Università degli studi di Pavia
Dipartimento di Matematica
Corso di Laurea Magistrale in Matematica



UNIVERSITÀ
DI PAVIA

Derived Categories with an Application to Semiorthogonal Decompositions of Enriques Surfaces

Tesi di Laurea Magistrale in Matematica

Relatore (Supervisor):
Prof. Riccardo Moschetti

Tesi di Laurea di:
Alessandro Marchetti
Matricola 564404

Anno Accademico 2025-2026

Abstract

In algebraic geometry, the bounded derived category $D^b(X)$ of a smooth projective variety X is an invariant that is finer than (co)homology; its semiorthogonal decompositions are a way of isolating parts of $D^b(X)$ which may carry a strong relationship with the geometry of X. In this thesis we develop the theory of derived categories, semiorthogonal decompositions, and Fourier–Mukai functors from the beginning, and we apply it to Enriques surfaces S: the numerical and lattice-theoretic properties of S produce a semiorthogonal decomposition $D^b(S) = \langle \mathcal{K}u(S, \mathcal{L}), \mathcal{L} \rangle$ (with $\mathcal{L}$ generated by exceptional line bundles), whose residual component $\mathcal{K}u(S, \mathcal{L})$ is an object of interest. We conclude by presenting the two derived Torelli theorems: $D^b(S)$, and even the residual component alone, determine S up to isomorphism.

Contents

1 Introduction **7**

2 Preliminaries **11**

2.1 (Quasi) Coherent Sheaves 11

2.2 Triangulated Categories 16

3 The Derived Category **23**

3.1 The Construction of $D(\mathcal{A})$ 23

3.2 Derived Functors 35

3.3 (Semi)orthogonal Decompositions 38

3.4 Fourier-Mukai functors 45

4 Enriques Surfaces **55**

4.1 The intersection form on divisors 55

4.2 K3 and Enriques Surfaces: Generalities 61

4.3 The Weyl group and (-2)-curves 65

4.4 The Derived Torelli Theorems 75

Chapter 1

Introduction

The derived category $D^b(X)$ of a smooth complex projective variety X is a fundamental invariant in modern algebraic geometry, as it provides an algebraic tool that generalizes (co)homology: derived categories are invariants of varieties that are finer than (co)homology, and are therefore widely used in the geometric context. (Co)homology groups are among the classical invariants that characterize varieties, but they often cancel information that we would prefer to preserve. To prevent this loss, which stems from applying the (co)homology functors, the idea of homological algebra is to shift the focus from the (co)homology groups back to the underlying complexes; the derived category $D^b(X)$ provides the precise framework to achieve this, allowing us to retain data that would otherwise vanish.

When applied to the geometry of Enriques surfaces, this categorical construction reveals its power as an algebraic invariant. The topological and numerical properties of these surfaces interact deeply with the categorical structure of $D^b(S)$, and allow for a semiorthogonal decomposition of $D^b(S)$ via exceptional sequences generated by line bundles. This decomposition isolates a subcategory, known as the residual component, which encodes the geometric data of the surface. The aim of this thesis is to present this construction and to show, through the derived Torelli theorems, to what extent the residual component determines the surface.

The thesis is structured in three chapters. The first two focus on the definition and study of derived categories in a general setting, while in the third one these tools are applied to the geometry of Enriques surfaces.

Chapter 2: Preliminaries. We begin in Section 2.1 by recalling the concept of coherent and quasi-coherent sheaf, in order to show that $\mathrm{Coh}(X)$, the category of coherent sheaves over a variety X, is abelian. In Section 2.2 the focus shifts to the structure of triangulated categories. A triangulated category $\mathcal{D}$ is an additive category endowed with a shift functor $T\colon \mathcal{D} \to \mathcal{D}$ and a set of distinguished triangles subject to certain axioms, which essentially allow us to compensate for the lack of an abelian structure on $\mathcal{D}$. For example, one of these axioms provides $\mathcal{D}$ with an operation called cone, which, in a way, defines the "cokernel" of a morphism in the triangulated category.

Chapter 3: The derived category. The derived category is endowed with this exact kind of triangulated structure. In Section 3.1, given an abelian category $\mathcal{A}$, we construct $D^b(\mathcal{A})$ starting from $K^b(\mathcal{A})$, the homotopy category of bounded complexes (i.e. complexes $A^\bullet$ such that $A^i = 0$ for $|i| \gg 0$), and forcing quasi-isomorphisms (i.e. morphisms of complexes which induce isomorphisms in cohomology) to be invertible. This is achieved by describing the morphisms in $D^b(\mathcal{A})$ as certain classes of diagrams called roofs, modulo an equivalence relation; this procedure is called localization, by analogy with the one for rings. One can define the derived category of any abelian category, but when X is a variety, $D^b(X)$ always indicates $D^b(\mathrm{Coh}(X))$, which is well defined since $\mathrm{Coh}(X)$ is abelian.

Section 3.2 is devoted to derived functors. A functor between abelian categories, such as the pushforward, the pullback or the tensor product of sheaves, is in general only left or right exact, and therefore does not preserve quasi-isomorphisms of complexes; thus, it does not induce a functor between the derived categories. We recall how one can repair this by replacing complexes with suitable resolutions, obtaining the derived functors, which are exact functors between triangulated categories and which recover the classical higher derived functors in cohomology. This is done also with a view to Section 3.4: the definition of a Fourier–Mukai functor only makes sense if the pushforward, the pullback and the tensor product are replaced by their derived versions.

In Section 3.3 we analyze (semi)orthogonal decompositions. Given a smooth projective variety X, a way to simplify the study of $D^b(X)$ is to use a decomposition of the form $D^b(X) = \langle \mathcal{A}, \mathcal{B} \rangle$, which indicates that objects of $D^b(X)$ can be expressed as "combinations" of objects of $\mathcal{A}$ and $\mathcal{B}$ through the shift and cone operations. The decomposition is semiorthogonal if $\mathrm{Hom}(B, A) = 0$ for all $B \in \mathcal{B}$ and $A \in \mathcal{A}$; it is orthogonal if, moreover, $\mathrm{Hom}(A, B) = 0$. Since X is projective, orthogonality is excessively restrictive: as shown by Bridgeland, X admits a nontrivial orthogonal decomposition if and only if it is disconnected, and the orthogonal components correspond to the connected components of X. This rigidity is the reason why our focus is on semiorthogonal decompositions.

The last part of this chapter, Section 3.4, is devoted to Fourier–Mukai functors. Let X and Y be smooth projective varieties, and let $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ be the projections. Given an object $E \in D^b(X \times Y)$, the Fourier–Mukai functor of kernel E is $\Phi_E(-) = q_*(p^*(-) \otimes E)$, where the functors are properly derived. Fourier–Mukai functors behave well in many respects: they are exact, they admit left and right adjoints (which are again Fourier–Mukai), and they can be composed to get another Fourier–Mukai functor. An important theorem by Orlov states that every equivalence $D^b(X) \to D^b(Y)$ between smooth projective varieties is of Fourier–Mukai type. Consequently, the study of such an equivalence is reduced to the study of its kernel, which is geometric in nature: it is a complex of coherent sheaves on $X \times Y$.

Chapter 4: Enriques surfaces. The link between derived categories and the geometry of varieties is further illustrated in the third chapter, where Enriques surfaces are studied. Section 4.1 recalls the intersection form on divisors, the Hodge index theorem, and the notions necessary to introduce the Néron–Severi group $\mathrm{NS}(S)$ and the lattice $\mathrm{Num}(S)$ with its intersection form; in this context we also define the nef cone of the surface, exploring its relation to the effective and ample cones via the Nakai–Moishezon criterion and Kleiman duality.

Section 4.2 presents the generalities of K3 and Enriques surfaces: an Enriques surface is

a smooth projective surface with $H^1(S, \mathcal{O}_S) = 0$ and a non-trivial canonical bundle K_S such that $K_S^{\otimes 2} \cong \mathcal{O}_S$, and every Enriques surface is the quotient of a K3 surface by a fixed-point-free involution.

Section 4.3 is the technical core of the thesis. We introduce the Weyl group $W(S)$ of $\mathrm{Num}(S)$ and its subgroup $W^{\mathrm{nod}}(S)$ generated by the reflections associated with (-2)-curves, and we characterize geometrically the nef cone of an Enriques surface as a fundamental chamber for the action of $W^{\mathrm{nod}}(S)$, whose bounding walls are defined by the reflections associated with (-2)-curves. A pivotal result is the following: an Enriques surface always admits a genus 1 fibration over the projective line, i.e. a map $S \to \mathbb{P}^1$ whose general fibre is an elliptic curve. Furthermore, any such fibration has exactly two multiple fibres, $2F$ and $2F'$. Taking one of the reduced fibres, say F, we obtain an effective divisor whose class in $\mathrm{NS}(S)$ we denote here $[F]_{\mathrm{NS}}$; similarly, its image under the quotient map $\mathrm{NS}(S) \to \mathrm{Num}(S)$ will be $[F]_{\mathrm{Num}}$. The length-one sequence $\{[F]_{\mathrm{Num}}\}$ can be extended to a sequence $\{[F]_{\mathrm{Num}}, [v_2], \ldots, [v_{10}]\} \subset \mathrm{Num}(S)$, which can be manipulated through the action of $W^{\mathrm{nod}}(S)$ in order to obtain a canonical expression, which depends on the non-degeneracy invariant of S. For a general Enriques surface, which admits no smooth rational curves, this expression is simply a sequence $\{[F_1], \ldots, [F_{10}]\}$. We then use this sequence to obtain a collection of subcategories of $D^b(S)$ that make $D^b(S) = \langle \mathcal{A}, \mathcal{F}_1, \ldots, \mathcal{F}_{10} \rangle$ a semiorthogonal decomposition of $D^b(S)$, where $\mathcal{A}$ is the orthogonal complement of the $\mathcal{F}_i$. Observe that, since the $\mathcal{F}_i$ are constructed naturally from S, the complexity of S is entirely encoded in the subcategory $\mathcal{A}$, often called the residual component of S. A series of examples in this section illustrates when such collections are only semiorthogonal and when they are fully orthogonal.

Finally, in Section 4.4 we state the derived Torelli theorems. The classical one asserts that an equivalence $D^b(S_1) \cong D^b(S_2)$ between the derived categories of two Enriques surfaces forces $S_1 \cong S_2$; the refined version shows that the same conclusion holds if one only assumes that the residual components of S_1 and S_2, associated with suitable collections of exceptional line bundles, are equivalent. We sketch the proof of the refined statement, which relies on the classification of 3-spherical objects in the residual component and on a criterion allowing us to extend a Fourier–Mukai equivalence between residual components to the whole derived category.

Chapter 2

Preliminaries

The aim of this thesis is to introduce the derived category and, in the end, study semiorthogonal decompositions of the bounded derived category $D^b(S)$ of an Enriques surface S. This requires two main kinds of preliminary material, which we collect in this chapter.

The first regards our main object of study. The category $D^b(S)$ is, by definition, the derived category of the abelian category $\mathrm{Coh}(S)$ of coherent sheaves on S. We therefore recall the basic properties of coherent sheaves that we will use throughout: if S is an Enriques surface, $\mathrm{Coh}(S)$ is abelian and k-linear, much like the larger category $\mathrm{QCoh}(S)$. Moreover, every coherent sheaf admits a finite locally free resolution, so that, in a way, $D^b(S)$ is controlled by the vector bundles over the variety. Indeed, among the most important objects in our study will be line bundles $\mathcal{O}_S(D)$, and much of the discussion about exceptional objects reduces to computing $\mathrm{Ext}^i(\mathcal{O}_S(A), \mathcal{O}_S(B)) \cong H^i(S, \mathcal{O}_S(B-A))$ by means of the Riemann–Roch theorem and Serre duality (see Section 4.3).

The second concerns the language in which the results are stated. The derived category is not abelian, but it carries the structure of a *triangulated category*; this is the right setting to speak of exceptional objects and semiorthogonal decompositions, two concepts that will prove essential for our purposes. Triangulated categories will be introduced in Section 2.2.

2.1 (Quasi) Coherent Sheaves

The notion of sheaf was introduced by Jean Leray (1906 – 1998) in the mid-1940s [Ler46], in the context of algebraic topology, as a tool to pass from local data on a space to global information. Sheaves of modules over a sheaf of rings, and in particular *coherent* sheaves, first appeared in analytic context, in the work of Kiyoshi Oka (1901 – 1978) [Oka50] and Henri Cartan (1904 – 2008) [Car50], while the passage to algebraic geometry is due to Jean-Pierre Serre (b. 1926) [Ser55]. In the same years, the work of Alexander Grothendieck (1928 – 2014) on abelian categories and homological algebra [Gro57] and on the theory of schemes [GD60] placed these ideas in the general framework in which we work today. There, *quasi-coherent* sheaves are the natural generalisation of modules over a ring, and coherent sheaves are those of finite type.

The reason why both notions appear in this section is the following. The category $\mathrm{Coh}(X)$ is the one we are really interested in, since it contains the aforementioned vector bundles.

The larger category $\mathrm{QCoh}(X)$, on the other hand, has enough injectives, which makes it the convenient place to construct derived functors (see Section 3.2). We will therefore work with the former, and use the properties of the latter only when needed.

Our references for this section are [Har77, Section II.5] and [Liu02, Chapter 5].

Recall that a *ringed space* is a pair $(X, \mathcal{O}_X)$ with X a topological space, and $\mathcal{O}_X$ a sheaf of rings over X (generally, whenever X is a variety, $\mathcal{O}_X$ is the sheaf of regular functions, also called *structure sheaf*).

Definition 2.1. Let $(X, \mathcal{O}_X)$ be a ringed space. A *sheaf of $\mathcal{O}_X$-modules*, or simply an *$\mathcal{O}_X$-module*, is a sheaf $\mathcal{F}$ on X such that, for each open set $U \subset X$, we have that:

i) $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module;
ii) for each inclusion of open sets $V \subset U$ the restriction morphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structures via the ring homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$.

In other words, point ii) requires that $(af)|_V = a|_V f|_V$ for every $a \in \mathcal{O}_X(U)$ and every $f \in \mathcal{F}(U)$.

A *morphism* of $\mathcal{O}_X$-modules $\mathcal{F} \to \mathcal{G}$ is a morphism of sheaves such that, for each open set $U \subset X$, the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$-modules.

Some classical examples of $\mathcal{O}_X$-modules include the structure sheaf $\mathcal{O}_X$ itself (along with the *free* sheaves $\mathcal{O}_X^{\oplus n}$), the sheaf of smooth sections $\mathcal{C}^\infty(E)$ of a vector bundle $E \to X$, and the sheaf of differential 1-forms Ω^1_X.

The category of $\mathcal{O}_X$-modules is denoted $\mathrm{Sh}_{\mathcal{O}_X}$. Having recalled this concept, we can now introduce:

Definition 2.2. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is said to be *quasi-coherent* if, for every $x \in X$, there exist an open neighbourhood U of x and an exact sequence of $\mathcal{O}_X$-modules:

$$\mathcal{O}_X^{(J)}|_U \to \mathcal{O}_X^{(I)}|_U \to \mathcal{F}|_U \to 0$$

for some index sets I and J.

Again, the structure sheaf and all of its powers $\mathcal{O}_X^{(I)}$ are quasi-coherent. Indeed, once we fix I we can always pick $J = \varnothing$ to obtain (for each open set $U \subset X$) the exact sequence:

$$0 \to \mathcal{O}_X^{(I)} \to \mathcal{O}_X^{(I)} \to 0.$$

Another classical example of a quasi-coherent sheaf is the sheaf of rational functions, $\mathcal{K}_X$. The category of quasi-coherent sheaves over a variety X is denoted by $\mathrm{QCoh}(X)$. We will now introduce coherent sheaves, which need a preliminary definition:

Definition 2.3. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is *finitely generated* if, for every $x \in X$, there exist an open neighbourhood U of x, an integer $n \geq 0$, and a surjective morphism $\mathcal{O}_X^n|_U \to \mathcal{F}|_U$.

Definition 2.4. Consider again a ringed space $(X, \mathcal{O}_X)$ with an $\mathcal{O}_X$-module $\mathcal{F}$. We say that $\mathcal{F}$ is *coherent* if it is finitely generated and if, for every open subset $U \subset X$, and every homomorphism:

$$\alpha \colon \mathcal{O}_X^n|_U \to \mathcal{F}|_U,$$

the kernel $\mathrm{Ker}(\alpha)$ is finitely generated (as an $\mathcal{O}_X|_U$-module).

The category of coherent sheaves over a variety X is denoted by $\mathrm{Coh}(X)$. Recall that a topological space X is said to be *noetherian* if it satisfies the descending chain condition for closed subsets, i.e., for any sequence $Y_1 \supset Y_2 \supset \cdots$ of closed subsets Y_i of X, there is an integer m such that $Y_m = Y_{m+1} = \cdots$.

Definition 2.5. A topological space X is *locally noetherian* if any point $x \in X$ admits a noetherian neighbourhood.

Any reasonable space one generally works with in algebraic geometry is both noetherian and locally noetherian; this is naturally the case for algebraic varieties. We can now state some useful characterizations of coherence.

Remark 2.6. If X is locally noetherian and $\mathcal{O}_X$ is coherent itself (in particular, this is the case if X is an algebraic variety), the coherence of an $\mathcal{O}_X$-module $\mathcal{F}$ can equivalently be defined as follows. We have that $\mathcal{F}$ is coherent if, for any point $x \in X$, there exist an open neighbourhood U of x, integers $n, m \geq 0$ and an exact sequence of $\mathcal{O}_X|_U$-modules:

$$\mathcal{O}_X^n|_U \to \mathcal{O}_X^m|_U \to \mathcal{F}|_U \to 0.$$

This equivalence clarifies immediately how coherence implies quasi-coherence in this context.

The following result gives another intuitive description of coherent sheaves.

Proposition 2.7. *Let X be an algebraic variety, with $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. The following properties are equivalent:*

i) $\mathcal{F}$ is coherent;
ii) $\mathcal{F}$ is finitely generated;
iii) for every affine open subset $U \subset X$, $\mathcal{F}(U)$ is a finitely generated module over $\mathcal{O}_X(U)$.

Over an algebraic variety, among the typical examples of coherent sheaves (which are equivalently finitely generated, by Proposition 2.7) we have, once again, the structure sheaf $\mathcal{O}_X$ and its finite direct sums $\mathcal{O}_X^{\oplus n}$ (coherence is obvious, by Remark 2.6). Other examples are line bundles (more precisely, the sheaves of the local sections of a given line bundle), the skyscraper sheaf $k(x)$ supported at a point $x \in X$, and the sheaf of differential 1-forms Ω_X^1. On the other hand, an easy example of a quasi-coherent sheaf which is not coherent is the direct sum of infinitely many copies of the structure sheaf:

$$\mathcal{F} = \bigoplus_{i \in I} \mathcal{O}_X \quad \text{with } |I| = \infty.$$

Another example in this sense is the sheaf of rational functions, $\mathcal{K}_X$.

Fix an *affine* algebraic variety X over an algebraically closed field. We want to relate $\mathrm{QCoh}(X)$ and $\mathrm{Coh}(X)$ to other categories with well-known properties, in order to draw

some conclusions on the former two. To start, A will denote $A[X] = \mathcal{O}_X(X)$, the ring of coordinates of X. When given a certain A-module, we want to devise a systematic way to construct a quasi-coherent sheaf. In other words, we are looking for a functor:

$$\widetilde{(-)}\colon A\text{-Mod} \to \mathrm{QCoh}(X), \quad M \mapsto \widetilde{M}.$$

We can achieve this in the following way. First, recall that the sets of the form:

$$D(f) = \{x \in X : f(x) \neq 0\}$$

for $f \in A$ constitute a base for the Zariski topology on X. Now let us fix an A-module M. Initially we define $\widetilde{M}$ on these base sets:

$$\widetilde{M}(D(f)) \coloneqq M_f = M \otimes_A A_f = \left\{ \frac{m}{f^k} \;\middle|\; m \in M, k \geq 0 \right\},$$

where M_f is the localization of the A-module M with respect to the multiplicative system $\{1, f, f^2, \dots\}$. Restrictions can be defined naturally as follows: consider the case $D(g) \subset D(f)$. By Hilbert's Nullstellensatz, there exist $a \in A$ and an integer $n \geq 1$ such that $g^n = af$, which means that f is invertible in the localized ring A_g. Therefore, we can define:

$$\rho_{D(g),D(f)} \left(\frac{m}{f^k} \right) \coloneqq \frac{m \cdot a^k}{(af)^k} = \frac{m \cdot a^k}{g^{nk}}.$$

This $\widetilde{M}$ is a sheaf on the basis $\{D(f)\}$, and gluing together the values of $\widetilde{M}$ in order to define it over any open set (and not just the $D(f)$) is a standard procedure. Furthermore, $\widetilde{M}$ inherits a structure of $\mathcal{O}_X$-module, since for any open set $U \subset X$ the ring $\mathcal{O}_X(U)$ acts on $\widetilde{M}(U)$ by multiplication. On an open set of the basis we have:

$$\mathcal{O}_X(U) \times \widetilde{M}(U) \to \widetilde{M}(U), \quad \left(\frac{a}{f^k}, \frac{m}{f^j} \right) \mapsto \frac{a \cdot m}{f^{k+j}}.$$

Finally, $\widetilde{(-)}$ sends a morphism of A-modules $\varphi\colon M \to N$ to a morphism of $\mathcal{O}_X$-modules $\widetilde{\varphi}\colon \widetilde{M} \to \widetilde{N}$, where:

$$\widetilde{\varphi}_{D(f)} \left(\frac{m}{f^k} \right) = \frac{\varphi(m)}{f^k}.$$

These maps commute with the restriction morphisms, and they glue together to define the global morphism of $\mathcal{O}_X$-modules $\widetilde{\varphi}\colon \widetilde{M} \to \widetilde{N}$. One can also show that $\widetilde{(-)}$ is exact, and that the constructed sheaf is, indeed, quasi-coherent.

