and is directly reflected in the structure of the CFFs. At leading twist, these can be written as convolutions of perturbatively calculable coefficient functions with the corresponding quark and gluon GPDs [22]. Schematically,

$$\mathcal{F}(\xi, t, Q^2) = \sum_{i=q,g} \int_{-1}^1 dx C_i \left( x, \xi, \frac{Q^2}{\mu^2}, \alpha_s(\mu) \right) F_i(x, \xi, t, \mu), \quad (1.21)$$

where the sum extends over all active quark flavours and the gluon channel. The function  $C_i$  is the hard-scattering coefficient, perturbatively calculable as an expansion in the QCD strong coupling  $\alpha_s$ , while  $F_i$  denotes the corresponding quark or gluon GPD. The factorisation scale  $\mu$  separates the short-distance dynamics encoded in  $C_i$  from the long-distance hadronic structure encoded in  $F_i$ .

Therefore, experiments provide only indirect access to GPDs: they constrain the CFFs, while recovering the full  $x$ -dependence of the underlying partonic distributions requires solving a non-trivial deconvolution problem. Their definition, properties, and dependence on the factorisation scale are discussed in the following sections.

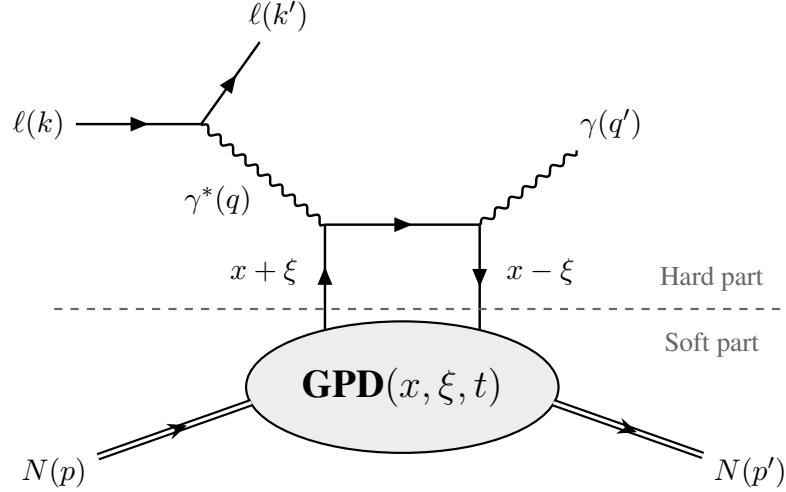


**Figure 1.4:** Handbag diagram representing the leading-twist factorisation of DVCS at leading order in perturbative QCD. The labels  $x + \xi$  and  $x - \xi$  denote the longitudinal momentum fractions of the active parton entering and leaving the hard subprocess.

## 1.2 Operator definition of GPDs

GPDs provide a partonic description of hadrons in non-forward kinematics and are associated with non-local quark and gluon correlation functions. Although the discussion of the previous section was specialised to DVCS on a nucleon target, the general operator definition of GPDs is not restricted to this particular process and can be formulated for a generic hadron  $H$ . We adopt the notation and kinematic convention introduced in Sec. 1.1.1, including the light-like vector  $n$ .

Before introducing the operator definition, it is useful to make explicit the partonic kinematics underlying the factorised picture discussed in Sec. 1.1.2. In the Bjorken limit relevant to DVCS and related hard exclusive processes, the leading contribution arises from partonic configurations collinear to the large hadron momentum. Within the leading-twist collinear description, the longitudinal momentum fraction of the active parton, i.e., the parton participating in the hard-scattering subprocess, is retained, whereas its intrinsic transverse momentum is integrated over. This collinear picture is naturally described using the same light-cone projections introduced in Sec. 1.1.1; their connection with high-energy parton kinematics is further discussed in App. A.

In forward kinematics, the active parton carries the same longitudinal momentum before and after the hard subprocess. In the non-forward kinematics relevant for GPDs, instead, momentum is transferred between the initial and final hadronic states, so that the parton entering and leaving the hard subprocess carries different longitudinal momenta. Denoting the corresponding parton momenta by  $k_{\text{in}}$  and  $k_{\text{out}}$ , the variable  $x$  parametrizes their average longitudinal momentum fraction with respect to the average hadron momentum  $P$ , according to

$$\frac{(k_{\text{in}} + k_{\text{out}}) \cdot n}{2} = x (P \cdot n). \quad (1.22)$$

The active parton also carries the momentum transferred between the initial and final hadronic states. Therefore, its longitudinal momentum changes by

$$(k_{\text{out}} - k_{\text{in}}) \cdot n = \Delta \cdot n = -2\xi P \cdot n. \quad (1.23)$$

Combining Eqs. (1.22) and (1.23) gives

$$k_{\text{in}} \cdot n = (x + \xi)P \cdot n, \quad k_{\text{out}} \cdot n = (x - \xi)P \cdot n. \quad (1.24)$$

The corresponding longitudinal momentum fractions are shown in the DVCS handbag diagram of Fig. 1.4.

The partonic information described above is encoded in hadronic matrix elements of bilocal quark and gluon operators, which define the corresponding GPD correlators. These bilocal operators are constructed from two field insertions separated along the light-like direction  $n$ . Introducing the symmetric coordinates

$$z_1^\mu = \frac{y}{2}n^\mu, \quad z_2^\mu = -\frac{y}{2}n^\mu, \quad (1.25)$$

the bare unpolarised quark and gluon GPD correlators of a generic hadron  $H$ , normalised according to the convention of Ref. [25], are defined as

$$\hat{F}_{q/H}(x, \xi, t) = \int \frac{dy}{2\pi} e^{-ix(P \cdot n)y} \left\langle p' \left| \bar{\psi}_q \left( \frac{yn}{2} \right) \frac{\not{n}}{2} \psi_q \left( -\frac{yn}{2} \right) \right| p \right\rangle, \quad (1.26)$$

$$\hat{F}_{g/H}(x, \xi, t) = \frac{n_\mu n_\nu}{x(P \cdot n)} \int \frac{dy}{2\pi} e^{-ix(P \cdot n)y} \left\langle p' \left| F_a^{\mu j} \left( \frac{yn}{2} \right) F_a^{\nu j} \left( -\frac{yn}{2} \right) \right| p \right\rangle,$$

where  $\psi_q$  is the quark field for the flavour  $q$  in the fundamental colour representation,  $\bar{\psi}_q \equiv \psi_q^\dagger \gamma^0$  is its Dirac adjoint, and  $F_a^{\mu\nu}$  is the gluon field strength for the colour configuration  $a$  in the adjoint representation. The integrals are understood to run between  $-\infty$  and  $+\infty$ . In addition, an average over the physical spin or helicity states of the initial hadron is understood. In the gluon correlator, the index  $j = 1, 2$  labels the transverse components and is implicitly summed over, together with the adjoint colour index  $a$ . Since  $n^2 = 0$ , the separation between the two fields is light-like,

$$(z_1 - z_2)^2 = y^2 n^2 = 0. \quad (1.27)$$

Furthermore, the Fourier phase in Eq. (1.26) makes explicit the connection with the partonic kinematics introduced above: the average longitudinal momentum of the active parton  $x(P \cdot n)$  is Fourier conjugate to the separation parameter  $y$ .

For a spin- $\frac{1}{2}$  hadron, the tensorial decomposition of the correlators in Eq. (1.26) defines the bare unpolarised GPDs  $\hat{H}_{i/H}$  and  $\hat{E}_{i/H}$  [25]

$$\begin{aligned} \hat{F}_{i/H}(x, \xi, t) = \frac{1}{P \cdot n} \left[ \hat{H}_{i/H}(x, \xi, t) \bar{u}(p') \frac{\not{n}}{2} u(p) \right. \\ \left. + \hat{E}_{i/H}(x, \xi, t) \bar{u}(p') \frac{i\sigma^{\mu\nu} n_\mu \Delta_\nu}{4M} u(p) \right], \end{aligned} \quad (1.28)$$

with  $i = q, g$ , where  $u$  is the spinor associated with the external hadron  $H$  and  $M$  is its mass.

The operator definitions in Eq. (1.26) are understood in the light-cone gauge,

$$n \cdot A = 0, \quad (1.29)$$

with  $A^\mu$  denoting the gluon gauge field. Although GPDs are not physical observables, they are gauge-invariant quantities. Since the two fields entering the bilocal operators are evaluated at different space-time points, their product is not gauge invariant by itself. Gauge invariance is restored by connecting the two fields with a Wilson line along the light-like direction  $n$ . In the light-cone gauge  $n \cdot A = 0$  adopted here, this Wilson line reduces to the identity and is therefore omitted in Eq. (1.26). This choice considerably simplifies the perturbative calculation of parton-in-parton GPDs discussed in the following sections.<sup>2</sup>

<sup>2</sup>In perturbative calculations, the Wilson line can be viewed as generating the emission of an arbitrary number of collinear gluons with scalar polarisation, thereby increasing the number of diagrams to be considered at a given perturbative order. Choosing the light-cone gauge eliminates these additional gauge-link contributions, significantly simplifying the diagrammatic expansion; see Ref. [25].

A schematic representation of the two parton-in-hadron GPDs is shown in Fig. 1.5. The external hadron enters with momentum  $P - \Delta/2$  and leaves with momentum  $P + \Delta/2$ , while the upper insertion represents the bilocal operator defining the quark or gluon GPD. Depending on the operator insertion, the correlator encodes either quark or gluon partonic information: a bilocal quark operator probes quark degrees of freedom, whereas a bilocal gluon operator probes gluon degrees of freedom.

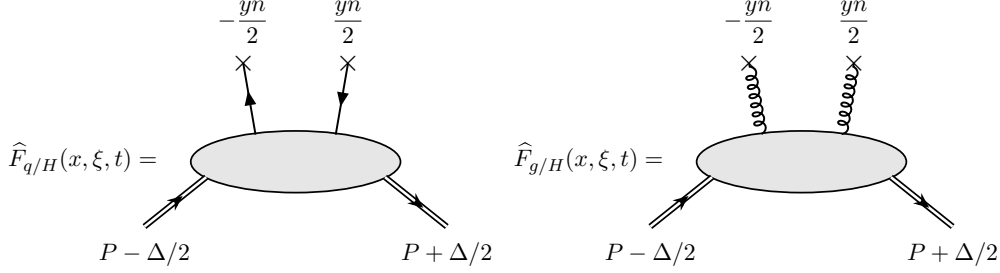


**Figure 1.5:** Graphical representation of the quark (left) and gluon (right) parton-in-hadron GPDs. Adapted from Ref. [25].

### 1.2.1 Partonic interpretation of GPDs

The physical meaning of the longitudinal variables entering the operator definitions becomes particularly transparent in a frame where the hadron carries a large longitudinal momentum. Using light-cone coordinates, with the conventions introduced in App. A, we choose the auxiliary light-like vector  $n$  as

$$n^\mu = n_-^\mu = (0, 1, \mathbf{0}_T), \quad (1.30)$$

so that

$$P \cdot n = P^+. \quad (1.31)$$

Therefore, the longitudinal components of the incoming and outgoing hadron momenta are, from Eq. (1.17),

$$p^+ = (1 + \xi)P^+, \quad p'^+ = (1 - \xi)P^+. \quad (1.32)$$

Correspondingly, the active parton entering and leaving the GPD correlator carry plus momenta

$$k_{\text{in}}^+ = (x + \xi)P^+, \quad k_{\text{out}}^+ = (x - \xi)P^+, \quad (1.33)$$

respectively, as follows from Eq. (1.24). Their difference,

$$k_{\text{out}}^+ - k_{\text{in}}^+ = \Delta^+ = -2\xi P^+, \quad (1.34)$$

is precisely the longitudinal momentum transferred between the initial and final hadronic states.

The support of quark and gluon GPDs is restricted to [37]

$$-1 \leq x \leq 1. \quad (1.35)$$

For definiteness, we consider  $\xi > 0$ <sup>3</sup>. Depending on the relative values of  $x$  and  $\xi$ , the signs of the momentum fractions in Eq. (1.33) lead to three distinct partonic configurations [6]. For

$$\xi < x < 1, \quad (1.36)$$

both  $x + \xi$  and  $x - \xi$  are positive. In the quark sector, the correlator therefore describes the emission of a quark from the incoming hadron with plus momentum

$$(x + \xi)P^+, \quad (1.37)$$

followed by its reabsorption into the outgoing hadron with plus momentum

$$(x - \xi)P^+. \quad (1.38)$$

Analogously, in the gluon sector, this region corresponds to the emission and reabsorption of a gluon carrying longitudinal momentum fractions  $x + \xi$  and  $x - \xi$ , respectively. This domain is referred to as the quark/gluon DGLAP region.

In the central interval

$$-\xi < x < \xi, \quad (1.39)$$

one has

$$x + \xi > 0, \quad x - \xi < 0. \quad (1.40)$$

The negative momentum fraction in the second parton line is reinterpreted as a positive momentum fraction  $\xi - x$  for an outgoing parton. In the quark sector, the correlator therefore describes the emission, or equivalently the absorption, of a quark–antiquark pair carrying plus momenta

$$(x + \xi)P^+ \quad \text{and} \quad (\xi - x)P^+, \quad (1.41)$$

whose total longitudinal momentum is

$$(x + \xi)P^+ + (\xi - x)P^+ = 2\xi P^+. \quad (1.42)$$

In the gluon sector, since gluons coincide with their own antiparticles, the same region represents the emission or absorption of a two-gluon state with total longitudinal momentum  $2\xi P^+$ . This central domain is known as the ERBL region. It has no direct counterpart in ordinary forward parton distributions.

Finally, for

$$-1 < x < -\xi, \quad (1.43)$$

---

<sup>3</sup>Due to time-reversal and parity symmetries, GPDs possess definite parity under  $\xi \rightarrow -\xi$ , so that the kinematics for negative skewness  $\xi < 0$  (corresponding to  $p'^+ > p^+$ ) contain no independent physical information.

both  $x + \xi$  and  $x - \xi$  are negative. For quarks, the correlator can then be interpreted in terms of antiquark emission and reabsorption, with the corresponding positive momentum fractions

$$-\xi - x \quad \text{and} \quad \xi - x. \quad (1.44)$$

This interval constitutes the antiquark DGLAP region. For gluons, the negative region  $-1 < x < -\xi$  describes the emission and reabsorption of a gluon state with reversed momentum flow. It is physically equivalent to the positive domain  $\xi < x < 1$ , with the corresponding positive momentum fractions given by  $\xi - x$  and  $-\xi - x$ .

The three regions can be summarised as

$$\begin{aligned} -1 < x < -\xi & : \text{ antiquark/gluon DGLAP region,} \\ -\xi < x < \xi & : \text{ ERL region (quark-antiquark or gluon-gluon pair),} \\ \xi < x < 1 & : \text{ quark/gluon DGLAP region.} \end{aligned} \quad (1.45)$$

For quark GPDs, the corresponding partonic configurations are illustrated schematically in Fig. 1.6.

Unlike ordinary PDFs, GPDs at non-zero skewness should not, in general, be interpreted as probability densities. The matrix elements defining them connect hadron states with different momenta and therefore encode off-diagonal quantum-mechanical correlations. In the DGLAP regions, they describe the interference between amplitudes in which a parton is removed from and reinserted into the hadron, whereas in the ERL region they involve a two-parton configuration, corresponding to a quark–antiquark pair in the quark sector or to a two-gluon state in the gluon sector.

The terminology DGLAP and ERL anticipates the different structures of the scale-evolution equations in the outer and central regions. These are discussed in Sec. 1.4. The special forward limit  $\xi \rightarrow 0$ , in which the connection with ordinary PDFs emerges, is instead considered in Sec. 1.3.

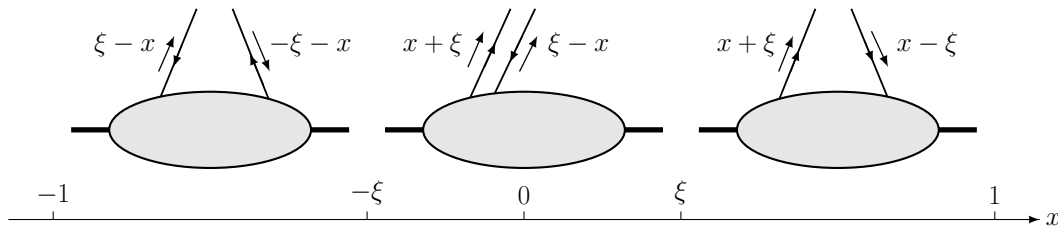


**Figure 1.6:** Partonic interpretation of quark GPDs in the three  $x$ -regions  $[-1, -\xi]$ ,  $[-\xi, \xi]$ , and  $[\xi, 1]$ . The two outer regions describe quark or antiquark emission and reabsorption, whereas the central region corresponds to the emission or absorption of a quark–antiquark pair. Reproduced from Ref. [6].

### 1.3 Properties

The operator definitions introduced in Sec. 1.2 imply a number of model-independent properties that strongly constrain the functional form of GPDs. In particular, GPDs reduce to ordinary PDFs in the forward limit, obey symmetry relations under changes of the longitudinal variables, and their Mellin moments are related to form factors of local operators.

In the following, we restrict the discussion to the unpolarised twist-two GPDs  $H_{i/H}$  and  $E_{i/H}$  introduced in Eq. (1.28). From this point onwards,  $F_{i/H}$ ,  $H_{i/H}$ , and  $E_{i/H}$  denote their renormalised counterparts. Their dependence on the factorisation scale  $\mu$  is left implicit and will be discussed explicitly in Sec. 1.4.

#### 1.3.1 Forward limit

The forward limit is obtained by letting the momentum transfer between the initial and final hadronic states vanish,

$$\Delta \longrightarrow 0. \quad (1.46)$$

According to the definitions in Eqs. (1.9) and (1.13), this implies

$$p' = p = P, \quad t \longrightarrow 0, \quad \xi \longrightarrow 0. \quad (1.47)$$

The non-forward matrix elements defining the GPDs then reduce to the forward matrix elements entering the definition of ordinary PDFs.

For the unpolarised quark GPD correlator of flavour  $q$ ,  $F_{q/H}(x, \xi, t)$ , one obtains [6, 22]

$$F_{q/H}(x, 0, 0) = H_{q/H}(x, 0, 0) = \theta(x) q_H(x) - \theta(-x) \bar{q}_H(-x), \quad (1.48)$$

where  $q_H(x)$  and  $\bar{q}_H(x)$ , defined for  $0 < x < 1$ , denote the usual quark and antiquark PDFs of the hadron  $H$ . Equivalently,

$$H_{q/H}(x, 0, 0) = q_H(x), \quad H_{q/H}(-x, 0, 0) = -\bar{q}_H(x), \quad 0 < x < 1. \quad (1.49)$$

The negative- $x$  region of a quark GPD therefore contains the corresponding antiquark distribution, consistently with the partonic interpretation discussed in Sec. 1.2.1.

With the normalisation adopted in Eq. (1.26), the forward limit of the unpolarised gluon GPD correlator  $F_{g/H}(x, \xi, t)$  reads <sup>4</sup>

$$F_{g/H}(x, 0, 0) = H_{g/H}(x, 0, 0) = \theta(x) g_H(x) - \theta(-x) g_H(-x), \quad (1.50)$$

where  $g_H(x)$ , defined for  $0 < x < 1$ , denotes the standard gluon PDF of the hadron  $H$ . Equivalently,

$$H_{g/H}(x, 0, 0) = g_H(x), \quad H_{g/H}(-x, 0, 0) = -g_H(x), \quad 0 < x < 1. \quad (1.51)$$

<sup>4</sup>The normalisation of the gluon GPD used here contains an explicit factor  $1/x$ , following Ref. [25]. In the convention commonly used in Refs. [6, 22], this factor is not extracted and the corresponding forward relation is written as  $H_{g/H}(x, 0, 0) = x g_H(x)$  for  $x > 0$  and  $H_{g/H}(x, 0, 0) = -x g_H(-x)$  for  $x < 0$ .

Unlike the quark case, the negative- $x$  region does not correspond to an independent antiparton distribution, since the gluon is its own antiparticle. Instead, it is fixed by the symmetry of the gluon GPD under  $x \rightarrow -x$ , which is discussed in the following subsection.

An important observation concerns the GPD  $E_{i/H}$ . From Eq. (1.28), its contribution to the correlator is multiplied by

$$i\sigma^{\mu\nu}n_\mu\Delta_\nu, \quad (1.52)$$

and therefore disappears when  $\Delta \rightarrow 0$ . Consequently,  $E_{q/H}$  and  $E_{g/H}$  have no analogue among the ordinary unpolarised PDFs. This does not imply that  $E_{i/H}(x, 0, 0)$  itself vanishes; rather, it decouples from the forward matrix element. The information encoded in  $E_{i/H}$  is therefore intrinsically associated with non-forward processes and cannot be obtained from inclusive deep-inelastic scattering alone.

### 1.3.2 Symmetry properties

Further constraints on GPDs follow from discrete symmetries. For a stable hadronic target, hermiticity and time-reversal invariance imply that the unpolarised GPDs considered here,  $H_{i/H}$  and  $E_{i/H}$ , are real-valued and even functions of the skewness [6, 22]:

$$\begin{aligned} H_{i/H}(x, \xi, t) &= H_{i/H}(x, -\xi, t) = H_{i/H}^*(x, \xi, t), \\ E_{i/H}(x, \xi, t) &= E_{i/H}(x, -\xi, t) = E_{i/H}^*(x, \xi, t), \end{aligned} \quad (1.53)$$

for both quarks and gluons. Hence, the region  $\xi < 0$  does not contain independent physical information, which justifies restricting the discussion to  $\xi \geq 0$ , as done in Sec. 1.2.1.

The behaviour under  $x \rightarrow -x$  depends on the parton species. Quark GPDs do not possess a definite parity under this transformation. As discussed in Sec. 1.2.1, the negative- $x$  region is associated with antiquark degrees of freedom. It is therefore customary to introduce, for  $x > 0$ , the corresponding antiquark GPDs through

$$\begin{aligned} H_{\bar{q}/H}(x, \xi, t) &\equiv -H_{q/H}(-x, \xi, t), \\ E_{\bar{q}/H}(x, \xi, t) &\equiv -E_{q/H}(-x, \xi, t). \end{aligned} \quad (1.54)$$

In the forward limit, the first of these definitions gives

$$H_{\bar{q}/H}(x, 0, 0) = \bar{q}_H(x), \quad 0 < x < 1, \quad (1.55)$$

consistently with Eq. (1.49).

Although quark GPDs are neither even nor odd in  $x$ , one can construct combinations corresponding to definite charge-conjugation parity in the crossed  $t$  channel [6, 22].<sup>5</sup> For

<sup>5</sup>The crossed  $t$ -channel is obtained by crossing the incoming hadron to the final state, so that  $\langle p' | \mathcal{O} | p \rangle$  is reinterpreted as a matrix element between the vacuum and a hadron-antihadron state,  $\langle H(p') \bar{H}(-p) | \mathcal{O} | 0 \rangle$ . The total momentum flowing through this channel is  $\Delta = p' - p$ , with  $\Delta^2 = t$ . Charge conjugation exchanges the hadron and antihadron, allowing the exchanged states to be classified by their charge-conjugation parity  $C = \pm 1$ .

the unpolarised GPDs considered here, these are

$$\begin{aligned} H_{q/H}^{(+)}(x, \xi, t) &\equiv H_{q/H}(x, \xi, t) - H_{q/H}(-x, \xi, t), \\ E_{q/H}^{(+)}(x, \xi, t) &\equiv E_{q/H}(x, \xi, t) - E_{q/H}(-x, \xi, t), \end{aligned} \quad (1.56)$$

corresponding to  $C = +1$ , and

$$\begin{aligned} H_{q/H}^{(-)}(x, \xi, t) &\equiv H_{q/H}(x, \xi, t) + H_{q/H}(-x, \xi, t), \\ E_{q/H}^{(-)}(x, \xi, t) &\equiv E_{q/H}(x, \xi, t) + E_{q/H}(-x, \xi, t), \end{aligned} \quad (1.57)$$

corresponding to  $C = -1$ . The superscripts  $(+)$  and  $(-)$  therefore denote even and odd charge-conjugation parity, respectively. Using Eq. (1.54), the  $C$ -even combination can equivalently be written, for  $x > 0$ , as

$$\begin{aligned} H_{q/H}^{(+)}(x, \xi, t) &= H_{q/H}(x, \xi, t) + H_{\bar{q}/H}(x, \xi, t), \\ E_{q/H}^{(+)}(x, \xi, t) &= E_{q/H}(x, \xi, t) + E_{\bar{q}/H}(x, \xi, t). \end{aligned} \quad (1.58)$$

For  $H$ , its forward limit therefore becomes

$$H_{q/H}^{(+)}(x, 0, 0) = q_H(x) + \bar{q}_H(x), \quad 0 < x < 1. \quad (1.59)$$

This combination is particularly relevant for the present discussion, since DVCS selects the  $C$ -even quark GPD combination [22].

For gluons, the situation is simpler because the gluon is its own antiparticle. With the normalisation adopted in Eq. (1.26), the unpolarised gluon GPDs are odd under  $x \rightarrow -x$  [6, 22],

$$\begin{aligned} H_{g/H}(-x, \xi, t) &= -H_{g/H}(x, \xi, t), \\ E_{g/H}(-x, \xi, t) &= -E_{g/H}(x, \xi, t). \end{aligned} \quad (1.60)$$

Thus, unlike the quark case, the negative- $x$  region does not contain an independent partonic distribution.

### 1.3.3 Sum rules and polynomiality

The connection between GPDs and ordinary PDFs is obtained by taking the forward limit, while retaining the dependence on the longitudinal momentum fraction  $x$ . A complementary reduction is obtained by integrating over  $x$ . In this case, the dependence on the partonic momentum fraction is removed, whereas the dependence on the momentum transfer  $t$  is retained. The resulting quantities are form factors of local QCD operators.

For the quark sector, integration over  $x$  turns the bilocal light-cone operator in Eq. (1.26) into the light-cone projection of the local flavour vector current

$$J_q^\mu(0) = \bar{\psi}_q(0) \gamma^\mu \psi_q(0). \quad (1.61)$$

More explicitly,

$$\int_{-1}^1 dx F_{q/H}(x, \xi, t) = \frac{1}{2P \cdot n} n_\mu \langle p' | J_q^\mu(0) | p \rangle. \quad (1.62)$$

For a spin- $\frac{1}{2}$  hadron  $H$ , the matrix element of this current is parametrised in terms of the flavour Dirac and Pauli form factors as [6]

$$\langle p' | J_q^\mu(0) | p \rangle = \bar{u}(p') \left[ \gamma^\mu F_1^q(t) + \frac{i\sigma^{\mu\nu} \Delta_\nu}{2M} F_2^q(t) \right] u(p). \quad (1.63)$$

Inserting Eq. (1.63) into Eq. (1.62) and comparing the resulting Dirac structures with the GPD decomposition in Eq. (1.28) gives the sum rules

$$\int_{-1}^1 dx H_{q/H}(x, \xi, t) = F_1^q(t), \quad (1.64)$$

and

$$\int_{-1}^1 dx E_{q/H}(x, \xi, t) = F_2^q(t). \quad (1.65)$$

Although the GPDs on the left-hand sides depend on  $\xi$ , their integrals do not. This follows from Lorentz invariance: after integration over  $x$ , the matrix element is expressed in terms of a local current and no longer depends on the particular light-cone direction used to define the skewness [6].

The electromagnetic current is obtained by weighting the flavour currents with the corresponding quark electric charges,

$$J_{\text{em}}^\mu = \sum_q e_q J_q^\mu, \quad (1.66)$$

so that the electromagnetic Dirac and Pauli form factors of the hadron are

$$F_{1,2}^H(t) = \sum_q e_q F_{1,2}^q(t). \quad (1.67)$$

For a nucleon target, these are the same electromagnetic form factors entering the BH amplitude discussed in Sec. 1.1.2.

The sum rules in Eqs. (1.64) and (1.65) can be generalised by considering the Mellin moments of a GPD  $F(x, \xi, t)$ , defined as

$$M_n(\xi, t) \equiv \int_{-1}^1 dx x^n F(x, \xi, t), \quad n = 0, 1, 2, \dots \quad (1.68)$$

In general, Mellin moments are related to matrix elements of local twist-two operators containing successive covariant derivatives. For quarks, the operator associated with the  $n$ -th Mellin moment has the form [6]

$$\mathcal{O}_q^{\mu_1 \dots \mu_n} = \mathcal{S} \bar{\psi}_q \gamma^\mu i \overleftrightarrow{D}^{\mu_1} \dots i \overleftrightarrow{D}^{\mu_n} \psi_q, \quad (1.69)$$

where  $\mathcal{S}$  denotes symmetrisation over the Lorentz indices and subtraction of traces,  $D^\mu$  is the QCD covariant derivative, and  $\overleftrightarrow{D}^\mu = \overrightarrow{D}^\mu - \overleftarrow{D}^\mu$ , the arrows indicating whether the derivative acts on the quark field to its right or on the conjugate quark field to its left. Lorentz covariance constrains the matrix elements of these local operators to a finite number of generalised form factors. As a consequence, the Mellin moments of GPDs are polynomials in the skewness  $\xi$ , a property known as *polynomiality*. For the unpolarised quark GPDs, one has [7]

$$\begin{aligned} \int_{-1}^1 dx x^n H_{q/H}(x, \xi, t) &= \sum_{\substack{k=0 \\ k \text{ even}}}^n (2\xi)^k A_{n+1,k}^q(t) + \epsilon_n (2\xi)^{n+1} C_{n+1}^q(t), \\ \int_{-1}^1 dx x^n E_{q/H}(x, \xi, t) &= \sum_{\substack{k=0 \\ k \text{ even}}}^n (2\xi)^k B_{n+1,k}^q(t) - \epsilon_n (2\xi)^{n+1} C_{n+1}^q(t), \end{aligned} \quad (1.70)$$

where

$$\epsilon_n \equiv \frac{1 - (-1)^n}{2} \quad (1.71)$$

is equal to 1 for odd  $n$  and 0 for even  $n$ . The coefficients  $A_{n+1,k}^q$ ,  $B_{n+1,k}^q$ , and  $C_{n+1}^q$  are generalised form factors of the corresponding local twist-two operators. Only even powers of  $\xi$  occur, consistently with the symmetry under  $\xi \rightarrow -\xi$  discussed in Sec. 1.3.2.

The lowest moment,  $n = 0$ , commonly referred to as the first Mellin moment, reproduces the sum rules in Eqs. (1.64) and (1.65). Indeed, setting  $n = 0$  in Eq. (1.70) gives

$$\int_{-1}^1 dx H_{q/H}(x, \xi, t) = A_{1,0}^q(t), \quad \int_{-1}^1 dx E_{q/H}(x, \xi, t) = B_{1,0}^q(t), \quad (1.72)$$

with

$$A_{1,0}^q(t) = F_1^q(t), \quad B_{1,0}^q(t) = F_2^q(t). \quad (1.73)$$

The next moment,  $n = 1$ , is the second Mellin moment and provides access to a different set of hadronic form factors. Setting  $n = 1$  in Eq. (1.70) gives

$$\begin{aligned} \int_{-1}^1 dx x H_{q/H}(x, \xi, t) &= A_{2,0}^q(t) + 4\xi^2 C_2^q(t), \\ \int_{-1}^1 dx x E_{q/H}(x, \xi, t) &= B_{2,0}^q(t) - 4\xi^2 C_2^q(t). \end{aligned} \quad (1.74)$$

The form factors  $A_{2,0}^q$ ,  $B_{2,0}^q$ , and  $C_2^q$  parametrize matrix elements of the quark contribution to the QCD energy–momentum tensor and are commonly referred to as *gravitational form factors* (see, e.g., Ref. [17] for a review and references therein). Adding the two relations in Eq. (1.74), the terms proportional to  $C_2^q$  cancel,

$$\int_{-1}^1 dx x [H_{q/H}(x, \xi, t) + E_{q/H}(x, \xi, t)] = A_{2,0}^q(t) + B_{2,0}^q(t), \quad (1.75)$$

so that the second Mellin moment of the combination  $H + E$  is independent of  $\xi$ .

For a spin- $\frac{1}{2}$  hadron, the quark contribution to the total angular momentum is related to these gravitational form factors by [15, 20]

$$J_q = \frac{1}{2} \left[ A_{2,0}^q(0) + B_{2,0}^q(0) \right]. \quad (1.76)$$

Combining this relation with Eq. (1.75) gives

$$J_q = \frac{1}{2} \int_{-1}^1 dx x \left[ H_{q/H}(x, \xi, 0) + E_{q/H}(x, \xi, 0) \right], \quad (1.77)$$

where  $J_q$  denotes the total angular momentum carried by quarks and antiquarks of flavour  $q$  in the hadron. Using the forward-limit relation derived in Sec. 1.3.1, the contribution involving  $H_q$  becomes

$$A_{2,0}^q(0) = \int_0^1 dx x \left[ q_H(x) + \bar{q}_H(x) \right], \quad (1.78)$$

which is the total longitudinal momentum fraction carried by quarks and antiquarks of flavour  $q$ . Thus, while the contribution of  $H_{q/H}$  is fixed in the forward limit by ordinary PDFs, the additional information required to determine  $J_q$  is contained in the GPD  $E_{q/H}$ .

The gluon sector exhibits an analogous polynomiality structure. Rewriting the relations of Ref. [6] in the normalisation used here gives

$$\begin{aligned} \int_0^1 dx x^n H_{g/H}(x, \xi, t) &= \sum_{\substack{k=0 \\ k \text{ even}}}^n (2\xi)^k A_{n+1,k}^g(t) + \epsilon_n (2\xi)^{n+1} C_{n+1}^g(t), \\ \int_0^1 dx x^n E_{g/H}(x, \xi, t) &= \sum_{\substack{k=0 \\ k \text{ even}}}^n (2\xi)^k B_{n+1,k}^g(t) - \epsilon_n (2\xi)^{n+1} C_{n+1}^g(t), \end{aligned} \quad (1.79)$$

for odd  $n$ . Although the moments in Eq. (1.79) are written over  $0 < x < 1$ , the restriction to odd values of  $n$  is consistent with the symmetry of the gluon GPDs over their full support  $-1 < x < 1$ . With the normalisation used here,  $H_{g/H}$  and  $E_{g/H}$  are odd under  $x \rightarrow -x$ . Therefore, for even  $n$  the integrands  $x^n H_{g/H}$  and  $x^n E_{g/H}$  are odd and their integrals over  $[-1, 1]$  vanish, whereas for odd  $n$  they are even, so that

$$\begin{aligned} \int_{-1}^1 dx x^n H_{g/H}(x, \xi, t) &= 2 \int_0^1 dx x^n H_{g/H}(x, \xi, t), \\ \int_{-1}^1 dx x^n E_{g/H}(x, \xi, t) &= 2 \int_0^1 dx x^n E_{g/H}(x, \xi, t), \end{aligned} \quad (1.80)$$

The negative- $x$  region therefore contains no independent information, and the non-vanishing gluon moments can be expressed entirely in terms of the region  $0 < x < 1$ .

An important difference from the quark sector appears already at the lowest moments. For quarks, the  $n = 0$  moment is associated with the local flavour vector current and gives the electromagnetic Dirac and Pauli form factors. No corresponding  $n = 0$  local twist-two operator exists in the gluon sector. The gluon tower starts at  $n = 1$ , since its lowest-rank gauge-invariant operator contains two field-strength tensors. Setting  $n = 1$  in Eq. (1.79) gives

$$\begin{aligned} \int_0^1 dx x H_{g/H}(x, \xi, t) &= A_{2,0}^g(t) + 4\xi^2 C_2^g(t), \\ \int_0^1 dx x E_{g/H}(x, \xi, t) &= B_{2,0}^g(t) - 4\xi^2 C_2^g(t). \end{aligned} \quad (1.81)$$

As in the quark sector, the form factors  $A_{2,0}^g$ ,  $B_{2,0}^g$ , and  $C_2^g$  parametrize matrix elements of the gluon contribution to the QCD energy-momentum tensor and are therefore gluon gravitational form factors. Adding the two relations, the terms proportional to  $C_2^g$  cancel,

$$\int_0^1 dx x [H_{g/H}(x, \xi, t) + E_{g/H}(x, \xi, t)] = A_{2,0}^g(t) + B_{2,0}^g(t). \quad (1.82)$$

For a spin- $\frac{1}{2}$  hadron, the gluon contribution to the total angular momentum is related to these gravitational form factors by [15, 20]

$$J_g = \frac{1}{2} [A_{2,0}^g(0) + B_{2,0}^g(0)], \quad (1.83)$$

which, together with Eq. (1.82), gives

$$J_g = \frac{1}{2} \int_0^1 dx x [H_{g/H}(x, \xi, 0) + E_{g/H}(x, \xi, 0)]. \quad (1.84)$$

In the forward limit,

$$A_{2,0}^g(0) = \int_0^1 dx x g_H(x), \quad (1.85)$$

which is the longitudinal momentum fraction carried by gluons. Thus, as for quarks, the contribution involving  $H_{g/H}$  is fixed in the forward limit by the ordinary gluon PDF, whereas the GPD  $E_{g/H}$  provides the additional information required to determine  $J_g$ .

Combining the quark and gluon contributions, the total spin of a spin- $\frac{1}{2}$  hadron can be written through Ji's sum rule as [15, 20]

$$\begin{aligned} \frac{1}{2} &= \sum_q J_q + J_g \\ &= \frac{1}{2} \sum_q \int_{-1}^1 dx x [H_{q/H}(x, \xi, 0) + E_{q/H}(x, \xi, 0)] \\ &\quad + \frac{1}{2} \int_0^1 dx x [H_{g/H}(x, \xi, 0) + E_{g/H}(x, \xi, 0)]. \end{aligned} \quad (1.86)$$

For a nucleon target, Ji's sum rule therefore makes GPDs a key tool for investigating the partonic origin of the nucleon spin and constraining orbital angular momentum contributions.

## 1.4 Scale evolution of GPDs

The bare GPD correlators introduced in Sec. 1.2 are matrix elements of bilocal light-cone operators. These operators develop ultraviolet (UV) divergences and therefore require renormalisation. As a consequence, the corresponding renormalised GPDs depend on an arbitrary renormalisation scale  $\mu$ . When GPDs enter the factorised description of hard scattering processes, this scale is identified with the so-called *factorisation scale*.

Controlling this scale dependence plays an important role in GPD phenomenology. A GPD parametrisation is typically defined at an initial scale  $\mu_0$ , whereas experimental measurements probe different values of the hard scale  $Q$ . In the case of DVCS, as discussed in Sec. 1.1.2, the GPDs enter the CFFs through a convolution with perturbatively calculable coefficient functions. Both the GPDs and the coefficient functions depend on the factorisation scale  $\mu$ , which separates the long-distance hadronic dynamics from the short-distance partonic subprocess. Since this separation is arbitrary, their  $\mu$ -dependence must compensate in the physical CFFs, up to the perturbative accuracy considered.

For DVCS, the factorisation scale is conventionally chosen to be of the same order as the hard scale of the process, set by the photon virtuality,

$$\mu^2 \sim Q^2, \quad (1.87)$$

thereby avoiding potentially large terms proportional to  $\ln(Q^2/\mu^2)$  in the perturbative coefficient functions. Consequently, before entering the convolution that determines the CFFs, the GPDs specified at the input scale  $\mu_0$  must be evolved to the factorisation scale  $\mu \sim Q$  associated with the process. The resulting CFFs parametrise the DVCS amplitude and are constrained through fits to exclusive photon leptonproduction cross sections. GPD evolution thus allows a common parametrisation defined at  $\mu_0$  to be consistently related to the different hard scales probed experimentally. This procedure is schematically illustrated in Fig. 1.7.

The relation between GPDs defined at different factorisation scales is governed by renormalisation-group equations, conventionally referred to as the GPD evolution equations. In the following, we derive these equations and introduce the corresponding evolution kernels.

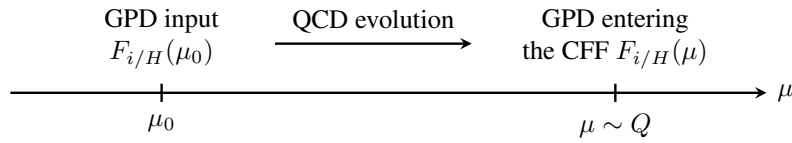


**Figure 1.7:** Schematic representation of GPD scale evolution. A GPD parametrisation specified at an initial scale  $\mu_0$  is evolved to the process scale  $Q$ , before entering the calculation of the CFFs.

### 1.4.1 Evolution equation and evolution kernels

The starting point for the derivation of the evolution equations is the scale independence of the bare GPD correlators. In the following, the dependence on the momentum transfer  $t$

will be left implicit, since  $t$  does not participate in the evolution.

In dimensional regularisation, with  $d = 4 - 2\epsilon$ , the bare GPD correlators  $\hat{F}_{i/H}$  can be renormalised in the  $\overline{\text{MS}}$  scheme as [25]

$$F_{i/H}(x, \xi, \mu) = \sum_{j=q,g} \int_{-1}^1 \frac{dy}{|y|} Z_{ij} \left( \frac{x}{y}, \frac{\xi}{x}, \alpha_s(\mu), \epsilon \right) \hat{F}_{j/H}(y, \xi, \epsilon), \quad i = q, g, \quad (1.88)$$

where the sum runs over the gluon and all active quark flavours at the scale  $\mu$ . The renormalisation kernels  $Z_{ij}$  remove the UV divergences of the bare correlators and allow for mixing between different partonic species. Exploiting the fact that the bare correlators do not depend on the arbitrary renormalisation scale,

$$\frac{d}{d \ln \mu^2} \hat{F}_{i/H}(x, \xi, \epsilon) = 0, \quad (1.89)$$

the logarithmic derivative with respect to  $\mu^2$  of Eq. (1.88) gives

$$\frac{dF_{i/H}(x, \xi, \mu)}{d \ln \mu^2} = \sum_{j=q,g} \int_{-1}^1 \frac{dy}{|y|} \frac{dZ_{ij} \left( \frac{x}{y}, \frac{\xi}{x}, \alpha_s(\mu), \epsilon \right)}{d \ln \mu^2} \hat{F}_{j/H}(y, \xi, \epsilon). \quad (1.90)$$

At this stage, the right-hand side still contains the bare correlators. To express the evolution entirely in terms of renormalised quantities, we introduce the inverse renormalisation kernel  $Z^{-1}$ , defined through [25]

$$\sum_{j=q,g} \int_{-1}^1 \frac{dx}{|x|} Z_{kj}^{-1} \left( \frac{z}{x}, \frac{\xi}{z}, \alpha_s, \epsilon \right) Z_{ji} \left( \frac{x}{y}, \frac{\xi}{x}, \alpha_s, \epsilon \right) = \delta_{ki} \delta \left( 1 - \frac{z}{y} \right). \quad (1.91)$$

The bare correlators can therefore be expressed in terms of the renormalised ones as

$$\hat{F}_{j/H}(y, \xi, \epsilon) = \sum_{k=q,g} \int_{-1}^1 \frac{dz}{|z|} Z_{jk}^{-1} \left( \frac{y}{z}, \frac{\xi}{y}, \alpha_s(\mu), \epsilon \right) F_{k/H}(z, \xi, \mu). \quad (1.92)$$

Substituting Eq. (1.92) into Eq. (1.90), we obtain an evolution equation entirely expressed in terms of renormalised GPD correlators,

$$\frac{dF_{i/H}(x, \xi, \mu)}{d \ln \mu^2} = \sum_{k=q,g} \int_{-1}^1 \frac{dz}{|z|} \mathcal{P}_{i/k} \left( \frac{x}{z}, \frac{\xi}{x}, \alpha_s(\mu) \right) F_{k/H}(z, \xi, \mu), \quad (1.93)$$

where  $\mathcal{P}_{i/k}$  are the GPD evolution kernels, or anomalous dimensions, defined by

$$\mathcal{P}_{i/k} \left( \frac{x}{z}, \frac{\xi}{x}, \alpha_s \right) = \lim_{\epsilon \rightarrow 0} \sum_{j=q,g} \int_{-1}^1 \frac{dy}{|y|} \frac{dZ_{ij} \left( \frac{x}{y}, \frac{\xi}{x}, \alpha_s(\mu), \epsilon \right)}{d \ln \mu^2} Z_{jk}^{-1} \left( \frac{y}{z}, \frac{\xi}{y}, \alpha_s, \epsilon \right). \quad (1.94)$$

These kernels are perturbatively calculable and admit an expansion in powers of the strong coupling  $\alpha_s$  [25],

$$\mathcal{P}_{i/k}\left(\frac{x}{z}, \frac{\xi}{x}, \alpha_s\right) = \sum_{n=0}^{\infty} a_s^{n+1} \mathcal{P}_{i/k}^{[n]}\left(\frac{x}{z}, \frac{\xi}{x}\right), \quad a_s \equiv \frac{\alpha_s}{4\pi}. \quad (1.95)$$

The perturbative accuracy of the GPD evolution is determined by the order at which the evolution kernels are retained. At leading order (LO), only the one-loop contribution, of order  $\alpha_s$ , is retained. At next-to-leading order (NLO), the two-loop contribution, of order  $\alpha_s^2$ , is included as well, with higher perturbative orders following analogously.

**DGLAP and ERBL structure.** The distinction between the DGLAP and ERBL regions introduced in Sec. 1.2.1 is directly reflected in the support of the GPD evolution kernels. In the outer DGLAP regions, only part of the GPD support contributes to the evolution at a fixed value of  $x$ . In particular, for  $x > \xi$ , the convolution in Eq. (1.93) is restricted to

$$x \leq z \leq 1, \quad (1.96)$$

whereas for  $x < -\xi$  it is restricted to

$$-1 \leq z \leq x. \quad (1.97)$$

Thus, in the DGLAP regions the evolution at a given  $x$  involves only longitudinal momentum fractions satisfying  $|z| \geq |x|$ . In the central ERBL region,  $|x| < \xi$ , the evolution at a fixed value of  $x$  can instead receive contributions from the entire support of the GPD,

$$-1 \leq z \leq 1. \quad (1.98)$$

The convolution in Eq. (1.93) therefore contains contributions both from the central region,  $|z| < \xi$ , and from the two outer regions,  $|z| > \xi$ . The corresponding support of the evolution kernel in the  $(x, z)$  plane is illustrated in Fig. 1.8. A more detailed discussion of the support properties of the GPD evolution equations can be found in Ref. [6].

The ordinary DGLAP and ERBL evolution equations are recovered as limiting cases of the general GPD evolution equation [6]. In the forward limit,

$$\xi \rightarrow 0, \quad (1.99)$$

the non-forward kernels reduce to the usual DGLAP splitting functions governing the scale dependence of PDFs,

$$\mathcal{P}_{i/k}\left(\frac{x}{z}, \frac{\xi}{x}, \alpha_s\right) \xrightarrow{\xi \rightarrow 0} P_{ik}\left(\frac{x}{z}, \alpha_s\right). \quad (1.100)$$

At the opposite endpoint,

$$|\xi| \rightarrow 1, \quad (1.101)$$

one of the external hadron states carries vanishing plus momentum, i.e.,

$$(1 - \xi)P^+ \rightarrow 0 \quad \text{or} \quad (1 + \xi)P^+ \rightarrow 0. \quad (1.102)$$

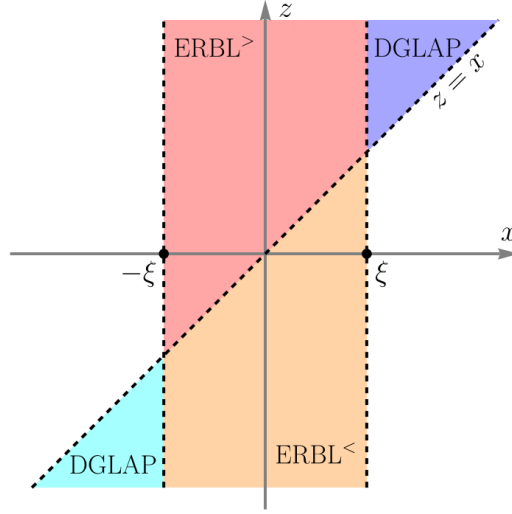


**Figure 1.8:** Support of the GPD evolution kernels in the  $(x, z)$  plane for positive skewness  $\xi$ . The outer regions correspond to DGLAP evolution, while the central region  $|x| < \xi$  corresponds to ERBL evolution. The dashed line denotes  $z = x$ .

In this limit, after an appropriate change of variables, the GPD evolution kernels reduce to the ERBL kernels governing the scale evolution of distribution amplitudes, which are non-perturbative quantities entering the description of exclusive processes. The GPD evolution equations therefore provide a common framework interpolating between the familiar DGLAP and ERBL limits.

As follows from Eq. (1.94), the evolution kernels are determined by the renormalisation kernels  $Z_{ij}$ , and hence by the UV renormalisation of the light-cone operators entering the definition of GPDs. Since this UV structure is universal, it does not depend on the external states between which the operators are evaluated. Therefore, instead of considering hadronic states, for which the GPD correlators are intrinsically non-perturbative, one can evaluate the same operators between quark or gluon states. The resulting quantities are referred to as *parton-in-parton GPDs* and provide the perturbatively calculable objects from which the renormalisation kernels  $Z_{ij}$ , and hence the GPD evolution kernels, can be extracted. We introduce these quantities in the following subsection.

### 1.4.2 Parton-in-parton GPDs

To extract the universal UV structure discussed above, we replace the external hadronic states by on-shell partonic states. We consider massless external partons with incoming and outgoing momenta

$$p_{\text{in}} = (1 + \eta)p, \quad p_{\text{out}} = (1 - \eta)p, \quad p^2 = 0. \quad (1.103)$$

The partonic states therefore reproduce the same form of longitudinal non-forward kinematics introduced for the hadronic GPDs, with  $p$  playing the role of the average partonic

momentum and  $\eta$  denoting the corresponding partonic skewness. The partonic skewness  $\eta$  should not be identified with the hadronic skewness  $\xi$  introduced in Sec. 1.1.1. Unlike the latter, the partonic variable  $\eta$  is not restricted to values smaller than unity; values  $\eta > 1$  correspond to ERBL kinematics. In the following,  $\eta$  is used whenever the kinematics refers to partonic external states or parton-level calculations. As in the hadronic case, we consider positive partonic skewness,  $\eta > 0$ .<sup>6</sup>

Unlike hadronic states, the external partons carry colour as well as spin or helicity quantum numbers. Since we consider unpolarised parton-in-parton GPDs, an average over these degrees of freedom has to be performed. The colour average depends on the representation of the external parton. For an external quark, which transforms in the fundamental representation of  $SU(N_c)$ , it is given by

$$\frac{1}{N_c} \text{Tr}_c[\dots], \quad (1.104)$$

whereas for an external gluon, which transforms in the adjoint representation, one has

$$\frac{1}{N_c^2 - 1} \text{Tr}_c[\dots]. \quad (1.105)$$

Here  $\text{Tr}_c$  denotes the trace over colour indices and  $N_c = 3$ . The different normalisation factors simply reflect the dimensions of the fundamental and adjoint colour representations,  $N_c$  and  $N_c^2 - 1$ , respectively. In addition, we average over the two physical spin or helicity states of the external parton,

$$\frac{1}{2} \sum_s, \quad (1.106)$$

where  $s$  denotes the spin index for an external quark and the helicity index for an external gluon. When Dirac indices are present, we denote the corresponding trace by  $\text{Tr}_D$ . In the following, for compactness,  $\text{Tr}$  will denote the traces over the relevant internal indices. Combining the colour and spin or helicity averages, the overall normalisation factors are therefore

$$\frac{1}{2N_c} \sum_s \text{Tr}[\dots] \quad \text{and} \quad \frac{1}{2(N_c^2 - 1)} \sum_s \text{Tr}[\dots] \quad (1.107)$$

for external quarks and gluons, respectively.

In the presence of more than one quark flavour, seven different bare parton-in-parton GPDs can be constructed by combining the quark and gluon operators with the possible external partonic states [25]. We first consider the gluon operator. Evaluating it between external gluon states, the gluon-in-gluon GPD is defined by

$$\begin{aligned} \hat{F}_{g/g}(x, \eta) &= \frac{n_\mu n_\nu}{2(N_c^2 - 1)x(n \cdot p)} \int \frac{dy}{2\pi} e^{-ix(n \cdot p)y} \\ &\times \sum_s \text{Tr} \left[ \left\langle (1 - \eta)p, s \left| F_a^{\mu j} \left( \frac{yn}{2} \right) F_a^{\nu j} \left( -\frac{yn}{2} \right) \right| (1 + \eta)p, s \right\rangle_g \right]. \end{aligned} \quad (1.108)$$

<sup>6</sup>Under  $\eta \rightarrow -\eta$ , the initial and final partonic states are interchanged, so that negative values of  $\eta$  contain no independent information.

The same gluon operator evaluated between external quark states instead defines the gluon-in-quark GPD,

$$\begin{aligned} \hat{F}_{g/q}(x, \eta) &= \frac{n_\mu n_\nu}{2N_c x(n \cdot p)} \int \frac{dy}{2\pi} e^{-ix(n \cdot p)y} \\ &\times \sum_s \text{Tr} \left[ \left\langle (1-\eta)p, s \left| F_a^{\mu j} \left( \frac{yn}{2} \right) F_a^{\nu j} \left( -\frac{yn}{2} \right) \right| (1+\eta)p, s \right\rangle_q \right]. \end{aligned} \quad (1.109)$$

Conversely, the quark operator of flavour  $q$  can be evaluated between external gluon states, giving the quark-in-gluon GPD,

$$\begin{aligned} \hat{F}_{q/g}(x, \eta) &= \frac{1}{2(N_c^2 - 1)} \int \frac{dy}{2\pi} e^{-ix(n \cdot p)y} \\ &\times \sum_s \text{Tr} \left[ \left\langle (1-\eta)p, s \left| \bar{\psi}_q \left( \frac{yn}{2} \right) \frac{\not{n}}{2} \psi_q \left( -\frac{yn}{2} \right) \right| (1+\eta)p, s \right\rangle_g \right]. \end{aligned} \quad (1.110)$$

Finally, the same quark operator can be evaluated between four different quark or antiquark external states. For an external quark of the same flavour  $q$ , the quark-in-quark GPD is defined by

$$\begin{aligned} \hat{F}_{q/q}(x, \eta) &= \frac{1}{2N_c} \int \frac{dy}{2\pi} e^{-ix(n \cdot p)y} \\ &\times \sum_s \text{Tr} \left[ \left\langle (1-\eta)p, s \left| \bar{\psi}_q \left( \frac{yn}{2} \right) \frac{\not{n}}{2} \psi_q \left( -\frac{yn}{2} \right) \right| (1+\eta)p, s \right\rangle_q \right]. \end{aligned} \quad (1.111)$$

For an external antiquark of the same flavour, one defines

$$\begin{aligned} \hat{F}_{q/\bar{q}}(x, \eta) &= \frac{1}{2N_c} \int \frac{dy}{2\pi} e^{-ix(n \cdot p)y} \\ &\times \sum_s \text{Tr} \left[ \left\langle (1-\eta)p, s \left| \bar{\psi}_q \left( \frac{yn}{2} \right) \frac{\not{n}}{2} \psi_q \left( -\frac{yn}{2} \right) \right| (1+\eta)p, s \right\rangle_{\bar{q}} \right]. \end{aligned} \quad (1.112)$$

If  $q' \neq q$  denotes a different quark flavour, the remaining two possibilities are

$$\begin{aligned} \hat{F}_{q/q'}(x, \eta) &= \frac{1}{2N_c} \int \frac{dy}{2\pi} e^{-ix(n \cdot p)y} \\ &\times \sum_s \text{Tr} \left[ \left\langle (1-\eta)p, s \left| \bar{\psi}_q \left( \frac{yn}{2} \right) \frac{\not{n}}{2} \psi_q \left( -\frac{yn}{2} \right) \right| (1+\eta)p, s \right\rangle_{q'} \right], \end{aligned} \quad (1.113)$$

and

$$\begin{aligned} \hat{F}_{q/\bar{q}'}(x, \eta) &= \frac{1}{2N_c} \int \frac{dy}{2\pi} e^{-ix(n \cdot p)y} \\ &\times \sum_s \text{Tr} \left[ \left\langle (1-\eta)p, s \left| \bar{\psi}_q \left( \frac{yn}{2} \right) \frac{\not{n}}{2} \psi_q \left( -\frac{yn}{2} \right) \right| (1+\eta)p, s \right\rangle_{\bar{q}'} \right]. \end{aligned} \quad (1.114)$$

A graphical representation of the bare parton-in-parton GPDs defined above is displayed in Fig. 1.9. In these graphs, the crosses represent the bilocal operator insertion together with the integration over the light-cone separation  $y$ , as illustrated in Fig. 1.10. The tensor  $-g_{\perp}^{\alpha\beta}$  projects onto the transverse gluon components and is equivalent to the  $\delta^{ij}$  notation when the transverse indices are written explicitly.

A useful simplification concerns the parton-in-parton GPDs involving the gluon operator. As discussed in Sec. 1.2, we work in the light-cone gauge  $n \cdot A = 0$ . In this gauge, the contraction of the field-strength tensor reduces to [25]

$$n_{\mu} F_a^{\mu j}(z) = (n \cdot \partial) A_a^j(z). \quad (1.115)$$

Using translational invariance together with the Fourier transform entering the GPD definition, the two derivatives generate the kinematic factor

$$(x^2 - \eta^2)(n \cdot p)^2. \quad (1.116)$$

The explicit derivation of this result is given in Sec. B.1. The gluon-in-gluon and gluon-in-quark GPDs can therefore be expressed directly in terms of the gluon fields as

$$\begin{aligned} \hat{F}_{g/k}(x, \eta) &= \frac{(n \cdot p)(x^2 - \eta^2)}{\mathcal{N}_k x} \int \frac{dy}{2\pi} e^{-ix(n \cdot p)y} \\ &\times \sum_s \text{Tr} \left[ \left\langle (1 - \eta)p, s \left| A_a^j \left( \frac{yn}{2} \right) A_a^j \left( -\frac{yn}{2} \right) \right| (1 + \eta)p, s \right\rangle_k \right], \end{aligned} \quad (1.117)$$

where  $k = g, q$  denotes the species of the external partonic state and

$$\mathcal{N}_g = 2(N_c^2 - 1), \quad \mathcal{N}_q = 2N_c. \quad (1.118)$$

Unlike their parton-in-hadron counterparts, parton-in-parton GPDs are perturbatively calculable, since the non-perturbative hadronic external states are replaced by asymptotic partonic states. Their matrix elements can therefore be evaluated order by order in perturbative QCD. Accordingly, the bare parton-in-parton GPDs admit the expansion

$$\hat{F}_{i/k}(x, \eta, \epsilon) = \sum_{n=0}^{\infty} a_s^n \hat{F}_{i/k}^{[n]}(x, \eta, \epsilon). \quad (1.119)$$

The fields entering Eqs. (1.108)–(1.114) are interacting fields. They can therefore radiate and absorb partons through QCD interactions before coupling to the external asymptotic states. Consequently, the parton species associated with the operator,  $i$ , does not in general need to coincide with that of the external state,  $k$ . Beyond tree level, this gives rise to off-diagonal parton-in-parton GPDs, such as the gluon-in-quark and quark-in-gluon distributions.

At tree level, however, no QCD interaction occurs between the external state and the operator insertion. The operator can therefore couple only to an external parton of the same species, so that only the diagonal quark-in-quark and gluon-in-gluon GPDs are non-vanishing. The corresponding diagrams are shown in Fig. 1.11.

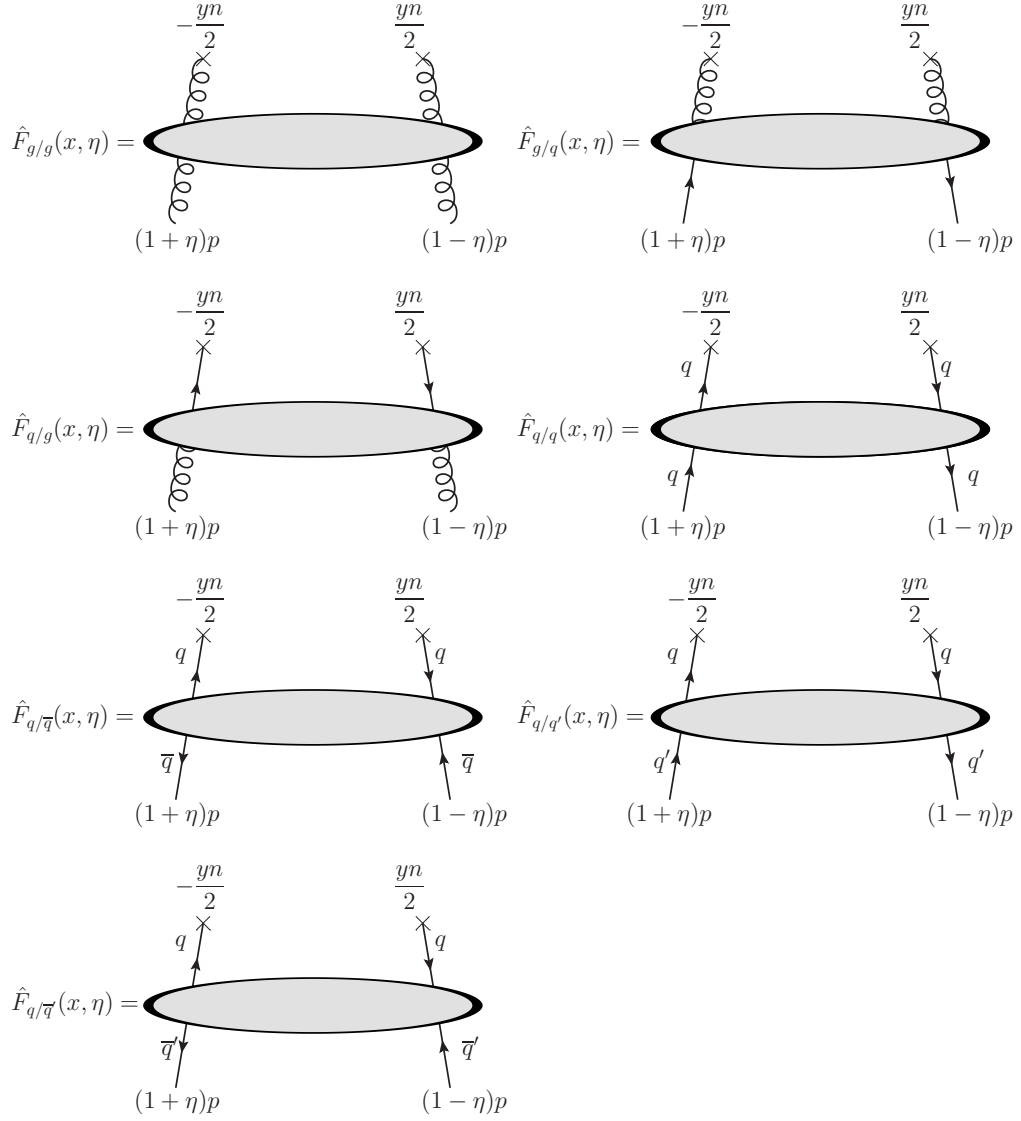


**Figure 1.9:** Schematic representation of the seven bare parton-in-parton GPDs. The upper lines represent the fields entering the bilocal quark or gluon operator, while the lower lines identify the external partonic states. The grey region represents the QCD interactions connecting the operator to the asymptotic states. Adapted from Ref. [25].

(a) Quark operator

(b) Gluon operator

$$\hat{F}_{q/q}^{[0]}(x, \eta) =$$

$$\hat{F}_{g/g}^{[0]}(x, \eta) =$$

From Eqs. (1.108) and (1.111), the two tree-level parton-in-parton GPDs are

$$\hat{F}_{q/q}^{[0]}(x, \eta) = \sqrt{1 - \eta^2} \delta(1 - x), \quad \hat{F}_{g/g}^{[0]}(x, \eta) = (1 - \eta^2) \delta(1 - x). \quad (1.120)$$

Their explicit evaluation is presented in App. B.2. For later convenience, we introduce the kinematic factors

$$D_q(\eta) \equiv \sqrt{1 - \eta^2}, \quad D_g(\eta) \equiv 1 - \eta^2, \quad (1.121)$$

in terms of which the tree-level result can be written compactly as

$$\hat{F}_{i/k}^{[0]}(x, \eta) = D_i(\eta) \delta_{ik} \delta(1 - x), \quad i, k = q, g. \quad (1.122)$$

At order  $\alpha_s$ , QCD interactions generate radiative corrections to these diagonal distributions and, at the same time, open the off-diagonal  $\hat{F}_{g/q}$  and  $\hat{F}_{q/g}$  channels. The remaining quark channels involving an antiquark or a different flavour contribute at higher orders.

The tree-level result in Eq. (1.122) provides the lowest-order contribution entering the renormalisation of the parton-in-parton GPDs. As already discussed in Sec. 1.4.1, the renormalisation of the light-cone operators is independent of the external states. Therefore, the same kernels  $Z_{ij}$  introduced in Eq. (1.88) also renormalise the parton-in-parton GPDs,

$$F_{i/k}(x, \eta, \mu) = \sum_{j=q,g} \int_{-1}^1 \frac{dy}{|y|} Z_{ij} \left( \frac{x}{y}, \frac{\eta}{x}, \alpha_s(\mu), \epsilon \right) \hat{F}_{j/k}(y, \eta, \epsilon). \quad (1.123)$$

Expanding the renormalised distributions and the renormalisation kernels in powers of  $a_s$  [25],

$$F_{i/k}(x, \eta, \mu) = \sum_{n=0}^{\infty} a_s^n F_{i/k}^{[n]}(x, \eta, \mu), \quad Z_{ij} \left( \frac{x}{y}, \frac{\eta}{x}, \alpha_s, \epsilon \right) = \sum_{n=0}^{\infty} a_s^n Z_{ij}^{[n]} \left( \frac{x}{y}, \frac{\eta}{x}, \epsilon \right), \quad (1.124)$$

and using the expansion of the bare distributions in Eq. (1.119), Eq. (1.123) becomes

$$\begin{aligned} \sum_{n=0}^{\infty} a_s^n F_{i/k}^{[n]}(x, \eta, \mu) &= \sum_{j=q,g} \int_{-1}^1 \frac{dy}{|y|} \left[ \sum_{r=0}^{\infty} a_s^r Z_{ij}^{[r]} \left( \frac{x}{y}, \frac{\eta}{x}, \epsilon \right) \right] \\ &\quad \times \left[ \sum_{m=0}^{\infty} a_s^m \hat{F}_{j/k}^{[m]}(y, \eta, \epsilon) \right]. \end{aligned} \quad (1.125)$$

Multiplying the two perturbative series on the right-hand side gives powers  $a_s^{r+m}$ . Therefore, at a fixed order  $a_s^n$ , the indices satisfy

$$r + m = n \implies m = n - r, \quad (1.126)$$

with  $r = 0, \dots, n$ , since  $m \geq 0$ . Equating the coefficients of the same power of  $a_s$  on both sides then yields

$$F_{i/k}^{[n]}(x, \eta, \mu) = \sum_{j=q,g} \sum_{r=0}^n \int_{-1}^1 \frac{dy}{|y|} Z_{ij}^{[r]} \left( \frac{x}{y}, \frac{\eta}{x}, \epsilon \right) \hat{F}_{j/k}^{[n-r]}(y, \eta, \epsilon). \quad (1.127)$$

This result therefore provides the renormalisation relation for the parton-in-parton GPDs order by order in perturbation theory. At a given order  $a_s^n$ , the renormalised contribution  $F_{i/k}^{[n]}$  is built from renormalisation kernels and bare parton-in-parton GPDs whose perturbative orders add up to  $n$ .