We are now interested in the opposite direction: given a quasi-coherent sheaf over X, we can easily obtain an A-module through the global sections functor:

$$\Gamma(X, -)\colon \mathrm{QCoh}(X) \to A\text{-Mod}, \quad \mathcal{F} \mapsto \Gamma(X, \mathcal{F}) = \mathcal{F}(X);$$

here, $\mathcal{F}(X)$ naturally inherits a structure of A-module given by the multiplication $A \times \mathcal{F}(X) \to \mathcal{F}(X)$.

Proposition 2.8. *Let X be an affine algebraic variety. The functors $\widetilde{(-)}$ and $\Gamma(X, -)$ are quasi-inverse to one another:*

$$\Gamma(X, \widetilde{M}) = \widetilde{M}(X) \cong M, \quad \widetilde{\Gamma(X, \mathcal{F})} \cong \mathcal{F}.$$

Therefore, $\mathrm{QCoh}(X)$ is equivalent to A-Mod. Furthermore, restricting the localization functor $\widetilde{(-)}$ provides the equivalence:

$$\mathrm{Coh}(X) \cong A\text{-Mod}_{\mathrm{fg}},$$

where A-$\mathrm{Mod}_{\mathrm{fg}}$ denotes the category of finitely generated A-modules.

Corollary 2.9. *Let X be an affine algebraic variety. Then* $\mathrm{QCoh}(X)$ *and* $\mathrm{Coh}(X)$ *are abelian categories.*

Proof. Being abelian is invariant under equivalence of categories, and A-Mod is abelian. The category A-$\mathrm{Mod}_{\mathrm{fg}}$ is abelian too: since A is noetherian, kernels and cokernels of morphisms of finitely generated A-modules are finitely generated. □

Remark 2.10. If X is not affine, $\mathrm{QCoh}(X)$ need not be equivalent to A-Mod. Nevertheless, $\mathrm{QCoh}(X)$ and $\mathrm{Coh}(X)$ are abelian subcategories of $\mathrm{Sh}_{\mathcal{O}_X}$ for every variety X. Indeed, kernels and cokernels in $\mathrm{Sh}_{\mathcal{O}_X}$ commute with restriction to open subsets, and quasi-coherence is a local property. So we may restrict to an affine open subset U, where $\mathcal{F}|_U$ is again quasi-coherent (resp. coherent), and conclude by Corollary 2.9 applied to U.

We will now close this section with a collection of results and remarks which will be used in the rest of the thesis.

Definition 2.11. Let $(X, \mathcal{O}_X)$ be a ringed space. A sheaf $\mathcal{F}$ on X is said to be *free of rank n* if it is isomorphic to $\mathcal{O}_X^{\oplus n}$. Moreover, $\mathcal{F}$ is *locally free of rank n* if there exists a covering $\{U_i\}$ of X such that $\mathcal{F}|_{U_i}$ is free of rank n for every i.

Observe that line bundles are, in particular, locally free sheaves of rank 1: the existence of a local frame over a given open set U makes the isomorphism $\mathcal{F}|_U \cong \mathcal{O}_X|_U$ immediately clear.

Proposition 2.12. *Let X be a quasi-projective variety. Any coherent sheaf $\mathcal{F}$ over X is a quotient of a locally free sheaf, and thus admits a* locally free resolution*, i.e. there exists an exact sequence of $\mathcal{O}_X$-modules:*

$$\cdots \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{L}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

where all the $\mathcal{L}_i$ are locally free. If, in addition, X is smooth, then $\mathcal{F}$ admits a finite locally free resolution of length at most $\dim(X)$*.*

The following result will be useful when characterizing the Serre functor of a given variety.

Proposition 2.13. *Tensoring by a locally free sheaf $\mathcal{M}$ gives an exact functor:*

$$\mathrm{Coh}(X) \longrightarrow \mathrm{Coh}(X), \quad \mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{M}.$$

One final property of $\mathrm{Coh}(X)$ and $\mathrm{QCoh}(X)$ that we want to state regards injective objects; this fact will come up when talking about derived functors (see section 3.2), and is among the reasons why we are also interested in $\mathrm{QCoh}(X)$.

Theorem 2.14. *Let X be an algebraic variety. Then* $\mathrm{QCoh}(X)$ *has enough injectives, i.e. every object $\mathcal{F} \in \mathrm{QCoh}(X)$ can be embedded in an injective object $\mathcal{G}$ through a monomorphism $\mathcal{F} \to \mathcal{G}$.*

Remark 2.15. Conversely, in general $\mathrm{Coh}(X)$ does not have enough injectives. Let $X = \mathbb{A}^1$, so that $A = k[t]$ and, by Proposition 2.8, $\mathrm{Coh}(X) \cong k[t]$-$\mathrm{Mod}_{\mathrm{fg}}$. We show that the only injective object of $\mathrm{Coh}(X)$ is 0; in particular, $\mathrm{Coh}(X)$ does not have enough injectives, as no non-zero object can be embedded in an injective object. Thanks to

the equivalence above, it suffices to work in $k[t]$-$\mathrm{Mod}_{\mathrm{fg}}$. Note that a morphism u in this category is a monomorphism if and only if it is injective: indeed, $\mathrm{Ker}(u)$ is finitely generated.

Let $I \neq 0$ be injective. First, we show that $I = fI$ for every nonzero $f \in k[t]$: let $x \in I$. The multiplication $\mu_f \colon k[t] \to k[t]$ is a monomorphism, and $\varphi \colon k[t] \to I$, $a \mapsto ax$, is a morphism. By injectivity there exists $\psi \colon k[t] \to I$ with $\psi \circ \mu_f = \varphi$, hence:

$$x = \varphi(1) = \psi(f \cdot 1) = f \cdot \psi(1) \in fI.$$

On the other hand, since $k[t]$ is a principal ideal domain and I is finitely generated:

$$I \cong k[t]^r \oplus \bigoplus_{i=1}^{s} k[t]/(p_i^{e_i}),$$

with p_i irreducible and $e_i \geq 1$. As $I \neq 0$, there is at least one summand C. Let $f = t$ if $C = k[t]$, and $f = p_i$ if $C = k[t]/(p_i^{e_i})$. Then $C/fC \cong k[t]/(f) \neq 0$, so $I/fI \cong \bigoplus_j C_j/fC_j \neq 0$, which contradicts $I = fI$. Hence $I = 0$.

2.2 Triangulated Categories

Historically, triangulated categories were first introduced independently by Dieter Puppe (1930 – 2005) [Pup62] and by Jean-Louis Verdier (1935 – 1989) in his doctoral thesis [Ver96], where he formalized ideas by his supervisor, Alexander Grothendieck. Verdier's main motivation for the introduction of this structure was to study and axiomatize the structure of the derived category of an abelian category, a construction which was first introduced in Verdier's thesis as well. Some early applications of these concepts include coherent duality, which generalizes Serre duality, and what would go on to be called Verdier duality, which extends Poincaré duality to singular manifolds. To be precise, the axioms stated by Puppe did not include TR4 (meaning that Puppe actually worked with what we now call *pretriangulated categories*), a kind of simplification which made sense in light of Puppe's focus on the stable homotopy category. The structure of triangulated category will prove essential in understanding and showing many results regarding the derived category, which, as was first noted by Verdier, is endowed with this exact structure. For the general theory of triangulated categories presented in this section, we refer to [Huy06, Chapter 1].

We begin by giving the formal definition of a triangulated category.

Definition 2.16. Let $\mathcal{A}$ be an additive category. A structure of *triangulated category* on $\mathcal{A}$ is given by two additional data: an additive equivalence $T \colon \mathcal{A} \to \mathcal{A}$, called *shift functor*, and a set of sequences of the following form:

$$A \longrightarrow B \longrightarrow C \longrightarrow T(A),$$

where A, B, and C are objects of $\mathcal{A}$. These sequences form the set of *distinguished triangles*, and are subject to a set of four axioms, which we will state shortly.

Before giving the axioms, let us introduce some notation. First of all, the shift functor T will be shortened by using only the notation [1], meaning that $A[1]$ will denote $T(A)$, and

more generally $A[n] \coloneqq T^n(A)$. This implies that the distinguished triangle above can be written as $A \longrightarrow B \longrightarrow C \longrightarrow A[1]$. Similarly, $f[1] \coloneqq T(f)$, and $f[n] \coloneqq T^n(f)$, for any morphism $f \in \mathrm{Hom}(A, B)$.
A morphism of distinguished triangles is given by a commutative diagram of the following form.

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \end{array}$$

Such a morphism is an isomorphism if f, g and h are isomorphisms. Note that this immediately implies that $f[1]$ is an isomorphism as well, since $f[1] = T(f)$ and T is an equivalence, and therefore sends isomorphisms to isomorphisms. The axioms that distinguished triangles are required to satisfy are the following.

TR1. The first axiom describes a set of sequences that are always distinguished triangles, and provides the class of distinguished triangles with some crucial properties. TR1 consists of three conditions:

i) Any sequence of the form $A \xrightarrow{\mathrm{id}} A \longrightarrow 0 \longrightarrow A[1]$ is a distinguished triangle;
ii) Any triangle (i.e. any four-term sequence) isomorphic to a distinguished triangle is a distinguished triangle;
iii) Any morphism $A \xrightarrow{f} B$ can be completed to a distinguished triangle:

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow A[1].$$

The completing object is called the *cone* of the morphism f, and can be denoted $C(f)$.

Therefore, the class of distinguished triangles is closed under isomorphism. Note that iii) acts as an analogue to the existence of cokernels in (pre)abelian categories; this is particularly relevant, because the derived category is generally not abelian. Furthermore, observe that TR1 does not postulate the uniqueness of the completing object C. However, as a consequence of the other axioms, C is unique up to a non-unique isomorphism. For further details, see Corollary 2.24.

TR2. The second axiom expresses the *rotation invariance* of the class of distinguished triangles: rotating a sequence preserves its status as a distinguished triangle. In particular, a triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ is distinguished if and only if the *rotated* triangle:

$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1]$$

is distinguished. We will come back to the minus sign in front of $f[1]$ eventually. For now, we will just observe that this sign is the reason why the initial category $\mathcal{A}$ must be assumed to be additive.

TR3. If there exists a commutative diagram of distinguished triangles with vertical arrows f and g:

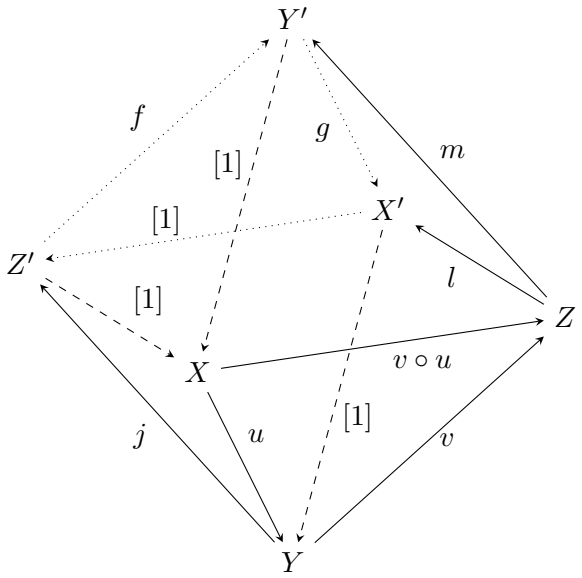
$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \vdots{\scriptstyle \exists h} & & \downarrow{\scriptstyle f[1]} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \end{array}$$

then it can be completed to a commutative diagram (thus, a morphism of distinguished triangles) by a morphism $h\colon C \to C'$. It is important to emphasize that while the axiom asserts the existence of such a morphism, it does not guarantee its uniqueness.

TR4. The fourth axiom is arguably the most complicated to formulate and, as we mentioned earlier, was omitted by Puppe in his original paper. Its primary purpose is to ensure that the cone construction behaves compatibly with the composition of morphisms. TR4 will not be employed in this thesis, but will be stated for completeness. Let $u\colon X \to Y$ and $v\colon Y \to Z$ be morphisms, and consider their composition $v \circ u\colon X \to Z$. Using TR1, we can find three distinguished triangles:

$$X \xrightarrow{u} Y \xrightarrow{j} Z' \longrightarrow X[1]$$
$$Y \xrightarrow{v} Z \xrightarrow{l} X' \longrightarrow Y[1]$$
$$X \xrightarrow{v\circ u} Z \xrightarrow{m} Y' \longrightarrow X[1]$$

The axiom states that the three completing objects X', Y' and Z' can be made into the vertices of the distinguished triangle $Z' \xrightarrow{f} Y' \xrightarrow{g} X' \longrightarrow Z'[1]$ in a way that makes "all the possible diagrams" commutative. To better visualize the requirement of TR4, the following diagram can be taken into consideration:



Some notational remarks need to be made. First, the dashed arrows have been marked with [1] to denote the fact that they represent maps from the completing object to the shift of the object they appear to land onto: for example, the arrow from Y' to X actually indicates the morphism $Y' \to X[1]$. The morphisms whose existence is imposed by TR4 are denoted by dotted arrows. Diagrams like the one above, also showcased in Verdier's original presentation, are the reason why TR4 is also called the *octahedral axiom*.

Remark 2.17. As we briefly mentioned in the introduction to this chapter, if we do not require the octahedral axiom TR4 to be satisfied, we obtain a so-called *pretriangulated category*. Obviously every triangulated category is pretriangulated, but the converse does not hold: an explicit example was recently exhibited by Chen et al. [CIL+26], relying

on techniques from representation theory and module categories over finite-dimensional algebras.

One could observe that, given a distinguished triangle $A \longrightarrow B \longrightarrow C \longrightarrow A[1]$, we have not required that the composition $A \longrightarrow C$ is 0. However, this follows automatically from TR1 and TR3. Indeed:

Proposition 2.18. *For any distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ in a triangulated category $\mathcal{A}$, the composition $A \to C$ is 0.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccccc} A & \xrightarrow{\mathrm{id}} & A & \xrightarrow{0} & 0 & \longrightarrow & A[1] \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f} & & \vdots & & \downarrow{\scriptstyle \mathrm{id}[1]=\mathrm{id}} \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & A[1] \end{array}$$

By TR1, the upper row is a distinguished triangle. Since the lower row is a distinguished triangle by hypothesis and the rest of the diagram obviously commutes, we can use TR3 to conclude that there must exist a morphism $0 \to C$ (necessarily the null morphism) such that the central square commutes as well, i.e. $g \circ f = 0$. □

Distinguished triangles and (long) exact sequences are closely related concepts. This is made clear by the following proposition and by its Corollary, which together tell us that distinguished triangles always induce a (long) exact sequence of morphism groups.

Proposition 2.19. *Let $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ be a distinguished triangle in a triangulated category $\mathcal{D}$. Then, for any object $A_0 \in \mathcal{D}$, the following sequences are exact:*

$$\mathrm{Hom}(A_0, A) \xrightarrow{u_*} \mathrm{Hom}(A_0, B) \xrightarrow{v_*} \mathrm{Hom}(A_0, C),$$
$$\mathrm{Hom}(C, A_0) \xrightarrow{v^*} \mathrm{Hom}(B, A_0) \xrightarrow{u^*} \mathrm{Hom}(A, A_0).$$

Proof. Consider the first sequence. In light of Proposition 2.18, it is clear that $\mathrm{Im}(u_*) \subseteq \mathrm{Ker}(v_*)$. Now suppose that $f\colon A_0 \to B$ composed with $v\colon B \to C$ is the trivial morphism $A_0 \to C$. We can apply TR2 to the triangle $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$, rotating it towards the left, to obtain:

$$B \xrightarrow{v} C \xrightarrow{w} A[1] \xrightarrow{-u[1]} B[1].$$

By TR1, the triangle $A_0 \xrightarrow{\mathrm{id}} A_0 \longrightarrow 0 \longrightarrow A_0[1]$ is distinguished as well; if we rotate it, we find that:

$$A_0 \longrightarrow 0 \longrightarrow A_0[1] \xrightarrow{-\mathrm{id}[1]} A_0[1]$$

is, once again, distinguished. We can now stack these two triangles on top of each other:

$$\begin{array}{ccccccc} A_0 & \longrightarrow & 0 & \longrightarrow & A_0[1] & \xrightarrow{-\mathrm{id}[1]} & A_0[1] \\ \downarrow{\scriptstyle f} & & \downarrow & & \vdots{\scriptstyle g} & & \downarrow{\scriptstyle f[1]} \\ B & \xrightarrow{v} & C & \xrightarrow{w} & A[1] & \xrightarrow{-u[1]} & B[1] \end{array}$$

The square on the left commutes, since we have $v \circ f = 0$ by hypothesis. Therefore, TR3 guarantees the existence of g, which makes the diagram into a morphism of distinguished

triangles. The commutativity of the right square implies that $f[1] \circ (-\mathrm{id}[1]) = -u[1] \circ g$, but since $\mathrm{id}[1] = \mathrm{id}$, we get:

$$u[1] \circ g = f[1].$$

Keeping in mind that the shift functor is an equivalence, and therefore an isomorphism of morphism groups, we can apply its inverse $[-1]$ to the previous identity in order to write that $u \circ g[-1] = f$. Thus, $f = u_*(g[-1])$, which implies $\mathrm{Ker}(v_*) \subseteq \mathrm{Im}(\mathrm{u}_*)$. The second assertion is proved in a similar fashion. □

Corollary 2.20. *Fix an object A_0 in a triangulated category $\mathcal{D}$. A distinguished triangle $A \longrightarrow B \longrightarrow C \longrightarrow A[1]$ always induces long exact sequences:*

$$\cdots \longrightarrow \mathrm{Hom}(A_0, A) \longrightarrow \mathrm{Hom}(A_0, B) \longrightarrow \mathrm{Hom}(A_0, C) \longrightarrow \mathrm{Hom}(A_0, A[1]) \longrightarrow \cdots,$$
$$\cdots \longrightarrow \mathrm{Hom}(A[1], A_0) \longrightarrow \mathrm{Hom}(C, A_0) \longrightarrow \mathrm{Hom}(B, A_0) \longrightarrow \mathrm{Hom}(A, A_0) \longrightarrow \cdots.$$

Proof. Proposition 2.19 shows the exactness of the two sequences at $\mathrm{Hom}(A_0, B)$ and $\mathrm{Hom}(B, A_0)$ respectively, but if we first use TR2 to rotate the triangle, we can immediately conclude exactness at every other node of the sequence. □

We will now focus on the uniqueness of the "completing object" whose existence is imposed by the third point of TR1. Preliminarily, we will need the following important result, which greatly generalizes Cayley's theorem for groups by extending it to any locally small category (i.e., any category whose morphisms are always sets, and never proper classes). Here, Nat will denote the category of natural transformations between two functors. For the next statement we refer to [BP20].

2.21 Yoneda Lemma. *Let $\mathcal{C}$ be a locally small category, with A any object in $\mathcal{C}$.*

i) If F is a functor $\mathcal{C} \to \mathrm{Set}$, then $\mathrm{Nat}(\mathrm{Hom}(A, -), F) \cong F(A)$.
ii) If G is a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, then $\mathrm{Nat}(\mathrm{Hom}(-, A), G) \cong G(A)$.