To make the UV structure of the renormalisation kernels explicit, their perturbative expansion can be further resolved according to their pole structure in dimensional regularisation. In the  $\overline{\text{MS}}$  scheme, one has [25]

$$Z_{ij} \left( \frac{x}{y}, \frac{\eta}{x}, \alpha_s, \epsilon \right) = \delta_{ij} \delta \left( 1 - \frac{x}{y} \right) + \sum_{n=1}^{\infty} a_s^n \sum_{p=1}^n \frac{1}{\epsilon^p} Z_{ij}^{[n,p]} \left( \frac{x}{y}, \frac{\eta}{x} \right). \quad (1.128)$$

Combining Eq. (1.128) with the tree-level result in Eq. (1.122), the first two orders of Eq. (1.127) explicitly read

$$F_{i/k}^{[0]}(x, \eta, \mu) = \widehat{F}_{i/k}^{[0]}(x, \eta, \epsilon) = D_i(\eta) \delta_{ik} \delta(1-x), \quad (1.129)$$

$$F_{i/k}^{[1]}(x, \eta, \mu) = \widehat{F}_{i/k}^{[1]}(x, \eta, \epsilon) + Z_{ik}^{[1]} \left( x, \frac{\eta}{x}, \epsilon \right) D_k(\eta). \quad (1.130)$$

The first non-trivial renormalisation relation is therefore the one-loop result in Eq. (1.130). At this order, the bare parton-in-parton GPDs contain UV divergences, which must be cancelled by the corresponding renormalisation counterterms in order for the renormalised distributions to remain finite. Therefore, the UV pole of the one-loop bare parton-in-parton GPDs determines the pole part of the renormalisation kernel  $Z_{ik}^{[1]}$ .

As shown in Sec. 1.4.1, the GPD evolution kernels are in turn determined by the renormalisation kernels. Together with the result above, this establishes a direct connection between the UV structure of the one-loop parton-in-parton GPDs and the one-loop evolution kernels  $\mathcal{P}_{i/k}^{[0]}$ , which govern GPD evolution at LO accuracy. In the next subsection, we make this connection explicit.

### 1.4.3 One-loop evolution kernels

We start from the perturbative expansion of the evolution kernels. In the partonic kinematics introduced in Sec. 1.4.2, Eq. (1.95) at LO reads

$$\mathcal{P}_{i/k} \left( \frac{x}{z}, \frac{\eta}{x}, \alpha_s \right) = a_s \mathcal{P}_{i/k}^{[0]} \left( \frac{x}{z}, \frac{\eta}{x} \right) + \mathcal{O}(\alpha_s^2). \quad (1.131)$$

At the same perturbative order, Eq. (1.128) gives

$$Z_{ij} \left( \frac{x}{y}, \frac{\eta}{x}, \alpha_s, \epsilon \right) = \delta_{ij} \delta \left( 1 - \frac{x}{y} \right) + \frac{a_s}{\epsilon} Z_{ij}^{[1,1]} \left( \frac{x}{y}, \frac{\eta}{x} \right) + \mathcal{O}(\alpha_s^2). \quad (1.132)$$

Since the renormalisation kernel depends on the scale  $\mu$  through the running coupling  $a_s(\mu)$ , its logarithmic derivative, with respect to  $\mu^2$ , can be written as

$$\frac{dZ_{ij}}{d \ln \mu^2} = \frac{da_s}{d \ln \mu^2} \frac{\partial Z_{ij}}{\partial a_s}. \quad (1.133)$$

In  $d = 4 - 2\epsilon$  dimensions, the running coupling satisfies [25]

$$\frac{da_s}{d \ln \mu^2} = -\epsilon a_s + \beta(a_s), \quad (1.134)$$

with  $\beta(a_s) = \mathcal{O}(\alpha_s^2)$ . On the other hand, differentiating Eq. (1.132) with respect to  $a_s$  gives

$$\frac{\partial Z_{ij}}{\partial a_s} = \frac{1}{\epsilon} Z_{ij}^{[1,1]} \left( \frac{x}{y}, \frac{\eta}{x} \right) + \mathcal{O}(\alpha_s). \quad (1.135)$$

Therefore,

$$\begin{aligned} \frac{d}{d \ln \mu^2} Z_{ij} \left( \frac{x}{y}, \frac{\eta}{x}, \alpha_s, \epsilon \right) &= [-\epsilon a_s + \mathcal{O}(\alpha_s^2)] \left[ \frac{1}{\epsilon} Z_{ij}^{[1,1]} \left( \frac{x}{y}, \frac{\eta}{x} \right) + \mathcal{O}(\alpha_s) \right] \\ &= -a_s Z_{ij}^{[1,1]} \left( \frac{x}{y}, \frac{\eta}{x} \right) + \mathcal{O}(\alpha_s^2). \end{aligned} \quad (1.136)$$

To determine the LO evolution kernel, Eq. (1.136) must now be combined with the inverse renormalisation kernel entering Eq. (1.94). Since  $dZ/d \ln \mu^2$  starts at order  $\alpha_s$ , only the zeroth-order contribution to  $Z^{-1}$  is required at this accuracy: higher-order terms in  $Z^{-1}$  would contribute to the evolution kernel starting at order  $\alpha_s^2$ . From Eq. (1.128), the renormalisation kernel reduces to the identity at zeroth order. The defining relation in Eq. (1.91) then implies that the inverse renormalisation kernel has the same zeroth-order form,

$$Z_{jk}^{-1} \left( \frac{y}{z}, \frac{\eta}{y}, \alpha_s, \epsilon \right) = \delta_{jk} \delta \left( 1 - \frac{y}{z} \right) + \mathcal{O}(\alpha_s). \quad (1.137)$$

Substituting Eqs. (1.136) and (1.137) into Eq. (1.94), we obtain

$$\begin{aligned} \mathcal{P}_{i/k} \left( \frac{x}{z}, \frac{\eta}{x}, \alpha_s \right) &= -a_s \sum_{j=q,g} \int_{-1}^1 \frac{dy}{|y|} Z_{ij}^{[1,1]} \left( \frac{x}{y}, \frac{\eta}{x} \right) \delta_{jk} \delta \left( 1 - \frac{y}{z} \right) + \mathcal{O}(\alpha_s^2) \\ &= -a_s Z_{ik}^{[1,1]} \left( \frac{x}{z}, \frac{\eta}{x} \right) + \mathcal{O}(\alpha_s^2), \end{aligned} \quad (1.138)$$

where the Kronecker delta fixes  $j = k$ , while the Dirac delta fixes  $y = z$ . Comparing this result with Eq. (1.131), we find

$$\mathcal{P}_{i/k}^{[0]} \left( \frac{x}{z}, \frac{\eta}{x} \right) = -Z_{ik}^{[1,1]} \left( \frac{x}{z}, \frac{\eta}{x} \right). \quad (1.139)$$

Each LO GPD evolution kernel is therefore determined by the coefficient  $Z_{ik}^{[1,1]}$  of the UV pole of the corresponding one-loop renormalisation kernel.

The connection with the one-loop parton-in-parton GPDs can now be made explicit. Let us denote by  $A_{i/k}(x, \eta)$  the coefficient of the UV pole of the one-loop bare parton-in-parton GPD,

$$\widehat{F}_{i/k}^{[1]}(x, \eta, \epsilon) = \frac{1}{\epsilon} A_{i/k}(x, \eta) + \widehat{F}_{i/k}^{[1], \text{fin}}(x, \eta) + \mathcal{O}(\epsilon). \quad (1.140)$$

The one-loop counterterm entering Eq. (1.130) is given by the simple-pole contribution of the renormalisation kernel in Eq. (1.132),

$$Z_{ik}^{[1]} \left( x, \frac{\eta}{x}, \epsilon \right) = \frac{1}{\epsilon} Z_{ik}^{[1,1]} \left( x, \frac{\eta}{x} \right). \quad (1.141)$$

Combining this expression with Eq. (1.140), the one-loop renormalisation relation becomes

$$F_{i/k}^{[1]}(x, \eta, \mu) = \frac{1}{\epsilon} \left[ A_{i/k}(x, \eta) + Z_{ik}^{[1,1]} \left( x, \frac{\eta}{x} \right) D_k(\eta) \right] + \text{finite}. \quad (1.142)$$

Since the renormalised parton-in-parton GPD must be finite in the limit  $\epsilon \rightarrow 0$ , the coefficient of the pole must vanish:

$$A_{i/k}(x, \eta) + Z_{ik}^{[1,1]} \left( x, \frac{\eta}{x} \right) D_k(\eta) = 0. \quad (1.143)$$

It follows that

$$Z_{ik}^{[1,1]} \left( x, \frac{\eta}{x} \right) = -A_{i/k}(x, \eta) D_k^{-1}(\eta). \quad (1.144)$$

Combining this relation with Eq. (1.139), we finally obtain

$$\mathcal{P}_{i/k}^{[0]} \left( x, \frac{\eta}{x} \right) = A_{i/k}(x, \eta) D_k^{-1}(\eta). \quad (1.145)$$

Thus, the determination of the LO evolution kernels reduces to the extraction of the UV pole of the corresponding one-loop bare parton-in-parton GPDs. The one-loop contribution  $\widehat{F}_{i/k}^{[1]}$  is obtained by evaluating the  $\mathcal{O}(\alpha_s)$  QCD diagrams that connect the bilocal operator defining parton  $i$  to the external partonic state  $k$ . The explicit one-loop calculation of the quark-in-quark channel, including the extraction of its UV pole, is presented in detail in Ref. [25], using the  $\overline{\text{MS}}$  renormalisation scheme and the light-cone gauge  $n \cdot A = 0$ . The same procedure can be applied to the remaining non-vanishing partonic channels.

It is convenient to organise the one-loop contributions into the conventionally denoted *real* and *virtual* parts. This terminology does not refer to whether the internal lines of the diagrams represent physical on-shell particles: both classes are perturbative contributions to the same partonic matrix element and contain internal propagators. Rather, the distinction refers to their kinematic structure in the evolution kernel. The so-called real contributions change the longitudinal momentum fraction flowing through the operator and therefore generate the non-trivial  $x$ -dependent part of the kernel. By contrast, virtual corrections leave this longitudinal momentum fraction unchanged and give contributions localised at  $x = 1$ , i.e. proportional to  $\delta(1 - x)$ .

Representative real and virtual one-loop topologies are shown in Figs. 1.12 and 1.13, respectively. The diagonal  $q/q$  and  $g/g$  channels receive both types of corrections, whereas the off-diagonal  $q/g$  and  $g/q$  parton-in-parton GPDs first appear at  $\mathcal{O}(\alpha_s)$ , consistently with the fact that their tree-level contributions vanish. Consequently, no virtual contributions proportional to  $\delta(1 - x)$  arise in the off-diagonal channels at this order.

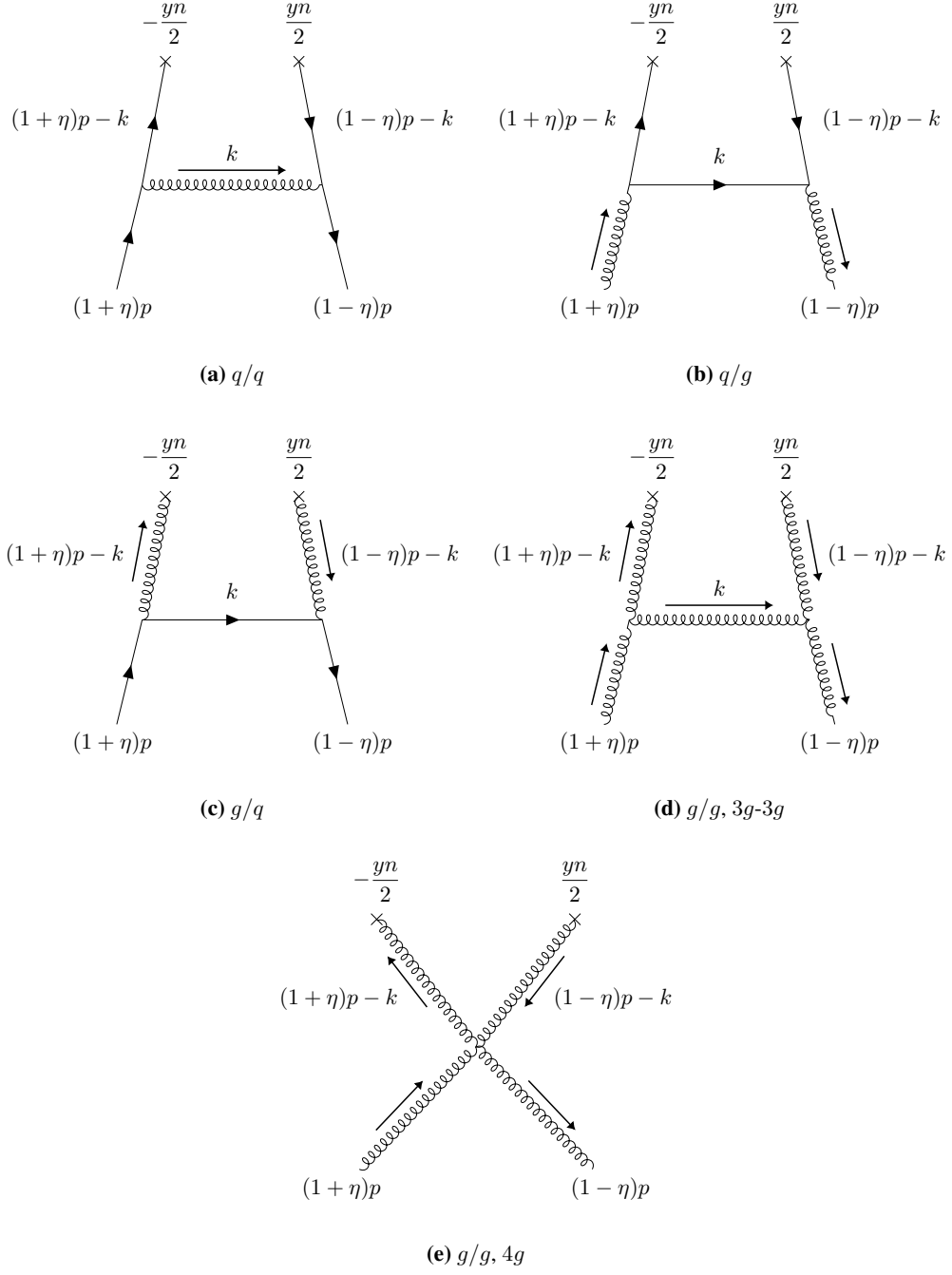


**Figure 1.12:** Real one-loop diagrams contributing to the non-vanishing parton-in-parton GPDs. Two distinct topologies contribute to the  $g/g$  channel: one containing two three-gluon vertices, and another with a four-gluon contact interaction.

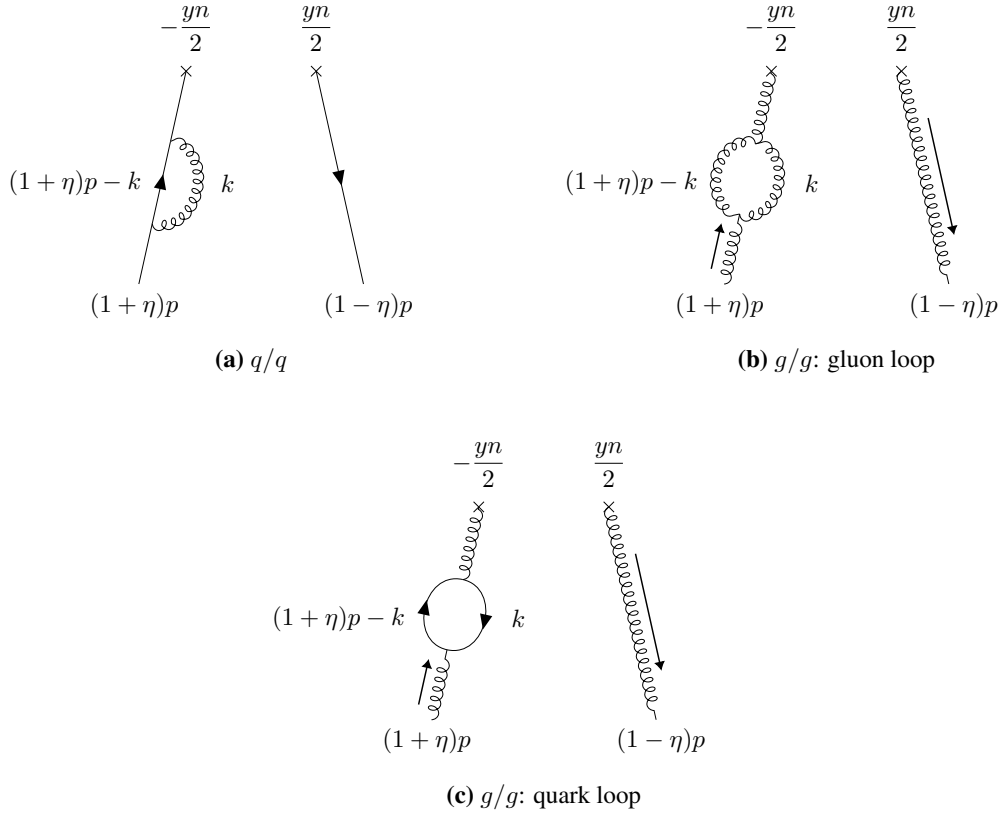


**Figure 1.13:** Representative one-loop virtual diagrams contributing to the diagonal parton-in-parton GPDs. Only the self-energy insertion on the left external leg is explicitly shown; the symmetric contribution on the right external leg is implicitly understood. Arrows on gluon lines indicate the choice of momentum flow.

Summing the one-loop contributions in each partonic channel, extracting the corresponding UV pole coefficients  $A_{i/k}$ , and using Eq. (1.145), one obtains the LO evolution kernels. In the conventions adopted here, they can be written in the compact form [25]

$$\mathcal{P}_{i/k}^{[0]} \left( x, \frac{\eta}{x} \right) = \theta(1-x) \left[ \theta(x+\eta) p_{ik} \left( x, \frac{\eta}{x} \right) + \theta(x-\eta) p_{ik} \left( x, -\frac{\eta}{x} \right) \right] + \delta_{ik} \delta(1-x) 2C_i \left[ K_i - 2 \int_0^1 \frac{dz}{1-z} - \ln \left( \left| 1 - \frac{\eta^2}{x^2} \right| \right) \right], \quad (1.146)$$

where  $p_{ik}(y, \kappa)$  are auxiliary functions of two dimensionless arguments. In Eq. (1.146), they are evaluated at  $y = x$  and  $\kappa = \pm\eta/x$ . Their explicit expressions for the four possible partonic channels are

$$p_{qq}(y, \kappa) = C_F \frac{(1+\kappa)(1-y+2\kappa y)}{\kappa(1+\kappa y)(1-y)}, \quad (1.147)$$

$$p_{qg}(y, \kappa) = T_R \frac{(1+\kappa)(1-2y+\kappa y)}{\kappa(1+\kappa y)(1-\kappa^2 y^2)}, \quad (1.148)$$

$$p_{gq}(y, \kappa) = C_F \frac{(1+\kappa)(2-y+\kappa y)}{\kappa y(1+\kappa y)}, \quad (1.149)$$

$$p_{gg}(y, \kappa) = -C_A \frac{1-\kappa^2}{\kappa(1-\kappa^2 y^2)} \left[ 1 - \frac{2\kappa y}{1-y} - \frac{2(1+y^2)}{y(1-\kappa)(1+\kappa y)} \right]. \quad (1.150)$$

The constants entering the diagonal contributions are

$$K_q = \frac{3}{2}, \quad K_g = \frac{11C_A - 4n_f T_R}{6C_A}, \quad (1.151)$$

with

$$C_q = C_F = \frac{N_c^2 - 1}{2N_c}, \quad C_g = C_A = N_c, \quad T_R = \frac{1}{2}, \quad N_c = 3, \quad (1.152)$$

so that  $C_F = 4/3$  and  $C_A = 3$ . The quantity  $n_f$  denotes the number of active quark flavours.

For the diagonal  $q/q$  and  $g/g$  channels, the real contributions contain an endpoint singularity as  $x \rightarrow 1$ , which combines with the singular part of the corresponding virtual contribution proportional to  $\delta(1-x)$ . The resulting expression has to be understood in the distributional sense by means of the so-called *plus prescription*.<sup>7</sup> This prescription ensures that the diagonal kernels are well defined when convoluted with a GPD.

<sup>7</sup>The plus distribution is defined through its action on a regular test function  $g(x)$  as

$$\int_0^1 dx \left( \frac{1}{1-x} \right)_+ g(x) = \int_0^1 dx \frac{g(x) - g(1)}{1-x}.$$

The derivation above establishes the partonic framework used in the following chapters. In the massless case, the UV pole of the one-loop parton-in-parton GPD determines the corresponding LO evolution kernel. For the heavy-quark threshold matching considered here, the analogous partonic calculation is performed keeping the heavy-quark mass finite, and the relevant matching information is instead encoded in the finite terms that remain after renormalisation.

## Chapter 2

# GPD evolution across heavy-quark thresholds

This chapter develops the theoretical framework for the evolution of GPDs across heavy-quark thresholds. We first discuss the fixed-flavour evolution below and above a heavy-quark threshold and introduce the matching condition connecting the GPDs defined with  $n_f$  and  $n_f + 1$  active quark flavours. The corresponding matching kernel is then combined with the evolution operators to describe the complete evolution across the threshold. The construction follows the standard heavy-quark threshold matching procedure developed for ordinary PDFs [27, 28], adapted here to the non-forward kinematics of GPDs. Finally, we formulate the matching problem at the partonic level, establishing the relation between the matching coefficients and massive parton-in-parton GPDs and providing the basis for the explicit one-loop calculation presented in the following chapter.

### 2.1 Flavour-number schemes and heavy-quark thresholds

As discussed in Sec. 1.4, GPDs depend on the factorisation scale  $\mu$ , and their scale dependence is determined by the corresponding evolution equations. For convenience, we recall the evolution equation introduced in Sec. 1.4.1 (Eq. (1.90)), which reads

$$\frac{dF_{i/H}(x, \xi, \mu)}{d \ln \mu^2} = \sum_{k=q_1, \dots, q_{n_f}, g} \int_{-1}^1 \frac{dz}{|z|} \mathcal{P}_{i/k} \left( \frac{x}{z}, \frac{\xi}{x}, \alpha_s(\mu) \right) F_{k/H}(z, \xi, \mu), \quad (2.1)$$

and is formulated for a fixed number  $n_f$  of active quark flavours: both  $i$  and  $k$  label the active partonic species, namely the gluon and the quark flavours  $q_1, \dots, q_{n_f}$ . For a given  $i$ , the equation determines the scale dependence of the corresponding renormalised GPD  $F_{i/H}$ , which receives contributions from the GPDs associated with all partonic species  $k$ , each weighted by the corresponding evolution kernel  $\mathcal{P}_{i/k}$ .

It is convenient to collect the GPDs associated with the active partonic species into a

vector,

$$F^{(n_f)}(x, \xi, \mu) = \begin{pmatrix} F_{q_1/H}(x, \xi, \mu) \\ F_{q_2/H}(x, \xi, \mu) \\ \vdots \\ F_{q_{n_f}/H}(x, \xi, \mu) \\ F_{g/H}(x, \xi, \mu) \end{pmatrix}, \quad (2.2)$$

where the superscript  $(n_f)$  indicates the number  $n_f$  of quark flavours treated as active degrees of freedom in GPD evolution. Correspondingly, the evolution kernels can be arranged into a matrix,

$$\mathcal{P}^{(n_f)} = \begin{pmatrix} \mathcal{P}_{q_1/q_1} & \mathcal{P}_{q_1/q_2} & \cdots & \mathcal{P}_{q_1/q_{n_f}} & \mathcal{P}_{q_1/g} \\ \mathcal{P}_{q_2/q_1} & \mathcal{P}_{q_2/q_2} & \cdots & \mathcal{P}_{q_2/q_{n_f}} & \mathcal{P}_{q_2/g} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{P}_{q_{n_f}/q_1} & \mathcal{P}_{q_{n_f}/q_2} & \cdots & \mathcal{P}_{q_{n_f}/q_{n_f}} & \mathcal{P}_{q_{n_f}/g} \\ \mathcal{P}_{g/q_1} & \mathcal{P}_{g/q_2} & \cdots & \mathcal{P}_{g/q_{n_f}} & \mathcal{P}_{g/g} \end{pmatrix}, \quad (2.3)$$

where each matrix element  $\mathcal{P}_{i/k}$  describes the contribution of the GPD component  $F_{k/H}$  to the evolution of  $F_{i/H}$ . Thus, the row index  $i$  identifies the GPD whose scale dependence is being determined, whereas the column index  $k$  identifies the partonic component from which the corresponding contribution originates.

Collecting the evolution equations for all the active partonic species, the notation introduced above allows one to write the complete set of equations in the compact form

$$\frac{d}{d \ln \mu^2} F^{(n_f)}(\mu) = \mathcal{P}^{(n_f)}(\mu) \otimes F^{(n_f)}(\mu), \quad (2.4)$$

where the dependence on the kinematic variables is left implicit, and the symbol  $\otimes$  includes both the convolution in the longitudinal momentum fraction and the sum over the active partonic species associated with the matrix multiplication. In particular, for a fixed row index  $i$  of the evolution kernel matrix  $\mathcal{P}^{(n_f)}$ , the equation above reproduces the evolution equation for the corresponding GPD  $F_{i/H}$  given in Eq. (2.1).

As long as the number of active flavours remains fixed, the solution of Eq. (2.4) can be formally written in terms of an evolution operator  $\Gamma^{(n_f)}$  as

$$F^{(n_f)}(\mu) = \Gamma^{(n_f)}(\mu, \mu_0) \otimes F^{(n_f)}(\mu_0), \quad (2.5)$$

where  $\Gamma^{(n_f)}(\mu, \mu_0)$  evolves the GPDs from the initial scale  $\mu_0$  to the scale  $\mu$ . Its dependence on the evolution kernels follows directly by substituting Eq. (2.5) into Eq. (2.4), which gives

$$\frac{d}{d \ln \mu^2} \Gamma^{(n_f)}(\mu, \mu_0) = \mathcal{P}^{(n_f)}(\mu) \otimes \Gamma^{(n_f)}(\mu, \mu_0), \quad (2.6)$$

where we have exploited the fact that  $F^{(n_f)}(\mu_0)$  is independent of the evolution scale  $\mu$  and that the relation must hold for an arbitrary initial GPD vector. The evolution operator

for a fixed number  $n_f$  of active flavours is fully determined by solving Eq. (2.6) with the boundary condition

$$\Gamma_{i/k}^{(n_f)}(\mu_0, \mu_0) = \delta_{ik} \delta(1-x). \quad (2.7)$$

The fixed- $n_f$  evolution introduced above is formulated with  $n_f$  active quark flavours, which enter GPD evolution as massless partonic degrees of freedom. At the factorisation scales relevant for GPD evolution, one works in the perturbative regime, where  $\mu$  is sufficiently large for the scale dependence of the distributions to be described by perturbatively calculable evolution kernels. In this regime, the light-quark masses satisfy  $m_{u,d,s} \ll \mu$  and can therefore be neglected. The  $u$ ,  $d$ , and  $s$  flavours can therefore be treated as massless active partonic degrees of freedom throughout the evolution. The situation is different for heavier quarks, such as charm and bottom. Their masses cannot, in general, be neglected over the whole range of scales relevant for the evolution: they introduce additional physical scales that must be consistently taken into account.

As discussed in Sec. 1.4, GPDs are evolved from an initial scale  $\mu_0$  to a factorisation scale  $\mu$ , typically chosen of the order of the hard scale  $Q$  of the process. If this range spans one or more heavy-quark mass scales, a description with a fixed number  $n_f$  of active flavours is no longer correct over the full evolution interval. One therefore adopts a zero-mass variable-flavour-number scheme, in which different numbers of active flavours are used in different scale regions and the number of active flavours changes when a heavy-quark threshold is crossed. Starting from the light flavours  $u$ ,  $d$ , and  $s$ , the first thresholds encountered as the factorisation scale is increased are those associated with the charm and bottom quarks.

To describe the change in the number of active flavours across a generic heavy-quark threshold, let us consider a heavy quark  $h$  with mass  $m_h$ . At scales of order or below  $m_h$ , the heavy-quark mass cannot be neglected, and it is therefore not appropriate to treat  $h$  as an additional massless parton entering GPD evolution. The heavy flavour is consequently not included among the active partonic species, and the GPDs evolve in a theory with  $n_f$  active quark flavours. This does not imply that heavy-quark effects are neglected altogether: they can still enter through perturbatively calculable mass-dependent short-distance contributions, while the GPD evolution itself is governed by the  $n_f$ -flavour theory. At scales well above  $m_h$ , by contrast, maintaining the same  $n_f$ -flavour description, in which the heavy quark does not participate as an active parton in GPD evolution, leads to logarithmic contributions in the perturbative coefficient functions of the hard process, of the form

$$\ln \frac{Q^2}{m_h^2}. \quad (2.8)$$

For DVCS, such terms appear in the perturbative hard-scattering coefficient functions entering the CFFs introduced in Sec. 1.1.2. When  $Q^2 \gg m_h^2$ , these logarithms become large, potentially spoiling the convergence of the perturbative expansion. It is then convenient to include the heavy quark among the active partonic degrees of freedom and treat it as effectively massless in the evolution. This is achieved by introducing the heavy-quark GPD at a matching scale  $\mu_h \sim m_h$ , with  $m_h$  setting the characteristic heavy-quark threshold scale

and  $\mu_h$  denoting the scale at which the  $n_f$ - and  $(n_f + 1)$ -flavour descriptions are matched. The full set of  $(n_f + 1)$ -flavour GPDs is subsequently evolved up to the factorisation scale  $\mu \sim Q$ ,

$$F^{(n_f+1)}(\mu) = \Gamma^{(n_f+1)}(\mu, \mu_h) \otimes F^{(n_f+1)}(\mu_h). \quad (2.9)$$

The change in the number of active flavours and the corresponding evolution below and above the matching scale are illustrated schematically in Fig. 2.1.

By introducing the heavy quark as an active partonic degree of freedom in GPD evolution, the logarithms associated with the hierarchy  $Q \gg m_h$  are reorganised and resummed through the scale evolution in the  $(n_f + 1)$ -flavour theory. This can be seen directly from the evolution operator  $\Gamma^{(n_f+1)}(\mu, \mu_h)$ . Schematically, neglecting the running of  $\alpha_s$  and the resulting scale dependence of the evolution kernel matrix, the solution of Eq. (2.6) is

$$\Gamma^{(n_f+1)}(\mu, \mu_h) \sim \exp_{\otimes} \left[ \mathcal{P}^{(n_f+1)} \ln \frac{\mu^2}{\mu_h^2} \right], \quad (2.10)$$

where the exponential is understood as an operator exponential with products defined through the convolution  $\otimes$ . Since the evolution kernel matrix starts at  $\mathcal{O}(\alpha_s)$ ,

$$\mathcal{P}^{(n_f+1)} = a_s \mathcal{P}^{[0],(n_f+1)} + \mathcal{O}(\alpha_s^2), \quad (2.11)$$

the perturbative expansion of the operator exponential generates logarithmically enhanced contributions of the form

$$\alpha_s^n \ln^n \frac{\mu^2}{\mu_h^2}, \quad (2.12)$$

which exhibits the leading-logarithmic structure in this approximation, together with sub-leading logarithms generated by higher-order contributions to the evolution kernel matrix. For  $\mu_h \sim m_h$  and  $\mu \sim Q$ , these terms correspond schematically to powers of  $\alpha_s \ln(Q^2/m_h^2)$ . The potentially large logarithms are therefore resummed by evolving the GPDs in the  $(n_f + 1)$ -flavour theory above the matching scale.

Despite this change of description, no physical discontinuity occurs at the matching scale  $\mu_h$ : the  $n_f$ - and  $(n_f + 1)$ -flavour descriptions provide two perturbative representations of the same underlying physics, organised in terms of different sets of active partonic

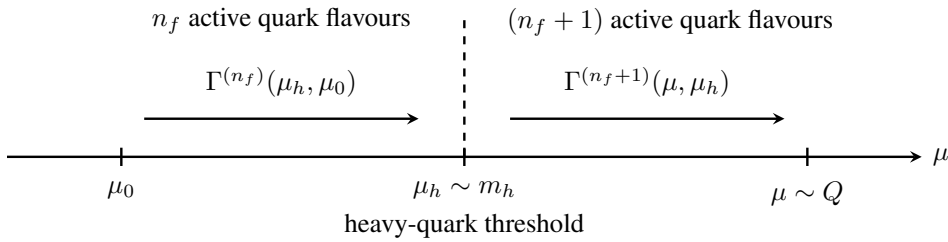


**Figure 2.1:** Schematic representation of GPD evolution across a heavy-quark threshold. Starting from the initial scale  $\mu_0$ , the GPDs evolve with  $n_f$  active quark flavours up to the matching scale  $\mu_h \sim m_h$ . Above this threshold, the heavy quark is included among the active flavours and the evolution proceeds within the  $(n_f + 1)$ -flavour theory up to the factorisation scale  $\mu \sim Q$ .

degrees of freedom. The GPDs themselves are continuous across the matching scale at LO and NLO, while discontinuities arise starting at NNLO in the zero-mass variable-flavour-number scheme considered here. While the evolution below and above  $\mu_h$  is governed by the corresponding fixed-flavour evolution equations, evolving across the matching scale requires a relation between the GPDs defined with  $n_f$  and  $n_f + 1$  active flavours. In particular, the distributions  $F^{(n_f)}(\mu_h)$ , obtained by evolving from scales below  $\mu_h$ , must be related to the distributions  $F^{(n_f+1)}(\mu_h)$ , which provide the initial condition for the evolution above it. This relation is supplied by the heavy-quark matching conditions introduced in the following section.

## 2.2 GPD matching across a heavy-quark threshold

Across the matching scale, the  $n_f$ - and  $(n_f + 1)$ -flavour descriptions differ in the treatment of the heavy-quark flavour  $h$ . As discussed in the previous section, below the threshold the heavy quark is not included among the  $n_f$  massless active partonic degrees of freedom entering GPD evolution, whereas above the threshold it becomes an active component of the  $(n_f + 1)$ -flavour description.

For the purpose of formulating the matching between the  $n_f$ - and  $(n_f + 1)$ -flavour GPDs, however, it is useful to distinguish the statement that the heavy quark is not an active flavour below the threshold from the stronger assumption that its GPD vanishes identically. The absence of  $h$  from the set of active flavours does not, by itself, require the matrix element of the corresponding heavy-quark light-cone operator to vanish. The heavy-quark GPD can therefore still be defined below the threshold, even though  $h$  is not treated as an active massless degree of freedom in the fixed- $n_f$  evolution. Accordingly, one may allow for a non-perturbative heavy-quark component  $F_{h/H}^{(n_f)}$  in the GPD vector defined with  $n_f$  active flavours. Such a contribution will be referred to as an intrinsic heavy-quark GPD. Its inclusion does not modify the fixed- $n_f$  evolution introduced in the previous section: the heavy flavour is not added to the sums over active partonic species, which continue to run over  $q_1, \dots, q_{n_f}, g$ .

It is therefore convenient, specifically for the formulation of the matching, to introduce an extended GPD vector by supplementing the active-flavour vector of Eq. (2.2) with this possible heavy-quark component. At the matching scale  $\mu_h$ , one has schematically

$$\tilde{F}^{(n_f)}(\mu_h) = \begin{pmatrix} F_{q_1/H}^{(n_f)} \\ \vdots \\ F_{q_{n_f}/H}^{(n_f)} \\ F_{h/H}^{(n_f)} \\ F_{g/H}^{(n_f)} \end{pmatrix}, \quad (2.13)$$

where the tilde distinguishes the extended vector entering the matching condition from the

active-flavour vector  $F^{(n_f)}$  introduced for the  $n_f$ -fixed-flavour evolution. The condition

$$F_{h/H}^{(n_f)}(\mu_h) = 0 \quad (2.14)$$

can be imposed as a particular case after deriving the general matching relations. In the  $n_f + 1$ -flavour description, the corresponding GPD vector is

$$F^{(n_f+1)}(\mu_h) = \begin{pmatrix} F_{q_1/H}^{(n_f+1)} \\ \vdots \\ F_{q_{n_f}/H}^{(n_f+1)} \\ F_{h/H}^{(n_f+1)} \\ F_{g/H}^{(n_f+1)} \end{pmatrix}, \quad (2.15)$$

where  $h$  is now an active partonic degree of freedom and subsequently participates in the  $(n_f + 1)$ -flavour evolution. Thus, at the matching scale the heavy-quark GPD is present in both flavour-number descriptions, but with a different dynamical origin: in the  $n_f$ -flavour description it represents a non-perturbative intrinsic component, whereas in the  $(n_f + 1)$ -flavour description it is generated perturbatively through the matching and subsequently evolves as an active partonic degree of freedom.

The GPDs defined in the two flavour-number descriptions are related by a matching kernel,

$$F^{(n_f+1)}(\mu_h) = \mathcal{A}^{(n_f+1 \leftarrow n_f)} \left( \frac{\mu_h^2}{m_h^2} \right) \otimes \tilde{F}^{(n_f)}(\mu_h), \quad (2.16)$$

where, as in the evolution equations, the symbol  $\otimes$  denotes both the convolution in the longitudinal momentum fraction and the sum over the partonic species associated with the matrix multiplication. For compactness, the kinematic arguments of the individual matching coefficients are left implicit and will be introduced explicitly in Sec. 2.3. Using the same partonic-channel notation as for the evolution kernels, the matching kernel can be written in a matrix form as

$$\mathcal{A}^{(n_f+1 \leftarrow n_f)} = \begin{pmatrix} \mathcal{A}_{q_1/q_1} & \cdots & \mathcal{A}_{q_1/q_{n_f}} & \mathcal{A}_{q_1/h} & \mathcal{A}_{q_1/g} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathcal{A}_{q_{n_f}/q_1} & \cdots & \mathcal{A}_{q_{n_f}/q_{n_f}} & \mathcal{A}_{q_{n_f}/h} & \mathcal{A}_{q_{n_f}/g} \\ \mathcal{A}_{h/q_1} & \cdots & \mathcal{A}_{h/q_{n_f}} & \mathcal{A}_{h/h} & \mathcal{A}_{h/g} \\ \mathcal{A}_{g/q_1} & \cdots & \mathcal{A}_{g/q_{n_f}} & \mathcal{A}_{g/h} & \mathcal{A}_{g/g} \end{pmatrix}. \quad (2.17)$$

Each entry  $\mathcal{A}_{i/k}$ , referred to as a *matching coefficient*, specifies how the GPD associated with the partonic species  $k$  in the  $n_f$ -flavour description contributes to the GPD associated with the species  $i$  in the  $(n_f + 1)$ -flavour description. Both indices therefore take values

$$i, k = q_1, \dots, q_{n_f}, h, g. \quad (2.18)$$

For a fixed partonic species  $i$ , this amounts to selecting the  $i$ -th row of the matching kernel matrix, and Eq. (2.16) takes the explicit form

$$F_{i/H}^{(n_f+1)}(\mu_h) = \sum_{k=q_1, \dots, q_{n_f}, h, g} \mathcal{A}_{i/k}^{(n_f+1 \leftarrow n_f)} \left( \frac{\mu_h^2}{m_h^2} \right) \otimes F_{k/H}^{(n_f)}(\mu_h), \quad (2.19)$$

where  $F_{h/H}^{(n_f)}$  is the intrinsic heavy-quark contribution introduced above.

The matching coefficients admit a perturbative expansion in powers of  $a_s$ ,

$$\mathcal{A}^{(n_f+1 \leftarrow n_f)} = \mathcal{A}^{[0]} + a_s(\mu_h) \mathcal{A}^{[1]} + a_s^2(\mu_h) \mathcal{A}^{[2]} + \dots \quad (2.20)$$

At tree level, no radiative heavy-quark effects contribute to the matching. Consequently, while an intrinsic heavy-quark component may be carried across the threshold, no heavy-quark contribution is generated perturbatively from the light-quark or gluon GPDs at the matching scale. The tree-level matching therefore simply identifies the boundary values of the corresponding GPD components in the two flavour-number descriptions,

$$F_{i/H}^{(n_f+1)}(\mu_h) = F_{i/H}^{(n_f)}(\mu_h), \quad i = q_1, \dots, q_{n_f}, h, g, \quad (2.21)$$

and the corresponding tree-level matching kernel reduces to the identity,

$$\mathcal{A}_{i/k}^{[0]}(x) = \delta_{ik} \delta(1-x), \quad i, k = q_1, \dots, q_{n_f}, h, g, \quad (2.22)$$

where  $\delta_{ik}$  is the identity in partonic space, while  $\delta(1-x)$  represents the identity with respect to the convolution over the longitudinal momentum fraction.

If no intrinsic heavy-quark component is assumed, no non-perturbative heavy-quark GPD is present in the  $n_f$ -flavour description. The heavy flavour is then assumed to be entirely generated perturbatively through QCD radiation. Since the heavy quark is not an active partonic degree of freedom in the  $n_f$ -flavour description, such perturbative generation does not occur in the fixed- $n_f$  evolution, and therefore

$$F_{h/H}^{(n_f)}(\mu_h) = 0. \quad (2.23)$$

At tree level, the matching kernel is the identity, so that this condition is directly inherited by the  $(n_f + 1)$ -flavour description,

$$F_{h/H}^{(n_f+1)}(\mu_h) = 0. \quad (2.24)$$

Hence, in the absence of an intrinsic heavy-quark component, the heavy-quark GPD vanishes on both sides of the matching relation at tree level and can become non-vanishing only through perturbative QCD corrections. The more general formulation introduced above is nevertheless useful because it retains the complete structure of the matching kernel and, in particular, allows the coefficients associated with the heavy-quark component of the  $n_f$ -flavour vector to be defined independently of any phenomenological assumption concerning the heavy-quark GPD below the threshold.

Beyond tree level, radiative corrections involving the heavy flavour make the matching across the heavy-quark threshold non-trivial. The matching kernel therefore acquires perturbative corrections and, up to one-loop accuracy, Eq. (2.16) becomes

$$F^{(n_f+1)}(\mu_h) = \left[ \mathcal{A}^{[0]} + a_s(\mu_h) \mathcal{A}^{[1]} \right] \otimes \tilde{F}^{(n_f)}(\mu_h) + \mathcal{O}(\alpha_s^2). \quad (2.25)$$

At this order, the heavy-flavour contributions can be classified according to the partonic species involved in the matching. The splitting  $g \rightarrow h\bar{h}$  generates a heavy-quark contribution from the gluon GPD and gives rise to the off-diagonal matching coefficient  $\mathcal{A}_{h/g}^{[1]}$ . In turn, gluon radiation from the heavy quark gives rise to the coefficient  $\mathcal{A}_{g/h}^{[1]}$ , while radiative corrections associated with the heavy-quark component itself contribute to the diagonal coefficient  $\mathcal{A}_{h/h}^{[1]}$ . Heavy-quark loop corrections to the gluon channel instead contribute to the diagonal coefficient  $\mathcal{A}_{g/g}^{[1]}$ . Since the QCD interaction is diagonal in quark flavour, a direct transition between a light quark and the heavy flavour is forbidden at this order, yielding

$$\mathcal{A}_{h/q_i}^{[1]} = \mathcal{A}_{q_i/h}^{[1]} = 0, \quad i = 1, \dots, n_f. \quad (2.26)$$

The one-loop matching involving the heavy-quark sector can therefore be restricted to the  $(h, g)$  block,

$$\mathcal{A}_{(h,g)} = \begin{pmatrix} \mathbf{1}_{\otimes} + a_s \mathcal{A}_{h/h}^{[1]} & a_s \mathcal{A}_{h/g}^{[1]} \\ a_s \mathcal{A}_{g/h}^{[1]} & \mathbf{1}_{\otimes} + a_s \mathcal{A}_{g/g}^{[1]} \end{pmatrix} + \mathcal{O}(\alpha_s^2), \quad (2.27)$$

where  $\mathbf{1}_{\otimes} \equiv \delta(1-x)$  denotes the identity with respect to the convolution in the longitudinal momentum fraction. The matching coefficients appearing in this block can be determined perturbatively from the corresponding partonic channels, namely  $h/h$ ,  $h/g$ ,  $g/h$ , and  $g/g$ . Their explicit relation to the one-loop parton-in-parton GPDs is discussed in Sec. 2.3.

At the hadronic level, the matching relations for the heavy-quark and gluon GPDs are obtained by selecting, respectively, the first and second rows of the  $(h, g)$  matching block in Eq. (2.27). For the heavy-quark component one finds

$$F_{h/H}^{(n_f+1)}(\mu_h) = F_{h/H}^{(n_f)}(\mu_h) + a_s(\mu_h) \left[ \mathcal{A}_{h/h}^{[1]} \otimes F_{h/H}^{(n_f)}(\mu_h) + \mathcal{A}_{h/g}^{[1]} \otimes F_{g/H}^{(n_f)}(\mu_h) \right] + \mathcal{O}(\alpha_s^2), \quad (2.28)$$

while the second row gives the corresponding gluon matching relation,

$$F_{g/H}^{(n_f+1)}(\mu_h) = F_{g/H}^{(n_f)}(\mu_h) + a_s(\mu_h) \left[ \mathcal{A}_{g/h}^{[1]} \otimes F_{h/H}^{(n_f)}(\mu_h) + \mathcal{A}_{g/g}^{[1]} \otimes F_{g/H}^{(n_f)}(\mu_h) \right] + \mathcal{O}(\alpha_s^2). \quad (2.29)$$

Equations (2.28) and (2.29) thus make explicit how the heavy-quark and gluon GPDs mix across the threshold via their diagonal  $(h/h, g/g)$  and off-diagonal  $(h/g, g/h)$  channels.

If the intrinsic heavy-quark contribution is neglected according to Eq. (2.23), the matching condition for the heavy-quark GPD reduces to

$$F_{h/H}^{(n_f+1)}(\mu_h) = a_s(\mu_h) \mathcal{A}_{h/g}^{[1]} \otimes F_{g/H}^{(n_f)}(\mu_h) + \mathcal{O}(\alpha_s^2). \quad (2.30)$$

Thus, in the absence of an intrinsic heavy component, the heavy-quark GPD in the  $(n_f + 1)$ -flavour description, where the heavy flavour is active, can be perturbatively generated at one loop from the gluon GPD through the  $g \rightarrow h\bar{h}$  channel. The coefficients  $\mathcal{A}_{h/h}^{[1]}$  and  $\mathcal{A}_{g/h}^{[1]}$  remain well-defined universal matching coefficients, but their contribution to the hadronic matching vanishes when they multiply  $F_{h/H}^{(n_f)} = 0$ .

So far, we have focused on the  $(h, g)$  sector, since, at one loop, it contains non-trivial heavy-quark matching contributions. The complete matching kernel also includes the light-quark sector. At this order, however, the light-quark GPDs are simply matched onto themselves, while the matching coefficients connecting them to the heavy-quark and gluon components vanish. This reflects the fact that one-loop corrections involving light quarks do not yet contain heavy-quark contributions and are therefore insensitive to the change in the number of active flavours across the threshold. The complete one-loop matching kernel therefore takes the form

$$\mathcal{A} = \begin{pmatrix} \mathbf{1}_{n_f} \mathbf{1}_{\otimes} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{\otimes} + a_s \mathcal{A}_{h/h}^{[1]} & a_s \mathcal{A}_{h/g}^{[1]} \\ \mathbf{0} & a_s \mathcal{A}_{g/h}^{[1]} & \mathbf{1}_{\otimes} + a_s \mathcal{A}_{g/g}^{[1]} \end{pmatrix} + \mathcal{O}(\alpha_s^2), \quad (2.31)$$

where  $\mathbf{1}_{n_f}$  denotes the identity in the light-flavour space, with  $\mathbf{1}_{\otimes}$  understood in each diagonal entry.

This simple structure of the light-quark sector is specific to one-loop accuracy. At higher perturbative orders, heavy-quark effects can also connect the light-quark sector to the heavy-quark and gluon components. For instance, at two-loop order, a light quark can radiate a gluon which subsequently splits into a heavy-quark pair, giving rise to  $\mathcal{A}_{h/q_i}^{[2]}$ , while heavy-quark corrections in the gluonic sector can generate  $\mathcal{A}_{g/q_i}^{[2]}$ .

The matching conditions introduced above ensure that the change in the number of active flavours does not affect physical predictions up to the perturbative accuracy considered. The GPDs defined with  $n_f$  and  $n_f + 1$  active flavours are distinct perturbative quantities and therefore need not coincide at the matching scale. Rather, their change across the threshold is accompanied by a corresponding change in the perturbative coefficient functions entering the factorised description of the hard process. In the case of DVCS, this compensation takes place between the GPDs and the hard-scattering coefficient functions entering the CFFs, so that the resulting physical predictions remain unchanged up to higher-order corrections.

The complete evolution across the heavy-quark threshold can then be constructed by combining the fixed-flavour evolution below and above the matching scale with the matching condition itself. Consider an initial scale  $\mu_0$  below the matching scale and a final scale  $\mu$  above it,

$$\mu_0 < \mu_h < \mu. \quad (2.32)$$

For notational convenience, the fixed- $n_f$  evolution operator can be extended to the enlarged vector  $\tilde{F}^{(n_f)}$  introduced above. We denote this operator by  $\tilde{\Gamma}^{(n_f)}$ . It acts as the usual  $n_f$ -flavour evolution operator on the active light-quark and gluon components, while acting trivially on the non-active heavy-quark component. Therefore,

$$\tilde{F}^{(n_f)}(\mu_h) = \tilde{\Gamma}^{(n_f)}(\mu_h, \mu_0) \otimes \tilde{F}^{(n_f)}(\mu_0). \quad (2.33)$$

This extension does not promote the heavy flavour to an active partonic degree of freedom below the threshold; it merely allows the evolution and matching steps to be written in the same enlarged partonic space. At the matching scale, the evolved  $n_f$ -flavour vector is related to the  $(n_f + 1)$ -flavour one through

$$F^{(n_f+1)}(\mu_h) = \mathcal{A}^{(n_f+1 \leftarrow n_f)} \left( \frac{\mu_h^2}{m_h^2} \right) \otimes \tilde{F}^{(n_f)}(\mu_h). \quad (2.34)$$

Above the threshold, all components of  $F^{(n_f+1)}$ , including the heavy-quark one, participate in the  $(n_f + 1)$ -flavour evolution. The complete evolution from  $\mu_0$  to  $\mu$  can therefore be written as

$$\begin{aligned} F^{(n_f+1)}(\mu) &= \Gamma^{(n_f+1)}(\mu, \mu_h) \otimes \mathcal{A}^{(n_f+1 \leftarrow n_f)} \left( \frac{\mu_h^2}{m_h^2} \right) \\ &\otimes \tilde{\Gamma}^{(n_f)}(\mu_h, \mu_0) \otimes \tilde{F}^{(n_f)}(\mu_0). \end{aligned} \quad (2.35)$$

### 2.3 Partonic framework for the matching calculation

Having established the general structure of the matching relation, we now turn to the extraction of the matching coefficients from partonic matrix elements. The matching kernel introduced in Eq. (2.16) originates from the matching of the corresponding bilocal light-cone operators and is therefore independent of the external states used to evaluate their matrix elements. The hadronic states entering the GPD definition can consequently be replaced by partonic states. The resulting parton-in-parton GPDs are perturbatively calculable and allow the individual entries of the matching kernel to be isolated, following the same strategy employed for the determination of the GPD evolution kernels in Sec. 1.4.

Evaluating the matching relation in Eq. (2.19) between partonic external states yields

$$F_{i/k}^{(n_f+1)}(\mu_h) = \sum_{j=q_1, \dots, q_{n_f}, h, g} \mathcal{A}_{i/j}^{(n_f+1 \leftarrow n_f)} \left( \frac{\mu_h^2}{m_h^2} \right) \otimes F_{j/k}^{(n_f)}(\mu_h), \quad (2.36)$$

where  $i$  identifies the partonic species associated with the bilocal GPD operator, while  $k$  specifies the external partonic state. Both indices can take the values

$$i, k = q_1, \dots, q_{n_f}, h, g. \quad (2.37)$$

The matching coefficients, as well as the renormalised parton-in-parton GPDs, admit perturbative expansions,

$$\mathcal{A}_{i/j} = \sum_{n=0}^{\infty} a_s^n \mathcal{A}_{i/j}^{[n]}, \quad F_{i/k} = \sum_{n=0}^{\infty} a_s^n F_{i/k}^{[n]}. \quad (2.38)$$

Substituting these expansions into Eq. (2.36) allows the partonic form of the matching condition to be imposed order by order in perturbation theory.

At tree level, omitting all arguments for brevity, one finds

$$F_{i/k}^{[0],(n_f+1)} = \sum_{j=q_1, \dots, q_{n_f}, h, g} \mathcal{A}_{i/j}^{[0]} \otimes F_{j/k}^{[0],(n_f)}. \quad (2.39)$$

Using the tree-level parton-in-parton GPD  $F_{j/k}^{[0]}(x, \eta) = D_k(\eta) \delta_{jk} \delta(1-x)$  from Eq. (1.122), with  $D_q(\eta) = \sqrt{1-\eta^2}$  and  $D_g(\eta) = 1 - \eta^2$  as defined in Eq. (1.121), the sum over intermediate species collapses to  $j = k$ :

$$F_{i/k}^{[0],(n_f+1)} = A_{i/k}^{[0]} \otimes [D_k(\eta) \delta(1-x)] = D_k(\eta) A_{i/k}^{[0]}. \quad (2.40)$$

Comparing this result with the direct tree-level evaluation yields

$$A_{i/k}^{[0]}(x) = \delta_{ik} \delta(1-x), \quad i, k = q_1, \dots, q_{n_f}, h, g. \quad (2.41)$$

Thus, the partonic projection reproduces exactly the tree-level matching kernel derived in Eq. (2.22). This provides the lowest-order consistency check of the partonic framework.

We now apply the same construction at one-loop accuracy. As discussed in the previous section, the non-trivial heavy-quark matching contributions at this order are contained in the four partonic channels

$$(i, k) = (h, h), (h, g), (g, h), (g, g), \quad (2.42)$$

which correspond to the four entries of the  $(h, g)$  matching block in Eq. (2.27). The purpose of the partonic calculation is therefore to determine the one-loop matching coefficients

$$\mathcal{A}_{h/h}^{[1]}, \quad \mathcal{A}_{h/g}^{[1]}, \quad \mathcal{A}_{g/h}^{[1]}, \quad \mathcal{A}_{g/g}^{[1]}. \quad (2.43)$$

To see how these coefficients are isolated, we first expand the partonic matching relation in Eq. (2.36) up to first order in  $\alpha_s$ . Using

$$\mathcal{A}_{i/j} = \mathcal{A}_{i/j}^{[0]} + a_s \mathcal{A}_{i/j}^{[1]} + \mathcal{O}(\alpha_s^2), \quad (2.44)$$

and

$$F_{i/k} = F_{i/k}^{[0]} + a_s F_{i/k}^{[1]} + \mathcal{O}(\alpha_s^2), \quad (2.45)$$

the terms of order  $\alpha_s$  give

$$F_{i/k}^{[1],(n_f+1)} = \sum_{j=q_1, \dots, q_{n_f}, h, g} \mathcal{A}_{i/j}^{[0]} \otimes F_{j/k}^{[1],(n_f)} + \sum_{j=q_1, \dots, q_{n_f}, h, g} \mathcal{A}_{i/j}^{[1]} \otimes F_{j/k}^{[0],(n_f)}. \quad (2.46)$$

The tree-level analysis above showed that

$$\mathcal{A}_{i/j}^{[0]} = \delta_{ij} \mathbf{1}_{\otimes}. \quad (2.47)$$

The first term in Eq. (2.46) therefore reduces to

$$\sum_{j=q_1, \dots, q_{n_f}, h, g} \mathcal{A}_{i/j}^{[0]} \otimes F_{j/k}^{[1], (n_f)} = F_{i/k}^{[1], (n_f)}, \quad (2.48)$$

and the one-loop partonic matching relation can be written as

$$F_{i/k}^{[1], (n_f+1)} - F_{i/k}^{[1], (n_f)} = \sum_{j=q_1, \dots, q_{n_f}, h, g} \mathcal{A}_{i/j}^{[1]} \otimes F_{j/k}^{[0], (n_f)}. \quad (2.49)$$

The difference on the left-hand side has a direct interpretation at the level of the one-loop partonic calculation. The contributions which are common to the two flavour-number descriptions cancel in the subtraction, leaving the contribution associated with the different treatment of the heavy flavour across the threshold. At the level of the contributions entering the partonic matrix element, this can be represented schematically as

$$F_{i/k}^{[1], (n_f+1)} = F_{i/k}^{[1], \text{common}} + F_{i/k}^{[1], m_h}, \quad (2.50)$$

where the first term of the sum collects the contributions which are unchanged across the threshold, while the second term denotes the contribution associated with the heavy quark of mass  $m_h$ . Correspondingly,

$$F_{i/k}^{[1], (n_f)} = F_{i/k}^{[1], \text{common}}. \quad (2.51)$$

Thus, the difference entering the one-loop matching condition in Eq. (2.49) isolates precisely the contribution to the parton-in-parton GPD associated with the finite heavy-quark mass  $m_h$ , which we refer to as the massive parton-in-parton contribution.

As an explicit example, consider the  $g/g$  channel. Contributions to the gluon-in-gluon GPD that are independent of the heavy flavour are common to the two descriptions and cancel in the subtraction. At the diagrammatic level, the remaining contribution  $F_{g/g}^{[1], m_h}$  coincides with the heavy-quark contribution to the gluon-in-gluon GPD, evaluated with the heavy-quark mass  $m_h$  kept finite. The same subtraction relation applies to the  $h/h$ ,  $h/g$ , and  $g/h$  channels. The explicit diagrammatic evaluation of the massive parton-in-parton contributions in all four channels is presented in Sec. 3.1. This observation motivates the following definition of the massive one-loop parton-in-parton GPD relevant for the threshold matching:

$$F_{i/k}^{[1], m_h} \equiv F_{i/k}^{[1], (n_f+1)} - F_{i/k}^{[1], (n_f)}, \quad (i, k) = (h, h), (h, g), (g, h), (g, g), \quad (2.52)$$

With this definition, Eq. (2.49) takes the compact form

$$F_{i/k}^{[1], m_h} = \sum_{j=q_1, \dots, q_{n_f}, h, g} \mathcal{A}_{i/j}^{[1]} \otimes F_{j/k}^{[0], (n_f)}. \quad (2.53)$$

Using the tree-level result

$$F_{j/k}^{[0], (n_f)} = D_k(\eta) \delta_{jk} \mathbf{1}_{\otimes}, \quad (2.54)$$

only the term with  $j = k$  survives in the sum, and Eq. (2.53) takes the form

$$F_{i/k}^{[1],m_h} = D_k(\eta) \mathcal{A}_{i/k}^{[1]}. \quad (2.55)$$

The one-loop matching coefficient associated with each of the four relevant partonic channels can therefore be extracted directly from the corresponding massive parton-in-parton GPD as

$$\mathcal{A}_{i/k}^{[1]} = F_{i/k}^{[1],m_h} D_k^{-1}(\eta), \quad (i, k) = (h, h), (h, g), (g, h), (g, g). \quad (2.56)$$

The quantity  $F_{i/k}^{[1],m_h}$  entering Eq. (2.56) is the renormalised massive parton-in-parton GPD. To determine it, we now consider its bare counterpart, which in dimensional regularisation can be expanded around  $\epsilon = 0$  as <sup>1</sup>

$$\widehat{F}_{i/k}^{[1],m_h} \left( x, \eta, \epsilon, \frac{\mu^2}{m_h^2} \right) = \frac{1}{\epsilon} \widehat{F}_{i/k}^{[1,-1]}(x, \eta) + \widehat{F}_{i/k}^{[1,0],m_h} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) + \mathcal{O}(\epsilon). \quad (2.57)$$

where the first entry in the superscript denotes the perturbative order in  $\alpha_s$ , while the second specifies the power of  $\epsilon$  in the expansion.

The UV pole is insensitive to the heavy-quark mass, since it originates from the region of loop momenta much larger than  $m_h$ . Its coefficient therefore coincides with that appearing in the corresponding massless parton-in-parton GPD in Eq. (1.140),

$$\widehat{F}_{i/k}^{[1,-1]}(x, \eta) = A_{i/k}(x, \eta), \quad (2.58)$$

so that the relation between the pole coefficient and the LO evolution kernel derived in Sec. 1.4.3 remains unchanged.

The bare massive GPD is renormalised in the  $\overline{\text{MS}}$  scheme by the same renormalisation kernel introduced in Sec. 1.4.2. At one loop,

$$F_{i/k}^{[1],m_h} = \widehat{F}_{i/k}^{[1],m_h} + Z_{ik}^{[1]} \left( x, \frac{\eta}{x}, \epsilon \right) D_k(\eta), \quad Z_{ik}^{[1]} = \frac{1}{\epsilon} Z_{ik}^{[1,1]}. \quad (2.59)$$

Substituting Eq. (2.57) into Eq. (2.59), the counterterm cancels the UV pole,

$$\widehat{F}_{i/k}^{[1,-1]}(x, \eta) + Z_{ik}^{[1,1]} \left( x, \frac{\eta}{x} \right) D_k(\eta) = 0. \quad (2.60)$$

Hence, in the four-dimensional limit, i.e.  $\epsilon \rightarrow 0$ , the renormalised massive parton-in-parton GPD is given by the finite part of the bare result,

$$F_{i/k}^{[1],m_h} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) = \widehat{F}_{i/k}^{[1,0],m_h} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) \equiv F_{i/k}^{[1,0],m_h} \left( x, \eta, \frac{\mu^2}{m_h^2} \right). \quad (2.61)$$

<sup>1</sup>The explicit  $\mu$ -dependence of the bare GPD is a consequence of the regularisation procedure. In dimensional regularisation,  $\mu$  is introduced to keep the renormalised strong coupling dimensionless in  $d = 4 - 2\epsilon$  dimensions; after renormalisation, the same scale is identified with the renormalisation scale.

Combining Eq. (2.61) with the partonic matching relation in Eq. (2.55), we finally obtain

$$\mathcal{A}_{i/k}^{[1]} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) = F_{i/k}^{[1,0],m_h} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) D_k^{-1}(\eta), \quad (2.62)$$

with

$$(i, k) = (h, h), (h, g), (g, h), (g, g). \quad (2.63)$$

Thus, apart from the tree-level normalisation factor  $D_k(\eta)$  associated with the external partonic state  $k$ , the finite part of each bare massive one-loop parton-in-parton GPD determines the corresponding one-loop matching coefficient. Explicitly,

$$\begin{aligned} \mathcal{A}_{h/h}^{[1]} &= F_{h/h}^{[1,0],m_h} D_h^{-1}, & \mathcal{A}_{g/h}^{[1]} &= F_{g/h}^{[1,0],m_h} D_h^{-1}, \\ \mathcal{A}_{h/g}^{[1]} &= F_{h/g}^{[1,0],m_h} D_g^{-1}, & \mathcal{A}_{g/g}^{[1]} &= F_{g/g}^{[1,0],m_h} D_g^{-1}, \end{aligned} \quad (2.64)$$

where the dependence on  $x$ ,  $\eta$ , and  $\mu^2/m_h^2$  is understood, and  $D_h \equiv D_q$ . Equivalently, introducing parton-in-parton GPDs normalised by their tree-level factors,

$$F'_{i/k} \equiv D_k^{-1}(\eta) F_{i/k}, \quad (2.65)$$

the relation in Eq. (2.62) takes the particularly simple form

$$\mathcal{A}_{i/k}^{[1]} = F'_{i/k}{}^{[1,0],m_h}. \quad (2.66)$$

In this normalisation, the finite part of the massive one-loop parton-in-parton GPD coincides directly with the corresponding one-loop matching coefficient.

The discussion above provides the general prescription for extracting the one-loop heavy-quark matching coefficients from massive partonic matrix elements. In the following chapter, this framework is applied to the four one-loop parton-in-parton GPDs associated with the four channels in Eq. (2.42). Their explicit diagrammatic calculation and UV renormalisation yield the finite contributions  $F_{i/k}^{[1,0],m_h}$ , from which the matching coefficients in Eq. (2.43) are obtained directly through Eq. (2.62).

## Chapter 3

# One-loop matching conditions for GPDs

This chapter presents the explicit one-loop calculation of the massive partonic contributions relevant for heavy-quark matching across flavour thresholds. We first evaluate the massive parton-in-parton GPDs in the heavy-quark–gluon sector, extending the massless calculation of Ch. 1 to a finite heavy-quark mass  $m_h$ . The complete off-diagonal contributions and the real part of the diagonal heavy-quark channel are derived directly from the non-forward calculation, while the virtual diagonal contributions are treated separately. We then use the renormalised finite terms to extract the corresponding matching coefficients and perform a series of consistency checks, including their UV structure and forward limit.

### 3.1 Massive parton-in-parton GPDs at one loop

As established in Sec. 2.3, the one-loop matching coefficients across a heavy-quark threshold can be determined from perturbatively calculable massive parton-in-parton GPDs. This calculation is the massive counterpart to the one-loop parton-in-parton calculation presented in Sec. 1.4.2. The underlying diagrammatic structure is therefore unchanged: the relevant real and virtual contributions are described by the same one-loop topologies shown in Figs. 1.12 and 1.13. The crucial difference with respect to the massless case is that the fermion lines corresponding to the heavy quark now carry a finite mass  $m_h$ , which affects both the on-shell kinematics of external heavy-quark states and the corresponding internal propagators. This leads to the four massive one-loop parton-in-parton GPDs,

$$F_{h/h}^{[1],m_h}, \quad F_{h/g}^{[1],m_h}, \quad F_{g/h}^{[1],m_h}, \quad F_{g/g}^{[1],m_h}, \quad (3.1)$$

which determine the four entries of the one-loop heavy-quark–gluon matching kernel in Eq. (2.31).

In the  $h/h$  channel, both the external quark states and the internal quark propagators carry the heavy-quark mass  $m_h$ . The same massive external-quark kinematics is used in the  $g/h$  channel, where the gluon operator is evaluated between heavy-quark states. In the

$h/g$  channel, instead, the external states are massless gluons, while the quark propagators connecting them to the heavy-quark operator carry the mass  $m_h$ . Finally, in the  $g/g$  channel the external gluons remain massless.

The real–virtual decomposition follows the same pattern as in the massless calculation for the  $h/h$ ,  $h/g$ , and  $g/h$  channels. The  $h/h$  channel receives both real and virtual one-loop contributions,

$$F_{h/h}^{[1],m_h} = F_{h/h}^{[1],m_h,\text{real}} + F_{h/h}^{[1],m_h,\text{virt}}, \quad (3.2)$$

whereas the off-diagonal  $h/g$  and  $g/h$  GPDs vanish at tree level, thereby receiving only real contributions at this order:

$$F_{h/g}^{[1],m_h} = F_{h/g}^{[1],m_h,\text{real}}, \quad F_{g/h}^{[1],m_h} = F_{g/h}^{[1],m_h,\text{real}}. \quad (3.3)$$

The  $g/g$  channel requires separate consideration. Although the complete one-loop gluon-in-gluon GPD contains both real and virtual contributions, the purely gluonic terms are common to both the  $n_f$ - and  $(n_f + 1)$ -flavour descriptions and therefore cancel in the subtraction defining  $F_{g/g}^{[1],m_h}$ , as discussed in Sec. 2.3. The heavy-quark contribution is therefore generated entirely by the virtual heavy-quark loop shown in Fig. 1.13,

$$F_{g/g}^{[1],m_h} = F_{g/g}^{[1],m_h,\text{virt}}. \quad (3.4)$$

We now specify the external-state kinematics used in the calculation. Two cases have to be distinguished, depending on whether the external state is a massless gluon or a massive heavy quark. For a massless external gluon, we retain the longitudinal non-forward kinematics introduced in Sec. 1.4.2 (Eq. (1.103)). Using the light-cone conventions introduced in App. A, we choose the external partonic states to be collinear to the  $+$  direction, with vanishing transverse momentum. Combined with the massless on-shell condition, the average partonic momentum simplifies to

$$p^\mu = (p^+, 0, \mathbf{0}_T), \quad (3.5)$$

which implies

$$p_{\text{in}}^2 = p_{\text{out}}^2 = 0. \quad (3.6)$$

This choice of kinematics applies directly to the  $h/g$  and  $g/g$  channels. When the external state is instead a heavy quark of mass  $m_h$ , the plus components are chosen with the same skewness dependence as in the massless case, while the minus components are fixed by the corresponding on-shell conditions:

$$\begin{aligned} p_{\text{in}}^\mu &= \left( (1 + \eta)p^+, \frac{m_h^2}{2(1 + \eta)p^+}, \mathbf{0}_T \right), \\ p_{\text{out}}^\mu &= \left( (1 - \eta)p^+, \frac{m_h^2}{2(1 - \eta)p^+}, \mathbf{0}_T \right), \end{aligned} \quad (3.7)$$

which ensures

$$p_{\text{in}}^2 = p_{\text{out}}^2 = m_h^2. \quad (3.8)$$

This kinematics applies to the  $h/h$  and  $g/h$  channels.