Furthermore, both isomorphisms are natural when considering both sides as functors from $\mathcal{C} \times \mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ (resp. from $\mathcal{C} \times \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$) to Set.

Remark 2.22. Let $C \in \mathcal{C}$ be an object in the locally small category $\mathcal{C}$. If we apply the covariant version of the Lemma to $F = \mathrm{Hom}(C, -)$ we get that, for any $C' \in \mathcal{C}$:

$$\mathrm{Hom}(C, C') \cong \mathrm{Nat}(\mathrm{Hom}(C', -), \mathrm{Hom}(C, -)), \quad \text{by} \quad f \mapsto f^*.$$

The contravariant version applied to $G = \mathrm{Hom}(-, C')$ yields a similar result:

$$\mathrm{Hom}(C, C') \cong \mathrm{Nat}(\mathrm{Hom}(-, C), \mathrm{Hom}(-, C')), \quad \text{by} \quad f \mapsto f_*.$$

This implies that:

$$f \text{ isomorphism} \iff f_* \text{ isomorphism} \iff f^* \text{ isomorphism}.$$

Proposition 2.23. *Consider the following morphism of distinguished triangles:*

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f[1]} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1], \end{array} \tag{2.1}$$

where A, B, C, A', B' and C' are objects of the triangulated category $\mathcal{D}$. If two of the vertical morphisms f, g and h are isomorphisms, then so is the third.

Proof. First, observe that it suffices to prove the result under the hypothesis that f and g are isomorphisms. Indeed, by TR2 we can rotate both triangles to get:

$$\begin{array}{ccccccc} B & \longrightarrow & C & \longrightarrow & A[1] & \longrightarrow & B[1] \\ {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle f[1]}\downarrow & & \downarrow{\scriptstyle g[1]} \\ B' & \longrightarrow & C' & \longrightarrow & A'[1] & \longrightarrow & B'[1], \end{array}$$

which solves the case g, h isomorphisms (the shift functor is an equivalence, thus $f[1]$ is an isomorphism if and only if f is). By rotating once again we get:

$$\begin{array}{ccccccc} C & \longrightarrow & A[1] & \longrightarrow & B[1] & \longrightarrow & C[1] \\ {\scriptstyle h}\downarrow & & {\scriptstyle f[1]}\downarrow & & {\scriptstyle g[1]}\downarrow & & \downarrow{\scriptstyle h[1]} \\ C' & \longrightarrow & A'[1] & \longrightarrow & B'[1] & \longrightarrow & C'[1], \end{array}$$

which takes care of the case where f and h are isomorphisms. Thus, we assume that f and g are isomorphisms. Fix any object $A_0 \in \mathcal{D}$, and apply the functor $\mathrm{Hom}(A_0, -)$ to diagram (2.1). By Corollary 2.20, we obtain a diagram whose rows are long exact sequences:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \mathrm{Hom}(A_0, A) & \longrightarrow & \mathrm{Hom}(A_0, B) & \longrightarrow & \mathrm{Hom}(A_0, C) & \longrightarrow & \mathrm{Hom}(A_0, A[1]) & \longrightarrow & \mathrm{Hom}(A_0, B[1]) \longrightarrow \cdots \\ & & {\scriptstyle f_*}\downarrow\cong & & {\scriptstyle g_*}\downarrow\cong & & \downarrow{\scriptstyle h_*} & & {\scriptstyle f[1]_*}\downarrow\cong & & {\scriptstyle g[1]_*}\downarrow\cong \\ \cdots & \longrightarrow & \mathrm{Hom}(A_0, A') & \longrightarrow & \mathrm{Hom}(A_0, B') & \longrightarrow & \mathrm{Hom}(A_0, C') & \longrightarrow & \mathrm{Hom}(A_0, A'[1]) & \longrightarrow & \mathrm{Hom}(A_0, B'[1]) \longrightarrow \cdots . \end{array}$$

Since f and g are isomorphisms and $[1]$ is an equivalence, we have that all the vertical morphisms in the resulting diagram are isomorphisms, except for h_*. But by the Five Lemma, we immediately find that h_* must be an isomorphism for all $A_0 \in \mathcal{D}$, and we conclude by Remark 2.22. □

Corollary 2.24. *Let $f\colon A \to B$ be a morphism between objects $A, B \in \mathcal{D}$, with $\mathcal{D}$ a triangulated category. Then the cone $C(f)$ is unique up to isomorphism.*

Proof. Suppose there are two cones C and C', i.e. both rows in the following diagram are distinguished triangles:

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \| & & \| & & \vdots\downarrow & & \| \\ A & \longrightarrow & B & \longrightarrow & C' & \longrightarrow & A[1]. \end{array}$$

The dotted morphism exists thanks to TR3, and by applying Proposition 2.23 we immediately conclude that it must be an isomorphism $C \cong C'$. □

Chapter 3

The Derived Category

This chapter is dedicated to constructing the central notion of this thesis, namely the derived category of an abelian category. As will be shown in Chapter 4, the reasons that motivate this kind of construction are largely geometric. When studying geometric objects, classical invariants like (co)homology groups often discard too much information. For instance, in classical topology, it is easy to find spaces that share the exact same (co)homology but are homotopically distinct.

Example 3.1. Take the torus $X = S^1 \times S^1$, and $Y = S^1 \vee S^1 \vee S^2$ (the wedge sum of two circumferences and a 2-sphere). The (co)homology groups of degree 0, 1 and 2 of both spaces are $\mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z}$, and $\mathbb{Z}$ respectively. However, their homotopy types differ, since $\pi_1(X) \cong \mathbb{Z} \times \mathbb{Z} \ncong \mathbb{Z} * \mathbb{Z} \cong \pi_1(Y)$.[1]

To prevent this loss of information, which stems from applying the (co)homology functors, the idea of homological algebra is to shift the focus from the (co)homology groups back to the underlying complexes. The derived category $D^b(X)$ provides the precise framework to achieve this, allowing us to retain data that would otherwise vanish.

The focus of the final chapter of the thesis is on the derived category of the category of coherent sheaves over a variety. This case also provides us with the most concrete evidence of the power of $D^b(X)$ as an invariant of geometric objects. However, we will introduce derived categories in a more general setting, and explain the construction for a generic abelian category, before shifting the focus to Fourier-Mukai functors and, eventually, to $D^b(\mathrm{Coh}(X))$.

3.1 The Construction of $D(\mathcal{A})$

The construction of the derived category presented in this section, along with the foundational results on roofs and quasi-isomorphisms, follows closely the treatment given by [Huy06, Section 2.1]. Let us begin by recalling the category of complexes of objects of an

[1] As a side note, while the derived category retains strictly more information than classical (co)homology, the fundamental group itself is not invariant under derived equivalence in the context of algebraic varieties (as shown by Schnell, [Sch12]), i.e. there exist varieties with equivalent derived categories but distinct fundamental groups.

abelian category $\mathcal{A}$. A *complex* in $\mathcal{A}$ is a diagram of the form:

$$\cdots \longrightarrow A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \longrightarrow \cdots$$

such that, for all $i \in \mathbb{Z}$, $\mathrm{Im}(d^{i-1}) \subset \mathrm{Ker}(d^i)$, or equivalently $d^i \circ d^{i-1} = 0$. A morphism $f\colon A^\bullet \to B^\bullet$ consists of a commutative diagram:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A^{i-1} & \xrightarrow{d_A^{i-1}} & A^i & \xrightarrow{d_A^i} & A^{i+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f^{i-1}} & & \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle f^{i+1}} & & \\
\cdots & \longrightarrow & B^{i-1} & \xrightarrow{d_B^{i-1}} & B^i & \xrightarrow{d_B^i} & B^{i+1} & \longrightarrow & \cdots
\end{array}$$

Definition 3.2. The *category of complexes* $C(\mathcal{A})$ of an abelian category $\mathcal{A}$ is the category whose objects are complexes in $\mathcal{A}$ and whose morphisms are morphisms of complexes, defined as above.

Proposition 3.3. *If $\mathcal{A}$ is an abelian category, then $C(\mathcal{A})$ is abelian as well.*

Proof. After observing that all the constructions can be carried out componentwise, the proof is straightforward, thus we will leave out the details. The additivity of $C(\mathcal{A})$ is proved by defining, for any two complex morphisms $f, g\colon A^\bullet \to B^\bullet$, $(f+g)^n = f^n + g^n$. Unsurprisingly, the null object is the complex $\cdots \to 0 \to 0 \to 0 \to \cdots$, and the biproduct of two complexes is given by $(A^\bullet \oplus B^\bullet)^n = A^n \oplus B^n$.
Likewise, preabelianity is proved by defining kernels and cokernels componentwise. If $f\colon A^\bullet \to B^\bullet$ is a complex morphism, then:

$$\begin{aligned} K^n &\coloneqq (\mathrm{Ker}(f))^n = \mathrm{Ker}(f^n) \\ C^n &\coloneqq (\mathrm{Coker}(f))^n = \mathrm{Coker}(f^n), \end{aligned}$$

with differentials $d_K^n\colon K^n \to K^{n+1}$, $d_C^n\colon C^n \to C^{n+1}$ induced by the universal properties of kernels and cokernels. From here, we can deduce abelianity by proving that the canonical map $s(f)\colon \mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism for any complex morphism f, a result that can be obtained, once again, by reasoning componentwise, and using the abelianity of $\mathcal{A}$. □

Before continuing with the construction, it is useful to clarify the role of $C(\mathcal{A})$ within the broader goal of this chapter, and to state the intentions behind what will follow. The category of chain complexes $C(\mathcal{A})$ is the natural setting for homological algebra, as it allows us to package collections of objects and morphisms between them into single objects. However, $C(\mathcal{A})$ alone is too rigid: when working with complexes of sheaves, we often want to consider two complexes as "equivalent" whenever there exists a map between them inducing an isomorphism on cohomology[2] (a *quasi-isomorphism*: see Definition 3.8), even if they are not isomorphic in $C(\mathcal{A})$. This is due to reasons that will be more thoroughly explained in the following pages (see Remark 3.9). Therefore, we want quasi-isomorphisms to become true isomorphisms. This process comes at a price: the abelian structure that $C(\mathcal{A})$ has just been shown to possess is lost along the way. To compensate, we will

[2]Note that having isomorphic cohomology objects $H^*(A^\bullet) \cong H^*(B^\bullet)$ is not sufficient for $A^\bullet$ and $B^\bullet$ to be quasi-isomorphic, as such an isomorphism of cohomologies may not be induced by any morphism between the original complexes.

endow the resulting category with a triangulated structure, where short exact sequences are replaced by distinguished triangles. In this regard, a shift functor $T\colon D(\mathcal{A}) \to D(\mathcal{A})$ needs to be introduced at some point. It turns out that said shift functor will be induced by the following functor, defined at the complex level:

Definition 3.4. Take a complex $A^\bullet \in C(\mathcal{A})$. Then $A^\bullet[1]$ is the complex with $(A^\bullet[1])^i \coloneqq A^{i+1}$, and differential $d^i_{A[1]} \coloneqq -d^{i+1}_A$. The shift $f[1]$ of a morphism of complexes $f\colon A^\bullet \to B^\bullet$ is the complex morphism $f[1]\colon A^\bullet[1] \to B^\bullet[1]$ given by $f[1]^i \coloneqq f^{i+1}$.

Again, we will come back to the minus sign in front of the shifted differential after introducing homotopy.

Proposition 3.5. *The shift functor above, $T\colon C(\mathcal{A}) \to C(\mathcal{A})$, $A^\bullet \mapsto A^\bullet[1]$, defines an equivalence of abelian categories.*

Proof. Rather than employing the classical characterization of equivalences, this result is more easily proved by showing an explicit quasi-inverse of T. Specifically, it is not difficult to show that $S\colon C(\mathcal{A}) \to C(\mathcal{A})$, $A^\bullet \mapsto A^\bullet[-1]$, with $(A^\bullet[-1])^i \coloneqq A^{i-1}$ and $d^i_{A[-1]} \coloneqq -d^{i-1}_A$, is a quasi-inverse to T. □

At this point, one could be tempted to use $T\colon C(\mathcal{A}) \to C(\mathcal{A})$ to define a triangulated structure on $C(\mathcal{A})$, which is additive as needed. To do so, we would need to define a class of distinguished triangles. The most obvious choice would be short exact sequences of complexes (note that Proposition 3.3 is what enables us to talk about exact sequences in $C(\mathcal{A})$). This, however, does not work.

Exercise 3.6. Let us try to define distinguished triangles in $C(\mathcal{A})$ as the class of sequences of the form $A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \xrightarrow{0} A^\bullet[1]$, where the objects $A^\bullet$, $B^\bullet$ and $C^\bullet$ form a short exact sequence $0 \longrightarrow A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \longrightarrow 0$. We want to show that this choice of distinguished triangles is incompatible with TR2. Consider a short exact sequence:

$$0 \longrightarrow A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \longrightarrow 0.$$

The choice we made implies that $A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \xrightarrow{0} A^\bullet[1]$ is distinguished. But by TR2, the rotated triangle $B^\bullet \xrightarrow{g} C^\bullet \xrightarrow{0} A^\bullet[1] \xrightarrow{-f[1]} B^\bullet[1]$ must be distinguished as well. This forces the exactness of the short sequence:

$$0 \longrightarrow B^\bullet \xrightarrow{g} C^\bullet \xrightarrow{0} A^\bullet[1] \longrightarrow 0,$$

as every distinguished triangle must come from a short exact sequence. Now, the surjectivity of 0 immediately yields the triviality of $A^\bullet[1]$, which in turn implies that g must be an isomorphism. This is obviously false in general; thus, TR2 cannot hold in this setting.

The exercise above tells us that some additional structure is needed. Recall the following, pivotal result:

3.7 Fundamental Theorem of Homological Algebra. *Any short exact sequence of complexes:*

$$0 \longrightarrow A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \longrightarrow 0$$

induces a long exact sequence in cohomology:

$$\cdots \longrightarrow H^i(A^\bullet) \xrightarrow{H^i(f)} H^i(B^\bullet) \xrightarrow{H^i(g)} H^i(C^\bullet) \xrightarrow{\delta^i} H^{i+1}(A^\bullet) \longrightarrow \cdots$$

where H^i denotes the i-th cohomology functor, defined on objects as the quotient:

$$H^i(A^\bullet) \coloneqq \frac{\mathrm{Ker}(d_A^i)}{\mathrm{Im}(d_A^{i-1})},$$

and $\delta^i \colon H^i(C^\bullet) \to H^{i+1}(A^\bullet)$ is the connecting homomorphism obtained via the standard diagram chase of the Snake Lemma.

A key observation is that any morphism $f \colon A^\bullet \to B^\bullet$ induces natural homomorphisms $H^i(f) \colon H^i(A^\bullet) \to H^i(B^\bullet)$ for every degree $i \in \mathbb{Z}$.

Definition 3.8. A morphism of complexes $f \colon A^\bullet \to B^\bullet$ is a *quasi-isomorphism* (qis) if the induced map $H^i(f) \colon H^i(A^\bullet) \to H^i(B^\bullet)$ is an isomorphism for all $i \in \mathbb{Z}$.

Quasi-isomorphisms play a central role in the definition of the derived category: to be precise, quasi-isomorphic complexes will be regarded as isomorphic objects in $D(\mathcal{A})$. Among the reasons why this is desirable is the following fact.

Remark 3.9. Let $\mathcal{F}$ be an object (in a certain abelian category) which admits a left resolution, given by:

$$\cdots \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0. \tag{3.1}$$

Denote by $\mathcal{F}^\bullet$ the complex with $(\mathcal{F}^\bullet)^i = 0$ for all $i \neq 0$, and $(\mathcal{F}^\bullet)^0 = \mathcal{F}$. Also, $\mathcal{E}^\bullet$ will indicate the complex obtained by adding $\mathcal{E}_i = 0$ for all $i < 0$ to the right of (3.1). $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ are quasi-isomorphic complexes. This is because $H^i(\mathcal{E}^\bullet) = 0 = H^i(\mathcal{F}^\bullet)$ for all degrees $i \neq 0$, and:

$$H^0(\mathcal{E}^\bullet) = \frac{\mathcal{E}_0}{\mathrm{Im}(\mathcal{E}_1 \to \mathcal{E}_0)} = \frac{\mathcal{E}_0}{\mathrm{Ker}(\mathcal{E}_0 \to \mathcal{F})} \cong \mathrm{Im}(\mathcal{E}_0 \to \mathcal{F}) = \mathcal{F} = H^0(\mathcal{F}^\bullet).$$

In other words, the complex associated to a certain object in an abelian category is quasi-isomorphic to the (left) resolutions of said object. This is particularly relevant in the geometric context: if we take $\mathcal{F} \in \mathrm{Coh}(X)$ a coherent sheaf on a smooth projective variety X, then by 2.12 $\mathcal{F}$ admits a locally free (left) resolution of length at most equal to $\dim(X)$, i.e. there exists an exact sequence of the form

$$0 \longrightarrow \mathcal{E}_n \longrightarrow \cdots \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0,$$

where all $\mathcal{E}_i$ are locally free coherent sheaves, and $n \leq \dim(X)$. Thus, $\mathcal{F}$ can then be studied in terms of this resolution; in fact, once made isomorphic in the derived category, studying one will be effectively equivalent to studying the other.

In conclusion, this remark tells us that quasi-isomorphisms are the "correct" notion of equivalence when studying objects via their (co)homology. A sheaf $\mathcal{F}$ and its resolution $\mathcal{E}^\bullet$ yield the same cohomological information, but are not isomorphic in the category of complexes. To identify an object with its resolution, a procedure that makes geometric calculations much less cumbersome, we must formally invert quasi-isomorphisms and turn

them into actual isomorphisms. We shall keep this philosophy in mind now that we want to construct morphisms in the derived category.

Suppose $C^\bullet \to A^\bullet$ is a quasi-isomorphism. For our purposes, any morphism of complexes $f\colon C^\bullet \to B^\bullet$ will have to "count as" a morphism $A^\bullet \to B^\bullet$ in the derived category[3]. This leads to the definition of morphisms in the derived category as diagrams of the form:

$$\begin{array}{ccc} & C^\bullet & \\ {\scriptstyle \text{qis}}\swarrow & & \searrow \\ A^\bullet & & B^\bullet \end{array} \tag{3.2}$$

where $C^\bullet \to A^\bullet$ is a quasi-isomorphism.

We would now like to provide these morphisms (called *roofs*) with some additional structure, and the natural setting to do so is the homotopy category. A reason why this setting makes sense is that a homotopic equivalence is always, in particular, a quasi-isomorphism, as shown in Remark 3.12.

Definition 3.10. Two morphisms of complexes $f, g\colon A^\bullet \to B^\bullet$ are said to be *homotopically equivalent*, denoted by $f \sim g$, if there exists a collection of morphisms $h^i\colon A^i \to B^{i-1}$, $i \in \mathbb{Z}$, such that

$$f^i - g^i = h^{i+1} \circ d_A^i + d_B^{i-1} \circ h^i.$$

This requirement can be more easily visualized with the following diagram:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & A^{i-1} & \xrightarrow{d_A^{i-1}} & A^i & \xrightarrow{d_A^i} & A^{i+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f^{i-1}} \downarrow{\scriptstyle g^{i-1}} & \swarrow{\scriptstyle h^i} & \downarrow{\scriptstyle f^i} \downarrow{\scriptstyle g^i} & \swarrow{\scriptstyle h^{i+1}} & \downarrow{\scriptstyle f^{i+1}} \downarrow{\scriptstyle g^{i+1}} & & \\ \cdots & \longrightarrow & B^{i-1} & \xrightarrow{d_B^{i-1}} & B^i & \xrightarrow{d_B^i} & B^{i+1} & \longrightarrow & \cdots \end{array}$$

Naturally, a morphism f is said to be *homotopically trivial* if it is homotopically equivalent to the trivial morphism, i.e. $f \sim 0$. Furthermore, a morphism $f\colon A^\bullet \to B^\bullet$ is a *homotopy equivalence* if there exists a morphism g, operating in the opposite direction, such that $f \circ g \sim \mathrm{id}_{B^\bullet}$ and $g \circ f \sim \mathrm{id}_{A^\bullet}$.
The *homotopy category of complexes* $K(\mathcal{A})$ is the category whose objects are the objects of $C(\mathcal{A})$, i.e., $\mathrm{Ob}(K(\mathcal{A})) = \mathrm{Ob}(C(\mathcal{A}))$, and whose morphisms are defined by

$$\mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet) := \mathrm{Hom}_{C(\mathcal{A})}(A^\bullet, B^\bullet)/\sim .$$

The following properties are all easily verified:

Proposition 3.11. *Let $\mathcal{A}$ be an abelian category.*

i) Homotopy equivalence between morphisms of complexes in $\mathcal{A}$ is an equivalence relation.

[3]The idea is the following: if a morphism $C^\bullet \xrightarrow{f} A^\bullet$ can become invertible, then we should be able to take any morphism $C^\bullet \to B^\bullet$ and construct a morphism $A^\bullet \to B^\bullet$; of course, if f really was invertible, we could consider the composition $A^\bullet \xrightarrow{f^{-1}} C^\bullet \longrightarrow B^\bullet$.

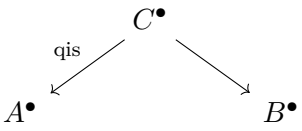
ii) Homotopically trivial morphisms $A^\bullet \to B^\bullet$ form an ideal in the morphism ring $\mathrm{Mor}_{C(\mathcal{A})}(A^\bullet, B^\bullet)$.
iii) If $f \sim g \colon A^\bullet \to B^\bullet$, then $H^i(f) = H^i(g)$ for all $i \in \mathbb{Z}$.
iv) If $f : A^\bullet \to B^\bullet$ and $g : B^\bullet \to A^\bullet$ are such that both their compositions are homotopically equivalent to the identity, i.e. $f \circ g \sim \mathrm{id}_B$ and $g \circ f \sim \mathrm{id}_A$, then f and g are quasi-isomorphisms, and $H^i(f)^{-1} = H^i(g)$.

Remark 3.12. A homotopy equivalence is, in particular, a quasi-isomorphism. This is easily shown by applying the cohomology functor H^i to the relations $f \circ g \sim \mathrm{id}_{B^\bullet}$ and $g \circ f \sim \mathrm{id}_{A^\bullet}$:

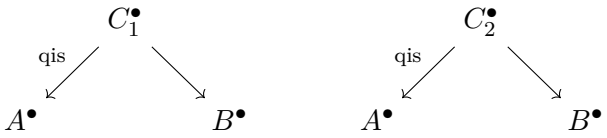
$$H^i(g) \circ H^i(f) = H^i(g \circ f) = H^i(\mathrm{id}_{A^\bullet}) = \mathrm{id}_{H^i(A^\bullet)},$$
$$H^i(f) \circ H^i(g) = H^i(f \circ g) = H^i(\mathrm{id}_{B^\bullet}) = \mathrm{id}_{H^i(B^\bullet)}.$$

This shows that $H^i(f)$ is an isomorphism, with inverse $H^i(g)$, for every degree i.

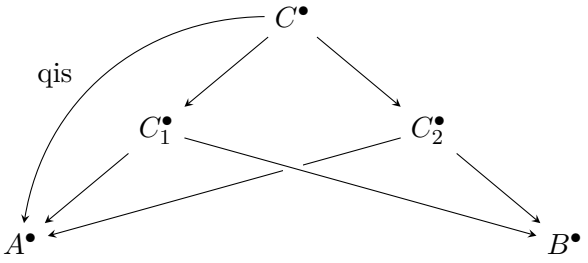
We are now finally ready to properly define $D(\mathcal{A})$, the derived category of an abelian category $\mathcal{A}$. First, we set $\mathrm{Ob}(D(\mathcal{A}))$ as $\mathrm{Ob}(K(\mathcal{A}))$, which is also equal to $\mathrm{Ob}(C(\mathcal{A}))$. We now need to define morphisms. As we mentioned earlier, we want to construct $\mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet)$ starting from the set of roofs defined above. Thus, we will take diagrams of the form:



modulo the following equivalence relation. We say that two diagrams:



are equivalent, meaning that they are the same morphism, if they are "dominated" in the homotopy category by a third, common roof, i.e., if there exists a commutative diagram of the form:



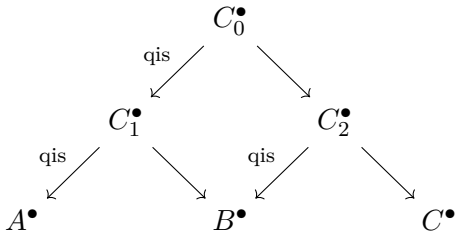
(3.3)

in the homotopy category $K(\mathcal{A})$. The requirement for the commutativity of the diagram can be reinterpreted in the following way. We are asking for the compositions $C^\bullet \to C_1^\bullet \to A^\bullet$ and $C^\bullet \to C_2^\bullet \to A^\bullet$, as well as $C^\bullet \to C_1^\bullet \to B^\bullet$ and $C^\bullet \to C_2^\bullet \to B^\bullet$, to be equal *in the homotopy category*, i.e. in $K(\mathcal{A})$. Therefore, we are requiring these two pairs of compositions to be homotopically equivalent.
The next step is describing how the composition of two (classes of) roofs works. Assume

we are given two roofs:



We want their composition to be given by a diagram of the form:



(3.4)

which is commutative in $K(\mathcal{A})$, i.e. such that $C_0^\bullet \to C_1^\bullet \to B^\bullet$ and $C_0^\bullet \to C_2^\bullet \to B^\bullet$ are homotopically equivalent. In order for this composition to make sense, we will need to prove that such a roof always exists, and that it is well-defined up to the equivalence relation on roofs that we talked about earlier. To achieve this, it is necessary to introduce the *mapping cone* of a complex morphism, which provides an explicit construction for the "completing" object whose existence, in a triangulated category, is required by the third point of TR1. Note that, while the mapping cone is defined for morphisms in the category of complexes, it is only the derived category that will be endowed with a triangulated structure: $C(\mathcal{A})$ is not a triangulated category.

Definition 3.13. Let $f\colon A^\bullet \to B^\bullet$ be a complex morphism. Its *mapping cone* $C(f)$ is the complex defined by:

$$\mathrm{C}(f)^i = A^{i+1} \oplus B^i \quad \text{and} \quad d^i_{\mathrm{C}(f)} := \begin{pmatrix} -d^{i+1}_A & 0 \\ f^{i+1} & d^i_B \end{pmatrix}.$$

It is easily checked that the mapping cone is a complex. In order for $C(f)$ to satisfy the requirement posed by TR1, we need to define morphisms:

$$\tau : B^\bullet \longrightarrow \mathrm{C}(f) \quad \text{and} \quad \pi : \mathrm{C}(f) \longrightarrow A^\bullet[1],$$

and establish that $A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{\tau} C(f) \xrightarrow{\pi} A^\bullet[1]$ is a distinguished triangle, along with any other sequence isomorphic to it. The morphisms τ and π are easily obtained from the canonical injection $B^i \to C(f)^i = A^{i+1} \oplus B^i$, and the canonical projection $C(f)^i = A^{i+1} \oplus B^i \to A^\bullet[1]^i = A^{i+1}$, respectively. It is also easy to show that the composition $B^\bullet \longrightarrow C(f) \longrightarrow A^\bullet[1]$ is trivial, and that the composition $A^\bullet \longrightarrow B^\bullet \longrightarrow C(f)$ is homotopic to the trivial morphism. The calculation that proves this latter property is the reason behind the sign in front of d^{i+1}_A, in the definition of the differential of the mapping cone.

We can now use the cone construction to prove that the composition of roofs proposed in (3.4) is well defined. First, we will need:

Lemma 3.14. *Let $f : A^\bullet \longrightarrow B^\bullet$ be a morphism of complexes, and let $C(f)$ be its mapping cone, with associated morphisms:*

$$\tau \colon B^\bullet \longrightarrow C(f), \quad \textit{and} \quad \pi : C(f) \longrightarrow A^\bullet[1].$$

Then there exists a complex morphism $g \colon A^\bullet[1] \longrightarrow C(\tau)$ which is an isomorphism in $\mathrm{K}(\mathcal{A})$, and which makes the following diagram commutative in $\mathrm{K}(\mathcal{A})$:

$$\begin{array}{ccccccc}
B^\bullet & \xrightarrow{\tau} & C(f) & \xrightarrow{\pi} & A^\bullet[1] & \xrightarrow{-f} & B^\bullet[1] \\
{\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle =}\downarrow \\
B^\bullet & \xrightarrow{\tau} & C(f) & \xrightarrow{\tau_\tau} & C(\tau) & \xrightarrow{\pi_\tau} & B^\bullet[1].
\end{array}$$

Proof. We define the morphism $g \colon A^\bullet[1] \longrightarrow C(\tau)$ as $(-f^{i+1}, \mathrm{id}, 0)$, since g more explicitly operates between these objects:

$$A^\bullet[1]^i = A^{i+1} \longrightarrow C(\tau)^i = B^{i+1} \oplus C(f)^i = B^{i+1} \oplus A^{i+1} \oplus B^i.$$

It is not difficult to check that this is a complex morphism. The inverse g^{-1} is just the projection onto the middle factor; furthermore, the way g is defined immediately tells us that the diagram

$$\begin{array}{ccc}
A^\bullet[1] & \xrightarrow{-f} & B^\bullet[1] \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
C(\tau) & \xrightarrow{\pi_\tau} & B^\bullet[1]
\end{array}$$

is commutative in $\mathrm{C}(\mathcal{A})$. The middle square in the diagram above, i.e.

$$\begin{array}{ccc}
C(f) & \xrightarrow{\pi} & A^\bullet[1] \\
{\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle g} \\
C(f) & \xrightarrow{\tau_\tau} & C(\tau)
\end{array}$$

commutes up to homotopy (but not necessarily in $\mathrm{C}(\mathcal{A})$), and this is shown by checking that $g \circ g^{-1}$ is homotopic to the identity, and by using the fact that $g^{-1} \circ \tau_\tau = \pi$. For further details, see [KS90, Section 1.4]. □

Remark 3.15. The Lemma shows that $K(\mathcal{A})$ satisfies TR2: the rotated triangle $B^\bullet \xrightarrow{\tau} C(f) \xrightarrow{\pi} A^\bullet[1] \xrightarrow{-f} B^\bullet[1]$ is isomorphic to a distinguished triangle, therefore it is distinguished as well.

Remark 3.16. Lemma 3.14 does not hold in the category of complexes $C(\mathcal{A})$: actually, in general, $A^\bullet[1]$ and $C(\tau)$ are not even isomorphic at all in $C(\mathcal{A})$, let alone through g. Indeed, for every $n \in \mathbb{Z}$ we have:

$$C(\tau)^n = B^{n+1} \oplus C(f)^n = B^{n+1} \oplus A^{n+1} \oplus B^n.$$

Take $A^\bullet = 0$ and $B^\bullet = B$ a non-zero object concentrated in degree 0, and $f = 0$. Then $A^\bullet[1] = 0$, whereas $C(\tau)^n \cong B^{n+1} \oplus B^n$ is non-zero for $n = -1, 0$: $C(\tau)$ is the complex $B \xrightarrow{\mathrm{id}} B$ in degrees $-1, 0$. This complex is non-zero in $C(\mathcal{A})$, even though it is acyclic (much like $A^\bullet[1]$), hence zero in $K(\mathcal{A})$ and in $D(\mathcal{A})$.

Proposition 3.17. *Consider a quasi-isomorphism $f\colon A^\bullet \to B^\bullet$, and an arbitrary morphism $g\colon C^\bullet \to B^\bullet$. There exists a commutative diagram:*

$$\begin{array}{ccc} C_0^\bullet & \xrightarrow{\text{qis}} & C^\bullet \\ \downarrow & & \downarrow{\scriptstyle g} \\ A^\bullet & \xrightarrow[\text{qis}]{f} & B^\bullet \end{array}$$

Proof. The existence of a commutative diagram like the one above is true in general, even without the hypothesis of f being a qis: since (see Proposition 3.3) the complex category is abelian, it admits every finite product, including the fiber product of f and g, which automatically provides us with the desired diagram. However, we also want the map $C_0^\bullet \longrightarrow C^\bullet$ to be a qis. The idea is to make use of the following commutative diagram:

$$\begin{array}{ccccccc} C(\tau\circ g)[-1] & \longrightarrow & C^\bullet & \xrightarrow{\tau\circ g} & C(f) & \longrightarrow & C(\tau\circ g) \\ \vdots & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle =} & & \vdots \\ A^\bullet & \xrightarrow{f} & B^\bullet & \xrightarrow{\tau} & C(f) & \longrightarrow & A^\bullet[1]. \end{array}$$

We want to construct the dotted morphism on the left. Thanks to the previous Lemma, we know that $B^\bullet \xrightarrow{\tau} C(f) \longrightarrow A^\bullet[1]$ is isomorphic to the triangle $B^\bullet \xrightarrow{\tau} C(f) \longrightarrow C(\tau)$ in $\mathrm{K}(\mathcal{A})$, therefore $C(\tau) \cong A^\bullet[1]$ in $\mathrm{K}(\mathcal{A})$. We can then find the required morphism by composing this isomorphism with the map $C(\tau \circ g) \longrightarrow C(\tau)$, which is given by:

$$C(\tau\circ g) = C^{i+1} \oplus C(f)^i \xrightarrow{(g^{i+1},\mathrm{id},\mathrm{id})} B^{i+1} \oplus C(f)^i = C(\tau),$$

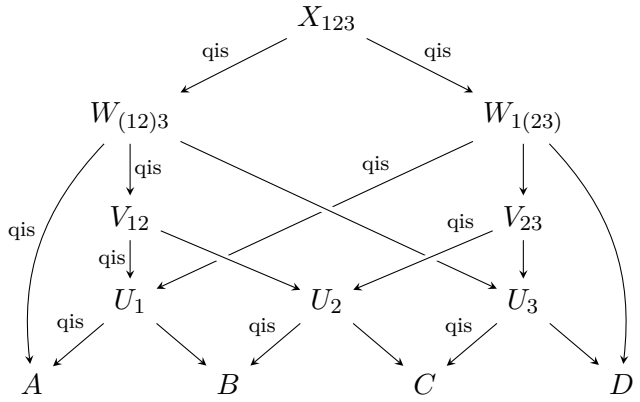
where $C(f)^i = A^{i+1} \oplus B^i$. Shifting backwards by 1 gives the desired map. One then concludes by using the long exact cohomology sequences to check that $C_0^\bullet \coloneqq C(\tau\circ g)[-1] \longrightarrow C^\bullet$ is a qis. □

Using the notation of (3.4), we can apply this Proposition to the quasi-isomorphism $C_2^\bullet \to B^\bullet$ and the morphism $C_1^\bullet \to B^\bullet$, to conclude the existence of the complex $C_0^\bullet$ and of the maps that close up the dominating roof, including the quasi-isomorphism $C_0^\bullet \to C_1^\bullet$.

One should also verify that the equivalence relation defined on roofs is compatible with this composition, making the morphisms in $D(\mathcal{A})$ well-defined. This requires proving that, regardless of the specific representatives chosen for certain classes of roofs $A^\bullet \to B^\bullet$ and $B^\bullet \to C^\bullet$, the resulting composition $A^\bullet \to C^\bullet$ remains the same. The proof of this fact consists of a rather lengthy and technical diagram chase in $K(\mathcal{A})$ relying on the structure introduced previously, and will be left out of this thesis. For a rigorous treatment of this

calculus of fractions and the well-definedness of roof composition, we refer the reader to [GM03, Section III.2].

Remark 3.18. Finally, one should also verify the associativity of roof composition. This property requires a similarly technical diagram chase in $K(\mathcal{A})$. Specifically, to compose three given roofs with vertices U_1, U_2, and U_3, one must construct a diagram taking the form of a five-story "tower" of roofs:



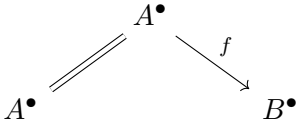
This is just a more complicated version of diagram (3.3). By repeatedly completing common roofs via the calculus of fractions in $K(\mathcal{A})$, it is possible to prove that the two ways of composing the three roofs yield two equivalent roofs, dominated by the apex object X_{123} up to homotopy. Specifically, the proof shows that the roofs given by the compositions:

$$(W_{(12)3} \to V_{12} \to U_1 \to A, W_{(12)3} \to U_3 \to D), \text{ and}$$
$$(W_{1(23)} \to U_1 \to A, W_{1(23)} \to V_{23} \to U_3 \to D)$$

are equivalent. Due to its length and complexity, this standard verification is omitted as well; the complete can also be found in [GM03, Section III.2].

Definition 3.19. Let $\mathcal{A}$ be an abelian category. Its *derived category* $D(\mathcal{A})$ is the category whose objects are $\mathrm{Ob}(D(\mathcal{A})) = \mathrm{Ob}(K(\mathcal{A})) = \mathrm{Ob}(C(\mathcal{A}))$, and whose morphisms are the roofs described in (3.2), with composition given by diagram (3.4).

To confirm that the construction of $D(\mathcal{A})$ fulfills its intended purpose, we perform two natural sanity checks. First, we need to introduce the localization functor $Q\colon K(\mathcal{A}) \to D(\mathcal{A})$, which is the identity on objects, and sends a morphism $f\colon A^\bullet \to B^\bullet$ to the roof:

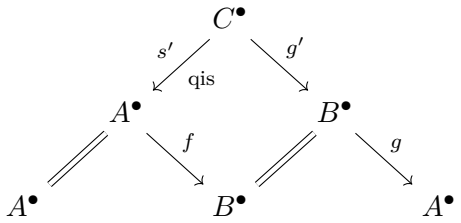


which will be denoted $(\mathrm{id}_{A^\bullet}, f)$ for brevity.

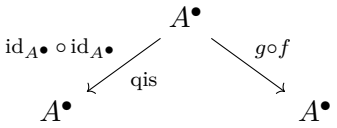
Proposition 3.20. *Let $\mathcal{A}$ be an abelian category, and let $Q\colon K(\mathcal{A}) \to D(\mathcal{A})$ be the canonical localization functor. If $f\colon A^\bullet \to B^\bullet$ is an isomorphism in $K(\mathcal{A})$ (or $C(\mathcal{A})$), then $Q(f)$ is an isomorphism in $D(\mathcal{A})$.*

Proof. Let $f\colon A^\bullet \longrightarrow B^\bullet$ be an isomorphism in $K(\mathcal{A})$. By definition, there exists a morphism $g\colon B^\bullet \longrightarrow A^\bullet$ in $K(\mathcal{A})$ such that $g \circ f = \mathrm{id}_{A^\bullet}$ and $f \circ g = \mathrm{id}_{B^\bullet}$ in $K(\mathcal{A})$. Under the localization functor $Q\colon K(\mathcal{A}) \to D(\mathcal{A})$, f and g are mapped to the roofs $Q(f) = (\mathrm{id}_{A^\bullet}, f)$ and $Q(g) = (\mathrm{id}_{B^\bullet}, g)$, respectively. We claim that $Q(g)$ is the inverse of $Q(f)$ in $D(\mathcal{A})$.

First, consider the composition $Q(g) \circ Q(f)\colon A^\bullet \longrightarrow A^\bullet$, which is given by the composition of the roofs $(\mathrm{id}_{A^\bullet}, f)$ and $(\mathrm{id}_{B^\bullet}, g)$:



The central square is trivially completed by choosing $C^\bullet = A^\bullet$, $s' = \mathrm{id}_{A^\bullet}$, and $g' = f$, since $\mathrm{id}_{B^\bullet} \circ f = f \circ \mathrm{id}_{A^\bullet}$. Thus, the composition is represented by the dominating roof:



Since $g \circ f = \mathrm{id}_{A^\bullet}$ in $K(\mathcal{A})$, this roof reduces directly to the identity roof $(\mathrm{id}_{A^\bullet}, \mathrm{id}_{A^\bullet}) = \mathrm{id}_{D(\mathcal{A})}(A^\bullet)$.

By an entirely symmetric argument, the composition $Q(f) \circ Q(g)\colon B^\bullet \longrightarrow B^\bullet$ yields the roof $(\mathrm{id}_{B^\bullet}, f \circ g) = (\mathrm{id}_{B^\bullet}, \mathrm{id}_{B^\bullet}) = \mathrm{id}_{D(\mathcal{A})}(B^\bullet)$.

Hence, $Q(f)$ is an isomorphism in $D(\mathcal{A})$ with inverse $Q(f)^{-1} = Q(g)$. □

As a final check, we will show that this construction is successful in making quasi-isomorphisms invertible, which was the initial objective.