All remaining conventions are kept identical to those in Sec. 1.4.2. In particular, we work in dimensional regularisation and in the light-cone gauge  $n \cdot A = 0$ , using the same colour and spin or helicity averages for the external partonic states. The explicit calculation of the massive real one-loop contributions is reported in App. C. The virtual terms entering the diagonal channels are discussed separately in their respective subsections. We now present the resulting massive parton-in-parton GPDs and discuss the key features relevant for the extraction of the matching coefficients.

### 3.1.1 Off-diagonal channels

We first consider the off-diagonal  $h/g$  and  $g/h$  channels. The bare one-loop results share the same support structure and can be written in the following compact form

$$\widehat{F}_{i/k}^{[1],m_h} \left( x, \eta, \epsilon, \frac{\mu^2}{m_h^2} \right) = \theta(1-x) \left[ \theta(x+\eta) \mathcal{H}_{i/k}(x, \eta) + \theta(x-\eta) \mathcal{H}_{i/k}(x, -\eta) \right], \quad (3.9)$$

with  $(i, k) = (h, g), (g, h)$ . The channel dependence is contained in the function  $\mathcal{H}_{i/k}$ . For brevity, the dependence of  $\mathcal{H}_{i/k}$  on  $\epsilon$  and  $\mu^2/m_h^2$  has been left implicit in Eq. (3.9). Its UV pole and finite contributions can then be separated as

$$\mathcal{H}_{i/k} \left( x, \eta, \epsilon, \frac{\mu^2}{m_h^2} \right) = \frac{1}{\epsilon} \mathcal{H}_{i/k}^{\text{pole}}(x, \eta) + \mathcal{H}_{i/k}^{\text{fin}} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) + \mathcal{O}(\epsilon). \quad (3.10)$$

For positive partonic skewness  $\eta$ , the Heaviside step functions in Eq. (3.9) make the support in  $x$  explicit. When  $0 < \eta < 1$ , the standard ERBL and DGLAP regions are recovered. In the ERBL region,  $-\eta < x < \eta$ , only the term proportional to  $\mathcal{H}_{i/k}(x, \eta)$  contributes, whereas in the positive DGLAP region,  $\eta < x < 1$ , both terms are present and combine as  $\mathcal{H}_{i/k}(x, \eta) + \mathcal{H}_{i/k}(x, -\eta)$ . The distributions vanish identically in the negative DGLAP region,  $x < -\eta$  as well as for  $x > 1$ .

For  $\eta > 1$ , the positive DGLAP region is absent, and the entire non-vanishing support,  $-\eta < x < 1$ , falls within the ERBL region. This region is not considered in the explicit results reported below, since the non-vanishing light-cone components of the outgoing external momentum  $p_{\text{out}}^\mu$  in Eqs. (1.103) and (3.7) become negative, requiring a separate kinematic treatment.

### Heavy-quark-in-gluon GPD

The  $h/g$  channel is obtained by evaluating the heavy-quark operator between massless external gluon states. The external momenta therefore obey the kinematics in Eq. (1.103), while the fermion propagators in the real diagram carry the heavy-quark mass  $m_h$ . The UV-pole coefficient of the function  $\mathcal{H}_{h/g}$  is

$$\mathcal{H}_{h/g}^{\text{pole}}(x, \eta) = T_R \frac{(x+\eta)(1+\eta-2x)}{\eta(1+\eta)}, \quad (3.11)$$

while the finite contribution reads

$$\mathcal{H}_{h/g}^{\text{fin}} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) = -T_R \frac{(x+\eta)(1+\eta-2x)}{\eta(1+\eta)} \ln \left( \frac{m_h^2}{\mu^2} \right), \quad (3.12)$$

where  $T_R = 1/2$ , as specified in Eq. (1.152). Using the general decomposition introduced in Eq. (2.57), the bare heavy-quark-in-gluon GPD can be written as

$$\widehat{F}_{h/g}^{[1],m_h} \left( x, \eta, \epsilon, \frac{\mu^2}{m_h^2} \right) = \frac{1}{\epsilon} \widehat{F}_{h/g}^{[1,-1]}(x, \eta) + \widehat{F}_{h/g}^{[1,0],m_h} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) + \mathcal{O}(\epsilon). \quad (3.13)$$

The pole and finite coefficients are obtained from Eq. (3.9) by replacing  $\mathcal{H}_{h/g}$  with  $\mathcal{H}_{h/g}^{\text{pole}}$  and  $\mathcal{H}_{h/g}^{\text{fin}}$ , respectively. The pole coefficient is

$$\widehat{F}_{h/g}^{[1,-1]}(x, \eta) = \theta(1-x) \left[ \theta(x+\eta) \mathcal{H}_{h/g}^{\text{pole}}(x, \eta) + \theta(x-\eta) \mathcal{H}_{h/g}^{\text{pole}}(x, -\eta) \right]. \quad (3.14)$$

As anticipated in Eq. (2.58), this coefficient coincides with the UV-pole coefficient of the corresponding massless quark-in-gluon GPD,

$$\widehat{F}_{h/g}^{[1,-1]}(x, \eta) = A_{q/g}(x, \eta). \quad (3.15)$$

Since  $D_g(\eta) = 1 - \eta^2$ , as defined in Eq. (1.121), the relation in Eq. (1.145) gives

$$\mathcal{P}_{q/g}^{[0]} \left( x, \frac{\eta}{x} \right) = A_{q/g}(x, \eta) D_g^{-1}(\eta) = \frac{\widehat{F}_{h/g}^{[1,-1]}(x, \eta)}{1 - \eta^2}, \quad (3.16)$$

which reproduces the LO quark-in-gluon kernel  $\mathcal{P}_{q/g}^{[0]}$  obtained in Eqs. (1.146) and (1.148).

The corresponding finite coefficient of the bare  $h/g$  parton-in-parton GPD is

$$\begin{aligned} \widehat{F}_{h/g}^{[1,0],m_h} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) &= \theta(1-x) \left[ \theta(x+\eta) \mathcal{H}_{h/g}^{\text{fin}} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) \right. \\ &\quad \left. + \theta(x-\eta) \mathcal{H}_{h/g}^{\text{fin}} \left( x, -\eta, \frac{\mu^2}{m_h^2} \right) \right]. \end{aligned} \quad (3.17)$$

After subtraction of the UV pole in the  $\overline{\text{MS}}$  scheme, this finite coefficient directly gives the renormalised massive parton-in-parton GPD,

$$F_{h/g}^{[1],m_h} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) = \widehat{F}_{h/g}^{[1,0],m_h} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) \equiv F_{h/g}^{[1,0],m_h} \left( x, \eta, \frac{\mu^2}{m_h^2} \right). \quad (3.18)$$

### Gluon-in-heavy-quark GPD

The  $g/h$  channel is obtained by evaluating the gluon operator between external heavy-quark states, whose momenta obey the massive on-shell kinematics in Eq. (3.7). The UV-pole coefficient of the function  $\mathcal{H}_{g/h}$  is

$$\mathcal{H}_{g/h}^{\text{pole}}(x, \eta) = C_F \sqrt{1 - \eta^2} \frac{(x + \eta)(2 - x + \eta)}{x\eta(1 + \eta)}, \quad (3.19)$$

while the finite contribution reads

$$\begin{aligned} \mathcal{H}_{g/h}^{\text{fin}} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) = & \frac{C_F \sqrt{1 - \eta^2}}{x\eta} \left[ \frac{x^2 - \eta^2}{1 + \eta} - \frac{(x + \eta)(2 - x + \eta)}{1 + \eta} \ln \left( \frac{(x + \eta)^2 m_h^2}{(1 + \eta)^2 \mu^2} \right) \right. \\ & \left. + \frac{(1 - x)(x^2 - \eta^2)}{x - \eta^2} \ln \left( \frac{(x + \eta)^2 (\eta^2 - 1)}{(1 + \eta)^2 (x^2 - \eta^2)} \right) \right], \end{aligned} \quad (3.20)$$

where  $C_F = 4/3$  for  $N_c = 3$ , as specified in Eq. (1.152). Inserting the decomposition in Eq. (3.10) into Eq. (3.9), the pole coefficient of the bare parton-in-parton GPD is

$$\hat{F}_{g/h}^{[1, -1]}(x, \eta) = \theta(1 - x) \left[ \theta(x + \eta) \mathcal{H}_{g/h}^{\text{pole}}(x, \eta) + \theta(x - \eta) \mathcal{H}_{g/h}^{\text{pole}}(x, -\eta) \right], \quad (3.21)$$

and coincides with the pole coefficient of the corresponding massless gluon-in-quark GPD,

$$\hat{F}_{g/h}^{[1, -1]}(x, \eta) = A_{g/q}(x, \eta). \quad (3.22)$$

For the heavy-quark external state, the quark normalisation factor is  $D_h(\eta) \equiv D_q(\eta) = \sqrt{1 - \eta^2}$ , with  $D_q$  defined in Eq. (1.121), so that Eq. (1.145) gives

$$\mathcal{P}_{g/q}^{[0]} \left( x, \frac{\eta}{x} \right) = A_{g/q}(x, \eta) D_h^{-1}(\eta) = \frac{\hat{F}_{g/h}^{[1, -1]}(x, \eta)}{\sqrt{1 - \eta^2}}, \quad (3.23)$$

which reproduces the LO gluon-in-quark evolution kernel  $\mathcal{P}_{g/q}^{[0]}$  obtained in Eqs. (1.146) and (1.149).

The corresponding finite coefficient of the bare  $g/h$  parton-in-parton GPD is

$$\begin{aligned} \hat{F}_{g/h}^{[1, 0], m_h} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) = & \theta(1 - x) \left[ \theta(x + \eta) \mathcal{H}_{g/h}^{\text{fin}} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) \right. \\ & \left. + \theta(x - \eta) \mathcal{H}_{g/h}^{\text{fin}} \left( x, -\eta, \frac{\mu^2}{m_h^2} \right) \right]. \end{aligned} \quad (3.24)$$

After subtraction of the UV pole in the  $\overline{\text{MS}}$  scheme, this finite coefficient directly gives the renormalised massive parton-in-parton GPD,

$$F_{g/h}^{[1], m_h} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) = \hat{F}_{g/h}^{[1, 0], m_h} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) \equiv F_{g/h}^{[1, 0], m_h} \left( x, \eta, \frac{\mu^2}{m_h^2} \right). \quad (3.25)$$

### 3.1.2 Diagonal channels

We now turn to the diagonal  $h/h$  and  $g/g$  channels.

#### Heavy-quark-in-heavy-quark GPD

As discussed above, the bare  $h/h$  GPD receives both real and virtual one-loop contributions,

$$\widehat{F}_{h/h}^{[1],m_h} = \widehat{F}_{h/h}^{[1],m_h,\text{real}} + \widehat{F}_{h/h}^{[1],m_h,\text{virt}}. \quad (3.26)$$

For  $0 < \eta < 1$ , the real contribution has the same ERBL–DGLAP support structure as the off-diagonal channels and reads

$$\begin{aligned} \widehat{F}_{h/h}^{[1],m_h,\text{real}} \left( x, \eta, \epsilon, \frac{\mu^2}{m_h^2} \right) &= \theta(1-x) \left[ \theta(x+\eta) \mathcal{H}_{h/h}(x, \eta) \right. \\ &\quad \left. + \theta(x-\eta) \mathcal{H}_{h/h}(x, -\eta) \right]. \end{aligned} \quad (3.27)$$

As for the off-diagonal channels, the partonic skewness is not restricted to the interval  $0 < \eta < 1$  in general:  $\eta > 1$  corresponds to ERBL kinematics, but this region is not explicitly considered in the contour calculation presented here.

The function  $\mathcal{H}_{h/h}$  is decomposed into its UV-pole and finite contributions as

$$\mathcal{H}_{h/h} \left( x, \eta, \epsilon, \frac{\mu^2}{m_h^2} \right) = \frac{1}{\epsilon} \mathcal{H}_{h/h}^{\text{pole}}(x, \eta) + \mathcal{H}_{h/h}^{\text{fin}} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) + \mathcal{O}(\epsilon), \quad (3.28)$$

where

$$\mathcal{H}_{h/h}^{\text{pole}}(x, \eta) = C_F \sqrt{1-\eta^2} \frac{(x+\eta)(1-x+2\eta)}{\eta(1-x)(1+\eta)}, \quad (3.29)$$

while the finite contribution is

$$\begin{aligned} \mathcal{H}_{h/h}^{\text{fin}} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) &= \frac{C_F \sqrt{1-\eta^2}}{\eta} \left[ -\frac{x+\eta}{1+\eta} - \frac{x+\eta}{1+\eta} \frac{1+x}{1-x} \ln \left( \frac{(1-x)^2 m_h^2}{(1+\eta)^2 \mu^2} \right) \right. \\ &\quad + \frac{x^2 - \eta^2}{1-x} \frac{1-\eta}{1+\eta} \ln \left( \frac{(1-x)^2 m_h^2}{(1+\eta)^2 \mu^2} \right) \\ &\quad + \frac{x^2 - \eta^2}{1-x} \ln \left( \frac{(1-x^2) m_h^2}{(1-\eta^2) \mu^2} \right) \\ &\quad \left. - \frac{2x+2\eta^2}{1-x} \ln \left( \frac{(1+x)(1+\eta)}{(1-x)(1-\eta)} \right) \right]. \end{aligned} \quad (3.30)$$

Substituting the decomposition in Eq. (3.28) into Eq. (3.27), the pole coefficient of the real contribution is

$$\begin{aligned} \widehat{F}_{h/h}^{[1,-1],\text{real}}(x, \eta) &= \theta(1-x) \left[ \theta(x+\eta) \mathcal{H}_{h/h}^{\text{pole}}(x, \eta) \right. \\ &\quad \left. + \theta(x-\eta) \mathcal{H}_{h/h}^{\text{pole}}(x, -\eta) \right]. \end{aligned} \quad (3.31)$$

As in the off-diagonal channels, the heavy-quark mass does not affect the UV pole, and the pole coefficient coincides with that of the corresponding massless quark-in-quark real contribution,

$$\widehat{F}_{h/h}^{[1,-1],\text{real}}(x, \eta) = A_{q/q}^{\text{real}}(x, \eta). \quad (3.32)$$

Hence, applying the relation in Eq. (1.145) yields

$$\mathcal{P}_{q/q}^{[0],\text{real}}\left(x, \frac{\eta}{x}\right) = A_{q/q}^{\text{real}}(x, \eta) D_h^{-1}(\eta) = \frac{\widehat{F}_{h/h}^{[1,-1],\text{real}}(x, \eta)}{\sqrt{1 - \eta^2}}, \quad (3.33)$$

which matches the real part of the LO quark-in-quark evolution kernel in Eqs. (1.146) and (1.147). Since the  $h/h$  channel is diagonal, the full kernel also contains a contribution localised at  $x = 1$ , generated by the corresponding virtual term.

The corresponding finite coefficient of the real contribution is

$$\begin{aligned} \widehat{F}_{h/h}^{[1,0],m_h,\text{real}}\left(x, \eta, \frac{\mu^2}{m_h^2}\right) &= \theta(1-x) \left[ \theta(x+\eta) \mathcal{H}_{h/h}^{\text{fin}}\left(x, \eta, \frac{\mu^2}{m_h^2}\right) \right. \\ &\quad \left. + \theta(x-\eta) \mathcal{H}_{h/h}^{\text{fin}}\left(x, -\eta, \frac{\mu^2}{m_h^2}\right) \right]. \end{aligned} \quad (3.34)$$

After subtraction of the UV pole in the  $\overline{\text{MS}}$  scheme, Eq. (3.34) provides the finite part of the real contribution. Unlike the off-diagonal channels, however, this term alone does not represent the full renormalised  $h/h$  parton-in-parton GPD, since the corresponding virtual contribution must also be included. The virtual contribution to the bare  $h/h$  GPD arises from the one-loop self-energy corrections to the external heavy-quark legs. Since these corrections do not change the longitudinal momentum fraction flowing through the operator, they are localised at  $x = 1$ ,

$$\widehat{F}_{h/h}^{[1],m_h,\text{virt}}\left(x, \eta, \epsilon, \frac{\mu^2}{m_h^2}\right) = \delta(1-x) \widehat{\mathcal{V}}_{h/h}^{[1],m_h}\left(\eta, \epsilon, \frac{\mu^2}{m_h^2}\right). \quad (3.35)$$

The finite real contribution is singular as  $x \rightarrow 1$  and therefore cannot be physically interpreted independently of the virtual correction. As in the massless  $q/q$  calculation discussed in Sec. 1.4.3, the endpoint-singular terms of the real contribution combine with the contribution proportional to  $\delta(1-x)$  from the virtual diagrams. Their sum can be recast in terms of plus distributions, so that the apparent endpoint divergence is replaced by a well-defined distributional structure.

After combining the real and virtual contributions and performing the  $\overline{\text{MS}}$  subtraction, the renormalised diagonal result can be written schematically as

$$F_{h/h}^{[1],m_h} = \left[ F_{h/h}^{[1,0],m_h,\text{real,sing}} \right]_+ + F_{h/h}^{[1,0],m_h,\text{real,reg}} + \delta(1-x) \mathcal{V}_{h/h}^{\text{fin}}\left(\eta, \frac{\mu^2}{m_h^2}\right), \quad (3.36)$$

where the first term denotes the plus-prescribed form of the endpoint-singular part of the real contribution, while  $F_{h/h}^{[1,0],m_h,\text{real,reg}}$  collects the remaining terms that are integrable at

$x = 1$ . The coefficient  $\mathcal{V}_{h/h}^{\text{fin}}$  denotes the residual finite coefficient of the  $\delta(1-x)$  term in the complete renormalised result, once the endpoint-singular part of the real contribution has been rewritten in terms of plus distributions.

The explicit virtual contribution is not evaluated here. Nevertheless, as in the forward PDF case, one expects it to be constrained by the lowest Mellin-moment sum rule discussed in Sec. 1.3.3. Since integration over  $x$  reduces the bilocal operator to the conserved local flavour vector current, the one-loop massive contribution relevant for the matching is expected to satisfy

$$\int_{-1}^1 dx F_{h/h}^{[1],m_h} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) = 0. \quad (3.37)$$

Substituting Eq. (3.36) into Eq. (3.37), and using the defining property of the plus distribution, one obtains

$$\mathcal{V}_{h/h}^{\text{fin}} \left( \eta, \frac{\mu^2}{m_h^2} \right) = - \int_{-1}^1 dx F_{h/h}^{[1,0],m_h,\text{real,reg}} \left( x, \eta, \frac{\mu^2}{m_h^2} \right). \quad (3.38)$$

### Heavy-quark contribution to the gluon-in-gluon GPD

The  $g/g$  channel does not require a separate non-forward calculation at this order. As discussed above, after subtracting the contributions common to the  $n_f$ - and  $(n_f+1)$ -flavour descriptions, the heavy-quark contribution is generated entirely by the virtual heavy-quark loop.

The simple structure of this contribution follows directly from the heavy-quark contribution to the gluon self-energy. Denoting by  $\Pi_h^{\mu\nu}(p)$  the gluon self-energy generated by a closed heavy-quark loop, gauge invariance implies the transversality of the heavy-quark vacuum polarisation, which, together with Lorentz invariance, yields the standard decomposition:

$$\Pi_h^{\mu\nu}(p) = (p^\mu p^\nu - g^{\mu\nu} p^2) \Pi_h(p^2, m_h^2), \quad (3.39)$$

where  $\Pi_h(p^2, m_h^2)$  is the corresponding scalar vacuum-polarisation function. Its dependence on the external gluon momentum enters only through the invariant virtuality  $p^2$ . Using the massless non-forward kinematics of Eq. (1.103), the incoming and outgoing gluons carry different longitudinal momenta, but satisfy  $p_{\text{in}}^2 = p_{\text{out}}^2 = 0$ . The scalar loop factor is therefore identical for both external legs and introduces no additional dependence on the skewness  $\eta$ . Moreover, being a purely virtual correction, it does not change the longitudinal momentum fraction flowing through the operator and is therefore localised at the tree-level support  $x = 1$ . The structure of the renormalised massive contribution is consequently proportional to the tree-level gluon-in-gluon GPD,

$$F_{g/g}^{[1],m_h} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) = D_g(\eta) \delta(1-x) B_{g/g}^{[1]} \left( \frac{\mu^2}{m_h^2} \right), \quad (3.40)$$

where  $B_{g/g}^{[1]}$  denotes the renormalised scalar coefficient generated by the heavy-quark loop. All the non-forward dependence of Eq. (3.40) is contained in the tree-level factor  $D_g(\eta)$ . In

the forward limit  $\eta \rightarrow 0$ , where  $D_g(0) = 1$ , the result reduces directly to the corresponding one-loop gluon-in-gluon parton-in-parton PDF.

Since the heavy-quark loop does not generate any additional dependence on the non-forward kinematics, the scalar coefficient  $B_{g/g}^{[1]}$  can be taken from the known forward calculation, without performing a new calculation. The corresponding heavy-quark contribution to the forward gluon-in-gluon partonic matrix element is computed at one loop in Ref. [27]. In the perturbative convention  $a_s = \alpha_s/(4\pi)$  adopted throughout this work, the resulting scalar coefficient reads

$$B_{g/g}^{[1]} \left( \frac{\mu^2}{m_h^2} \right) = \frac{4}{3} T_R \ln \left( \frac{m_h^2}{\mu^2} \right). \quad (3.41)$$

Substituting Eq. (3.41) into Eq. (3.40) yields the massive gluon-in-gluon parton-in-parton GPD relevant for the threshold matching:

$$F_{g/g}^{[1],m_h} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) = \frac{4}{3} T_R D_g(\eta) \delta(1-x) \ln \left( \frac{m_h^2}{\mu^2} \right). \quad (3.42)$$

This simple structure should be contrasted with the virtual contribution in the  $h/h$  channel. In the light-cone gauge  $n \cdot A = 0$ , the gluon propagator contains terms proportional to the light-like gauge vector  $n^\mu$ , so that the heavy-quark self-energy can depend on the longitudinal projection  $p \cdot n$  in addition to  $p^2$ . Since the incoming and outgoing heavy quarks carry different longitudinal momenta, proportional to  $1 + \eta$  and  $1 - \eta$ , respectively, as shown in Eq. (3.7), the corresponding external-leg corrections retain a non-trivial dependence on the skewness  $\eta$ , as reflected by the  $\eta$ -dependent virtual coefficient in Eq. (3.35).

## 3.2 One-loop matching coefficients

We now use the renormalised massive parton-in-parton GPDs obtained in Sec. 3.1 to extract the corresponding one-loop matching coefficients. The general relation between the finite massive partonic matrix element and the matching coefficient was derived in Sec. 2.3. In particular, according to Eq. (2.62),  $\mathcal{A}_{i/k}^{[1]} = F_{i/k}^{[1,0],m_h} D_k^{-1}$ , where the tree-level normalisation factors are defined in Eq. (1.121). The results obtained below determine the entries of the heavy-quark–gluon block of the one-loop matching kernel introduced in Eq. (2.31). We first consider the off-diagonal coefficients, which are completely determined by the real one-loop contributions, and then discuss the diagonal channels.

### 3.2.1 Off-diagonal coefficients

We begin with the coefficient  $\mathcal{A}_{h/g}^{[1]}$ . Using the renormalised massive heavy-quark-in-gluon GPD obtained in Eq. (3.18) and dividing by the gluon normalisation factor  $D_g(\eta)$ , we find

$$\mathcal{A}_{h/g}^{[1]} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) = \frac{1}{1 - \eta^2} \theta(1 - x) \left[ \theta(x + \eta) \mathcal{H}_{h/g}^{\text{fin}} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) + \theta(x - \eta) \mathcal{H}_{h/g}^{\text{fin}} \left( x, -\eta, \frac{\mu^2}{m_h^2} \right) \right]. \quad (3.43)$$

The function  $\mathcal{H}_{h/g}^{\text{fin}}$  is given explicitly in Eq. (3.12). Comparing it with the pole coefficient in Eq. (3.11), one sees that the complete finite contribution is proportional to  $\ln(\mu^2/m_h^2)$ , with the same  $x$ - and  $\eta$ -dependent coefficient that multiplies the UV pole. The latter was shown in Eq. (3.16) to reproduce the LO quark-in-gluon evolution kernel. Equation (3.43) can therefore be written in the compact form

$$\mathcal{A}_{h/g}^{[1]} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) = \mathcal{P}_{q/g}^{[0]} \left( x, \frac{\eta}{x} \right) \ln \left( \frac{\mu^2}{m_h^2} \right). \quad (3.44)$$

The coefficient  $\mathcal{A}_{h/g}^{[1]}$  is therefore completely determined by the same kernel that governs the gluon contribution to the LO evolution of a quark GPD.

We next consider the gluon-from-heavy-quark coefficient  $\mathcal{A}_{g/h}^{[1]}$ . Dividing Eq. (3.25) by the external-state normalisation  $D_h(\eta)$  yields

$$\mathcal{A}_{g/h}^{[1]} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) = \frac{1}{\sqrt{1 - \eta^2}} \theta(1 - x) \left[ \theta(x + \eta) \mathcal{H}_{g/h}^{\text{fin}} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) + \theta(x - \eta) \mathcal{H}_{g/h}^{\text{fin}} \left( x, -\eta, \frac{\mu^2}{m_h^2} \right) \right]. \quad (3.45)$$

The explicit expression for  $\mathcal{H}_{g/h}^{\text{fin}}$  is given in Eq. (3.20). In contrast to the  $h/g$  channel, the finite result is not purely proportional to the heavy-mass logarithm. Its dependence on the matching scale can nevertheless be isolated by decomposing  $\ln[(x + \eta)^2 m_h^2 / ((1 + \eta)^2 \mu^2)]$  in Eq. (3.20) into  $-\ln(\mu^2/m_h^2)$  and a scale-independent kinematic logarithm, and analogously for the contribution obtained under  $\eta \rightarrow -\eta$ . The coefficient of  $\ln(\mu^2/m_h^2)$  is then precisely the combination of pole terms identified with the LO gluon-in-quark evolution kernel in Eq. (3.23). The full result can thus be organised as

$$\mathcal{A}_{g/h}^{[1]} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) = \mathcal{P}_{g/q}^{[0]} \left( x, \frac{\eta}{x} \right) \ln \left( \frac{\mu^2}{m_h^2} \right) + \mathcal{R}_{g/h}^{[1]}(x, \eta). \quad (3.46)$$

Here  $\mathcal{R}_{g/h}^{[1]}(x, \eta)$  denotes the finite, matching-scale-independent remainder. Equivalently, it is obtained by evaluating Eq. (3.45) at  $\mu = m_h$ . Unlike the  $h/g$  channel, the massive gluon-in-heavy-quark calculation therefore contains genuinely finite non-forward terms beyond the logarithm dictated by the LO evolution kernel.

These two off-diagonal results illustrate a general feature of threshold matching calculations: the coefficient of the logarithm involving the heavy-quark mass is fixed by the same short-distance contribution that generates the UV pole of the corresponding bare parton-in-parton GPD.

### 3.2.2 Diagonal coefficients

The diagonal channels require additional care because their real and virtual contributions have to be combined before obtaining a well-defined distribution in  $x$ .

For the  $h/h$  channel, the renormalised massive GPD has the distributional structure discussed in Eq. (3.36). Applying the general matching relation of Eq. (2.62) gives

$$\begin{aligned} \mathcal{A}_{h/h}^{[1]} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) &= \frac{1}{\sqrt{1-\eta^2}} \left[ F_{h/h}^{[1,0],m_h,\text{real},\text{sing}} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) \right]_+ \\ &+ \frac{1}{\sqrt{1-\eta^2}} F_{h/h}^{[1,0],m_h,\text{real},\text{reg}} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) \\ &+ \delta(1-x) \frac{\mathcal{V}_{h/h}^{\text{fin}} \left( \eta, \frac{\mu^2}{m_h^2} \right)}{\sqrt{1-\eta^2}}. \end{aligned} \quad (3.47)$$

The finite massive real contribution entering the matching coefficient is given in Eq. (3.34). According to Eq. (2.62), it is divided by the tree-level external-quark normalisation factor  $D_h(\eta)$ . Its endpoint-singular part is then combined with the virtual contribution and written in plus-distribution form.

In particular, collecting the terms proportional to the heavy-mass logarithm in Eq. (3.30), one finds that the coefficient of  $\ln(\mu^2/m_h^2)$  coincides exactly with the pole coefficient  $\mathcal{H}_{h/h}^{\text{pole}}$  given in Eq. (3.29). After restoring the complete support structure and dividing by the external-state normalisation factor, this coefficient reproduces the real part of the LO quark-in-quark evolution kernel identified in Eq. (3.33). The real  $h/h$  contribution thus exhibits the same relation between the heavy-mass logarithm and the UV pole found in the off-diagonal channels. Since the virtual contribution has not been evaluated explicitly here, the complete one-loop coefficient  $\mathcal{A}_{h/h}^{[1]}$  cannot be given, although its real part and associated logarithmic dependence are fully fixed.

The  $g/g$  coefficient follows directly from the massive gluon-in-gluon parton-in-parton GPD obtained in Eq. (3.42). Dividing by the tree-level gluon normalisation factor  $D_g(\eta)$ , as prescribed by Eq. (2.62), gives

$$\mathcal{A}_{g/g}^{[1]} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) = \frac{4}{3} T_R \delta(1-x) \ln \left( \frac{m_h^2}{\mu^2} \right). \quad (3.48)$$

The cancellation of the factor  $D_g(\eta)$  makes explicit that the heavy-quark contribution to the one-loop  $g/g$  matching coefficient does not acquire any additional dependence on the skewness  $\eta$ . It therefore coincides with the corresponding forward PDF matching coefficient.

### 3.2.3 Matching-scale dependence

The one-loop matching contributions discussed above are defined at a generic factorisation scale  $\mu$ , whose dependence enters through logarithms of the ratio  $\mu^2/m_h^2$ . For later convenience, we define

$$L_h(\mu) \equiv \ln\left(\frac{\mu^2}{m_h^2}\right). \quad (3.49)$$

As shown in Eqs. (3.44) and (3.46), the coefficient of  $L_h(\mu)$  in the off-diagonal channels is given by the corresponding LO GPD evolution kernel, in addition to possible finite terms. The same relation is found for the real part of the  $h/h$  channel, whose logarithmic coefficient reproduces the real part of the LO quark-in-quark kernel, as discussed below Eq. (3.47). In particular, the  $h/g$  coefficient is purely logarithmic, whereas the  $g/h$  coefficient and the real  $h/h$  contribution also contain finite, scale-independent terms.

The matching scale  $\mu_h$  is introduced by evaluating these coefficients at  $\mu = \mu_h$  to connect the  $n_f$ - and  $(n_f + 1)$ -flavour GPDs across the heavy-quark threshold. The logarithms then become  $L_h(\mu_h) = \ln(\mu_h^2/m_h^2)$ , making explicit the theoretical motivation for the choice  $\mu_h \sim m_h$  introduced in Ch. 2: keeping the matching scale close to the heavy-quark mass prevents these logarithms from becoming large and keeps the fixed-order matching coefficients under perturbative control. For the canonical choice  $\mu_h = m_h$ ,  $L_h(\mu_h) = 0$ . The  $h/g$  matching coefficient therefore vanishes identically, whereas finite non-logarithmic contributions survive in the  $g/h$  channel and in the real part of the  $h/h$  channel. Thus, choosing the matching scale equal to the heavy-quark mass removes the heavy-mass logarithms without necessarily eliminating all one-loop matching corrections.

## 3.3 Consistency checks and forward limit

The results derived above provide two complementary consistency checks. The first concerns the relation, already observed channel by channel in Sec. 3.2, between the UV poles of the bare massive parton-in-parton GPDs and the heavy-mass logarithms appearing in their finite parts. The second is provided by the forward limit, in which the non-forward matching coefficients must reduce to the corresponding heavy-flavour PDF matching coefficients.

### 3.3.1 UV poles and heavy-mass logarithms

As already observed in Sec. 3.2, the coefficient multiplying  $L_h(\mu)$ , defined in Eq. (3.49), coincides with the corresponding UV-pole coefficient. This equivalence follows directly from the structure of the dimensionally regulated one-loop integrals. Schematically, the  $\epsilon$ -expansion of a UV-divergent contribution with pole coefficient  $\mathcal{A}_{i/k}(x, \eta)$  yields

$$\left(\frac{\mu^2}{m_h^2}\right)^\epsilon \frac{\mathcal{A}_{i/k}(x, \eta)}{\epsilon} = \frac{\mathcal{A}_{i/k}(x, \eta)}{\epsilon} + \mathcal{A}_{i/k}(x, \eta) L_h(\mu) + \mathcal{O}(\epsilon). \quad (3.50)$$

Additional functions of the partonic kinematics may generate finite kinematic logarithms, but they do not modify the coefficient of  $L_h(\mu)$ . Once the  $\overline{\text{MS}}$  subtraction removes the

UV pole, this same coefficient therefore remains in front of the heavy-mass logarithm. The explicit results obtained above satisfy this relation, providing an internal consistency check of the massive one-loop calculation.

### 3.3.2 Forward limit

A second consistency check is provided by the forward limit, where the non-forward matching coefficients must reduce to the corresponding heavy-flavour PDF matching coefficients. The latter are known from the variable-flavour-number matching calculation without intrinsic heavy flavour [27]. When a non-vanishing heavy-quark distribution is retained below threshold, the additional heavy-quark-initiated matching coefficients are given in Ref. [38]. In the collinear kinematics used throughout the calculation, the forward limit corresponds to  $\eta \rightarrow 0$ . For fixed  $0 < x < 1$ , the theta functions appearing in the parton-in-parton GPDs of Eqs. (3.9) and (3.27) reduce to unity in the forward limit, while the tree-level normalisation factors defined in Eq. (1.121) satisfy  $D_h(\eta) \rightarrow 1$  and  $D_g(\eta) \rightarrow 1$ .

The  $h/g$  channel provides the simplest forward-limit check. Combining Eqs. (3.14) and (3.16), and using the expression for  $\mathcal{H}_{h/g}^{\text{pole}}$  in Eq. (3.11), one finds

$$\begin{aligned} \lim_{\eta \rightarrow 0} \mathcal{P}_{q/g}^{[0]} \left( x, \frac{\eta}{x} \right) &= \lim_{\eta \rightarrow 0} \frac{\mathcal{H}_{h/g}^{\text{pole}}(x, \eta) + \mathcal{H}_{h/g}^{\text{pole}}(x, -\eta)}{1 - \eta^2} \\ &= 2T_R [x^2 + (1 - x)^2]. \end{aligned} \quad (3.51)$$

Although the two terms  $\mathcal{H}_{h/g}^{\text{pole}}(x, \pm\eta)$  are individually singular as  $\eta \rightarrow 0$ , their sum has a finite limit; the forward result must therefore be taken after combining the two contributions. Using Eq. (3.44), this gives

$$\mathcal{A}_{h/g}^{[1]} \left( x, 0, \frac{\mu^2}{m_h^2} \right) = 2T_R [x^2 + (1 - x)^2] \ln \left( \frac{\mu^2}{m_h^2} \right), \quad (3.52)$$

in agreement with the corresponding one-loop PDF matching coefficient of Ref. [27].

The  $g/h$  channel provides a more stringent test, since its matching coefficient contains a finite, matching-scale-independent remainder in addition to the heavy-mass logarithm. Applying the same forward-limit procedure to Eq. (3.45) gives

$$\mathcal{A}_{g/h}^{[1]} \left( x, 0, \frac{\mu^2}{m_h^2} \right) = 2C_F \frac{1 + (1 - x)^2}{x} \left[ \ln \left( \frac{\mu^2}{m_h^2} \right) - 2 \ln x - 1 \right]. \quad (3.53)$$

This reproduces the heavy-quark-initiated PDF matching coefficient of Ref. [38]. This agreement thus provides a robust check on both the heavy-mass logarithm and the finite, matching-scale-independent remainder of the non-forward calculation.

For the diagonal  $h/h$  channel, the same procedure can be applied to the real contribution away from the endpoint  $x = 1$ . Using Eq. (3.34), one obtains

$$\mathcal{A}_{h/h}^{[1], \text{real}} \left( x, 0, \frac{\mu^2}{m_h^2} \right) = -2C_F \frac{1 + x^2}{1 - x} \left[ \ln \left( \frac{(1 - x)^2 m_h^2}{\mu^2} \right) + 1 \right]. \quad (3.54)$$

This agrees with the real part of the corresponding one-loop heavy-quark-initiated PDF matching coefficient given in Ref. [38]. A complete forward-limit check of the diagonal  $h/h$  matching coefficient, however, also requires the virtual contribution localised at  $x = 1$ . Since this contribution has not been determined in the present calculation, only the real part of the  $h/h$  forward limit can be explicitly verified.

The  $g/g$  channel does not provide an independent forward-limit check. As discussed in Sec. 3.1.2, the heavy-quark contribution is purely virtual and its non-forward dependence is entirely contained in the tree-level factor  $D_g(\eta)$ , which cancels when extracting the matching coefficient. The resulting coefficient is independent of  $\eta$  and reduces trivially to the forward result. Since  $B_{g/g}^{[1]}$  is taken directly from the known massive PDF calculation of Ref. [27], this agreement reflects a structural consistency rather than an independent check.

The status of these consistency checks is summarised in Tab. 3.1. Taken together, they provide a non-trivial validation of the extension of the heavy-flavour matching calculation to non-zero skewness.

| Channel | UV-mass-log check                 | Forward-limit check |
|---------|-----------------------------------|---------------------|
| $h/g$   | complete                          | complete            |
| $g/h$   | complete                          | complete            |
| $h/h$   | real contribution                 | real contribution   |
| $g/g$   | inherited from the forward result | structural          |

**Table 3.1:** Summary of the consistency checks performed on the one-loop heavy-quark matching coefficients.

# Conclusion

The aim of this thesis was to investigate the one-loop matching conditions required for the evolution of GPDs across heavy-quark mass thresholds. The analysis was formulated in terms of parton-in-parton GPDs. The massless one-loop calculation was first carried out to recover the known leading-order evolution kernels and to validate the conventions and renormalisation procedure adopted throughout this work. This approach was then extended to include a finite heavy-quark mass in order to determine the contributions entering the threshold matching.

The calculation was carried out in the heavy-quark–gluon sector. The off-diagonal matching coefficients were completely determined at one loop. Their logarithmic dependence on the heavy-quark mass is controlled by the corresponding leading-order evolution kernels, while the massive calculation generates additional finite terms not fixed by the evolution alone. In the diagonal heavy-quark channel, the real contribution and its distributional structure were fully derived, whereas the finite virtual contribution remains to be evaluated. The heavy-quark contribution to the diagonal gluon channel is instead purely virtual at this order and reduces to the corresponding ordinary PDF result after the non-forward dependence is factorised. The present calculation therefore completes the off-diagonal sector and identifies the remaining contribution needed to complete the one-loop heavy-quark–gluon matching.

The resulting expressions satisfy the expected consistency checks. Their logarithmic structure agrees with that dictated by GPD evolution, and in the forward limit the calculated contributions reproduce the established heavy-flavour matching results for ordinary PDFs. These checks support the consistency of extending the flavour-matching framework from forward to non-forward kinematics.

Taken together, these results show that the extension of heavy-flavour matching from PDFs to GPDs preserves the underlying evolution structure while introducing genuinely non-forward information via finite matching terms. They therefore provide important analytical ingredients for a consistent treatment of heavy-quark mass thresholds in GPD evolution. A natural continuation of this work is the explicit evaluation of the remaining finite virtual contribution in the diagonal heavy-quark channel. Once completed, these full one-loop matching conditions can be integrated with fixed-flavour GPD evolution into a numerical implementation across heavy-quark thresholds. This will allow the impact of heavy-flavour effects to be assessed quantitatively in phenomenological studies of hard exclusive processes.



## Appendix A

# Light-cone coordinates and high-energy kinematics

Light-cone coordinates provide a particularly convenient description of high-energy processes in which a large momentum singles out a preferred spatial direction. In the context of deep-inelastic and hard exclusive scattering, this direction is typically chosen along the  $z$ -axis, where the target hadron is taken to carry a large momentum. In this appendix, we introduce the light-cone conventions used throughout this work and discuss their connection with the infinite-momentum frame and the leading-twist collinear description of partonic dynamics.

### A.1 Light-cone coordinates

For a generic four-vector

$$v^\mu = (v^0, v^1, v^2, v^3), \quad (\text{A.1})$$

we define the light-cone components as [6]

$$v^+ \equiv \frac{v^0 + v^3}{\sqrt{2}}, \quad v^- \equiv \frac{v^0 - v^3}{\sqrt{2}}, \quad \mathbf{v}_T \equiv (v^1, v^2). \quad (\text{A.2})$$

It is convenient to introduce the two light-like vectors

$$n_+^\mu = \frac{1}{\sqrt{2}}(1, 0, 0, 1), \quad n_-^\mu = \frac{1}{\sqrt{2}}(1, 0, 0, -1), \quad (\text{A.3})$$

which satisfy

$$n_+^2 = n_-^2 = 0, \quad n_+ \cdot n_- = 1. \quad (\text{A.4})$$

A four-vector can then be decomposed as

$$v^\mu = v^+ n_+^\mu + v^- n_-^\mu + v_T^\mu, \quad (\text{A.5})$$

with

$$v^+ = v \cdot n_-, \quad v^- = v \cdot n_+, \quad v_T^\mu = (0, \mathbf{v}_T, 0). \quad (\text{A.6})$$

With the metric convention  $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ , the scalar product of two four-vectors  $v$  and  $w$  takes the form

$$v \cdot w = v^+ w^- + v^- w^+ - \mathbf{v}_T \cdot \mathbf{w}_T, \quad (\text{A.7})$$

which yields for the invariant mass squared

$$v^2 = 2v^+ v^- - \mathbf{v}_T^2. \quad (\text{A.8})$$

## A.2 Fast-moving hadrons and infinite-momentum frame

The advantages of light-cone coordinates become particularly transparent when considering a hadron moving with a large positive momentum along the  $z$ -direction. The physical picture is most intuitive in such a frame and, in particular, in the infinite-momentum frame (IMF), where the longitudinal hadron momentum becomes arbitrarily large, i.e.,  $p^3 \rightarrow \infty$  [6, 29].

Consider a particle of mass  $M$  and transverse momentum  $\mathbf{p}_T$ . Its on-shell condition,

$$p^2 = M^2, \quad (\text{A.9})$$

can be written in light-cone coordinates as

$$2p^+ p^- - \mathbf{p}_T^2 = M^2, \quad (\text{A.10})$$

and hence

$$p^- = \frac{M^2 + \mathbf{p}_T^2}{2p^+}. \quad (\text{A.11})$$

For a particle moving rapidly along the  $+z$ -direction, i.e.,

$$p^3 \gg M, \quad |\mathbf{p}_T| \ll p^3, \quad (\text{A.12})$$

one has

$$p^+ \simeq \sqrt{2} p^3, \quad p^- \simeq \frac{M^2 + \mathbf{p}_T^2}{2p^+} \ll p^+. \quad (\text{A.13})$$

In light-cone coordinates, the four-momentum is therefore dominated by its plus component,

$$p^\mu \simeq p^+ n_+^\mu, \quad (\text{A.14})$$

up to corrections suppressed by the large longitudinal momentum. Thus, for a highly energetic hadron moving in the  $+z$ -direction, the ordinary longitudinal momentum  $p^3$  is, at leading power in the large momentum expansion, equivalently encoded in the light-cone plus component. In this sense,  $p^+$  is the natural variable to describe the large longitudinal

momentum of the hadron. Since  $p^+ = p \cdot n_-$ , the dominant longitudinal component is obtained by projecting the four-momentum onto the conjugate light-like direction  $n_-$ , defined in Eq. (A.3) and given, in light-cone components, by

$$n_-^\mu = (0, 1, \mathbf{0}_T). \quad (\text{A.15})$$

The IMF can be regarded as the limiting realisation of this kinematical hierarchy. In the limit  $p^3 \rightarrow +\infty$ , with  $M$  and  $\mathbf{p}_T$  kept fixed, one has

$$p^+ \rightarrow \infty, \quad p^- \rightarrow 0, \quad \frac{|\mathbf{p}_T|}{p^+} \rightarrow 0. \quad (\text{A.16})$$

Hence, the dominance of the plus component becomes asymptotically exact, and the direction of the hadron momentum approaches the light-like direction  $n_+$ ,

$$\frac{p^\mu}{p^+} \longrightarrow n_+^\mu. \quad (\text{A.17})$$

The role of the IMF is mainly interpretative: it makes the physical meaning of the light-cone variables particularly transparent, but the definition of GPDs is not restricted to this specific reference frame. Once expressed in terms of light-cone components and Lorentz-invariant quantities, the formalism can be applied in any convenient reference frame [6].

A further useful property of light-cone variables is their simple transformation behaviour under longitudinal Lorentz boosts. Under a boost of rapidity  $\omega$  along the  $z$ -axis, the components transform as

$$p^+ \longrightarrow e^\omega p^+, \quad p^- \longrightarrow e^{-\omega} p^-, \quad \mathbf{p}_T \longrightarrow \mathbf{p}_T. \quad (\text{A.18})$$

Therefore, ratios of plus components are invariant under longitudinal boosts. In particular, the longitudinal momentum fraction

$$x = \frac{k^+}{P^+} \quad (\text{A.19})$$

is boost-invariant, since both  $k^+$  and  $P^+$  scale with the same exponential factor  $e^\omega$ . This makes  $x$  a natural kinematic variable for describing the fraction of the large longitudinal momentum carried by a parton.

### A.3 Collinear partonic kinematics

The hierarchy among the light-cone momentum components discussed for a fast-moving hadron also characterizes the partonic momenta in the collinear sector associated with the hadron. Let the parent hadron carry a large plus momentum  $P^+$ . A generic parton momentum can be decomposed as

$$k^\mu = k^+ n_+^\mu + k^- n_-^\mu + k_T^\mu. \quad (\text{A.20})$$

Its plus component is conventionally expressed in terms of a longitudinal momentum fraction  $x$ ,

$$k^+ = xP^+. \quad (\text{A.21})$$

For a parton collinear to a fast-moving hadron, the intrinsic transverse momentum and virtuality are associated with non-hard scales. These can be characterised by a typical hadronic scale  $\Lambda_{\text{had}}$  [6, 23], so that schematically

$$|\mathbf{k}_T| \sim \Lambda_{\text{had}}, \quad |k^2| \sim \Lambda_{\text{had}}^2, \quad (\text{A.22})$$

while the plus momentum of the parton is large compared with the hadronic scale,

$$k^+ = xP^+ \gg \Lambda_{\text{had}}. \quad (\text{A.23})$$

Using the light-cone relation

$$k^2 = 2k^+k^- - \mathbf{k}_T^2, \quad (\text{A.24})$$

one finds schematically

$$k^- \sim \frac{\Lambda_{\text{had}}^2}{k^+} = \frac{\Lambda_{\text{had}}^2}{xP^+}. \quad (\text{A.25})$$

The different light-cone components therefore obey the kinematic hierarchy

$$\frac{|\mathbf{k}_T|}{k^+} \sim \frac{\Lambda_{\text{had}}}{xP^+} \ll 1, \quad \frac{k^-}{k^+} \sim \frac{\Lambda_{\text{had}}^2}{x^2(P^+)^2} \ll 1. \quad (\text{A.26})$$

The momentum of a collinear parton is therefore dominated by its plus component and is approximately aligned with the direction of the parent hadron,

$$k^\mu \simeq k^+ n_+^\mu = xP^+ n_+^\mu. \quad (\text{A.27})$$

Within the collinear description of a hard process characterised by a large momentum transfer  $Q^2 \gg \Lambda_{\text{had}}^2$ , the dependence of the hard-scattering subprocess on the small minus and transverse components of the parton momentum gives corrections suppressed by powers of the ratio  $\Lambda_{\text{had}}/Q$ . In the Bjorken limit, these sub-leading corrections become negligible, so that, at leading power in  $\Lambda_{\text{had}}/Q$ , the hard subprocess depends only on the collinear projection given in Eq. (A.27). The components  $k^-$  and  $\mathbf{k}_T$  are not assumed to vanish inside the hadron; rather, they are not resolved as independent variables by the hard probe and are integrated over in the definition of collinear parton distributions.

In coordinate space, the same collinear framework is formulated in terms of bilocal operators separated along a light-like direction. A Fourier transform with respect to this separation provides the representation of the distribution in terms of the conjugate longitudinal momentum variable. For GPDs, this is the average plus momentum of the active parton,

$$k_{\text{av}}^+ \equiv \frac{k_{\text{in}}^+ + k_{\text{out}}^+}{2} = xP^+, \quad (\text{A.28})$$

where  $x$  denotes the corresponding longitudinal momentum fraction.

The coordinate variable conjugate to this plus momentum is the minus light-cone separation. Accordingly, with the conventions used in Chap. 1, the separation vector entering the GPD operator is written as

$$z^\mu = y n^\mu, \quad n^\mu = n_-^\mu, \quad (\text{A.29})$$

so that

$$z^+ = 0, \quad z_T = 0, \quad z^2 = 0. \quad (\text{A.30})$$

The Fourier duality between the longitudinal momentum and the field separation can be made explicit by writing, at leading power,

$$k_{\text{av}}^\mu \simeq x P^+ n_+^\mu, \quad z^\mu = y n_-^\mu. \quad (\text{A.31})$$

Since  $n_+ \cdot n_- = 1$ , one obtains

$$k_{\text{av}} \cdot z = x P^+ y = x(P \cdot n)y, \quad n = n_-, \quad (\text{A.32})$$

where  $P \cdot n = P^+$ . Hence, the average plus momentum  $xP^+$  and the minus light-cone separation  $y$  form the Fourier-conjugate pair entering the GPD definition. Schematically, the quark GPD correlator takes the form

$$F^q(x, \xi, t) \sim \int dy e^{-ix(P \cdot n)y} \langle p' | \bar{\psi} \left( \frac{yn}{2} \right) \Gamma \mathcal{W} \psi \left( -\frac{yn}{2} \right) | p \rangle, \quad n = n_-, \quad (\text{A.33})$$

with the precise Dirac structure and normalisation given in Eq. (1.26).

The appearance of the light-like vector  $n = n_-$  in the GPD definition is therefore directly related to the leading-twist collinear description. The longitudinal momentum variable retained in the collinear description is the average plus momentum  $xP^+$ ; correspondingly, in coordinate space the bilocal operator defining the correlator is separated along the conjugate minus light-cone direction  $n_-^\mu$ .



## Appendix B

# Parton-in-parton GPDs: technical details

This appendix derives the fundamental field-operator identities and tree-level expressions for unpolarised parton-in-parton GPDs. First, we establish an identity recasting the gluon-in-parton correlator directly in terms of transverse gauge fields in light-cone gauge. Second, by evaluating the matrix elements over free asymptotic particle states, we compute the tree-level quark-in-quark and gluon-in-gluon GPDs, showing how their leading-twist momentum dependence collapses onto the tree-level delta distribution  $\delta(1 - x)$  multiplied by explicit kinematic factors in the skewness.

### B.1 Gluon operator in the light-cone gauge

In this section, we derive the relation used in Eq. (1.117) to express the parton-in-parton GPDs involving the gluon operator directly in terms of gluon fields. For  $k = g, q$ , the corresponding bare parton-in-parton GPD can be written in the compact form

$$\begin{aligned} \hat{F}_{g/k}(x, \eta) &= \frac{n_\mu n_\nu}{\mathcal{N}_k x(n \cdot p)} \int \frac{dy}{2\pi} e^{-ix(n \cdot p)y} \\ &\times \sum_s \text{Tr} \left[ \left\langle p_{\text{out}}, s \left| F_a^{\mu j} \left( \frac{yn}{2} \right) F_a^{\nu j} \left( -\frac{yn}{2} \right) \right| p_{\text{in}}, s \right\rangle_k \right], \end{aligned} \quad (\text{B.1})$$

where the incoming and outgoing momenta are parameterised as

$$p_{\text{in}} = (1 + \eta)p, \quad p_{\text{out}} = (1 - \eta)p, \quad (\text{B.2})$$

and the colour normalisation factors are

$$\mathcal{N}_g = 2(N_c^2 - 1), \quad \mathcal{N}_q = 2N_c. \quad (\text{B.3})$$

In the light-cone gauge  $n \cdot A = 0$ , the contraction of the field-strength tensor with the light-like vector  $n^\mu$  simplifies to a directional derivative of the transverse gauge potential:

$$n_\mu F_a^{\mu j}(z) = (n \cdot \partial) A_a^j(z). \quad (\text{B.4})$$

Therefore, Eq. (B.1) becomes

$$\begin{aligned} \widehat{F}_{g/k}(x, \eta) &= \frac{1}{\mathcal{N}_k x(n \cdot p)} \int \frac{dy}{2\pi} e^{-ix(n \cdot p)y} \\ &\times \sum_s \text{Tr} \left[ \left\langle p_{\text{out}}, s \left| (n \cdot \partial) A_a^j \left( \frac{yn}{2} \right) (n \cdot \partial) A_a^j \left( -\frac{yn}{2} \right) \right| p_{\text{in}}, s \right\rangle_k \right]. \end{aligned} \quad (\text{B.5})$$

To determine the effect of the two directional derivatives, we insert a complete set of intermediate momentum eigenstates,

$$\mathbf{1} = \sum_X |X\rangle \langle X|, \quad (\text{B.6})$$

where the sum is understood to include the appropriate phase-space integrations. For compactness, we temporarily suppress the external spin label and the trace over internal indices. The matrix element in Eq. (B.5) can then be written as

$$\begin{aligned} &\left\langle p_{\text{out}} \left| (n \cdot \partial) A_a^j \left( \frac{yn}{2} \right) (n \cdot \partial) A_a^j \left( -\frac{yn}{2} \right) \right| p_{\text{in}} \right\rangle \\ &= \sum_X \left\langle p_{\text{out}} \left| (n \cdot \partial) A_a^j \left( \frac{yn}{2} \right) \right| X \right\rangle \left\langle X \left| (n \cdot \partial) A_a^j \left( -\frac{yn}{2} \right) \right| p_{\text{in}} \right\rangle. \end{aligned} \quad (\text{B.7})$$

Using space-time translational invariance,

$$A_a^j(z) = e^{i\widehat{P} \cdot z} A_a^j(0) e^{-i\widehat{P} \cdot z}, \quad (\text{B.8})$$

the two matrix elements without derivatives become

$$\left\langle p_{\text{out}} \left| A_a^j \left( \frac{yn}{2} \right) \right| X \right\rangle = e^{\frac{iy}{2} n \cdot (p_{\text{out}} - p_X)} \left\langle p_{\text{out}} \left| A_a^j(0) \right| X \right\rangle, \quad (\text{B.9})$$

$$\left\langle X \left| A_a^j \left( -\frac{yn}{2} \right) \right| p_{\text{in}} \right\rangle = e^{\frac{iy}{2} n \cdot (p_{\text{in}} - p_X)} \left\langle X \left| A_a^j(0) \right| p_{\text{in}} \right\rangle. \quad (\text{B.10})$$

The product of the two translational phase factors simplifies to

$$\begin{aligned} &e^{\frac{iy}{2} n \cdot (p_{\text{out}} - p_X)} e^{\frac{iy}{2} n \cdot (p_{\text{in}} - p_X)} \\ &= e^{\frac{iy}{2} n \cdot (p_{\text{out}} + p_{\text{in}} - 2p_X)} = e^{iy n \cdot (p - p_X)}, \end{aligned} \quad (\text{B.11})$$

where we used the average momentum relation

$$p_{\text{out}} + p_{\text{in}} = 2p. \quad (\text{B.12})$$

Acting with the derivative operators  $(n \cdot \partial)$  brings down explicit momentum factors:

$$\left\langle p_{\text{out}} \left| (n \cdot \partial) A_a^j \left( \frac{yn}{2} \right) \right| X \right\rangle = i n \cdot (p_{\text{out}} - p_X) \left\langle p_{\text{out}} \left| A_a^j \left( \frac{yn}{2} \right) \right| X \right\rangle, \quad (\text{B.13})$$

$$\left\langle X \left| (n \cdot \partial) A_a^j \left( -\frac{yn}{2} \right) \right| p_{\text{in}} \right\rangle = -i n \cdot (p_{\text{in}} - p_X) \left\langle X \left| A_a^j \left( -\frac{yn}{2} \right) \right| p_{\text{in}} \right\rangle. \quad (\text{B.14})$$

Therefore, Eq. (B.7) becomes

$$\begin{aligned} & \left\langle p_{\text{out}} \left| (n \cdot \partial) A_a^j \left( \frac{yn}{2} \right) (n \cdot \partial) A_a^j \left( -\frac{yn}{2} \right) \right| p_{\text{in}} \right\rangle \\ &= \sum_X [n \cdot (p_{\text{out}} - p_X)] [n \cdot (p_{\text{in}} - p_X)] \\ & \times \left\langle p_{\text{out}} \left| A_a^j \left( \frac{yn}{2} \right) \right| X \right\rangle \left\langle X \left| A_a^j \left( -\frac{yn}{2} \right) \right| p_{\text{in}} \right\rangle. \end{aligned} \quad (\text{B.15})$$

We now consider the Fourier integral appearing in Eq. (B.5). Using Eq. (B.11), the dependence on the light-cone separation  $y$  is

$$e^{-ix(n \cdot p)y} e^{iy n \cdot (p - p_X)} = e^{iy[(1-x)(n \cdot p) - n \cdot p_X]}. \quad (\text{B.16})$$

Consequently,

$$\int \frac{dy}{2\pi} e^{iy[(1-x)(n \cdot p) - n \cdot p_X]} = \delta((1-x)(n \cdot p) - n \cdot p_X). \quad (\text{B.17})$$

The Dirac delta function fixes the longitudinal component of the intermediate state  $X$  to

$$n \cdot p_X = (1-x)(n \cdot p). \quad (\text{B.18})$$

Using Eq. (B.2), the two momentum factors in Eq. (B.15) simplify to:

$$n \cdot (p_{\text{out}} - p_X) = (x - \eta)(n \cdot p), \quad (\text{B.19})$$

$$n \cdot (p_{\text{in}} - p_X) = (x + \eta)(n \cdot p). \quad (\text{B.20})$$

Their product is therefore independent of  $p_X$ :

$$\begin{aligned} & n \cdot (p_{\text{out}} - p_X) n \cdot (p_{\text{in}} - p_X) \\ &= (x - \eta)(x + \eta)(n \cdot p)^2 = (x^2 - \eta^2)(n \cdot p)^2. \end{aligned} \quad (\text{B.21})$$

Substituting Eqs. (B.17) and (B.21) into the Fourier transform of Eq. (B.15) yields

$$\begin{aligned} & \int \frac{dy}{2\pi} e^{-ix(n \cdot p)y} \left\langle p_{\text{out}} \left| (n \cdot \partial) A_a^j \left( \frac{yn}{2} \right) (n \cdot \partial) A_a^j \left( -\frac{yn}{2} \right) \right| p_{\text{in}} \right\rangle \\ &= \sum_X [n \cdot (p_{\text{out}} - p_X)] [n \cdot (p_{\text{in}} - p_X)] \\ & \times \delta((1-x)(n \cdot p) - n \cdot p_X) \left\langle p_{\text{out}} \left| A_a^j(0) \right| X \right\rangle \left\langle X \left| A_a^j(0) \right| p_{\text{in}} \right\rangle. \end{aligned} \quad (\text{B.22})$$

On the support of the delta function, Eq. (B.21) applies, so that

$$\begin{aligned} & \int \frac{dy}{2\pi} e^{-ix(n \cdot p)y} \left\langle p_{\text{out}} \left| (n \cdot \partial) A_a^j \left( \frac{yn}{2} \right) (n \cdot \partial) A_a^j \left( -\frac{yn}{2} \right) \right| p_{\text{in}} \right\rangle \\ &= (x^2 - \eta^2)(n \cdot p)^2 \sum_X \delta((1-x)(n \cdot p) - n \cdot p_X) \\ & \times \left\langle p_{\text{out}} \left| A_a^j(0) \right| X \right\rangle \left\langle X \left| A_a^j(0) \right| p_{\text{in}} \right\rangle. \end{aligned} \quad (\text{B.23})$$

Writing the delta distribution again in its Fourier representation gives

$$\begin{aligned}
 & \int \frac{dy}{2\pi} e^{-ix(n \cdot p)y} \left\langle p_{\text{out}} \left| (n \cdot \partial) A_a^j \left( \frac{yn}{2} \right) (n \cdot \partial) A_a^j \left( -\frac{yn}{2} \right) \right| p_{\text{in}} \right\rangle \\
 &= (x^2 - \eta^2)(n \cdot p)^2 \int \frac{dy}{2\pi} e^{-ix(n \cdot p)y} \sum_X e^{iy n \cdot (p - p_X)} \\
 & \quad \times \left\langle p_{\text{out}} \left| A_a^j(0) \right| X \right\rangle \left\langle X \left| A_a^j(0) \right| p_{\text{in}} \right\rangle.
 \end{aligned} \tag{B.24}$$

Reabsorbing the translation phases via Eqs. (B.9) and (B.10),

$$\begin{aligned}
 & e^{iy n \cdot (p - p_X)} \left\langle p_{\text{out}} \left| A_a^j(0) \right| X \right\rangle \left\langle X \left| A_a^j(0) \right| p_{\text{in}} \right\rangle \\
 &= \left\langle p_{\text{out}} \left| A_a^j \left( \frac{yn}{2} \right) \right| X \right\rangle \left\langle X \left| A_a^j \left( -\frac{yn}{2} \right) \right| p_{\text{in}} \right\rangle.
 \end{aligned} \tag{B.25}$$

Hence, Eq. (B.24) becomes

$$\begin{aligned}
 & \int \frac{dy}{2\pi} e^{-ix(n \cdot p)y} \left\langle p_{\text{out}} \left| (n \cdot \partial) A_a^j \left( \frac{yn}{2} \right) (n \cdot \partial) A_a^j \left( -\frac{yn}{2} \right) \right| p_{\text{in}} \right\rangle \\
 &= (x^2 - \eta^2)(n \cdot p)^2 \int \frac{dy}{2\pi} e^{-ix(n \cdot p)y} \sum_X \left\langle p_{\text{out}} \left| A_a^j \left( \frac{yn}{2} \right) \right| X \right\rangle \left\langle X \left| A_a^j \left( -\frac{yn}{2} \right) \right| p_{\text{in}} \right\rangle.
 \end{aligned} \tag{B.26}$$

Finally, from the completeness relation in Eq. (B.6), we obtain

$$\begin{aligned}
 & \int \frac{dy}{2\pi} e^{-ix(n \cdot p)y} \left\langle p_{\text{out}} \left| (n \cdot \partial) A_a^j \left( \frac{yn}{2} \right) (n \cdot \partial) A_a^j \left( -\frac{yn}{2} \right) \right| p_{\text{in}} \right\rangle \\
 &= (x^2 - \eta^2)(n \cdot p)^2 \int \frac{dy}{2\pi} e^{-ix(n \cdot p)y} \left\langle p_{\text{out}} \left| A_a^j \left( \frac{yn}{2} \right) A_a^j \left( -\frac{yn}{2} \right) \right| p_{\text{in}} \right\rangle.
 \end{aligned} \tag{B.27}$$

Restoring the external spin sum and the traces over the relevant internal indices, and substituting Eq. (B.27) into Eq. (B.5), we arrive at the identity quoted in Eq. (1.117):

$$\begin{aligned}
 \hat{F}_{g/k}(x, \eta) &= \frac{(n \cdot p)(x^2 - \eta^2)}{\mathcal{N}_k x} \int \frac{dy}{2\pi} e^{-ix(n \cdot p)y} \\
 & \quad \times \sum_s \text{Tr} \left[ \left\langle p_{\text{out}}, s \left| A_a^j \left( \frac{yn}{2} \right) A_a^j \left( -\frac{yn}{2} \right) \right| p_{\text{in}}, s \right\rangle_k \right].
 \end{aligned} \tag{B.28}$$

## B.2 Tree-level massless parton-in-parton GPDs

In this section, we explicitly derive the tree-level parton-in-parton GPDs introduced in Sec. 1.4.2. At this order, no QCD interaction connects the bilocal field operators to external partonic states. Therefore, only the diagonal quark-in-quark and gluon-in-gluon distributions are non-vanishing.

### B.2.1 Quark-in-quark GPD

We start from the definition of the quark-in-quark GPD:

$$\begin{aligned} \hat{F}_{q/q}(x, \eta) &= \frac{1}{2N_c} \int \frac{dy}{2\pi} e^{-ix(n \cdot p)y} \\ &\times \sum_s \text{Tr} \left[ \left\langle (1-\eta)p, s \left| \bar{\psi}_q \left( \frac{yn}{2} \right) \frac{\not{y}}{2} \psi_q \left( -\frac{yn}{2} \right) \right| (1+\eta)p, s \right\rangle_q \right]. \end{aligned} \quad (\text{B.29})$$

At tree level, only the asymptotic free-field components of the interacting fields contribute to the matrix element. The interacting quark fields can therefore be replaced by their free counterparts when acting on the external quark states. The action of the free fields on the corresponding asymptotic states is given by

$$\psi_q^{(0)}(z) |k, s\rangle_q = e^{-ik \cdot z} u_{q,s}(k) |0\rangle, \quad (\text{B.30})$$

$${}_q \langle k, s | \bar{\psi}_q^{(0)}(z) = \langle 0 | \bar{u}_{q,s}(k) e^{ik \cdot z}, \quad (\text{B.31})$$

where  $u_{q,s}(k)$  is the quark spinor of flavour  $q$ , momentum  $k$ , and spin  $s$ , while  $\bar{u}_{q,s}(k)$  denotes the corresponding Dirac-conjugate spinor.