Proposition 3.21. *Let $\mathcal{A}$ be an abelian category, and let $Q\colon K(\mathcal{A}) \to D(\mathcal{A})$ be the canonical localization functor. If $s\colon A^\bullet \to B^\bullet$ is a quasi-isomorphism, then $Q(s)$ is an isomorphism in $D(\mathcal{A})$, with explicit inverse given by the roof $(s, \mathrm{id}_{A^\bullet})$.*

Proof. Let $s\colon A^\bullet \longrightarrow B^\bullet$ be a quasi-isomorphism. By definition of the localization functor, its image in $D(\mathcal{A})$ is represented by the roof:

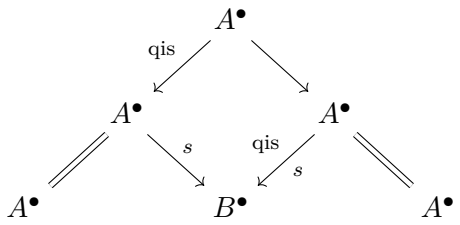
$$Q(s) = (\mathrm{id}_{A^\bullet}, s) = \left(A^\bullet \xleftarrow{\mathrm{id}_{A^\bullet}} A^\bullet \xrightarrow{s} B^\bullet \right).$$

We claim that $Q(s)$ is an isomorphism in $D(\mathcal{A})$, with explicit inverse given by the roof:

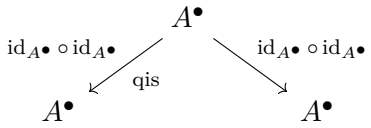
$$r = (s, \mathrm{id}_{A^\bullet}) = \left(B^\bullet \xleftarrow{s} A^\bullet \xrightarrow{\mathrm{id}_{A^\bullet}} A^\bullet \right).$$

First, let us compute the composition $r \circ Q(s)\colon A^\bullet \longrightarrow A^\bullet$. This corresponds to the

composition of roofs:

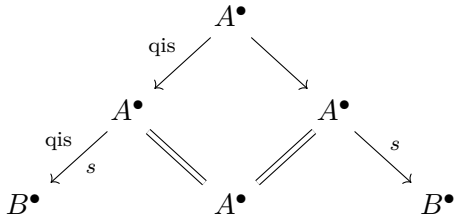


The central square completes trivially by taking the top vertex to be $A^\bullet$ with both downward arrows equal to $\mathrm{id}_{A^\bullet}$, since $s \circ \mathrm{id}_{A^\bullet} = s \circ \mathrm{id}_{A^\bullet}$. Thus, the composition is given by the roof:

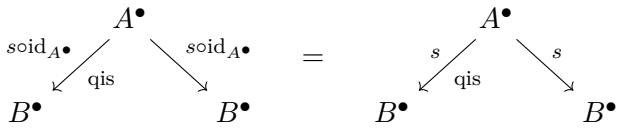


which is precisely the identity roof $(\mathrm{id}_{A^\bullet}, \mathrm{id}_{A^\bullet}) = \mathrm{id}_{D(\mathcal{A})}(A^\bullet)$.

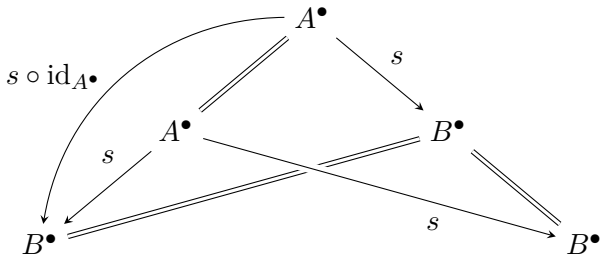
Conversely, consider the composition $Q(s) \circ r \colon B^\bullet \longrightarrow B^\bullet$:



Again, setting the top vertex to $A^\bullet$ with identity maps makes the central square commute ($\mathrm{id}_{A^\bullet} \circ \mathrm{id}_{A^\bullet} = \mathrm{id}_{A^\bullet} \circ \mathrm{id}_{A^\bullet}$). The dominating roof is:



This roof is equivalent to the identity roof $(\mathrm{id}_{B^\bullet}, \mathrm{id}_{B^\bullet})$ on $B^\bullet$ via the morphism $s \colon A^\bullet \to B^\bullet$ (more precisely, via the roof $Q(s) = (\mathrm{id}_{A^\bullet}, s)$), as shown by the commutative diagram:



Therefore, $Q(s) \circ r = \mathrm{id}_{D(\mathcal{A})}(B^\bullet)$, and we conclude that $Q(s)$ is an isomorphism in $D(\mathcal{A})$ with inverse $Q(s)^{-1} = (s, \mathrm{id}_{A^\bullet})$. □

It is often easier to work in special subcategories of $D(\mathcal{A})$, where we impose boundedness conditions on the cohomology of complexes. In particular:

Definition 3.22. Consider $\mathcal{A}$ an abelian category. Let $A^\bullet$ be a complex in $D(\mathcal{A})$, and denote by $H^i(A^\bullet)$ its i-th cohomology group.

i) $D^+(\mathcal{A})$ is the *bounded below derived category*: it consists of complexes $A^\bullet$ such that $H^i(A^\bullet) = 0$ for $i \ll 0$ (i.e., the cohomology vanishes in sufficiently low degrees).

ii) $D^-(\mathcal{A})$ is the *bounded above derived category*: it consists of complexes $A^\bullet$ such that $H^i(A^\bullet) = 0$ for $i \gg 0$ (i.e., the cohomology vanishes in sufficiently high degrees).

iii) $D^b(\mathcal{A})$ is the *bounded derived category*: it consists of complexes $A^\bullet$ whose cohomology is bounded in both directions, meaning $H^i(A^\bullet) = 0$ for all $|i| \gg 0$.

Remark 3.23. As a consequence of Remark 3.15, the subcategories $D^+(\mathcal{A})$, $D^-(\mathcal{A})$, and $D^b(\mathcal{A})$ introduced in Definition 3.22 satisfy TR2. While we will not explicitly verify this fact, the remaining three triangulated axioms TR1, TR3, and TR4 hold as well: this ensures that $D^+(\mathcal{A})$, $D^-(\mathcal{A})$, and $D^b(\mathcal{A})$ are all full *triangulated subcategories* of $D(\mathcal{A})$. For the definition of triangulated subcategory, see Definition 3.33.

3.2 Derived Functors

The material and results presented in this section are based on the standard treatment in Huybrechts [Huy06, Section 2.2].

Now that we have constructed the derived category, in this section we investigate how functors between abelian categories can be extended to this new setting. Consider $\mathcal{A}$ and $\mathcal{B}$ two abelian categories, with $F\colon \mathcal{A} \to \mathcal{B}$ a functor between them; we want F to induce a functor between the derived categories. If F is not exact, one can show that a naive extension of F to $D(\mathcal{A})$ might send an acyclic complex (i.e., a complex with null cohomology, and thus a trivial element of $D(\mathcal{A})$) to a non-acyclic one: therefore, blindly extending F to complexes in $D(\mathcal{A})$ does not make sense, unless F is exact.

The construction we will carry out, which leads to the notion of *derived functor*, calls for a hypothesis of left- or right-exactness on F. If $F\colon \mathcal{A} \to \mathcal{B}$ is left-exact, we obtain the *right-derived functor*:

$$\mathbf{R}F\colon D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B}),$$

and if F is right-exact we get the *left-derived functor*:

$$\mathbf{L}F\colon D^-(\mathcal{A}) \longrightarrow D^-(\mathcal{B}).$$

Since the two constructions are very similar, we will only discuss the right-derived functor, which requires[4] one last hypothesis on $\mathcal{A}$: we will assume it to contain enough injectives. Conversely, derivation on the left requires enough projectives.

Let us denote by $\mathcal{I}_\mathcal{A}$ the class of injective objects of $\mathcal{A}$.

[4]This hypothesis can actually be weakened. The precise construction is beyond the scope of this thesis and will therefore be omitted. Briefly, it is enough to find a class of objects $\mathcal{I}_F \subset \mathcal{A}$ (called *F-adapted* objects) that possesses specific properties enabling it to act as the class of injective objects of a category that satisfies the original, more restrictive hypotheses.

Proposition 3.24. *Let $\mathcal{A}$ be an abelian category that contains enough injectives. Then, the localization functor $Q^+\colon K^+(\mathcal{A}) \to D^+(\mathcal{A})$ naturally restricts to a functor $\iota\colon K^+(\mathcal{I}_\mathcal{A}) \to D^+(\mathcal{A})$ which is an equivalence.*

A quasi-inverse to ι, which we will denote by ι^{-1}, is given by associating to any bounded below complex $A^\bullet \in D^+(\mathcal{A})$ a quasi-isomorphic complex of injective objects $I^\bullet \in K^+(\mathcal{I}_\mathcal{A})$. Such a complex $I^\bullet$ is called an *injective resolution* of $A^\bullet$. We have the following diagram:

$$\begin{array}{ccccc} K^+(\mathcal{I}_\mathcal{A}) & \hookrightarrow & K^+(\mathcal{A}) & \xrightarrow{K(F)} & K^+(\mathcal{B}) \\ & \overset{\iota^{-1}}{\nwarrow}\ \overset{\iota}{\searrow} & \downarrow{\scriptstyle Q_\mathcal{A}} & & \downarrow{\scriptstyle Q_\mathcal{B}} \\ & & D^+(\mathcal{A}) & \xrightarrow{\mathbf{R}F} & D^+(\mathcal{B}), \end{array}$$

where $K(F)$ denotes the functor (induced by F) which sends a complex $(\cdots \to A^{i-1} \to A^i \to A^{i+1} \to \cdots)$ to $(\cdots \to F(A^{i-1}) \to F(A^i) \to F(A^{i+1}) \to \cdots)$, which is well-defined in the homotopy category.

Definition 3.25. Under the hypotheses mentioned above, the *right-derived functor* of F is the functor:

$$\mathbf{R}F \coloneqq Q_\mathcal{B} \circ K(F) \circ \iota^{-1}\colon D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B}).$$

We will now list some of the main properties of the right-derived functor. First, we need a fundamental definition.

Definition 3.26. An additive functor $F\colon \mathcal{D} \to \mathcal{D}'$ between triangulated categories is *exact* if $F \circ T_\mathcal{D} \cong T_{\mathcal{D}'} \circ F$, and any distinguished triangle $A \longrightarrow B \longrightarrow C \longrightarrow A[1]$ in $\mathcal{D}$ is mapped to a distinguished triangle $F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow F(A)[1]$.

Note that applying F to all the terms of the first triangle would yield $F(A[1])$, but this can be identified with $F(A)[1]$ thanks to the natural isomorphism imposed by the same definition.

Proposition 3.27. *The right-derived functor of $F\colon \mathcal{A} \to \mathcal{B}$ satisfies the following properties.*

i) There exists a natural morphism of functors $Q_\mathcal{B} \circ K(F) \to \mathbf{R}F \circ Q_\mathcal{A}$.
ii) The right-derived functor $\mathbf{R}F\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is an exact functor of triangulated categories.
iii) Let $G\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$ be an exact functor. Then any morphism of functors $Q_\mathcal{B} \circ K(F) \to G \circ Q_\mathcal{A}$ factorizes through a unique functor morphism $\mathbf{R}F \to G$.

Similar facts hold true for the left-derived functor $\mathbf{L}F\colon D^-(\mathcal{A}) \to D^-(\mathcal{B})$ (under the hypothesis that F is right-exact). Furthermore, these properties determine $\mathbf{R}F$ and $\mathbf{L}F$ up to a unique isomorphism.

The construction we delineated here is sometimes called the *total* left/right derived functor of F, in contrast with the i-th derived functors R^iF and L_iF, defined at the level of the abelian categories where F operates. These can be recovered by applying cohomology:

Definition 3.28. Let $\mathbf{R}F\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$ be the (total) right derived functor of a left-exact functor $F\colon \mathcal{A} \to \mathcal{B}$. Then, for any complex $A^\bullet \in D^+(\mathcal{A})$, we have:

$$R^iF(A^\bullet) = H^i(\mathbf{R}F(A^\bullet)) \in \mathcal{B};$$

the induced additive functors $R^iF\colon \mathcal{A} \to \mathcal{B}$ are the *higher derived functors* of F.

Remark 3.29. Consider an object $A \in \mathcal{A}$, and an injective resolution $(A \to I^0 \to I^1 \to \cdots)$. It always holds true that $R^iF(A) = 0$ for $i < 0$, and $R^0F(A) \cong F(A)$. Indeed:

$$R^iF(A) = H^i(\cdots \to F(I^0) \to F(I^1) \to \cdots),$$

which implies that $R^0F(A) = \mathrm{Ker}(F(I^0) \to F(I^1)) = F(A)$, since F is left-exact.

Examples 3.30. We will now quickly state exactness results for some key functors that will need to be derived in the rest of the thesis.

i) Let $\mathcal{R}$ be a sheaf of commutative rings over a topological space X, and consider the abelian category of sheaves of $\mathcal{R}$-modules $\mathrm{Sh}_{\mathcal{R}}(X)$. If $\mathcal{F} \in \mathrm{Sh}_{\mathcal{R}}(X)$, then the tensor product:
$$\mathcal{F} \otimes_{\mathcal{R}} -\colon \mathrm{Sh}_{\mathcal{R}}(X) \longrightarrow \mathrm{Sh}_{\mathcal{R}}(X)$$
is a right-exact functor. Moreover, if $(X, \mathcal{O}_X)$ is a ringed space, this also holds true for the restrictions $\mathrm{QCoh}(X) \to \mathrm{QCoh}(X)$ and $\mathrm{Coh}(X) \to \mathrm{Coh}(X)$, provided that $\mathcal{F} \in \mathrm{QCoh}(X)$ or $\mathcal{F} \in \mathrm{Coh}(X)$, respectively. Therefore, the tensor product of sheaves will need to be derived on the left. In particular, derivation of the restriction to coherent sheaves yields:
$$\mathcal{F}^\bullet \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_X} -\colon D^-(X) \longrightarrow D^-(X),$$
where $(\mathcal{F}^\bullet)^i$ is $\mathcal{F}$ for $i = 0$, and 0 in all other degrees.

ii) Let $f\colon X \to Y$ be a continuous map. The inverse image of f gives an exact functor:
$$f^{-1}\colon \mathrm{Sh}_{\mathcal{R}}(Y) \longrightarrow \mathrm{Sh}_{f^{-1}(\mathcal{R})}(X),$$
with $\mathcal{R}$ any sheaf of rings on Y. If $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, we define the pullback of f as:
$$f^* \coloneqq (\mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} -) \circ f^{-1}\colon \mathrm{Qcoh}(Y) \longrightarrow \mathrm{Qcoh}(X),$$
which is a right-exact functor, since $(\mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} -)$ is right exact by i). If X and Y are smooth projective varieties, the pullback maps a coherent sheaf to a coherent sheaf, meaning that f^* restricts to a functor $\mathrm{Coh}(Y) \to \mathrm{Coh}(X)$; when we derive on the left, we obtain:
$$\mathbf{L}f^*\colon D^-(Y) \longrightarrow D^-(X).$$

iii) Again, let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces, with X and Y smooth projective varieties. Then the pushforward $f_*\colon \mathrm{Coh}(X) \to \mathrm{Coh}(Y)$ is left-exact; therefore, derivation on the right yields[5]:
$$\mathbf{R}f_*\colon D^+(X) \longrightarrow D^+(Y).$$

[5]Strictly speaking, the construction of the derived functor requires the abelian category to have enough injectives, which $\mathrm{Coh}(X)$ does not in general: see Remark 2.15. The derived functor is nonetheless well-defined: one constructs $\mathbf{R}f_* : D^+(\mathrm{QCoh}(X)) \to D^+(\mathrm{QCoh}(Y))$ using injective resolutions in $\mathrm{QCoh}(X)$, which does have enough injectives by Theorem 2.14, and then uses that, since f is proper (as X and Y are projective), the higher direct images R^if_* of a coherent sheaf are again coherent (see e.g. [Har77, Theorem III.8.8]). This ensures that $\mathbf{R}f_*$ restricts to a well-defined functor $D^+(\mathrm{Coh}(X)) \to D^+(\mathrm{Coh}(Y))$.

iv) Let $\mathcal{A}$ be an abelian category with enough injectives, and let $A \in \mathcal{A}$. The functor:

$$\mathrm{Hom}_{\mathcal{A}}(A, -)\colon \mathcal{A} \longrightarrow \mathbf{Ab}$$

is left-exact. Deriving it on the right yields the total right-derived functor:

$$\mathbf{R}\,\mathrm{Hom}_{\mathcal{A}}(A, -)\colon D^+(\mathcal{A}) \longrightarrow D^+(\mathbf{Ab}).$$

The higher derived functors of $\mathbf{R}\,\mathrm{Hom}(A, -)$ are precisely the classical extension groups:

$$R^i\,\mathrm{Hom}_{\mathcal{A}}(A, -) = H^i(\mathbf{R}\,\mathrm{Hom}_{\mathcal{A}}(A, -)) = \mathrm{Ext}^i_{\mathcal{A}}(A, -).$$

In the following, whenever the abelian category $\mathcal{A}$ is obvious we will omit it, and simply write $\mathrm{Ext}^i(A, -)$.

We will close this section with an important result which links the extension groups with the morphism groups of objects in the derived category. Note that, when considering an object $A \in \mathcal{A}$ as an object in $D(\mathcal{A})$, the symbol A is used to denote the complex concentrated in degree 0, i.e. the complex that is 0 in any degree $i \neq 0$ and is A for $i = 0$. We first need a Lemma:

Lemma 3.31. *Let $A^\bullet, I^\bullet \in K^+(\mathcal{A})$, with $\mathcal{A}$ an abelian category, and I^i injective for every i. Then:*

$$\mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, I^\bullet) = \mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, I^\bullet).$$

Theorem 3.32. *Let $A, B \in \mathcal{A}$ be objects in an abelian category with enough injectives. Then there are natural isomorphisms:*

$$\mathrm{Ext}^i_{\mathcal{A}}(A, B) \cong \mathrm{Hom}_{D(\mathcal{A})}(A, B[i]).$$

Proof. Suppose $(B \xrightarrow{d} I^0 \xrightarrow{d_0} I^1 \xrightarrow{d_1} \cdots)$ is an injective resolution of B. Then, by the previous construction:

$$\mathrm{Ext}^i_{\mathcal{A}}(A, B) = H^i(\mathrm{Hom}(A, I^0) \xrightarrow{d_{0*}} \mathrm{Hom}(A, I^1) \xrightarrow{d_{1*}} \cdots).$$

Now, a morphism $f \in \mathrm{Hom}(A, I^i)$ belongs to $\mathrm{Ker}(d_{i*})$ if and only if f defines a morphism of complexes $A \to I^\bullet[i]$ (recall that by A we mean the complex concentrated in degree 0). Moreover, a morphism $f \in \mathrm{Hom}(A, I^i)$ is in $\mathrm{Im}(d_{i-1*})$ if and only if it is homotopically trivial. Therefore:

$$\mathrm{Ext}^i_{\mathcal{A}}(A, B) \cong \mathrm{Hom}_{K(\mathcal{A})}(A, I^\bullet[i]) = \mathrm{Hom}_{D(\mathcal{A})}(A, I^\bullet[i]),$$

where we use Lemma 3.31 for the last equality. Then we conclude, since $B \cong I^\bullet$ as objects of $D^+(\mathcal{A})$ (see Remark 3.9). □

3.3 (Semi)orthogonal Decompositions

The derived category of a variety[6] can be a very complicated object. One way to simplify its study, and to gain some intuition about its structure, is to make use of certain *decompositions*, which we will properly introduce and explore in this section. Again, for the

[6]Unless specified otherwise, by *variety* we always mean a smooth, projective variety (over an algebraically closed field of characteristic 0, e.g., $\mathbb{C}$). In the literature, many of the results we will state actually concern *schemes*; we will not introduce this concept. For the purposes of this thesis, it suffices to know that smooth projective varieties are, in particular, (noetherian, projective, ...) schemes, and thus satisfy all the hypotheses generally required.

theory introduced here we refer to [Huy06, Sections 1.3 and 1.4]; for a modern overview of the examples that we showcase, see [PS22, Section 1.2]. First, we need to state a very important definition.

Definition 3.33. Let $\mathcal{D}$ be a triangulated category. A subcategory $\mathcal{D}' \subset \mathcal{D}$ is a *triangulated subcategory* if the structure of triangulated category induced on $\mathcal{D}'$ makes the inclusion exact.

In the case of a full subcategory $\mathcal{D}' \subset \mathcal{D}$, this amounts to requiring that $\mathcal{D}'$ is invariant under the action of the shift functor, and for any distinguished triangle $A \longrightarrow B \longrightarrow C \longrightarrow A[1]$ in $\mathcal{D}$ with $A, B \in \mathcal{D}'$ the object C is isomorphic to an object in $\mathcal{D}'$.

Definition 3.34. A full triangulated subcategory $\mathcal{D}' \subset \mathcal{D}$ is called *(right) admissible* if the inclusion has a right adjoint $\pi \colon \mathcal{D} \longrightarrow \mathcal{D}'$, i.e. there exist functorial isomorphisms $\mathrm{Hom}_{\mathcal{D}}(i(A), B) \simeq \mathrm{Hom}_{\mathcal{D}'}(A, \pi(B))$ for all $A \in \mathcal{D}'$ and $B \in \mathcal{D}$.

Definition 3.35. The *(right) orthogonal complement* of a(n admissible) subcategory $\mathcal{D}' \subset \mathcal{D}$ is the full subcategory $\mathcal{D}'^{\perp}$ of all objects $C \in \mathcal{D}$ such that $\mathrm{Hom}(B, C) = 0$ for all $B \in \mathcal{D}'$.

For an example of a non-admissible subcategory, see Example 3.40. The following result provides a very useful characterization of admissibility, and gives us a way to grasp its usefulness.

Proposition 3.36. *A full triangulated subcategory $\mathcal{D}' \subset \mathcal{D}$ is admissible if and only if, for all $A \in \mathcal{D}$, there exists a distinguished triangle $B \longrightarrow A \longrightarrow C \longrightarrow B[1]$, with $B \in \mathcal{D}'$ and $C \in \mathcal{D}'^{\perp}$.*

In other words, this proposition tells us that admissible subcategories allow us to systematically "split" any object of the ambient category into two independent building blocks, one from $\mathcal{D}'$ and one from its orthogonal, tied together by a distinguished triangle.

Remark 3.37. The orthogonal complement of an admissible subcategory $\mathcal{D}'$ is a triangulated subcategory. First, observe that the condition $\mathrm{Hom}(B, C) = 0$ for all $B \in \mathcal{D}'$ implies that

$$\mathrm{Hom}(B, C[i]) \simeq \mathrm{Hom}(B[-i], C) = 0, \tag{3.5}$$

using that $\mathcal{D}'$ is invariant under shift. This shows that $\mathcal{D}'^{\perp}$ is shift-invariant as well. Furthermore, if $C_1 \longrightarrow C_2 \longrightarrow C_3 \longrightarrow C_1[1]$ is a distinguished triangle with $C_1, C_2 \in \mathcal{D}'^{\perp}$, then we can apply the functor $\mathrm{Hom}(B, -)$ to obtain, thanks to Corollary 2.20, the long exact sequence:

$$\cdots \longrightarrow \mathrm{Hom}(B, C_1) \longrightarrow \mathrm{Hom}(B, C_2) \longrightarrow \mathrm{Hom}(B, C_3) \longrightarrow \mathrm{Hom}(B, C_1[1]) \longrightarrow \cdots$$

The first two terms are 0 because $C_1, C_2 \in \mathcal{D}'^{\perp}$, and the fourth one is also 0 due to (3.5). Thus, $C_3 \in \mathcal{D}'^{\perp}$.

Now that we have introduced admissible subcategories and orthogonal complements, we are ready to construct the notion of decomposition that we mentioned earlier, in the preamble to this section.

Definition 3.38. A sequence of full admissible triangulated subcategories

$$\mathcal{D}_1, \ldots, \mathcal{D}_n \subset \mathcal{D}$$

is *semiorthogonal* if we have, for all $i > j$, $\mathcal{D}_j \subset \mathcal{D}_i^{\perp}$. Such a sequence defines a *semiorthogonal decomposition* of $\mathcal{D}$ if $\mathcal{D}$ is generated by the $\mathcal{D}_i$, i.e. $\mathcal{D}$ is equivalent, via the inclusion, to the smallest full triangulated subcategory of $\mathcal{D}$ containing all of the $\mathcal{D}_i$.

Example 3.39. Let $\mathcal{D}' \subset \mathcal{D}$ be an admissible full triangulated subcategory. Call $\mathcal{D}_1 \coloneqq \mathcal{D}'^{\perp}$, and $\mathcal{D}_2 \coloneqq \mathcal{D}'$. Then $\mathcal{D}_1, \mathcal{D}_2 \subset \mathcal{D}$ is a semiorthogonal decomposition of $\mathcal{D}$. It is clear that they constitute a semiorthogonal sequence. To see that they generate $\mathcal{D}$, consider any object $A \in \mathcal{D}$. The admissibility of $\mathcal{D}'$ and Proposition 3.36 guarantee the existence of a distinguished triangle:

$$B \longrightarrow A \longrightarrow C \longrightarrow B[1],$$

with $B \in \mathcal{D}'$ and $C \in \mathcal{D}'^{\perp}$. We can now rotate the triangle and obtain:

$$C[-1] \longrightarrow B \longrightarrow A \longrightarrow C,$$

thus making A isomorphic (thanks to TR3) to the cone of the morphism $C[-1] \longrightarrow B$. Now consider $\mathcal{T} \coloneqq \langle \mathcal{D}', \mathcal{D}'^{\perp} \rangle$, the smallest full triangulated subcategory containing $\mathcal{D}'$ and $\mathcal{D}'^{\perp}$. We have that $B \in \mathcal{D}' \subset \mathcal{T}$, and $C[-1] \in \mathcal{D}'^{\perp} \subset \mathcal{T}$, again thanks to the shift-invariance of $\mathcal{D}'^{\perp}$. Now, since $\mathcal{T}$ is triangulated, it is closed under the cone operation, therefore $A \in \mathcal{T}$, which implies $\mathcal{D} \subset \mathcal{T}$.

Example 3.40. While exceptional objects will provide a systematic method to construct admissible subcategories (see Lemma 3.45), it is important to emphasize that not every full triangulated subcategory is admissible.

A concrete geometric counterexample is provided by the bounded derived category $D^b(X)$ of a K3 surface X. As we shall prove in Theorem 4.22, $D^b(X)$ admits no non-trivial semiorthogonal decompositions. Since any proper, non-trivial admissible subcategory $\mathcal{D}' \subset D^b(X)$ would automatically induce a non-trivial semiorthogonal decomposition $D^b(X) = \langle \mathcal{D}'^{\perp}, \mathcal{D}' \rangle$ (as shown in Example 3.39), every non-trivial, proper full triangulated subcategory of $D^b(X)$ is non-admissible.

We will now introduce a property that, whenever exhibited by a sequence of objects, immediately yields a semiorthogonal decomposition. Before doing so, we need one more fundamental definition in category theory.

Definition 3.41. A *k-linear category* is an additive category $\mathcal{A}$ such that the morphism groups $\mathrm{Hom}(A, B)$ are k-vector spaces for all objects $A, B \in \mathcal{A}$, and all compositions are k-bilinear. An additive functor F between k-linear categories $\mathcal{A}$ and $\mathcal{B}$ is said to be *k-linear* if, for any two objects $A, B \in \mathcal{A}$, the map $F\colon \mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$ is k-linear.

Definition 3.42. An object $E \in \mathcal{D}$ in a k-linear triangulated category $\mathcal{D}$ is called *exceptional* if its space of endomorphisms is isomorphic to k and there are no non-zero morphisms from E to any non-zero shift of E. In other words, we require:

$$\mathrm{Hom}(E, E[\ell]) = \begin{cases} k & \text{if } \ell = 0 \\ 0 & \text{if } \ell \neq 0. \end{cases}$$

An *exceptional sequence* is a sequence $E_1, \dots, E_n$ of exceptional objects such that we have $\mathrm{Hom}(E_i, E_j[\ell]) = 0$ for all $i > j$ and all ℓ. Equivalently:

$$\mathrm{Hom}(E_i, E_j[\ell]) = \begin{cases} k & \text{if } \ell = 0, i = j \\ 0 & \text{if } i > j \text{ or if } \ell \neq 0, i = j. \end{cases}$$

Note that there is no requirement for when $j > i$. An exceptional sequence is *full* if $\mathcal{D}$ is generated by $\{E_i\}$, i.e. any full triangulated subcategory containing all the objects E_i is equivalent to $\mathcal{D}$ (via the inclusion).

Let us study a quick example of exceptionality.

Example 3.43. Let k be a field. We will denote by p the variety consisting of a single point. A coherent sheaf over p is a sheaf of $\mathcal{O}_p$-modules. Since $\mathcal{O}_p(p) = k$, the category $\mathrm{Coh}(p)$ is equivalent to $\mathrm{Vec}_f(k)$, the category of finite-dimensional vector spaces over k. This implies that

$$D^b(p) \cong D^b(\mathrm{Vec}_f(k)).$$

Now, observe that $\mathrm{Vec}_f(k)$ is a *semisimple* category, i.e. every short exact sequence splits: this implies that every complex is isomorphic to its cohomology complex (with null boundary). It follows that $D^b(p)$ is equivalent to the category of graded vector spaces $V^\bullet = \bigoplus_{n\in\mathbb{Z}} V^n$, with shift functor given by $V[1]^n = V^{n+1}$. In particular, every object is isomorphic to a finite direct sum:

$$\bigoplus_n k[n]^{\oplus d_n},$$

and the only objects that cannot be decomposed are the shifts of k, which satisfy:

$$\mathrm{Hom}(k, k[i]) = \begin{cases} k & i = 0, \\ 0 & i \neq 0; \end{cases}$$

hence, $k \in D^b(p)$ is exceptional.

Moreover, we have the following result:

Proposition 3.44. *Let $\mathcal{D}$ be a k-linear triangulated category and $E \in \mathcal{D}$ an exceptional object. Then the smallest full triangulated subcategory $\langle E\rangle \subset \mathcal{D}$ containing E is equivalent, as a triangulated category, to the bounded derived category of a point.*

With the following Lemma we want to show that there is always a way to construct an admissible triangulated subcategory, starting from a given exceptional object.

Lemma 3.45. *Let $\mathcal{D}$ be a k-linear triangulated category which is also* Hom-finite*, meaning that for any two objects $A, B \in \mathcal{D}$ the vector space $\bigoplus_i \mathrm{Hom}(A, B[i])$ is finite-dimensional. If $E \in \mathcal{D}$ is exceptional, then the objects $\bigoplus E[i]^{\oplus j_i}$ form an admissible triangulated subcategory $\langle E\rangle$ of $\mathcal{D}$.*

Proof. Proposition 3.44 implies that $\langle E\rangle$ is a triangulated category. In order to see that $\langle E\rangle$ is admissible, fix any object $A \in \mathcal{D}$. Since $\mathrm{Hom}(E, A[i]) \simeq \mathrm{Hom}(E[-i], A)$, we can easily define[7] a morphism

$$\bigoplus \left(\mathrm{Hom}(E, A[i]) \otimes E[-i]\right) \longrightarrow A,$$

[7]Since $\mathcal{D}$ is Hom-finite, the vector space $\bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}(E, A[i])$ is finite-dimensional. Thus, the domain $\bigoplus_{i\in\mathbb{Z}} \big(\mathrm{Hom}(E, A[i]) \otimes E[-i]\big) \cong \bigoplus_{i\in\mathbb{Z}} E[-i]^{\oplus d_i}$ (where $d_i = \dim_k \mathrm{Hom}(E, A[i])$) is a finite direct sum of objects in $\mathcal{D}$, and hence is a well-defined object of $\mathcal{D}$.

which, for any fixed value of i, is simply an evaluation morphism. This can be completed to a distinguished triangle

$$\bigoplus (\mathrm{Hom}(E, A[i]) \otimes E[-i]) \longrightarrow A \longrightarrow B.$$

Using that E is exceptional, one finds $\mathrm{Hom}(E, B[i]) = 0$. Hence, $B \in \langle E \rangle^{\perp}$, using 3.36 and the fact that the first term is in $\langle E \rangle$. □

Example 3.46. Let $E_1, \ldots, E_n$ be an exceptional sequence in $\mathcal{D}$. Then the triangulated subcategories

$$\mathcal{D}_1 := \langle E_1 \rangle, \ldots, \mathcal{D}_n := \langle E_n \rangle$$

are admissible, thanks to Lemma 3.45, and they also form a semiorthogonal sequence. If the exceptional sequence is full, then $\mathcal{D}_1, \ldots, \mathcal{D}_n \subset \mathcal{D}$ is a semiorthogonal decomposition.

We will conclude this section with a useful criterion to tell whether a semiorthogonal sequence generates the entire category, thus yielding a semiorthogonal decomposition.

Lemma 3.47. *A semiorthogonal sequence of full admissible triangulated subcategories $\mathcal{D}_1, \ldots, \mathcal{D}_n \subset \mathcal{D}$ generates $\mathcal{D}$, thus giving a semiorthogonal decomposition of $\mathcal{D}$, if and only if any object $A \in \mathcal{D}$ with $A \in \mathcal{D}_i^{\perp}$ for all $i = 1, \ldots, n$ is trivial.*

Proof. [⇒] Suppose $\mathcal{D}_1, \ldots \mathcal{D}_n$ is a semiorthogonal decomposition. For any $A \in \mathcal{D}$, define ${}^{\perp}A$ as the full triangulated subcategory of all objects $B \in \mathcal{D}$ with $\mathrm{Hom}(B, A[i]) = 0$ for all $i \in \mathbb{Z}$. By hypothesis, $A \in \bigcap \mathcal{D}_i^{\perp}$, which implies $\mathcal{D}_1, \ldots \mathcal{D}_n \subset {}^{\perp}A$. Thus ${}^{\perp}A = \mathcal{D}$, and in particular $A \in {}^{\perp}A$. This, in turn, means that $\mathrm{Hom}(A, A) = 0$, therefore $A \cong 0$.

[⇐] Now, assume $\bigcap \mathcal{D}_i^{\perp} = \{0\}$. For simplicity, we will also assume that $n = 2$; the general case is analogous. We want to show that $\mathcal{D} \subset \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$, thus we start by taking $A \in \mathcal{D}$. Since $\mathcal{D}_2$ is admissible, using Proposition 3.36 we can find a distinguished triangle:

$$B \longrightarrow A \longrightarrow C \longrightarrow B[1],$$

with $B \in \mathcal{D}_2$ and $C \in \mathcal{D}_2^{\perp}$. We can decompose C even further by using the admissibility of $\mathcal{D}_1$, obtaining the distinguished triangle:

$$D \longrightarrow C \longrightarrow D' \longrightarrow D[1],$$

with $D \in \mathcal{D}_1$ and $D' \in \mathcal{D}_1^{\perp}$. Note that $D \in \mathcal{D}_1 \subset \mathcal{D}_2^{\perp}$ (using the semiorthogonality of the sequence), and $C \in \mathcal{D}_2^{\perp}$: by reasoning just like in Remark 3.37, we can conclude that $D' \in \mathcal{D}_2^{\perp}$. Thus $D' \in \mathcal{D}_1^{\perp} \cap \mathcal{D}_2^{\perp} = \{0\}$, which implies $C \cong D \in \mathcal{D}_1$. Then A sits in a distinguished triangle between an object in $\mathcal{D}_2$ and an object in $\mathcal{D}_1$. □

We will now briefly describe a more restrictive kind of decomposition. The requirement for semiorthogonality is that there are no non-zero morphisms from objects in higher-index subcategories towards objects in lower-index subcategories, i.e., $\mathcal{D}_j \subset \mathcal{D}_i^{\perp}$ for all $i > j$. If we require no non-zero morphism in either direction, we obtain the notion of *orthogonal* (or *fully orthogonal*) *decomposition*. For the sake of simplicity we will only consider the case of two-term decompositions.

Definition 3.48. A triangulated category $\mathcal{D}$ is *orthogonally decomposed* into triangulated subcategories $\mathcal{D}_1, \mathcal{D}_2 \subset \mathcal{D}$ if:

i) $\mathcal{D}_1$ and $\mathcal{D}_2$ both contain objects non-isomorphic to 0.
ii) For any object $A \in \mathcal{D}$ there exists a distinguished triangle

$$B_1 \longrightarrow A \longrightarrow B_2 \longrightarrow B_1[1]$$

with $B_1 \in \mathcal{D}_1$ and $B_2 \in \mathcal{D}_2$.
iii) $\mathrm{Hom}(B_1, B_2) = \mathrm{Hom}(B_2, B_1) = 0$ for all $B_1 \in \mathcal{D}_1$ and $B_2 \in \mathcal{D}_2$.

A triangulated category is said to be *indecomposable* if it does not admit an orthogonal decomposition.

It turns out that, in the geometric context, the requirement for full orthogonality is excessively restrictive. The following result is the reason why our focus will be exclusively on semiorthogonal decompositions: orthogonal ones always bring us back to the derived categories of the connected components of the variety.

Proposition 3.49. *Let X be a smooth projective variety and let $D^b(X)$ be its bounded derived category of coherent sheaves, i.e. $D^b(\mathrm{Coh}(X))$. Then $D^b(X)$ is an indecomposable triangulated category if and only if X is connected.*

Remark 3.50. In fact, a stronger result holds true. Let X be a disconnected smooth projective variety, with X_i its connected components for $i = 1, \dots, n$. By Proposition 3.49, $D^b(X)$ admits a fully orthogonal decomposition. As shown by Bridgeland in [Bri99], the orthogonal components of the decomposition correspond one-to-one to the connected components X_i, and more precisely:

$$D^b(X) \cong \bigoplus_{i=1}^{n} D^b(X_i).$$

Now that we have introduced the necessary formalism, we will dedicate the rest of this section to the study of some important examples of semiorthogonal decompositions. First, we will consider the sequence

$$\mathcal{B} \coloneqq \{\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n)\} \subset D^b(\mathbb{P}^n), \tag{3.6}$$

where $\mathcal{O}$ denotes the structure sheaf of $\mathbb{P}^n$. We will prove that $\mathcal{B}$ is an exceptional sequence, and a result by Beilinson [Bei78] will ensure that it provides a semiorthogonal decomposition of $D^b(\mathbb{P}^n)$ by the same construction of Example 3.46. First, we need the following Lemma:

Lemma 3.51. *Let X be a smooth projective variety and let L be a line bundle on X. The tensor product functor*

$$- \otimes L \colon D^b(X) \longrightarrow D^b(X)$$

is an autoequivalence of categories. In particular, for any pair of objects $A^\bullet, B^\bullet \in D^b(X)$ and for all $i \in \mathbb{Z}$, it induces canonical isomorphisms:

$$\mathrm{Hom}(A^\bullet, B^\bullet) \cong \mathrm{Hom}(A^\bullet \otimes L, B^\bullet \otimes L),$$
$$\mathrm{Ext}^i(A^\bullet, B^\bullet) \cong \mathrm{Ext}^i(A^\bullet \otimes L, B^\bullet \otimes L).$$

Proof. A quasi-inverse to $- \otimes L$ is clearly given by $- \otimes L^\vee$. As to the identities: an equivalence is fully faithful, thus the first isomorphism is obvious. The second one follows from Theorem 3.32:

$$\begin{aligned}\operatorname{Ext}^i(A^\bullet, B^\bullet) \cong \operatorname{Hom}_{D^b(X)}(A^\bullet, B^\bullet[i]) &\cong \operatorname{Hom}_{D^b(X)}(A^\bullet \otimes L, B^\bullet[i] \otimes L) \\ &\cong \operatorname{Hom}_{D^b(X)}\left(A^\bullet \otimes L, (B^\bullet \otimes L)[i]\right) \\ &\cong \operatorname{Ext}^i(A^\bullet \otimes L, B^\bullet \otimes L),\end{aligned}$$

where the third isomorphism uses the fact that the tensor product commutes with the shift functor. □

Remark 3.52. Take any sheaf $\mathcal{F}$ over the smooth projective variety X, whose structure sheaf we denote by $\mathcal{O}_X$. Then a morphism of sheaves $\varphi\colon \mathcal{O}_X \to \mathcal{F}$ determines a global section $s \coloneqq \varphi(X)(1)$. By restricting s to open subsets of X and using the linearity of sheaf morphisms, it is clear that φ is uniquely determined by s, which is just an element of $\mathcal{F}(X)$. Therefore:

$$\operatorname{Hom}(\mathcal{O}_X, \mathcal{F}) \cong \Gamma(X, \mathcal{F}) = H^0(X, \mathcal{F}). \tag{3.7}$$