Applying these relations to the incoming and outgoing states appearing in the quark-in-quark GPD, one obtains

$$\psi_q^{(0)} \left( -\frac{yn}{2} \right) \left| (1+\eta)p, s \right\rangle_q = e^{i(1+\eta)p \cdot \frac{yn}{2}} u_{q,s}((1+\eta)p) |0\rangle, \quad (\text{B.32})$$

$${}_q \langle (1-\eta)p, s | \bar{\psi}_q^{(0)} \left( \frac{yn}{2} \right) = \langle 0 | \bar{u}_{q,s}((1-\eta)p) e^{i(1-\eta)p \cdot \frac{yn}{2}}. \quad (\text{B.33})$$

The matrix element evaluates to

$$\begin{aligned} &{}_q \langle (1-\eta)p, s | \bar{\psi}_q^{(0)} \left( \frac{yn}{2} \right) \frac{\not{y}}{2} \psi_q^{(0)} \left( -\frac{yn}{2} \right) | (1+\eta)p, s \rangle_q \\ &= e^{i(1-\eta)p \cdot \frac{yn}{2}} e^{i(1+\eta)p \cdot \frac{yn}{2}} \bar{u}_{q,s}((1-\eta)p) \frac{\not{y}}{2} u_{q,s}((1+\eta)p) \\ &= e^{i(n \cdot p)y} \bar{u}_{q,s}((1-\eta)p) \frac{\not{y}}{2} u_{q,s}((1+\eta)p). \end{aligned} \quad (\text{B.34})$$

Substituting this result into Eq. (B.29), one gets

$$\begin{aligned} \hat{F}_{q/q}(x, \eta) &= \frac{1}{2N_c} \int \frac{dy}{2\pi} e^{i(1-x)(n \cdot p)y} \\ &\times \sum_s \text{Tr} \left[ \bar{u}_{q,s}((1-\eta)p) \frac{\not{y}}{2} u_{q,s}((1+\eta)p) \right]. \end{aligned} \quad (\text{B.35})$$

The spin sum can be performed using the completeness relation for quark spinors,

$$\sum_s u_{q,s}((1+\eta)p) \bar{u}_{q,s}((1-\eta)p) = \sqrt{1-\eta^2} \not{p}. \quad (\text{B.36})$$

Making the colour and Dirac traces explicit, one finds

$$\sum_s \text{Tr} \left[ \bar{u}_{q,s}((1-\eta)p) \frac{\not{p}}{2} u_{q,s}((1+\eta)p) \right] = \sqrt{1-\eta^2} \text{Tr}_c(I_c) \text{Tr}_D \left[ \frac{\not{p}}{2} \right]. \quad (\text{B.37})$$

The colour trace gives

$$\text{Tr}_c(I_c) = N_c, \quad (\text{B.38})$$

while the Dirac trace can be evaluated using

$$\text{Tr}_D(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}, \quad (\text{B.39})$$

which gives

$$\text{Tr}_D \left( \frac{\not{p}}{2} \right) = 2 n \cdot p. \quad (\text{B.40})$$

In Eq. (B.35), the colour trace cancels the factor  $1/N_c$ , while the remaining factor  $1/2$  combines with the Dirac trace. Hence,

$$\hat{F}_{q/q}(x, \eta) = \sqrt{1-\eta^2} (n \cdot p) \int \frac{dy}{2\pi} e^{i(1-x)(n \cdot p)y}. \quad (\text{B.41})$$

The Fourier integral evaluates to

$$\int \frac{dy}{2\pi} e^{i(1-x)(n \cdot p)y} = \delta((1-x)(n \cdot p)). \quad (\text{B.42})$$

Using the standard scaling property of the Dirac delta distribution,

$$\delta(az) = \frac{1}{|a|} \delta(z), \quad (\text{B.43})$$

together with the conventional choice  $n \cdot p > 0$ , one obtains

$$\delta((1-x)(n \cdot p)) = \frac{1}{n \cdot p} \delta(1-x). \quad (\text{B.44})$$

The prefactor  $(n \cdot p)$  cancels out completely, giving the leading-twist tree-level result:

$$\hat{F}_{q/q}^{[0]}(x, \eta) = \sqrt{1-\eta^2} \delta(1-x). \quad (\text{B.45})$$

## B.2.2 Gluon-in-gluon GPD

Using the gauge field formulation derived in Eq. (B.28), the gluon-in-gluon GPD is expressed as

$$\begin{aligned} \hat{F}_{g/g}(x, \eta) &= \frac{(n \cdot p)(x^2 - \eta^2)}{2(N_c^2 - 1)x} \int \frac{dy}{2\pi} e^{-ix(n \cdot p)y} \\ &\times \sum_s \text{Tr} \left[ \left\langle (1-\eta)p, s \left| A_a^j \left( \frac{yn}{2} \right) A_a^j \left( -\frac{yn}{2} \right) \right| (1+\eta)p, s \right\rangle_g \right]. \end{aligned} \quad (\text{B.46})$$

At tree level, only the asymptotic free-field components of the interacting gluon fields contribute to the matrix element. The action of the free gluon field on the corresponding asymptotic states is given by

$$A_a^{j(0)}(z) |k, s\rangle_g = e^{-ik \cdot z} e_{a,s}^j(k) |0\rangle, \quad (\text{B.47})$$

$${}_g\langle k, s| A_a^{j(0)}(z) = \langle 0| e_{a,s}^{j*}(k) e^{ik \cdot z}, \quad (\text{B.48})$$

where  $e_{a,s}^j(k)$  denotes the  $j$ -th transverse component of the gluon polarisation vector with momentum  $k$ , colour index  $a$ , and helicity  $s$ .

Applying these relations to the incoming and outgoing gluon states, one obtains

$$A_a^{j(0)}\left(-\frac{yn}{2}\right) \left| (1+\eta)p, s \right\rangle_g = e^{i(1+\eta)p \cdot \frac{yn}{2}} e_{a,s}^j((1+\eta)p) |0\rangle, \quad (\text{B.49})$$

$${}_g\left\langle (1-\eta)p, s \right| A_a^{j(0)}\left(\frac{yn}{2}\right) = \langle 0| e_{a,s}^{j*}((1-\eta)p) e^{i(1-\eta)p \cdot \frac{yn}{2}}. \quad (\text{B.50})$$

Hence,

$$\begin{aligned} & {}_g\left\langle (1-\eta)p, s \right| A_a^{j(0)}\left(\frac{yn}{2}\right) A_a^{j(0)}\left(-\frac{yn}{2}\right) \left| (1+\eta)p, s \right\rangle_g \\ &= e^{i(1-\eta)p \cdot \frac{yn}{2}} e^{i(1+\eta)p \cdot \frac{yn}{2}} e_{a,s}^{j*}((1-\eta)p) e_{a,s}^j((1+\eta)p) \\ &= e^{i(n \cdot p)y} e_{a,s}^{j*}((1-\eta)p) e_{a,s}^j((1+\eta)p). \end{aligned} \quad (\text{B.51})$$

Substituting this result into Eq. (B.46), one obtains

$$\begin{aligned} \hat{F}_{g/g}(x, \eta) &= \frac{(n \cdot p)(x^2 - \eta^2)}{2(N_c^2 - 1)x} \int \frac{dy}{2\pi} e^{i(1-x)(n \cdot p)y} \\ &\quad \times \sum_s \text{Tr} \left[ e_{a,s}^{j*}((1-\eta)p) e_{a,s}^j((1+\eta)p) \right]. \end{aligned} \quad (\text{B.52})$$

The sum over the external gluon helicities can be performed using the orthogonality relation for physical gluon polarisation vectors,

$$\begin{aligned} \sum_s e_{a,s}^{i*}((1-\eta)p) e_{a,s}^j((1+\eta)p) &= -g_{\perp}^{ij} = \delta^{ij}, \\ \sum_s e_{a,s}^{j*}((1-\eta)p) e_{a,s}^i((1+\eta)p) &= \delta_{ij} \sum_s e_{a,s}^{i*}((1-\eta)p) e_{a,s}^j((1+\eta)p) \\ &= \delta_{ij} \delta^{ij} = 2, \end{aligned} \quad (\text{B.53})$$

where, in the last equality, the repeated transverse indices  $i, j$  are summed over the two transverse directions,  $i, j = 1, 2$ . The colour trace in the adjoint representation gives

$$\text{Tr}_c(I_{\text{adj}}) = N_c^2 - 1. \quad (\text{B.54})$$

The polarisation and colour factors therefore cancel the normalisation factor  $2(N_c^2 - 1)$  in Eq. (B.52), leaving

$$\hat{F}_{g/g}(x, \eta) = \frac{(n \cdot p)(x^2 - \eta^2)}{x} \int \frac{dy}{2\pi} e^{i(1-x)(n \cdot p)y}. \quad (\text{B.55})$$

As in the quark case, the Fourier integral evaluates to

$$\int \frac{dy}{2\pi} e^{i(1-x)(n \cdot p)y} = \delta((1-x)(n \cdot p)) = \frac{1}{n \cdot p} \delta(1-x). \quad (\text{B.56})$$

Therefore,

$$\hat{F}_{g/g}(x, \eta) = \frac{x^2 - \eta^2}{x} \delta(1-x). \quad (\text{B.57})$$

Since the Dirac delta distribution fixes  $x = 1$ , the prefactor can be evaluated at  $x = 1$ :

$$\frac{x^2 - \eta^2}{x} \delta(1-x) = (1 - \eta^2) \delta(1-x). \quad (\text{B.58})$$

Thus, the final tree-level gluon-in-gluon GPD takes the concise form:

$$\hat{F}_{g/g}^{[0]}(x, \eta) = (1 - \eta^2) \delta(1-x). \quad (\text{B.59})$$

## Appendix C

# One-loop massive parton-in-parton GPDs: calculation details

This appendix collects the details of the massive one-loop parton-in-parton calculations entering the results of Sec. 3.1. We consider the real contributions to the  $h/g$ ,  $g/h$ , and  $h/h$  channels, following the same operator definitions and light-cone-gauge conventions used for the massless calculation in Ref. [25].

### C.1 Heavy-quark-in-gluon GPD

We begin with the real one-loop contribution to the heavy-quark-in-gluon GPD. The external gluons remain massless, while all quark propagators in the loop carry the heavy-quark mass  $m_h$ . The external momenta are therefore

$$p_{\text{in}}^\mu = (1 + \eta)p^\mu, \quad p_{\text{out}}^\mu = (1 - \eta)p^\mu, \quad p^\mu = (p^+, 0, \mathbf{0}_T), \quad (\text{C.1})$$

with

$$p_{\text{in}}^2 = p_{\text{out}}^2 = 0. \quad (\text{C.2})$$

These are the same massless non-forward kinematics introduced in Eq. (1.103). The bare heavy-quark-in-gluon GPD is defined by

$$\begin{aligned} \hat{F}_{h/g}(x, \eta) &= \frac{1}{2(N_c^2 - 1)} \int \frac{dy}{2\pi} e^{-ix(n \cdot p)y} \\ &\times \sum_s \text{Tr} \left[ \left\langle p_{\text{out}}, s \left| \bar{\psi}_h \left( \frac{yn}{2} \right) \frac{\not{y}}{2} \psi_h \left( -\frac{yn}{2} \right) \right| p_{\text{in}}, s \right\rangle_g \right], \end{aligned} \quad (\text{C.3})$$

where  $n^\mu = (0, 1, \mathbf{0}_T)$ , so that  $n \cdot p = p^+$ . At tree level, this matrix element vanishes. The first non-vanishing contribution is therefore generated by the real diagram at  $\mathcal{O}(\alpha_s)$ , shown in Fig. 1.12.

Applying the Feynman rules in light-cone gauge and working in dimensional regularisation, the one-loop contribution reads

$$\begin{aligned}
 a_s \widehat{F}_{h/g}^{[1],m_h}(x, \eta) &= \frac{1}{2(N_c^2 - 1)} \int \frac{dy}{2\pi} e^{-ix(n \cdot p)y} \sum_s \varepsilon_b^{\nu*}(p_{\text{out}}, s) \varepsilon_a^\mu(p_{\text{in}}, s) \\
 &\times \int \frac{d^d k}{(2\pi)^d} e^{i(p-k) \cdot ny} \text{Tr} \left[ (-ig_s \mu^\epsilon \gamma_\mu t^a) \frac{i((1+\eta)\not{p} - \not{k} + m_h)}{((1+\eta)p - k)^2 - m_h^2 + i\epsilon} \right. \\
 &\times \frac{\not{p}}{2} \frac{i((1-\eta)\not{p} - \not{k} + m_h)}{((1-\eta)p - k)^2 - m_h^2 + i\epsilon} (-ig_s \mu^\epsilon \gamma_\nu t^b) \frac{i(\not{k} - m_h)}{k^2 - m_h^2 + i\epsilon} \left. \right],
 \end{aligned} \tag{C.4}$$

where  $d = 4 - 2\epsilon$  and  $a_s = \alpha_s/(4\pi) = g_s^2/(16\pi^2)$ .

The factor  $1/2$  in Eq. (C.3) denotes the usual average over the two physical gluon helicities. In dimensional regularisation, when evaluating one-loop diagrams, this average is continued to the  $d - 2$  transverse polarisation states,

$$\frac{1}{2} \sum_{s=1}^2 \varepsilon_a^\mu(p_{\text{in}}, s) \varepsilon_b^{\nu*}(p_{\text{out}}, s) \longrightarrow \frac{1}{d-2} \sum_{s=1}^{d-2} \varepsilon_a^\mu(p_{\text{in}}, s) \varepsilon_b^{\nu*}(p_{\text{out}}, s). \tag{C.5}$$

The corresponding polarisation sum is

$$\sum_{s=1}^{d-2} \varepsilon_a^\mu(p_{\text{in}}, s) \varepsilon_b^{\nu*}(p_{\text{out}}, s) = -\delta_{ab} g_{\perp}^{\mu\nu}, \tag{C.6}$$

so that

$$\frac{1}{d-2} \sum_{s=1}^{d-2} \varepsilon_a^\mu(p_{\text{in}}, s) \varepsilon_b^{\nu*}(p_{\text{out}}, s) = -\frac{\delta_{ab} g_{\perp}^{\mu\nu}}{d-2} = -\frac{\delta_{ab} g_{\perp}^{\mu\nu}}{2(1-\epsilon)}. \tag{C.7}$$

The Fourier integral over the light-cone separation  $y$  gives

$$\int \frac{dy}{2\pi} e^{i(1-x)p^+ y} e^{-ik^+ y} = \delta((1-x)p^+ - k^+). \tag{C.8}$$

Decomposing the  $d$ -dimensional loop measure into light-cone coordinates,

$$d^{4-2\epsilon} k = dk^+ dk^- d^{2-2\epsilon} \mathbf{k}_T, \tag{C.9}$$

the  $k^+$  integration can be performed directly using the delta function, which fixes

$$k^+ = (1-x)p^+. \tag{C.10}$$

At this stage, the colour and Dirac structures can be separated. Using the Kronecker delta arising from the polarisation sum and contracting colour indices via

$$\delta_{ab} \text{Tr}_c[t^a t^b] = T_R(N_c^2 - 1), \tag{C.11}$$

the average over the external gluon colour states reduces to an overall factor  $T_R$ .

Collecting the results above, the one-loop contribution in Eq. (C.4) takes the form

$$\hat{F}_{h/g}^{[1],m_h}(x,\eta) = \frac{i T_R \mu^{2\epsilon}}{1-\epsilon} \int \frac{d^{2-2\epsilon} \mathbf{k}_T}{(2\pi)^{2-2\epsilon}} dk^- \frac{N_{h/g}}{D_0 D_+ D_-} \Big|_{k^+=(1-x)p^+}, \quad (\text{C.12})$$

where the factor  $1/(1-\epsilon)$  originates from the average over the external gluon transverse polarisations, while the numerator and the three propagator denominators are defined as

$$\begin{aligned} N_{h/g} &= \sum_{j=1}^{d-2} \text{Tr}[\gamma^j ((1+\eta)\not{p} - \not{k} + m_h) \\ &\quad \times \not{p} ((1-\eta)\not{p} - \not{k} + m_h) \gamma^j (\not{k} - m_h)], \\ D_0 &= k^2 - m_h^2 + i\epsilon, \\ D_+ &= ((1+\eta)p - k)^2 - m_h^2 + i\epsilon, \\ D_- &= ((1-\eta)p - k)^2 - m_h^2 + i\epsilon. \end{aligned} \quad (\text{C.13})$$

**Evaluation of the numerator** Using  $k^+ = (1-x)p^+$  and evaluating the Dirac trace explicitly, one obtains

$$\begin{aligned} N_{h/g} &= 4p^+ \left[ 2(d-2)p^+ k^- (x^2 - \eta^2) \right. \\ &\quad \left. + (dx + d - 6x - 2) \mathbf{k}_T^2 + (d-2)(x+1)m_h^2 \right]. \end{aligned} \quad (\text{C.14})$$

**Evaluation of the denominator** The dependence of the three propagator denominators in Eq. (C.12) on  $k^-$  can be factorised as

$$D_0 D_+ D_- = 8(p^+)^3 (1-x)(x^2 - \eta^2) (k^- - k_1^-) (k^- - k_2^-) (k^- - k_3^-), \quad (\text{C.15})$$

where the poles are located at

$$\begin{aligned} k_1^- &= \frac{\mathbf{k}_T^2 + m_h^2}{2(1-x)p^+} - i(1-x)\epsilon, \\ k_2^- &= -\frac{\mathbf{k}_T^2 + m_h^2}{2(x+\eta)p^+} + i(x+\eta)\epsilon, \\ k_3^- &= -\frac{\mathbf{k}_T^2 + m_h^2}{2(x-\eta)p^+} + i(x-\eta)\epsilon. \end{aligned} \quad (\text{C.16})$$

After fixing  $k^+$ , the remaining longitudinal integration in Eq. (C.12) is the integral over  $k^-$ . Since the denominators are linear in  $k^-$  and the numerator in Eq. (C.14) is at most

linear in  $k^-$ , this step can be carried out by contour integration in the complex  $k^-$  plane. It is convenient to define the master integrals

$$I_{h/g}(\mathbf{k}_T^2) \equiv \int_{-\infty}^{+\infty} \frac{dk^-}{(k^- - k_1^-)(k^- - k_2^-)(k^- - k_3^-)}, \quad (\text{C.17})$$

and

$$J_{h/g}(\mathbf{k}_T^2) \equiv \int_{-\infty}^{+\infty} \frac{k^- dk^-}{(k^- - k_1^-)(k^- - k_2^-)(k^- - k_3^-)}. \quad (\text{C.18})$$

The position of the poles in Eq. (C.16) determines both the support of the GPD and the different contributions to these integrals in the ERBL and DGLAP regions. For  $0 < \eta < 1$ , one finds

|                    | $k_1^-$ | $k_2^-$ | $k_3^-$ |  |
|--------------------|---------|---------|---------|--|
| $x < -\eta$        | lower   | lower   | lower   |  |
| $-\eta < x < \eta$ | lower   | upper   | lower   |  |
| $\eta < x < 1$     | lower   | upper   | upper   |  |
| $x > 1$            | upper   | upper   | upper   |  |

(C.19)

where “upper” and “lower” specify the complex half-plane relative to the real axis containing each pole. Since all poles lie on the same side of the real axis for  $x < -\eta$  and  $x > 1$ , the contour integrals vanish in these regions. In the ERBL region,  $-\eta < x < \eta$ , the pole  $k_2^-$  is separated from  $k_1^-$  and  $k_3^-$ , whereas in the positive DGLAP region,  $\eta < x < 1$ , the pole  $k_1^-$  is separated from the pair  $k_2^-, k_3^-$ . The pole configuration in the two regions with non-vanishing support is illustrated in Fig. C.1. Closing the contour in the upper half-plane, the ERBL region picks up the residue at  $k_2^-$ , while the positive DGLAP region receives contributions from the poles at  $k_2^-$  and  $k_3^-$ .

Evaluating  $I_{h/g}$  and  $J_{h/g}$  by the residue theorem gives

$$I_{h/g}(\mathbf{k}_T^2) = - \frac{4\pi i (p^+)^2 (1-x)(x^2 - \eta^2)}{\eta(\mathbf{k}_T^2 + m_h^2)^2} \theta(1-x) \times \left[ \theta(x+\eta) \frac{x+\eta}{1+\eta} - \theta(x-\eta) \frac{x-\eta}{1-\eta} \right], \quad (\text{C.20})$$

$$J_{h/g}(\mathbf{k}_T^2) = \frac{2\pi i p^+ (1-x)(x^2 - \eta^2)}{\eta(\mathbf{k}_T^2 + m_h^2)} \theta(1-x) \times \left[ \theta(x+\eta) \frac{1}{1+\eta} - \theta(x-\eta) \frac{1}{1-\eta} \right]. \quad (\text{C.21})$$

The longitudinal integration is thus completed by substituting Eqs. (C.20) and (C.21) into Eq. (C.12). The remaining integration is purely transverse and can be reduced to two master integrals over the  $d-2 = 2-2\epsilon$  transverse dimensions,

$$I_{T,h/g} \equiv \mu^{2\epsilon} \int \frac{d^{2-2\epsilon} \mathbf{k}_T}{(2\pi)^{2-2\epsilon}} \frac{1}{\mathbf{k}_T^2 + m_h^2}, \quad (\text{C.22})$$

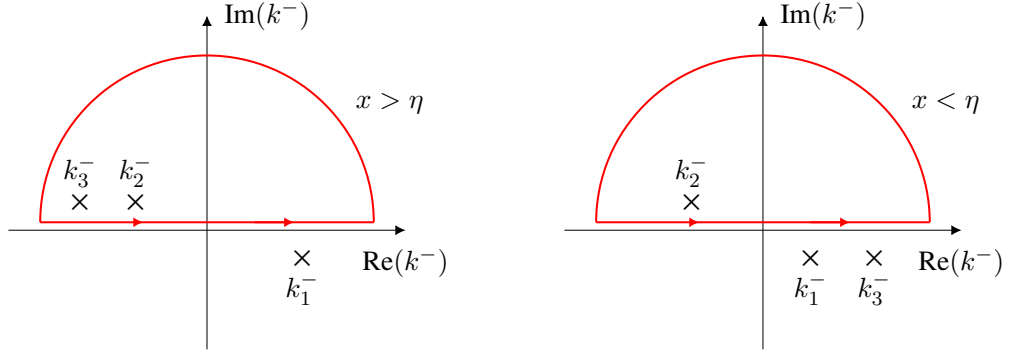


**Figure C.1:** Position of the poles in the complex  $k^-$  plane. The red contour indicates the integration path.

and

$$J_{T,h/g} \equiv \mu^{2\epsilon} \int \frac{d^{2-2\epsilon} \mathbf{k}_T}{(2\pi)^{2-2\epsilon}} \frac{1}{(\mathbf{k}_T^2 + m_h^2)^2}. \quad (\text{C.23})$$

In dimensional regularisation, their relevant expansions around  $\epsilon = 0$  read [39]

$$I_{T,h/g} = \frac{S_\epsilon}{4\pi} \left[ \frac{1}{\epsilon} - \ln \left( \frac{m_h^2}{\mu^2} \right) \right] + \mathcal{O}(\epsilon), \quad J_{T,h/g} = \frac{S_\epsilon}{4\pi m_h^2} + \mathcal{O}(\epsilon), \quad (\text{C.24})$$

where, following the convention of Ref. [25],

$$S_\epsilon = \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} = 1 + \epsilon (\ln 4\pi - \gamma_E) + \mathcal{O}(\epsilon^2). \quad (\text{C.25})$$

In the  $\overline{\text{MS}}$  scheme, UV poles occur in the combination  $S_\epsilon/\epsilon$ . Consequently, the finite contribution generated by the  $\mathcal{O}(\epsilon)$  expansion of  $S_\epsilon$  multiplying the UV pole is absorbed in the  $\overline{\text{MS}}$  subtraction. Hereafter, the conventional factor  $S_\epsilon$  is kept implicit.

At this point, the support obtained from the  $k^-$  contour integration can be made explicit in the full bare GPD. For  $0 < \eta < 1$ , the one-loop result can be written as

$$\begin{aligned} \widehat{F}_{h/g}^{[1],m_h} \left( x, \eta, \epsilon, \frac{\mu^2}{m_h^2} \right) &= \theta(1-x) \left[ \theta(x+\eta) \mathcal{H}_{h/g} \left( x, \eta, \epsilon, \frac{\mu^2}{m_h^2} \right) \right. \\ &\quad \left. + \theta(x-\eta) \mathcal{H}_{h/g} \left( x, -\eta, \epsilon, \frac{\mu^2}{m_h^2} \right) \right]. \end{aligned} \quad (\text{C.26})$$

Before expanding in  $\epsilon$ , the function multiplying  $\theta(x+\eta)$  can be written in terms of the transverse master integrals as

$$\begin{aligned} \mathcal{H}_{h/g} \left( x, \eta, \epsilon, \frac{\mu^2}{m_h^2} \right) &= \frac{T_R(x+\eta)}{\eta(1+\eta)(1-\epsilon)} \\ &\times \left\{ 4\pi \left[ \frac{d-2}{2}(1+\eta) - 2x \right] I_{T,h/g} + 8\pi x m_h^2 J_{T,h/g} \right\}. \end{aligned} \quad (\text{C.27})$$

The contribution proportional to  $\theta(x - \eta)$  is obtained from the same expression by the replacement  $\eta \rightarrow -\eta$ .

Substituting Eq. (C.24) into Eq. (C.27), one obtains

$$\begin{aligned} \mathcal{H}_{h/g} \left( x, \eta, \epsilon, \frac{\mu^2}{m_h^2} \right) &= \frac{T_R(x + \eta)}{\eta(1 + \eta)(1 - \epsilon)} \left\{ \left[ \frac{d - 2}{2}(1 + \eta) - 2x \right] \right. \\ &\quad \times \left[ \frac{1}{\epsilon} - \ln \left( \frac{m_h^2}{\mu^2} \right) \right] + 2x \left. \right\} + \mathcal{O}(\epsilon). \end{aligned} \quad (\text{C.28})$$

Using  $d - 2 = 2(1 - \epsilon)$ , this becomes

$$\begin{aligned} \mathcal{H}_{h/g} \left( x, \eta, \epsilon, \frac{\mu^2}{m_h^2} \right) &= \frac{T_R(x + \eta)}{\eta(1 + \eta)(1 - \epsilon)} \left\{ [(1 - \epsilon)(1 + \eta) - 2x] \right. \\ &\quad \times \left[ \frac{1}{\epsilon} - \ln \left( \frac{m_h^2}{\mu^2} \right) \right] + 2x \left. \right\}. \end{aligned} \quad (\text{C.29})$$

Using

$$\frac{1}{1 - \epsilon} = 1 + \epsilon + \mathcal{O}(\epsilon^2), \quad (\text{C.30})$$

and expanding up to finite order in  $\epsilon$ , one obtains

$$\mathcal{H}_{h/g} \left( x, \eta, \epsilon, \frac{\mu^2}{m_h^2} \right) = \frac{T_R(x + \eta)(1 + \eta - 2x)}{\eta(1 + \eta)} \left[ \frac{1}{\epsilon} - \ln \left( \frac{m_h^2}{\mu^2} \right) \right] + \mathcal{O}(\epsilon). \quad (\text{C.31})$$

The pole and finite contributions are thus isolated as

$$\mathcal{H}_{h/g}^{\text{pole}}(x, \eta) = T_R \frac{(x + \eta)(1 + \eta - 2x)}{\eta(1 + \eta)}, \quad (\text{C.32})$$

and

$$\mathcal{H}_{h/g}^{\text{fin}} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) = -T_R \frac{(x + \eta)(1 + \eta - 2x)}{\eta(1 + \eta)} \ln \left( \frac{m_h^2}{\mu^2} \right). \quad (\text{C.33})$$

Substituting these results back into Eq. (C.26), the bare one-loop GPD can be decomposed into pole and finite contributions as

$$\widehat{F}_{h/g}^{[1], m_h} = \frac{1}{\epsilon} \widehat{F}_{h/g}^{[1, -1]} + \widehat{F}_{h/g}^{[1, 0], m_h} + \mathcal{O}(\epsilon), \quad (\text{C.34})$$

where

$$\widehat{F}_{h/g}^{[1, -1]}(x, \eta) = \theta(1 - x) \left[ \theta(x + \eta) \mathcal{H}_{h/g}^{\text{pole}}(x, \eta) + \theta(x - \eta) \mathcal{H}_{h/g}^{\text{pole}}(x, -\eta) \right], \quad (\text{C.35})$$

and

$$\begin{aligned} \widehat{F}_{h/g}^{[1, 0], m_h} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) &= \theta(1 - x) \left[ \theta(x + \eta) \mathcal{H}_{h/g}^{\text{fin}} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) \right. \\ &\quad \left. + \theta(x - \eta) \mathcal{H}_{h/g}^{\text{fin}} \left( x, -\eta, \frac{\mu^2}{m_h^2} \right) \right]. \end{aligned} \quad (\text{C.36})$$

## C.2 Gluon-in-heavy-quark GPD

We now consider the real one-loop contribution to the gluon-in-heavy-quark GPD. In this channel, the external states are massive heavy quarks, while the two gluon propagators attached to the bilocal operator remain massless. The external momenta are chosen as in Eq. (3.7),

$$\begin{aligned} p_{\text{in}}^\mu &= \left( (1+\eta)p^+, \frac{m_h^2}{2(1+\eta)p^+}, \mathbf{0}_T \right), \\ p_{\text{out}}^\mu &= \left( (1-\eta)p^+, \frac{m_h^2}{2(1-\eta)p^+}, \mathbf{0}_T \right), \end{aligned} \quad (\text{C.37})$$

so that

$$p_{\text{in}}^2 = p_{\text{out}}^2 = m_h^2. \quad (\text{C.38})$$

The bare gluon-in-heavy-quark GPD is defined by

$$\begin{aligned} \hat{F}_{g/h}(x, \eta) &= \frac{(n \cdot p)(x^2 - \eta^2)}{2N_c x} \int \frac{dy}{2\pi} e^{-ix(n \cdot p)y} \\ &\times \sum_s \text{Tr} \left[ \left\langle p_{\text{out}}, s \left| A_a^j \left( \frac{yn}{2} \right) A_a^j \left( -\frac{yn}{2} \right) \right| p_{\text{in}}, s \right\rangle_h \right], \end{aligned} \quad (\text{C.39})$$

where the transverse index  $j$  is summed over. This matrix element vanishes at tree level. The first non-vanishing contribution is therefore the real diagram at  $\mathcal{O}(\alpha_s)$ , shown in Fig. 1.12.

Applying the Feynman rules in light-cone gauge and working in dimensional regularisation, the one-loop contribution reads

$$\begin{aligned} a_s \hat{F}_{g/h}^{[1], m_h}(x, \eta) &= \frac{(n \cdot p)(x^2 - \eta^2)}{2N_c x} \int \frac{dy}{2\pi} e^{-ix(n \cdot p)y} \int \frac{d^d k}{(2\pi)^d} e^{i(p_{\text{out}} - k) \cdot yn/2} e^{i(p_{\text{in}} - k) \cdot yn/2} \\ &\times \sum_s \text{Tr} \left[ (-ig_s \mu^\epsilon \gamma^\mu t_a) \frac{i(\not{k} + m_h)}{k^2 - m_h^2 + i\epsilon} iD_{\mu j}(p_{\text{in}} - k) iD_{j\nu}(p_{\text{out}} - k) \right. \\ &\quad \left. \times (-ig_s \mu^\epsilon \gamma^\nu t_a) u_s(p_{\text{in}}) \bar{u}_s(p_{\text{out}}) \right], \end{aligned} \quad (\text{C.40})$$

with  $d = 4 - 2\epsilon$ . The two exponential factors combine according to

$$e^{i(p_{\text{out}} - k) \cdot yn/2} e^{i(p_{\text{in}} - k) \cdot yn/2} = e^{i(p^+ - k^+)y}. \quad (\text{C.41})$$

The Fourier integration then enforces momentum conservation:

$$\int \frac{dy}{2\pi} e^{i(1-x)p^+y} e^{-ik^+y} = \delta((1-x)p^+ - k^+), \quad (\text{C.42})$$

fixing

$$k^+ = (1-x)p^+. \quad (\text{C.43})$$

At this stage, the colour and Dirac structures can be factorised. The colour trace reads

$$\text{Tr}_c[t_a t_a] = N_c C_F, \quad (\text{C.44})$$

reducing the average over the external colour states to the overall factor  $C_F$ .

The spinor structure is slightly more involved because the external states are massive and non-forward. For the kinematics in Eq. (C.37), the corresponding spin sum is

$$\begin{aligned} \sum_s u_s(p_{\text{in}}) \bar{u}_s(p_{\text{out}}) &= \frac{m_h}{\sqrt{1-\eta^2}} \mathbf{1} + \frac{m_h^2}{2p^+ \sqrt{1-\eta^2}} \gamma^+ + p^+ \sqrt{1-\eta^2} \gamma^- \\ &\quad + \frac{m_h \eta}{2\sqrt{1-\eta^2}} (\gamma^- \gamma^+ - \gamma^+ \gamma^-), \end{aligned} \quad (\text{C.45})$$

whose four terms will be denoted by  $\mathcal{P}_1$ ,  $\mathcal{P}_+$ ,  $\mathcal{P}_-$ , and  $\mathcal{P}_\sigma$ , respectively.

Finally, the two gluon propagators in Eq. (C.40) are written in light-cone gauge as

$$D_{\alpha\beta}(\ell) = \frac{1}{\ell^2 + i\epsilon} \left[ -g_{\alpha\beta} + \frac{\ell_\alpha n_\beta + \ell_\beta n_\alpha}{\ell \cdot n} \right]. \quad (\text{C.46})$$

Defining

$$\ell_L = p_{\text{in}} - k, \quad \ell_R = p_{\text{out}} - k, \quad (\text{C.47})$$

the constraint in Eq. (C.43) immediately gives

$$\ell_L \cdot n = (x + \eta)p^+, \quad \ell_R \cdot n = (x - \eta)p^+, \quad (\text{C.48})$$

which fix the light-cone denominators entering the Dirac numerator.

**Evaluation of the numerator** For each of the four spin structures  $X = 1, +, -, \sigma$  entering the massive spin sum in Eq. (C.45), the corresponding numerator can be written as

$$N_{g/h}^X = \sum_{j=1}^{d-2} \text{Tr} \left[ \left( \gamma^j - \frac{k_j}{(x+\eta)p^+} \not{k} \right) (\not{k} + m_h) \left( \gamma^j - \frac{k_j}{(x-\eta)p^+} \not{k} \right) \mathcal{P}_X \right]. \quad (\text{C.49})$$

Evaluating the Dirac traces explicitly, one finds

$$N_{g/h}^1 = -\frac{4(d-2)m_h^2}{\sqrt{1-\eta^2}}, \quad (\text{C.50})$$

$$N_{g/h}^+ = \frac{2(d-2)(1-x)m_h^2}{\sqrt{1-\eta^2}}, \quad (\text{C.51})$$

$$N_{g/h}^- = 4(d-2)p^+ \sqrt{1-\eta^2} k^- + \frac{8\sqrt{1-\eta^2}}{x^2 - \eta^2} \mathbf{k}_T^2, \quad (\text{C.52})$$

$$N_{g/h}^\sigma = 0. \quad (\text{C.53})$$

Thus, only three components of the massive non-forward spin sum contribute.