Recall that the extension groups $\operatorname{Ext}^i(\mathcal{O}_X, -)$ are the right derived functors of $\operatorname{Hom}(\mathcal{O}_X, -)$, while the cohomology groups $H^i(X, -)$ are the right derived functors of $\Gamma(X, -)$. Thus, by (3.7), we also have:

$$\operatorname{Ext}^i(\mathcal{O}_X, \mathcal{F}) = R^i \operatorname{Hom}(\mathcal{O}_X, \mathcal{F}) \cong R^i\Gamma(X, \mathcal{F}) = H^i(X, \mathcal{F}).$$

Proposition 3.53. *The collection* $\{\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n)\} \subset D^b(\mathbb{P}^n)$ *constitutes a semiorthogonal, exceptional sequence.*

Proof. By definition, we have to verify that each object $\mathcal{O}(i)$ is exceptional, and that $\operatorname{Hom}_{D^b(\mathbb{P}^n)}(\mathcal{O}(i), \mathcal{O}(j)[\ell]) = 0$ for all $i > j$ and for all $\ell \in \mathbb{Z}$. We can treat both conditions simultaneously. For any $i, j \in \{0, \ldots, n\}$ and for any $\ell \in \mathbb{Z}$, we have:

$$\begin{aligned}\operatorname{Hom}_{D^b(\mathbb{P}^n)}(\mathcal{O}(i), \mathcal{O}(j)[\ell]) &\cong \operatorname{Ext}^\ell(\mathcal{O}(i), \mathcal{O}(j)) && \text{(Theorem 3.32)} \\ &\cong \operatorname{Ext}^\ell(\mathcal{O}, \mathcal{O}(j-i)) && \text{(Lemma 3.51)} \\ &\cong H^\ell(\mathbb{P}^n, \mathcal{O}(j-i)) && \text{(Remark 3.52).}\end{aligned}$$

Therefore, the proof reduces to showing that:

$$H^\ell(\mathbb{P}^n, \mathcal{O}(j-i)) = \begin{cases} k & \text{if } i = j, \ell = 0, \\ 0 & \text{if } i = j, \ell \neq 0, \\ 0 & \text{if } i > j \text{ and any } \ell \in \mathbb{Z}. \end{cases}$$

For $i = j$ and $\ell = 0$, $H^0(\mathbb{P}^n, \mathcal{O})$ are just the global sections of the trivial bundle, which are only the constants, since $\mathbb{P}^n$ is projective. Moreover, a classical result by Serre (see [Har77, Theorem III.5.1]) guarantees that $H^\ell(\mathbb{P}^n, \mathcal{O}) = 0$ for all $\ell > 0$. As for the case $i > j$, $H^\ell(\mathbb{P}^n, \mathcal{O}(j-i))$ is 0 for $\ell = 0$, since a non-zero element would be a homogeneous polynomial with negative degree. Triviality in all degrees $0 < \ell < n$ follows from another classical computation, while $\ell = n$ is obtained from the case $\ell = 0$ by Serre duality. □

By Example 3.46, if we were able to show that sequence (3.6) is also full, we would obtain a semiorthogonal decomposition of the bounded derived category of the projective space $\mathbb{P}^n$. This highly non-trivial fact was proved by Alexander Beilinson in [Bei78], using a locally free resolution of the diagonal in $\mathbb{P}^n \times \mathbb{P}^n$:

3.54 Beilinson's Theorem. *The exceptional sequence* $\{\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n)\}$ *is full in* $D^b(\mathbb{P}^n)$.

Corollary 3.55. *The bounded derived category* $D^b(\mathbb{P}^n)$ *admits the semiorthogonal decomposition:*

$$D^b(\mathbb{P}^n) = \langle \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n) \rangle.$$

Another important example of semiorthogonal decomposition is given by quadrics. First, let us establish the following notion.

Definition 3.56. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of $\mathcal{O}_{\mathbb{P}^{n+1}}$-modules. For any $d \in \mathbb{Z}$, we denote by $\phi(d)$ the morphism obtained by composing ϕ with the d-th *Serre twist functor* $- \otimes_{\mathcal{O}_{\mathbb{P}^{n+1}}} \mathcal{O}_{\mathbb{P}^{n+1}}(d)$, i.e.:

$$\phi(d) \coloneqq \phi \otimes \mathrm{id}_{\mathcal{O}_{\mathbb{P}^{n+1}}(d)} : \mathcal{F}(d) = \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^{n+1}}(d) \longrightarrow \mathcal{G}(d) = \mathcal{G} \otimes \mathcal{O}_{\mathbb{P}^{n+1}}(d).$$

Example 3.57. What follows can be found in [Kap88]. Let $Q \subset \mathbb{P}^{n+1}$ be an n-dimensional smooth quadric defined by an equation $\{q = 0\}$. The structure sheaf of Q will be denoted $\mathcal{O}_Q$. The bounded derived category $D^b(Q)$ admits a semiorthogonal decomposition generated by exceptional bundles, whose explicit form depends on the parity of n:

- **Odd dimension:** let $n = 2m + 1$. Then there exists a morphism of $\mathcal{O}_{\mathbb{P}^{n+1}}$-modules:

  $$\phi \colon \mathcal{O}_{\mathbb{P}^{n+1}}(-1)^{\oplus 2^{m+1}} \longrightarrow \mathcal{O}_{\mathbb{P}^{n+1}}^{\oplus 2^{m+1}}$$

  such that $\phi \circ (\phi(-1)) = q \cdot \mathrm{id} \colon \mathcal{O}_{\mathbb{P}^{n+1}}(-2)^{\oplus 2^{m+1}} \to \mathcal{O}_{\mathbb{P}^{n+1}}^{\oplus 2^{m+1}}$. Observe that this makes sense, since multiplication by q maps sections of $\mathcal{O}_{\mathbb{P}^{n+1}}(-2)$ to sections of $\mathcal{O}_{\mathbb{P}^{n+1}}$; in other words, $q \cdot \mathrm{id}$ can be thought of as the morphism given by the $2^{m+1} \times 2^{m+1}$ diagonal matrix where all the diagonal entries are the degree-2 homogeneous polynomial q.

  The derived category admits the following semiorthogonal decomposition:

  $$D^b(Q) = \langle \mathcal{S}, \mathcal{O}_Q, \mathcal{O}_Q(1), \ldots, \mathcal{O}_Q(n-1) \rangle,$$

  where $\mathcal{S}$ denotes the *spinor bundle* over Q, defined as $\mathrm{Coker}(\phi|_Q(-1))$. For an exhaustive treatment of these objects, see [Ott88].

- **Even dimension:** now, let $n = 2m$. In this case, there exist two morphisms of $\mathcal{O}_{\mathbb{P}^{n+1}}$-modules:

  $$\phi, \psi \colon \mathcal{O}_{\mathbb{P}^{n+1}}(-1)^{\oplus 2^m} \longrightarrow \mathcal{O}_{\mathbb{P}^{n+1}}^{\oplus 2^m}$$

  such that $\phi \circ (\psi(-1)) = \psi \circ (\phi(-1)) = q \cdot \mathrm{id} \colon \mathcal{O}_{\mathbb{P}^{n+1}}(-2)^{\oplus 2^m} \to \mathcal{O}_{\mathbb{P}^{n+1}}^{\oplus 2^m}$. If we now denote by $\mathcal{S}^- \coloneqq \mathrm{Coker}(\phi|_Q(-1))$ and $\mathcal{S}^+ \coloneqq \mathrm{Coker}(\psi|_Q(-1))$ the two spinor bundles over Q, we can write a semiorthogonal decomposition of $D^b(Q)$ as:

  $$D^b(Q) = \langle \mathcal{S}^-, \mathcal{S}^+, \mathcal{O}_Q, \mathcal{O}_Q(1), \ldots, \mathcal{O}_Q(n-1) \rangle.$$

3.4 Fourier-Mukai functors

This section introduces the concept of Fourier-Mukai functor between the derived categories of smooth projective varieties. This kind of construction was first introduced by

Shigeru Mukai (b. 1953) in [Muk81], where he proved the equivalence between $D(X)$, with X an abelian variety, and $D(\hat{X})$, where $\hat{X}$ denotes the dual abelian variety of X. Indeed, the explicit equivalence that Mukai exhibited is what we now call a Fourier-Mukai functor. The name of Joseph Fourier (1768 – 1830) is motivated by an analogy drawn by Mukai himself between the conclusion of his own paper, and the famous result which states that the classical Fourier transform gives an isometry between $L^2(V)$ and $L^2(V^\vee)$, with V a real vector space and $V^\vee$ its dual.

Fourier-Mukai functors are extremely important and frequent in the geometric context; for example, we will see that an exact equivalence between the bounded derived categories of smooth projective varieties is of Fourier-Mukai type (see Corollary 3.75). These functors exhibit many elegant properties: for instance, they are exact, admit left and right adjoints, and can be composed in order to obtain another Fourier-Mukai functor. In this thesis, Fourier-Mukai functors (and in particular Orlov's celebrated result, see Theorem 3.74) will be employed to give a rough idea of the proof of Torelli's theorem for Enriques surfaces, in section 4.4. For the general theory of Serre functors and Fourier-Mukai transforms presented in this section, we refer to [Huy06, Sections 1.1 and 5.1].

Before formalizing Fourier-Mukai functors, it is useful to introduce the very important notion of *Serre functors*, which provide a categorical formulation of Serre duality. In the context of Fourier-Mukai functors, Serre functors can be utilized to give a more compact expression for the left and right adjoints. Furthermore, they will prove crucial when showing that the derived category of a K3 surface admits no semiorthogonal decomposition (see section 3.2).

Definition 3.58. Let $\mathcal{A}$ be a k-linear category. A *Serre functor* is a k-linear equivalence $S : \mathcal{A} \to \mathcal{A}$ such that, for any two objects $A, B \in \mathcal{A}$, there exists an isomorphism of k-vector spaces:

$$\eta_{A,B} \colon \operatorname{Hom}(A, B) \xrightarrow{\sim} \operatorname{Hom}(B, S(A))^*$$

which is functorial in A and in B.

Naturally, this induces the pairing $\operatorname{Hom}(B, S(A)) \times \operatorname{Hom}(A, B) \to k$, which sends (f, g) to $\eta_{A,B}(g)(f)$. A useful property of Serre functors is the following:

Proposition 3.59. *Let $\mathcal{A}$ and $\mathcal{B}$ be k-linear categories with finite dimensional morphism groups. If $\mathcal{A}$ admits a Serre functor $S_\mathcal{A}$, and $\mathcal{B}$ has a Serre functor $S_\mathcal{B}$, then any k-linear equivalence $F \colon \mathcal{A} \to \mathcal{B}$ commutes with the Serre functors, i.e. $F \circ S_\mathcal{A} \cong S_\mathcal{B} \circ F$.*

Proposition 3.59 is particularly relevant when working with the derived category of a variety, i.e. $D^b(\operatorname{Coh}(X))$: since $\operatorname{Coh}(X)$ is k-linear, and the shift functor is always a k-linear equivalence, in this setting the eventual Serre functor (see Proposition 3.62 below) commutes with the shift functor.

Proposition 3.60. *Let $F \colon \mathcal{A} \to \mathcal{B}$ be a functor between k-linear categories endowed with Serre functors $S_\mathcal{A}$ and $S_\mathcal{B}$, respectively. Then:*

$$G \dashv F \implies F \dashv S_\mathcal{A} \circ G \circ S_\mathcal{B}^{-1},$$
$$F \dashv H \implies S_\mathcal{A}^{-1} \circ H \circ S_\mathcal{B} \dashv F.$$

Proof. The result follows simply from applying the definitions of adjoint and Serre functor:

$$\begin{aligned}\mathrm{Hom}(A_1, (S_{\mathcal{A}} \circ G \circ S_{\mathcal{B}}^{-1})(A_2)) &\simeq \mathrm{Hom}((G \circ S_{\mathcal{B}}^{-1})(A_2), A_1)^* \\ &\simeq \mathrm{Hom}(S_{\mathcal{B}}^{-1}(A_2), F(A_1))^* \\ &\simeq \mathrm{Hom}(F(A_1), S_{\mathcal{B}}(S_{\mathcal{B}}^{-1}(A_2))) \\ &\simeq \mathrm{Hom}(F(A_1), A_2);\end{aligned}$$

The second identity is proved similarly. □

Remark 3.61. Let $\mathcal{A}$ be an abelian category with finite-dimensional morphism groups. If $\mathcal{A}$ admits a Serre functor S, then it is unique up to a unique isomorphism. Indeed, suppose S' is another Serre functor for $\mathcal{A}$. Then, for any $A, B \in \mathcal{A}$:

$$\mathrm{Hom}(B, S(A))^* \cong \mathrm{Hom}(A, B) \cong \mathrm{Hom}(B, S'(A))^*.$$

By dualizing, we can infer that $\mathrm{Hom}(-, S(A))$ and $\mathrm{Hom}(-, S'(A))$ are isomorphic. Therefore, by Yoneda's Lemma (more precisely, by the results of Remark 2.22), $S(A) \cong S'(A)$ for any $A \in \mathcal{A}$, which implies $S \cong S'$.

As we anticipated, the bounded derived category of a smooth projective variety X always admits a Serre functor, which we will now explicitly construct. First, we will denote by K_X the canonical bundle of X. Recall that, unlike what happens with general coherent sheaves, the tensor product by a locally free sheaf $\mathcal{M}$ gives an exact functor (see Proposition 2.13):

$$\mathrm{Coh}(X) \longrightarrow \mathrm{Coh}(X), \quad \mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{M},$$

which immediately induces an exact functor between the derived categories. The shift functors $[i]$ are also exact for any $i \in \mathbb{Z}$. Therefore:

Proposition 3.62. *Let X be a smooth projective variety. Then, the composition:*

$$D^*(X) \xrightarrow{K_X \otimes -} D^*(X) \xrightarrow{[\dim(X)]} D^*(X),$$

where $* = b, +$ or $-$, is an exact functor, denoted as S_X. Furthermore, the functor $S_X \colon D^b(X) \to D^b(X)$ is a Serre functor.

In light of the previous Proposition, S_X is called the *Serre functor* of the variety X. The Proposition also provides an alternative proof of the classic Serre duality for line bundles:

3.63 Serre Duality for line bundles. *Let L be a line bundle over the smooth projective variety X. Denote $n \coloneqq \dim(X)$. Then, for all $0 \leq i \leq n$:*

$$H^i(X, L) \cong H^{n-i}(X, K_X \otimes L^\vee)^*.$$

Proof. By Remark 3.52, $H^i(X, L) \cong \mathrm{Ext}^i(\mathcal{O}_X, L)$, which is isomorphic to $\mathrm{Hom}(\mathcal{O}_X, L[i])$ by Theorem 3.32. If we apply the definition of Serre functor S_X we obtain:

$$\begin{aligned}\mathrm{Hom}(\mathcal{O}_X, L[i]) \cong \mathrm{Hom}(L[i], K_X[n])^* &\cong \mathrm{Hom}(L, K_X[n-i])^* \\ &\cong \mathrm{Ext}^{n-i}(L, K_X)^* \\ &\cong \mathrm{Ext}^{n-i}(\mathcal{O}_X, K_X \otimes L^\vee)^* \\ &\cong H^{n-i}(X, K_X \otimes L^\vee)^*,\end{aligned}$$

where we used Lemma 3.51 for the second and the fourth isomorphisms. □

Remark 3.64. Notice how the properties of the Serre functor S_X are tied to the geometry of the canonical bundle K_X. For instance, if a variety possesses a trivial canonical bundle (such as K3 surfaces or abelian varieties), its Serre functor is simply the shift by the dimension of the variety, $S_X \cong [\dim(X)]$. On the other hand, if a variety has a torsion canonical bundle of order m (meaning $K_X^{\otimes m} \cong \mathcal{O}_X$), applying the Serre functor m times yields a pure shift: $S_X^m \cong [m \dim(X)]$. This is precisely what happens in the case of Enriques surfaces: since $m = 2$ and $\dim(X) = 2$, its Serre functor acts as a "square root" of the shift [4].

We are now ready to shift our focus to Fourier-Mukai functors. Let X and Y be smooth projective varieties, and denote the canonical projections by:

$$p\colon X \times Y \to X, \quad q\colon X \times Y \to Y.$$

Definition 3.65. Let $\mathcal{P} \in D^b(X \times Y)$. The corresponding *Fourier-Mukai functor* (or *Fourier-Mukai transform*) is the functor $\Phi_{\mathcal{P}}\colon D^b(X) \to D^b(Y)$ defined by:

$$\Phi_{\mathcal{P}}(\mathcal{E}^\bullet) = q_*(p^*(\mathcal{E}^\bullet) \otimes \mathcal{P}).$$

The object $\mathcal{P}$ is called the *Fourier-Mukai kernel* of the Fourier-Mukai functor $\Phi_{\mathcal{P}}$.

Remark 3.66. In order for the definition to make sense, the functors involved need to be properly derived before being composed together. Specifically (see also Examples 3.30):

1. The pushforward of the projection q is left-exact, thus it needs to be derived on the right.
2. In general, the pullback of a map between smooth projective varieties is only right-exact, but the projection p is flat. Therefore, p^* needs no derivation on the left at all.
3. The tensor product of coherent sheaves is right-exact, therefore we need to derive $\otimes$ on the left.

In conclusion, a more rigorous definition of the Fourier-Mukai functor of kernel $\mathcal{P}$ would call for the following expression:

$$\Phi_{\mathcal{P}}(\mathcal{E}^\bullet) = \mathbf{R}q_*(p^*(\mathcal{E}^\bullet) \overset{\mathbf{L}}{\otimes} \mathcal{P}).$$

For the sake of notational simplicity, we will omit the derivation symbols in the rest of the thesis, and denote $\mathbf{R}q_*$ and $\overset{\mathbf{L}}{\otimes}$ simply by q_* and $\otimes$ respectively, at least in the context of Fourier-Mukai functors. Moreover, note that the object $\mathcal{P} \in D^b(X \times Y)$ could also be used to define a Fourier-Mukai functor in the opposite direction, $D^b(Y) \to D^b(X)$. For maximal clarity, the notation is the following:

$$\Phi_{\mathcal{P}}^{X \to Y}\colon D^b(X) \to D^b(Y), \quad \text{and} \quad \Phi_{\mathcal{P}}^{Y \to X}\colon D^b(Y) \to D^b(X).$$

Remark 3.67. A Fourier-Mukai functor is given by the composition of three functors which, after proper derivation, are exact functors between triangulated categories, namely:

$$\begin{gathered} p^*\colon D^b(X) \to D^b(X \times Y), \\ - \otimes \mathcal{P}\colon D^b(X \times Y) \to D^b(X \times Y), \\ q_*\colon D^b(X \times Y) \to D^b(Y). \end{gathered}$$

For this reason, $\Phi_{\mathcal{P}}$ is exact for all $\mathcal{P} \in D^b(X \times Y)$.

A functor $F\colon D^b(X) \to D^b(Y)$ is said to be of *Fourier-Mukai type* if there exists an object $\mathcal{P} \in D^b(X \times Y)$ such that F is isomorphic to $\Phi_{\mathcal{P}}$. Many of the functors that we already introduced are of Fourier-Mukai type, as the following examples show.