**Evaluation of the denominator** The three propagator denominators in Eq. (C.40) are

$$\begin{aligned} D_0 &= k^2 - m_h^2 + i\epsilon, \\ D_+ &= \ell_L^2 + i\epsilon, \\ D_- &= \ell_R^2 + i\epsilon. \end{aligned} \quad (\text{C.54})$$

With  $k^+ = (1-x)p^+$ , their dependence on  $k^-$  factorises as

$$D_0 D_+ D_- = 8(p^+)^3 (1-x)(x^2 - \eta^2) (k^- - k_1^-) (k^- - k_2^-) (k^- - k_3^-), \quad (\text{C.55})$$

where

$$\begin{aligned} k_1^- &= \frac{\mathbf{k}_T^2 + m_h^2}{2(1-x)p^+} - i(1-x)\epsilon, \\ k_2^- &= \frac{1}{2(x+\eta)p^+} \left[ \frac{x+\eta}{1+\eta} m_h^2 - \mathbf{k}_T^2 \right] + i(x+\eta)\epsilon, \\ k_3^- &= \frac{1}{2(x-\eta)p^+} \left[ \frac{x-\eta}{1-\eta} m_h^2 - \mathbf{k}_T^2 \right] + i(x-\eta)\epsilon. \end{aligned} \quad (\text{C.56})$$

With the numerator and denominator structures now explicit, the contribution associated with each spin structure can be written as

$$\widehat{F}_{g/h}^{[1],X}(x, \eta) = 2i C_F \frac{p^+(x^2 - \eta^2)}{x} \mu^{2\epsilon} \int \frac{d^{2-2\epsilon} \mathbf{k}_T}{(2\pi)^{2-2\epsilon}} dk^- \frac{N_{g/h}^X}{D_0 D_+ D_-} \Big|_{k^+=(1-x)p^+}. \quad (\text{C.57})$$

Summing over all contributions yields the full one-loop result:

$$\widehat{F}_{g/h}^{[1],m_h} = \sum_{X=1,+,-,\sigma} \widehat{F}_{g/h}^{[1],X}, \quad (\text{C.58})$$

with the  $\sigma$  contribution vanishing identically according to Eq. (C.53).

The pole structure in Eq. (C.55) allows the  $k^-$  integration to be performed by contour integration in the complex  $k^-$  plane. To perform the  $k^-$  integration, we introduce the master integrals

$$I_{g/h}(\mathbf{k}_T^2) \equiv \int_{-\infty}^{+\infty} \frac{dk^-}{(k^- - k_1^-)(k^- - k_2^-)(k^- - k_3^-)}, \quad (\text{C.59})$$

and

$$J_{g/h}(\mathbf{k}_T^2) \equiv \int_{-\infty}^{+\infty} \frac{k^- dk^-}{(k^- - k_1^-)(k^- - k_2^-)(k^- - k_3^-)}. \quad (\text{C.60})$$

For  $0 < \eta < 1$ , these are evaluated using the complex pole structures identified in Sec. C.1. Since all poles lie on the same side of the real axis for  $x < -\eta$  and  $x > 1$ , the integrals vanish in these regions. In the regions with non-vanishing support, the pole distribution in the upper and lower half-planes is the same as that illustrated in Fig. C.1, although their real

parts are channel dependent. Closing the contour in the upper half-plane, the ERBL contribution is generated by the residue at  $k_2^-$ , while the positive DGLAP contribution receives the residues at  $k_2^-$  and  $k_3^-$ .

For compactness, we define

$$\begin{aligned} B_0 &= (1 - \eta^2) \mathbf{k}_T^2 - (x^2 - \eta^2) m_h^2, \\ C_\pm &= (1 \pm \eta)^2 \mathbf{k}_T^2 + (x \pm \eta)^2 m_h^2, \\ A_\pm &= (1 \pm \eta) \mathbf{k}_T^2 - (x \pm \eta) m_h^2. \end{aligned} \quad (\text{C.61})$$

Evaluating the integrals in Eqs. (C.59) and (C.60) by the residue theorem gives

$$\begin{aligned} I_{g/h}(\mathbf{k}_T^2) &= - \frac{4\pi i (p^+)^2 (1-x)(1-\eta^2)(x^2-\eta^2)}{\eta B_0} \theta(1-x) \\ &\times \left[ \theta(x+\eta) \frac{(x+\eta)(1+\eta)}{C_+} - \theta(x-\eta) \frac{(x-\eta)(1-\eta)}{C_-} \right], \end{aligned} \quad (\text{C.62})$$

and

$$\begin{aligned} J_{g/h}(\mathbf{k}_T^2) &= \frac{2\pi i p^+ (1-x)(1-\eta^2)(x^2-\eta^2)}{\eta B_0} \theta(1-x) \\ &\times \left[ \theta(x+\eta) \frac{A_+}{C_+} - \theta(x-\eta) \frac{A_-}{C_-} \right]. \end{aligned} \quad (\text{C.63})$$

Substituting these results into Eq. (C.57) completes the longitudinal integration. The resulting transverse integrand contains terms proportional to

$$\frac{1}{B_0 C_\pm}, \quad \frac{\mathbf{k}_T^2}{B_0 C_\pm}, \quad \frac{A_\pm}{B_0 C_\pm}. \quad (\text{C.64})$$

These structures are not linearly independent. In fact, the terms with explicit factors of  $\mathbf{k}_T^2$  or  $A_\pm$  in the numerator reduce to the two basic structures  $1/C_\pm$  and  $1/(B_0 C_\pm)$  via

$$\begin{aligned} \frac{\mathbf{k}_T^2}{B_0 C_\pm} &= \frac{1}{1-\eta^2} \frac{1}{C_\pm} + \frac{(x^2-\eta^2)m_h^2}{1-\eta^2} \frac{1}{B_0 C_\pm}, \\ \frac{A_\pm}{B_0 C_\pm} &= \frac{1}{1 \mp \eta} \frac{1}{C_\pm} + \frac{(x \pm \eta)(x-1)m_h^2}{1 \mp \eta} \frac{1}{B_0 C_\pm}. \end{aligned} \quad (\text{C.65})$$

It is therefore sufficient to introduce the transverse master integrals

$$I_{T,g/h}^\pm \equiv \mu^{2\epsilon} \int \frac{d^{2-2\epsilon} \mathbf{k}_T}{(2\pi)^{2-2\epsilon}} \frac{1}{C_\pm}, \quad (\text{C.66})$$

and

$$J_{T,g/h}^\pm \equiv \mu^{2\epsilon} \int \frac{d^{2-2\epsilon} \mathbf{k}_T}{(2\pi)^{2-2\epsilon}} \frac{1}{B_0 C_\pm}. \quad (\text{C.67})$$

The first master integral gives [39]

$$I_{T,g/h}^{\pm} = \frac{1}{4\pi(1 \pm \eta)^2} \left[ \frac{1}{\epsilon} - \ln \left( \frac{(x \pm \eta)^2 m_h^2}{(1 \pm \eta)^2 \mu^2} \right) \right] + \mathcal{O}(\epsilon), \quad (\text{C.68})$$

where the conventional factor  $S_\epsilon$  is kept implicit, as specified in the previous section.

The second master integral is UV finite. Since both  $B_0$  and  $C_\pm$  are linear in  $k_T^2$ , the integrand can be decomposed into partial fractions as

$$\frac{1}{B_0 C_\pm} = \frac{1 \mp \eta}{2m_h^2(x \pm \eta)(x - \eta^2)} \frac{1}{B_0} - \frac{1 \pm \eta}{2m_h^2(x \pm \eta)(x - \eta^2)} \frac{1}{C_\pm}. \quad (\text{C.69})$$

The partial-fraction decomposition also requires the transverse integral associated with  $B_0$ , defined as

$$I_{T,g/h}^0 \equiv \mu^{2\epsilon} \int \frac{d^{2-2\epsilon} \mathbf{k}_T}{(2\pi)^{2-2\epsilon}} \frac{1}{B_0}. \quad (\text{C.70})$$

Its relevant expansion around  $\epsilon = 0$  is [39]

$$I_{T,g/h}^0 = \frac{1}{4\pi(1 - \eta^2)} \left[ \frac{1}{\epsilon} - \ln \left( \frac{(x^2 - \eta^2)m_h^2}{(\eta^2 - 1)\mu^2} \right) \right] + \mathcal{O}(\epsilon). \quad (\text{C.71})$$

Combining Eqs. (C.68), (C.69), and (C.71), Eq. (C.67) gives

$$J_{T,g/h}^{\pm} = \frac{1}{8\pi m_h^2(1 \pm \eta)(x \pm \eta)(x - \eta^2)} \ln \left( \frac{(x \pm \eta)^2(\eta^2 - 1)}{(1 \pm \eta)^2(x^2 - \eta^2)} \right) + \mathcal{O}(\epsilon). \quad (\text{C.72})$$

At this point, the support obtained from the  $k^-$  contour integration can be made explicit in the full bare GPD. For  $0 < \eta < 1$ , the one-loop result can be written as

$$\begin{aligned} \widehat{F}_{g/h}^{[1],m_h} \left( x, \eta, \epsilon, \frac{\mu^2}{m_h^2} \right) = \theta(1 - x) \left[ \theta(x + \eta) \mathcal{H}_{g/h} \left( x, \eta, \epsilon, \frac{\mu^2}{m_h^2} \right) \right. \\ \left. + \theta(x - \eta) \mathcal{H}_{g/h} \left( x, -\eta, \epsilon, \frac{\mu^2}{m_h^2} \right) \right]. \end{aligned} \quad (\text{C.73})$$

Before expanding in  $\epsilon$ , the function multiplying  $\theta(x + \eta)$  can be written in terms of the transverse master integrals as

$$\begin{aligned} \mathcal{H}_{g/h} \left( x, \eta, \epsilon, \frac{\mu^2}{m_h^2} \right) = \frac{2\pi(x + \eta)(1 + \eta)C_F\sqrt{1 - \eta^2}}{x\eta} \left\{ \left[ 4 - (d - 2)(x - \eta) \right] I_{T,g/h}^+ \right. \\ \left. + \left[ 4 - 2x(d - 2) \right] (x^2 - \eta^2)m_h^2 J_{T,g/h}^+ \right\}. \end{aligned} \quad (\text{C.74})$$

The contribution proportional to  $\theta(x - \eta)$  is obtained from the same expression by the replacement  $\eta \rightarrow -\eta$ .

Substituting Eqs. (C.68) and (C.72) into Eq. (C.74), and using  $d - 2 = 2(1 - \epsilon)$ , one obtains

$$\begin{aligned} \mathcal{H}_{g/h} \left( x, \eta, \epsilon, \frac{\mu^2}{m_h^2} \right) &= \frac{C_F \sqrt{1 - \eta^2}}{\eta} \left\{ \frac{x + \eta}{x(1 + \eta)} \left[ 2 - (1 - \epsilon)(x - \eta) \right] \right. \\ &\quad \times \left[ \frac{1}{\epsilon} - \ln \left( \frac{(x + \eta)^2 m_h^2}{(1 + \eta)^2 \mu^2} \right) \right] \\ &\quad \left. + \left[ 1 - x(1 - \epsilon) \right] \frac{x^2 - \eta^2}{x(x - \eta^2)} \ln \left( \frac{(x + \eta)^2 (\eta^2 - 1)}{(1 + \eta)^2 (x^2 - \eta^2)} \right) \right\}. \end{aligned} \quad (\text{C.75})$$

Expanding through  $\mathcal{O}(\epsilon^0)$ , one obtains

$$\mathcal{H}_{g/h} \left( x, \eta, \epsilon, \frac{\mu^2}{m_h^2} \right) = \frac{1}{\epsilon} \mathcal{H}_{g/h}^{\text{pole}}(x, \eta) + \mathcal{H}_{g/h}^{\text{fin}} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) + \mathcal{O}(\epsilon), \quad (\text{C.76})$$

where

$$\mathcal{H}_{g/h}^{\text{pole}}(x, \eta) = C_F \sqrt{1 - \eta^2} \frac{(x + \eta)(2 - x + \eta)}{x\eta(1 + \eta)}, \quad (\text{C.77})$$

and

$$\begin{aligned} \mathcal{H}_{g/h}^{\text{fin}} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) &= \frac{C_F \sqrt{1 - \eta^2}}{x\eta} \left[ \frac{x^2 - \eta^2}{1 + \eta} - \frac{(x + \eta)(2 - x + \eta)}{1 + \eta} \ln \left( \frac{(x + \eta)^2 m_h^2}{(1 + \eta)^2 \mu^2} \right) \right. \\ &\quad \left. + \frac{(1 - x)(x^2 - \eta^2)}{x - \eta^2} \ln \left( \frac{(x + \eta)^2 (\eta^2 - 1)}{(1 + \eta)^2 (x^2 - \eta^2)} \right) \right]. \end{aligned} \quad (\text{C.78})$$

Substituting these results back into Eq. (C.73), the bare one-loop GPD can be decomposed into pole and finite contributions as

$$\widehat{F}_{g/h}^{[1], m_h} = \frac{1}{\epsilon} \widehat{F}_{g/h}^{[1, -1]} + \widehat{F}_{g/h}^{[1, 0], m_h} + \mathcal{O}(\epsilon), \quad (\text{C.79})$$

where

$$\widehat{F}_{g/h}^{[1, -1]}(x, \eta) = \theta(1 - x) \left[ \theta(x + \eta) \mathcal{H}_{g/h}^{\text{pole}}(x, \eta) + \theta(x - \eta) \mathcal{H}_{g/h}^{\text{pole}}(x, -\eta) \right], \quad (\text{C.80})$$

and

$$\begin{aligned} \widehat{F}_{g/h}^{[1, 0], m_h} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) &= \theta(1 - x) \left[ \theta(x + \eta) \mathcal{H}_{g/h}^{\text{fin}} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) \right. \\ &\quad \left. + \theta(x - \eta) \mathcal{H}_{g/h}^{\text{fin}} \left( x, -\eta, \frac{\mu^2}{m_h^2} \right) \right]. \end{aligned} \quad (\text{C.81})$$

### C.3 Real heavy-quark-in-heavy-quark GPD

We finally consider the real one-loop contribution to the diagonal heavy-quark-in-heavy-quark GPD. In this channel, both the external states and the internal quark propagators carry the heavy-quark mass  $m_h$ . The external momenta are the same as in the  $g/h$  calculation, given in Eq. (C.37). The bare heavy-quark-in-heavy-quark GPD is defined by

$$\begin{aligned} \hat{F}_{h/h}(x, \eta) &= \frac{1}{2N_c} \int \frac{dy}{2\pi} e^{-ix(n \cdot p)y} \\ &\times \sum_s \text{Tr} \left[ \left\langle p_{\text{out}}, s \left| \bar{\psi}_h \left( \frac{yn}{2} \right) \frac{\not{y}}{2} \psi_h \left( -\frac{yn}{2} \right) \right| p_{\text{in}}, s \right\rangle_h \right]. \end{aligned} \quad (\text{C.82})$$

Unlike the two off-diagonal channels considered above, this GPD is non-vanishing at tree level. At one loop, it receives both real and virtual corrections. While the virtual corrections have already been addressed in Chapter 3, this section is dedicated only to the real contribution generated by the diagram in Fig. 1.12.

Applying the Feynman rules in light-cone gauge and working in dimensional regularisation, the real one-loop contribution reads

$$\begin{aligned} a_s \hat{F}_{h/h}^{[1], m_h, \text{real}}(x, \eta) &= \frac{1}{2N_c} \int \frac{dy}{2\pi} e^{-ix(n \cdot p)y} \int \frac{d^d k}{(2\pi)^d} e^{i(p_{\text{out}} - k) \cdot yn/2} e^{i(p_{\text{in}} - k) \cdot yn/2} \\ &\times \sum_s \text{Tr} \left[ iD_{\mu\nu}(k) (-ig_s \mu^\epsilon \gamma^\mu t_a) \frac{i(\not{p}_{\text{in}} - \not{k} + m_h)}{(p_{\text{in}} - k)^2 - m_h^2 + i\epsilon} \frac{\not{y}}{2} \right. \\ &\quad \times \left. \frac{-i(\not{p}_{\text{out}} - \not{k} + m_h)}{(p_{\text{out}} - k)^2 - m_h^2 + i\epsilon} (ig_s \mu^\epsilon \gamma^\nu t_a) u_s(p_{\text{in}}) \bar{u}_s(p_{\text{out}}) \right], \end{aligned} \quad (\text{C.83})$$

with  $d = 4 - 2\epsilon$ . The two exponential factors combine into

$$e^{i(p_{\text{out}} - k) \cdot yn/2} e^{i(p_{\text{in}} - k) \cdot yn/2} = e^{i(p^+ - k^+)y}. \quad (\text{C.84})$$

The Fourier integration over the light-cone separation  $y$  therefore gives

$$\int \frac{dy}{2\pi} e^{i(1-x)p^+y} e^{-ik^+y} = \delta((1-x)p^+ - k^+), \quad (\text{C.85})$$

so that the  $k^+$  integration fixes

$$k^+ = (1-x)p^+. \quad (\text{C.86})$$

After extracting the perturbative coefficient and evaluating the colour trace, the real one-loop contribution in Eq. (C.83) can be cast in the form

$$\hat{F}_{h/h}^{[1], m_h, \text{real}}(x, \eta) = i C_F \mu^{2\epsilon} \int \frac{d^{2-2\epsilon} \mathbf{k}_T}{(2\pi)^{2-2\epsilon}} dk^- \frac{N_{h/h}}{D_0 D_+ D_-} \Big|_{k^+ = (1-x)p^+}, \quad (\text{C.87})$$

where the numerator and the three propagator denominators are defined as

$$\begin{aligned}
 N_{h/h} &= d_{\mu\nu}(k) \text{Tr} \left[ \gamma^\mu (\not{p}_{\text{in}} - \not{k} + m_h) \not{n} (\not{p}_{\text{out}} - \not{k} + m_h) \gamma^\nu S_0 \right], \\
 D_0 &= k^2 + i\epsilon, \\
 D_+ &= (p_{\text{in}} - k)^2 - m_h^2 + i\epsilon, \\
 D_- &= (p_{\text{out}} - k)^2 - m_h^2 + i\epsilon.
 \end{aligned} \tag{C.88}$$

Here,  $S_0$  denotes the non-forward heavy-quark spin sum in Eq. (C.45), while

$$d_{\mu\nu}(k) = -g_{\mu\nu} + \frac{k_\mu n_\nu + k_\nu n_\mu}{k \cdot n} \tag{C.89}$$

is the numerator of the light-cone-gauge gluon propagator,

$$D_{\mu\nu}(k) = \frac{d_{\mu\nu}(k)}{k^2 + i\epsilon}. \tag{C.90}$$

**Evaluation of the numerator** As in the  $g/h$  calculation, the massive spin sum can be decomposed as

$$S_0 = \mathcal{P}_1 + \mathcal{P}_+ + \mathcal{P}_- + \mathcal{P}_\sigma. \tag{C.91}$$

Accordingly, the numerator in Eq. (C.88) can be written as

$$N_{h/h} = \sum_{X=1,+, -, \sigma} N_{h/h}^X, \quad N_{h/h}^X = N_{h/h}^{X,(g)} + N_{h/h}^{X,(n)}, \tag{C.92}$$

where  $N_{h/h}^{X,(g)}$  and  $N_{h/h}^{X,(n)}$  denote the contributions generated by the  $-g_{\mu\nu}$  and  $n^\mu$ -dependent terms of the light-cone-gauge gluon propagator, respectively. Using  $k^+ = (1-x)p^+$  and evaluating the Dirac traces in  $d = 4 - 2\epsilon$  dimensions, the contributions proportional to  $-g_{\mu\nu}$  are

$$\begin{aligned}
 N_{h/h}^{1,(g)} &= -\frac{8d m_h^2 p^+ x}{\sqrt{1 - \eta^2}}, \\
 N_{h/h}^{+,(g)} &= \frac{4(d-2)m_h^2 p^+ (x^2 - \eta^2)}{\sqrt{1 - \eta^2}}, \\
 N_{h/h}^{-,(g)} &= 4(d-2)p^+ \sqrt{1 - \eta^2} (\mathbf{k}_T^2 + m_h^2), \\
 N_{h/h}^{\sigma,(g)} &= -\frac{8(d-4)m_h^2 p^+ \eta^2}{\sqrt{1 - \eta^2}}.
 \end{aligned}$$

The contributions generated by the  $n^\mu$ -dependent part of the propagator are instead

$$\begin{aligned} N_{h/h}^{1,(n)} &= \frac{16m_h^2 p^+ x}{\sqrt{1-\eta^2}}, \\ N_{h/h}^{+,(n)} &= 0, \\ N_{h/h}^{-,(n)} &= \frac{16p^+ \sqrt{1-\eta^2}}{1-x} [2p^+ k^- (x^2 - \eta^2) + x \mathbf{k}_T^2], \\ N_{h/h}^{\sigma,(n)} &= -\frac{16m_h^2 p^+ \eta^2}{\sqrt{1-\eta^2}}. \end{aligned}$$

**Evaluation of the denominator** After fixing  $k^+ = (1-x)p^+$ , the dependence on  $k^-$  of the three propagator denominators in Eq. (C.88) factorises as

$$D_0 D_+ D_- = 8(p^+)^3 (1-x)(x^2 - \eta^2) (k^- - k_1^-) (k^- - k_2^-) (k^- - k_3^-), \quad (\text{C.93})$$

where the pole positions are given by

$$\begin{aligned} k_1^- &= \frac{\mathbf{k}_T^2}{2(1-x)p^+} - i(1-x)\epsilon, \\ k_2^- &= -\frac{\mathbf{k}_T^2 + \frac{1-x}{1+\eta}m_h^2}{2(x+\eta)p^+} + i(x+\eta)\epsilon, \\ k_3^- &= -\frac{\mathbf{k}_T^2 + \frac{1-x}{1-\eta}m_h^2}{2(x-\eta)p^+} + i(x-\eta)\epsilon. \end{aligned} \quad (\text{C.94})$$

The integration  $k^-$  in Eq. (C.87) is evaluated via contour integration in the complex  $k^-$  plane. Since the numerator is at most linear in  $k^-$ , the longitudinal integration can be expressed in terms of two basic contour integrals:

$$I_{h/h}(\mathbf{k}_T^2) \equiv \int_{-\infty}^{+\infty} \frac{dk^-}{(k^- - k_1^-)(k^- - k_2^-)(k^- - k_3^-)}, \quad (\text{C.95})$$

and

$$J_{h/h}(\mathbf{k}_T^2) \equiv \int_{-\infty}^{+\infty} \frac{k^- dk^-}{(k^- - k_1^-)(k^- - k_2^-)(k^- - k_3^-)}. \quad (\text{C.96})$$

The pole positions in Eq. (C.94) determine which residues contribute in the different kinematic regions and hence the support of the real contribution. For  $0 < \eta < 1$ , one finds

|                    | $k_1^-$ | $k_2^-$ | $k_3^-$ |        |
|--------------------|---------|---------|---------|--------|
| $x < -\eta$        | lower   | lower   | lower   |        |
| $-\eta < x < \eta$ | lower   | upper   | lower   |        |
| $\eta < x < 1$     | lower   | upper   | upper   |        |
| $x > 1$            | upper   | upper   | upper   | (C.97) |

where “upper” and “lower” refer to the half-plane in which the pole lies with respect to the real axis. As in the previous channels, the integral vanishes for  $x < -\eta$  and  $x > 1$ , where all poles lie in the same half-plane. Within the non-vanishing support, the pole configuration follows the pattern illustrated in Fig. C.1, with real parts determined by Eq. (C.94). Closing the contour in the upper half-plane therefore selects  $k_2^-$  in the ERBL region and  $k_2^-, k_3^-$  in the positive DGLAP region.

It is convenient to introduce the transverse quantities

$$\begin{aligned} B_0 &= (1 - \eta^2) \mathbf{k}_T^2 + (1 - x^2) m_h^2, \\ C_\pm &= (1 \pm \eta)^2 \mathbf{k}_T^2 + (1 - x)^2 m_h^2, \\ A_\pm &= (1 \pm \eta) \mathbf{k}_T^2 + (1 - x) m_h^2. \end{aligned} \quad (\text{C.98})$$

In terms of these quantities, the contour integrations give

$$\begin{aligned} I_{h/h}(\mathbf{k}_T^2) &= -\frac{4\pi i (p^+)^2 (1 - x)(1 - \eta^2)(x^2 - \eta^2)}{\eta B_0} \theta(1 - x) \\ &\quad \times \left[ \theta(x + \eta) \frac{(x + \eta)(1 + \eta)}{C_+} - \theta(x - \eta) \frac{(x - \eta)(1 - \eta)}{C_-} \right], \end{aligned} \quad (\text{C.99})$$

and

$$\begin{aligned} J_{h/h}(\mathbf{k}_T^2) &= \frac{2\pi i p^+ (1 - x)(1 - \eta^2)(x^2 - \eta^2)}{\eta B_0} \theta(1 - x) \\ &\quad \times \left[ \theta(x + \eta) \frac{A_+}{C_+} - \theta(x - \eta) \frac{A_-}{C_-} \right]. \end{aligned} \quad (\text{C.100})$$

Substituting Eqs. (C.99) and (C.100) into Eq. (C.87), the terms in the numerator independent of  $k^-$  multiply  $I_{h/h}$ , whereas those linear in  $k^-$  multiply  $J_{h/h}$ . The remaining transverse dependence can be expressed in terms of the structures

$$\frac{1}{B_0 C_\pm}, \quad \frac{\mathbf{k}_T^2}{B_0 C_\pm}, \quad \frac{A_\pm}{B_0 C_\pm}. \quad (\text{C.101})$$

The structures containing explicit factors of  $\mathbf{k}_T^2$  or  $A_\pm$  in the numerator can be reduced using

$$\frac{\mathbf{k}_T^2}{B_0 C_\pm} = \frac{1}{1 - \eta^2} \frac{1}{C_\pm} - \frac{(1 - x^2) m_h^2}{1 - \eta^2} \frac{1}{B_0 C_\pm}, \quad (\text{C.102})$$

and

$$\frac{A_\pm}{B_0 C_\pm} = \frac{1}{2} \left( \frac{1}{C_\pm} + \frac{1}{B_0} \right). \quad (\text{C.103})$$

The transverse integration is thus reduced to the master integrals

$$I_{T,h/h}^\pm \equiv \mu^{2\epsilon} \int \frac{d^{2-2\epsilon} \mathbf{k}_T}{(2\pi)^{2-2\epsilon}} \frac{1}{C_\pm}, \quad J_{T,h/h}^\pm \equiv \mu^{2\epsilon} \int \frac{d^{2-2\epsilon} \mathbf{k}_T}{(2\pi)^{2-2\epsilon}} \frac{1}{B_0 C_\pm}, \quad (\text{C.104})$$

and

$$I_{T,h/h}^0 \equiv \mu^{2\epsilon} \int \frac{d^{2-2\epsilon} \mathbf{k}_T}{(2\pi)^{2-2\epsilon}} \frac{1}{B_0}, \quad (\text{C.105})$$

The two single-denominator master integrals can be evaluated directly in dimensional regularisation, yielding [39]

$$I_{T,h/h}^\pm = \frac{1}{4\pi(1 \pm \eta)^2} \left[ \frac{1}{\epsilon} - \ln \left( \frac{(1-x)^2 m_h^2}{(1 \pm \eta)^2 \mu^2} \right) \right] + \mathcal{O}(\epsilon), \quad (\text{C.106})$$

$$I_{T,h/h}^0 = \frac{1}{4\pi(1-\eta^2)} \left[ \frac{1}{\epsilon} - \ln \left( \frac{(1-x^2) m_h^2}{(1-\eta^2) \mu^2} \right) \right] + \mathcal{O}(\epsilon), \quad (\text{C.107})$$

where, as in the previous sections, the conventional factor  $S_\epsilon$  is kept implicit.

The master integral  $J_{T,h/h}^\pm$  is UV finite. Performing a partial-fraction decomposition in  $\mathbf{k}_T^2$ , one finds

$$\frac{1}{B_0 C_\pm} = -\frac{1 \mp \eta}{2m_h^2(1-x)(x \pm \eta)} \frac{1}{B_0} + \frac{1 \pm \eta}{2m_h^2(1-x)(x \pm \eta)} \frac{1}{C_\pm}. \quad (\text{C.108})$$

Consequently,

$$J_{T,h/h}^\pm = -\frac{1 \mp \eta}{2m_h^2(1-x)(x \pm \eta)} I_{T,h/h}^0 + \frac{1 \pm \eta}{2m_h^2(1-x)(x \pm \eta)} I_{T,h/h}^\pm. \quad (\text{C.109})$$

Substituting Eqs. (C.106) and (C.107) into Eq. (C.109), the UV poles cancel and one finds

$$J_{T,h/h}^\pm = \frac{1}{8\pi m_h^2(1-x)(1 \pm \eta)(x \pm \eta)} \ln \left( \frac{(1+x)(1 \pm \eta)}{(1-x)(1 \mp \eta)} \right) + \mathcal{O}(\epsilon). \quad (\text{C.110})$$

At this point, the support obtained from the  $k^-$  contour integration can be made explicit. For  $0 < \eta < 1$ , the real contribution takes the form

$$\begin{aligned} \widehat{F}_{h/h}^{[1],m_h,\text{real}} \left( x, \eta, \epsilon, \frac{\mu^2}{m_h^2} \right) = \theta(1-x) & \left[ \theta(x+\eta) \mathcal{H}_{h/h} \left( x, \eta, \epsilon, \frac{\mu^2}{m_h^2} \right) \right. \\ & \left. + \theta(x-\eta) \mathcal{H}_{h/h} \left( x, -\eta, \epsilon, \frac{\mu^2}{m_h^2} \right) \right]. \quad (\text{C.111}) \end{aligned}$$

The function multiplying  $\theta(x+\eta)$  can be expressed in terms of the three master integrals

$I_{T,h/h}^+$ ,  $I_{T,h/h}^0$ , and  $J_{T,h/h}^+$ . Substituting their explicit expressions yields

$$\begin{aligned} \mathcal{H}_{h/h} \left( x, \eta, \epsilon, \frac{\mu^2}{m_h^2} \right) = & \frac{C_F \sqrt{1-\eta^2}}{\eta} \left\{ \frac{x+\eta}{1+\eta} \frac{1+x-\epsilon(1-x)}{1-x} \left[ \frac{1}{\epsilon} - \ln \left( \frac{(1-x)^2 m_h^2}{(1+\eta)^2 \mu^2} \right) \right] \right. \\ & - \frac{x^2 - \eta^2}{1-x} \left[ \frac{1-\eta}{1+\eta} \left[ \frac{1}{\epsilon} - \ln \left( \frac{(1-x)^2 m_h^2}{(1+\eta)^2 \mu^2} \right) \right] \right. \\ & \quad \left. \left. + \left[ \frac{1}{\epsilon} - \ln \left( \frac{(1-x^2) m_h^2}{(1-\eta^2) \mu^2} \right) \right] \right] \right. \\ & \left. - \frac{2x + 2\eta^2 + \epsilon(x^2 - x - 2\eta^2)}{1-x} \right. \\ & \quad \left. \times \ln \left( \frac{(1+x)(1+\eta)}{(1-x)(1-\eta)} \right) \right\} + \mathcal{O}(\epsilon). \end{aligned} \quad (\text{C.112})$$

Expanding through  $\mathcal{O}(\epsilon^0)$ , one obtains

$$\mathcal{H}_{h/h} \left( x, \eta, \epsilon, \frac{\mu^2}{m_h^2} \right) = \frac{1}{\epsilon} \mathcal{H}_{h/h}^{\text{pole}}(x, \eta) + \mathcal{H}_{h/h}^{\text{fin}} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) + \mathcal{O}(\epsilon). \quad (\text{C.113})$$

The UV-pole coefficient is

$$\mathcal{H}_{h/h}^{\text{pole}}(x, \eta) = C_F \sqrt{1-\eta^2} \frac{(x+\eta)(1-x+2\eta)}{\eta(1-x)(1+\eta)} \quad (\text{C.114})$$

and the corresponding finite part reads

$$\begin{aligned} \mathcal{H}_{h/h}^{\text{fin}} \left( x, \eta, \frac{\mu^2}{m_h^2} \right) = & \frac{C_F \sqrt{1-\eta^2}}{\eta} \left[ - \frac{x+\eta}{1+\eta} - \frac{x+\eta}{1+\eta} \frac{1+x}{1-x} \ln \left( \frac{(1-x)^2 m_h^2}{(1+\eta)^2 \mu^2} \right) \right. \\ & + \frac{x^2 - \eta^2}{1-x} \frac{1-\eta}{1+\eta} \ln \left( \frac{(1-x)^2 m_h^2}{(1+\eta)^2 \mu^2} \right) \\ & + \frac{x^2 - \eta^2}{1-x} \ln \left( \frac{(1-x^2) m_h^2}{(1-\eta^2) \mu^2} \right) \\ & \left. - \frac{2x + 2\eta^2}{1-x} \ln \left( \frac{(1+x)(1+\eta)}{(1-x)(1-\eta)} \right) \right]. \end{aligned} \quad (\text{C.115})$$

The contribution proportional to  $\theta(x-\eta)$  is obtained from the same expression by making the replacement  $\eta \rightarrow -\eta$ .

Substituting Eqs. (C.114) and (C.115) back into Eq. (C.111), the real bare one-loop GPD decomposes into pole and finite contributions:

$$\widehat{F}_{h/h}^{[1], m_h, \text{real}} = \frac{1}{\epsilon} \widehat{F}_{h/h}^{[1, -1], \text{real}} + \widehat{F}_{h/h}^{[1, 0], m_h, \text{real}} + \mathcal{O}(\epsilon), \quad (\text{C.116})$$

where

$$\begin{aligned} \widehat{F}_{h/h}^{[1,-1],\text{real}}(x, \eta) = \theta(1-x) & \left[ \theta(x+\eta) \mathcal{H}_{h/h}^{\text{pole}}(x, \eta) \right. \\ & \left. + \theta(x-\eta) \mathcal{H}_{h/h}^{\text{pole}}(x, -\eta) \right], \end{aligned} \quad (\text{C.117})$$

and

$$\begin{aligned} \widehat{F}_{h/h}^{[1,0],m_h,\text{real}}\left(x, \eta, \frac{\mu^2}{m_h^2}\right) = \theta(1-x) & \left[ \theta(x+\eta) \mathcal{H}_{h/h}^{\text{fin}}\left(x, \eta, \frac{\mu^2}{m_h^2}\right) \right. \\ & \left. + \theta(x-\eta) \mathcal{H}_{h/h}^{\text{fin}}\left(x, -\eta, \frac{\mu^2}{m_h^2}\right) \right]. \end{aligned} \quad (\text{C.118})$$



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