Examples 3.68. i) The identity $\mathrm{id}\colon D^b(X) \to D^b(X)$ is of Fourier-Mukai type, as it is naturally isomorphic to the Fourier-Mukai functor $\Phi_{\mathcal{O}_\Delta}$. Here, $\mathcal{O}_\Delta$ denotes the structure sheaf of the diagonal $\Delta \subset X \times X$. To see this, we first need to state the so-called *projection formula*, which we will not prove. Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. If $\mathcal{F}$ is an $\mathcal{O}_X$-module and $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank, then there exists a natural isomorphism:

$$f_*(\mathcal{F} \otimes f^*(\mathcal{E})) \cong f_*(\mathcal{F}) \otimes \mathcal{E}.$$

Showing that the inclusion $\iota\colon X \xrightarrow{\sim} \Delta \subset X \times X$ and the sheaves $\mathcal{O}_X$ and $p^*(\mathcal{E})$ satisfy the hypotheses of the projection formula is an easy check. In the derived world, the projection formula generalizes to the so-called *derived projection formula*:

$$\mathbf{R}f_*(\mathcal{F}^\bullet) \overset{\mathbf{L}}{\otimes} \mathcal{E}^\bullet \cong \mathbf{R}f_*(\mathcal{F}^\bullet \overset{\mathbf{L}}{\otimes} \mathbf{L}f^*(\mathcal{E}^\bullet)). \tag{3.8}$$

Therefore, we have that:

$$\begin{aligned} \Phi_{\mathcal{O}_\Delta}(\mathcal{E}^\bullet) &= q_*(p^*\mathcal{E}^\bullet \otimes \mathcal{O}_\Delta) = q_*(p^*\mathcal{E}^\bullet \otimes \iota_*\mathcal{O}_X) \\ &\cong q_*(\iota_*(\iota^* p^* \mathcal{E}^\bullet \otimes \mathcal{O}_X)) && \text{(derived projection formula)} \\ &\cong (q \circ \iota)_*(p \circ \iota)^* \mathcal{E}^\bullet \cong \mathcal{E}^\bullet && (\text{as } q \circ \iota = \mathrm{id} = p \circ \iota), \end{aligned}$$

for all $\mathcal{E}^\bullet \in D^b(X)$, which implies $\mathrm{id} \cong \Phi_{\mathcal{O}_\Delta}$.

ii) Let $f\colon X \to Y$ be a regular map between smooth projective varieties. Then we have that the pushforward of f is of Fourier-Mukai type. Specifically:

$$f_* \cong \Phi^{X \to Y}_{\mathcal{O}_{\Gamma_f}}\colon D^b(X) \to D^b(Y),$$

where Γ_f denotes the graph of f, and $\mathcal{O}_{\Gamma_f}$ is the structure sheaf of the graph. Analogously, the very same object $\mathcal{O}_{\Gamma_f}$ can be used as the Fourier-Mukai kernel of a Fourier-Mukai functor in the opposite direction. It turns out that what we obtain is isomorphic to the pullback of f:

$$f^* \cong \Phi^{Y \to X}_{\mathcal{O}_{\Gamma_f}}\colon D^b(Y) \to D^b(X).$$

iii) The shift functor $T\colon D^b(X) \to D^b(X)$ is isomorphic to the Fourier-Mukai functor whose kernel is the shift of the structure sheaf over the diagonal $\Delta \subset X \times X$. In other words, $T \cong \Phi_{\mathcal{O}_\Delta[1]}$, which makes the shift functor Fourier-Mukai.

iv) Again, consider the diagonal embedding $\iota\colon X \xrightarrow{\sim} \Delta \subset X \times X$. Then:

$$\Phi_{\iota_*(K_X)} \cong S_X[-\dim(X)],$$

where S_X denotes the Serre functor of X (see Proposition 3.62). Therefore, S_X is the composition of the Fourier-Mukai functor above with the shift $[\dim(X)]$, which is Fourier-Mukai by iii). As we will see in Proposition 3.72, this implies that the Serre functor is Fourier-Mukai as well.

Among the useful properties of Fourier-Mukai functors that we mentioned at the beginning of this section is the existence of left and right adjoints for every Fourier-Mukai functor. In fact, the two adjoints are Fourier-Mukai functors themselves, whose kernel can be described explicitly.

Definition 3.69. Consider an object $\mathcal{P} \in D^b(X \times Y)$. We will set:

$$\mathcal{P}_L \coloneqq (\mathcal{P}^\vee \otimes q^* K_Y)[\dim(Y)], \quad \text{and} \quad \mathcal{P}_R \coloneqq (\mathcal{P}^\vee \otimes p^* K_X)[\dim(X)];$$

both objects belong to $D^b(X \times Y)$.[8]

Remark 3.70. The Fourier-Mukai morphisms induced by $\mathcal{P}_L$ and $\mathcal{P}_R$ can be expressed more directly by using the Serre functors associated to the varieties X and Y. In particular, we have that:

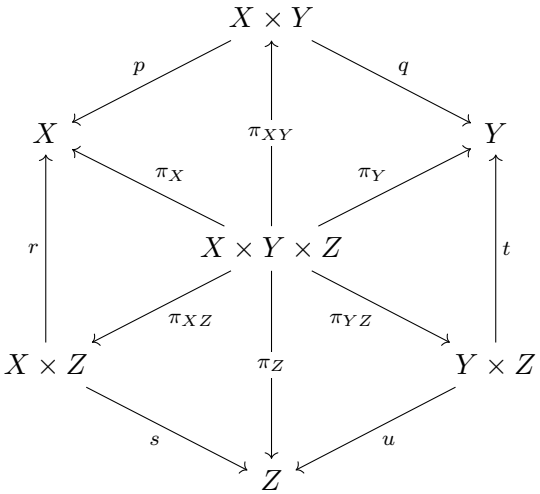
$$\Phi_{\mathcal{P}_L} \cong \Phi_{\mathcal{P}^\vee} \circ S_Y \colon D^b(Y) \to D^b(X),$$
$$\Phi_{\mathcal{P}_R} \cong S_X \circ \Phi_{\mathcal{P}^\vee} \colon D^b(Y) \to D^b(X).$$

Proposition 3.71. *Let $\Phi_{\mathcal{P}} \colon D^b(X) \to D^b(Y)$ be the Fourier-Mukai functor of kernel $\mathcal{P}$. Then $\Phi_{\mathcal{P}_L}$ and $\Phi_{\mathcal{P}_R}$ are, respectively, left and right adjoint to $\Phi_{\mathcal{P}}$, i.e. $\Phi_{\mathcal{P}_L} \dashv \Phi_{\mathcal{P}}$, and $\Phi_{\mathcal{P}} \dashv \Phi_{\mathcal{P}_R}$.*

The final key property of Fourier-Mukai functors regards composition: specifically, the composition of Fourier-Mukai functors is still Fourier-Mukai, and once again we are able to give an explicit description of the Fourier-Mukai kernel of the resulting functor. Let X, Y and Z be smooth projective varieties, and consider objects $\mathcal{P} \in D^b(X \times Y)$ and $\mathcal{Q} \in D^b(Y \times Z)$. Then we will define the object $\mathcal{R} \in D^b(X \times Z)$, which is going to be the Fourier-Mukai kernel of the composition, as

$$\mathcal{R} \coloneqq \pi_{XZ*}(\pi_{XY}^* \mathcal{P} \otimes \pi_{YZ}^* \mathcal{Q}), \tag{3.9}$$

where the projections follow the notation established by the following diagram, which also gives names to all the other possible projections:



[8] When we write $\mathcal{P}^\vee$ we are implicitly making use of the notion of *derived dual*. Its construction is too long and intricate for our purposes; very briefly, $\mathcal{F}^{\bullet\vee} \coloneqq \mathbf{R}\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{O}_X)$, where we employ $\mathcal{H}om$, a generalization of the local morphism sheaf to the case of complexes.

Proposition 3.72. *The composition* $D^b(X) \xrightarrow{\Phi_{\mathcal{P}}} D^b(Y) \xrightarrow{\Phi_{\mathcal{Q}}} D^b(Z)$ *is isomorphic to the Fourier-Mukai functor* $\Phi_{\mathcal{R}} \colon D^b(X) \longrightarrow D^b(Z)$.

Proof. Fix an object $\mathcal{E}^\bullet \in D^b(X)$. Then:

$$\Phi_{\mathcal{R}}(\mathcal{E}^\bullet) = s_*(r^*\mathcal{E}^\bullet \otimes \mathcal{R}) = s_*(r^*\mathcal{E}^\bullet \otimes \pi_{XZ*}(\pi_{XY}^*\mathcal{P} \otimes \pi_{YZ}^*\mathcal{Q}))$$

We can now apply the derived projection formula (3.8) to morphism π_{XZ} and objects $r^*\mathcal{E}^\bullet$ and $\pi_{XY}^*\mathcal{P} \otimes \pi_{YZ}^*\mathcal{Q}$ to obtain:

$$s_*(\pi_{XZ*}(\pi_X^*\mathcal{E}^\bullet \otimes \pi_{XY}^*\mathcal{P} \otimes \pi_{YZ}^*\mathcal{Q})),$$

which is equal to $\pi_{Z*}(\pi_X^*\mathcal{E}^\bullet \otimes \pi_{XY}^*\mathcal{P} \otimes \pi_{YZ}^*\mathcal{Q}))$, since $s \circ \pi_{XZ} = \pi_Z$. We will now employ another formula which we will not demonstrate: if $f \colon X \to Y$ is a regular map between smooth projective varieties, and $\mathcal{F}^\bullet, \mathcal{E}^\bullet \in D^b(Y)$, then:

$$\mathbf{L}f^*(\mathcal{F}^\bullet) \overset{\mathbf{L}}{\otimes} \mathbf{L}f^*(\mathcal{E}^\bullet) \cong \mathbf{L}f^*(\mathcal{F}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{E}^\bullet). \tag{3.10}$$

Here, π_{XY} will play the role of f, and since projections are flat, we can ignore the left-derivation symbol and apply the formula to the objects $p^*\mathcal{E}^\bullet$ and $\mathcal{P}$, yielding:

$$\pi_{Z*}(\pi_{XY}^*(p^*\mathcal{E}^\bullet \otimes \mathcal{P}) \otimes \pi_{YZ}^*\mathcal{Q}).$$

This is equal to $u_*\pi_{YZ*}(\pi_{XY}^*(p^*\mathcal{E}^\bullet \otimes \mathcal{P}) \otimes \pi_{YZ}^*\mathcal{Q})$, since $u \circ \pi_{YZ} = \pi_Z$. Another, more intuitive application of the projection formula gives:

$$u_*(\pi_{YZ*}\pi_{XY}^*(p^*\mathcal{E}^\bullet \otimes \mathcal{P}) \otimes \mathcal{Q}).$$

A further identity involving derived functors[9], along with the flatness of projections, enables us to say that $\pi_{YZ*} \circ \pi_{XY}^* = t^* \circ q_*$, a fact that lets us conclude that what we obtained is isomorphic to:

$$u_*(t^*q_*(p^*\mathcal{E}^\bullet \otimes \mathcal{P}) \otimes \mathcal{Q}) = u_*(t^*\Phi_{\mathcal{P}}(\mathcal{E}^\bullet) \otimes \mathcal{Q}) = \Phi_{\mathcal{Q}}(\Phi_{\mathcal{P}}(\mathcal{E}^\bullet)),$$

which proves that $\Phi_{\mathcal{R}} \cong \Phi_{\mathcal{Q}} \circ \Phi_{\mathcal{P}}$. □

Remark 3.73. If the composition $\Phi_{\mathcal{Q}} \circ \Phi_{\mathcal{P}}$ is not an equivalence, its kernel is not, in general, unique. However, the choice made in (3.9) is the natural one with respect to the two adjoint functors. Indeed, if we follow the notation of Definition 3.69, we find that:

$$\mathcal{R}_L \cong \pi_{XZ*}(\pi_{XY}^*\mathcal{P}_L \otimes \pi_{YZ}^*\mathcal{Q}_L), \quad \text{and} \quad \mathcal{R}_R \cong \pi_{XZ*}(\pi_{XY}^*\mathcal{P}_R \otimes \pi_{YZ}^*\mathcal{Q}_R),$$

i.e., we just need to replace $\mathcal{P}, \mathcal{Q}$ with $\mathcal{P}_L, \mathcal{Q}_L$ or $\mathcal{P}_R, \mathcal{Q}_R$ in the definition of $\mathcal{R}$.

We will close this section with a pivotal result in the context of Fourier-Mukai functors. It was proved by Dmitri Orlov (b. 1966) in [Orl97] (see also his comprehensive survey [Orl03] for a complete treatment), providing a partial answer to the following questions:

[9] Consider a fibre product diagram:

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{v} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[u]{} & Z \end{array}$$

with u flat and f proper. Then $u^*Rf_*\mathcal{F}^\bullet \cong Rg_*v^*\mathcal{F}^\bullet$ for any $\mathcal{F}^\bullet \in D(\mathbf{Qcoh}(Y))$.

i) Let X and Y be smooth projective varieties. Are all exact functors $D^b(X) \longrightarrow D^b(Y)$ of Fourier-Mukai type?

ii) In the same context, is the Fourier-Mukai kernel unique up to isomorphism?

Under additional hypotheses on the functor $D^b(X) \to D^b(Y)$, Orlov's theorem answers both questions positively.

3.74 Orlov's Theorem. *Let X and Y be smooth projective varieties, and let $F\colon D^b(X) \longrightarrow D^b(Y)$ be a fully faithful exact functor. If F admits a left and a right adjoint, then there exists an object $\mathcal{P} \in D^b(X \times Y)$, which is unique up to isomorphism, such that $F \cong \Phi_{\mathcal{P}}$.*

As equivalences always admit left and right adjoints (given by their quasi-inverse), we immediately find the following consequence of Orlov's theorem, which is actually one of its most common use cases.

Corollary 3.75. *Let $F\colon D^b(X) \longrightarrow D^b(Y)$ be an equivalence between the bounded derived categories of two smooth projective varieties. Then there exists an object $\mathcal{P} \in D^b(X \times Y)$, unique up to isomorphism, such that $F \cong \Phi_{\mathcal{P}}$.*

As a quick and powerful application of Orlov's theorem, we will prove the following geometric result.

Proposition 3.76. *Let X and Y be smooth projective varieties with equivalent bounded derived categories. Then* $\dim(X) = \dim(Y)$.

Proof. Let us call $F\colon D^b(X) \xrightarrow{\sim} D^b(Y)$ an equivalence. By Orlov's theorem, F must be of the form $\Phi_{\mathcal{P}}$ for a certain $\mathcal{P} \in D^b(X \times Y)$. Furthermore, by Proposition 3.71, F admits left and right adjoints given by the Fourier-Mukai functors of kernel $\mathcal{P}_L = (\mathcal{P}^\vee \otimes q^* K_Y)[\dim(Y)]$ and $\mathcal{P}_R = (\mathcal{P}^\vee \otimes p^* K_X)[\dim(X)]$, respectively. Since F is an equivalence, both of its adjoints are quasi-inverses to F, therefore $\Phi_{\mathcal{P}_L}$ and $\Phi_{\mathcal{P}_R}$ are equivalences as well. Hence we can apply Corollary 3.75 to conclude that $\mathcal{P}_L$ and $\mathcal{P}_R$ must be isomorphic objects in $D^b(X \times Y)$, i.e.

$$(\mathcal{P}^\vee \otimes q^* K_Y)[\dim(Y)] \cong (\mathcal{P}^\vee \otimes p^* K_X)[\dim(X)].$$

The tensor product and the shift functor commute: by applying the shift $[-\dim(Y)]$ and tensoring by $q^* K_Y^\vee$ we can isolate the term $\mathcal{P}^\vee$, and find that:

$$\mathcal{P}^\vee \cong \mathcal{P}^\vee \otimes ((p^* K_X \otimes q^* K_Y^\vee)[\dim(X) - \dim(Y)]).$$

To simplify the notation, let us denote $d \coloneqq \dim(X) - \dim(Y)$ and $\mathcal{L} \coloneqq p^* K_X \otimes q^* K_Y^\vee$. Commutation with the shift functor is an important property of the derived tensor product, therefore, we have:

$$\mathcal{P}^\vee \cong (\mathcal{P}^\vee \otimes \mathcal{L})[d]. \tag{3.11}$$

Now, since $\mathcal{P}^\vee \in D^b(X \times Y)$, and $\mathcal{P}^\vee \neq 0$, there exists an integer k which is the largest such that $H^k(\mathcal{P}^\vee) \neq 0$. Equation (3.11) implies that:

$$H^k(\mathcal{P}^\vee) \cong H^k((\mathcal{P}^\vee \otimes \mathcal{L})[d]) \cong H^{k+d}(\mathcal{P}^\vee \otimes \mathcal{L}).$$

Another standard property of $\overset{\mathbf{L}}{\otimes}$ tells us that, since $\mathcal{L}$ is a line bundle, the functor $- \otimes \mathcal{L}$ is exact, and thus commutes with cohomology. Therefore:

$$H^k(\mathcal{P}^\vee) \cong H^{k+d}(\mathcal{P}^\vee) \otimes \mathcal{L}.$$

If $d > 0$, then $H^{k+d}(\mathcal{P}^\vee)$ must be 0, by the maximality of k. But this would imply $H^k(\mathcal{P}^\vee) = 0$, which contradicts the very definition of k. The case $d < 0$ is verified to be false in a similar way, this time by taking k to be the minimal integer which makes cohomology not zero. □

Remark 3.77. To conclude this chapter, we mention a celebrated result by Bondal and Orlov [BO01], which heavily relies on Orlov's theorem. They proved that if the canonical bundle of a smooth projective variety X is ample (see Definition 4.9) or anti-ample (i.e., $K_X^\vee$ is ample), then the variety can be uniquely reconstructed from its derived category; in other words, if $D^b(X) \cong D^b(Y)$, then $X \cong Y$. Fano varieties and curves of genus $g \geq 2$ satisfy these hypotheses. For K3 surfaces, however, the canonical bundle is trivial. This lack of ampleness allows for a more interesting scenario: the existence of non-isomorphic, yet derived equivalent, surfaces, often called *Fourier-Mukai partners*. Enriques surfaces, instead, have no non-trivial Fourier-Mukai partners, as we will see in Section 4.4.

Chapter 4

Enriques Surfaces

In this chapter we finally showcase the power of the derived category as an algebraic invariant of surfaces. We will focus on the case of Enriques surfaces (in characteristic 0), whose topological and numerical properties interact deeply with the categorical structure of $D^b(X)$, and allow for a semiorthogonal decomposition of $D^b(X)$ via exceptional sequences generated by line bundles. This algebraic decomposition will enable us to isolate a subcategory, known as the *residual component*, which encodes the geometric data of the surface; indeed, the final theorem of this chapter demonstrates how the global isomorphism class of an Enriques surface can be entirely recovered from this component.

Before moving on to the formal definition of an Enriques surface, we will introduce some preliminary geometric structures, namely the Néron-Severi group $\mathrm{NS}(X)$ of a surface X, and its canonical lattice $\mathrm{Num}(X)$, along with its associated intersection form. Initially, this will be done in a more general setting: throughout section 3.1, by *surface* we will mean smooth, projective, complex variety of dimension 2.

4.1 The intersection form on divisors

The material of this section is standard. The intersection form on divisors, the Hodge index theorem and the Nakai–Moishezon criterion follow [Har77, Section V.1]; the Néron–Severi group and the exponential sequence are treated in [Har77, BHPVdV04], and the basic lattice theory we use can be found in [CDL25, Section 0.8] and also in [BHPVdV04]. Finally, the introduction to the cones of divisors follows the exposition in [Laz04, Chapter 1].

In this section we will develop some rudiments of intersection theory, in order to properly establish how the product of two divisors is defined. By *curve* we will mean any effective divisor on the surface, i.e. any formal sum of codimension-1 subvarieties where all the coefficients are positive.

Let X be a smooth projective surface. Given two divisors C and D over X, we want to define a product $C.D$ which, in a way, generalizes the concept of counting the intersections of the two curves over the surface.

Definition 4.1. Let C and D be curves on X, with $P \in C \cap D$ a point of intersection. Let f and g be the local equations (at P) of C and D, respectively. We say that C and D meet *transversally* at P if f and g generate the maximal ideal $\mathfrak{m}_P \subset \mathcal{O}_{P,X}$.

Geometrically, this algebraic condition means that the two curves are nonsingular at P and do not share a common tangent line. This ensures that their intersection at that point is as simple as possible, without tangencies or higher multiplicities. Now, if C and D are two nonsingular curves which meet transversally at r points $P_1, \ldots, P_r$, we would want the intersection number $C.D$ to be equal to r. This is one of the properties that we impose on the intersection pairing; other ones appear to be as natural, if we keep thinking of this pairing as a generalization of counting intersections.

Theorem 4.2. *Let X be a smooth projective surface. There exists a unique pairing $\mathrm{Div}(X) \times \mathrm{Div}(X) \to \mathbb{Z}$, denoted by $C.D$, such that:*

(i) if C and D are nonsingular curves meeting only transversally, then $C.D$ counts their intersections, i.e. $C.D = \#(C \cap D)$;
(ii) it is symmetric: $C.D = D.C$ for all $C, D \in \mathrm{Div}(X)$;
(iii) it is additive: $(C_1 + C_2).D = C_1.D + C_2.D$, and $C.(D_1 + D_2) = C.D_1 + C.D_2$;
(iv) it depends only on the linear equivalence class of the curves involved: if $C_1 \equiv C_2$, then $C_1.D = C_2.D$ for all $D \in \mathrm{Div}(X)$.

If C is a divisor over the surface X, then its *self-intersection* number is $C.C$, also denoted as C^2.

Recall that, with respect to the tensor product, the set of holomorphic line bundles (up to isomorphism) over X forms a group $\mathrm{Pic}(X)$, called the *Picard group*. This group is canonically isomorphic to $\mathrm{Div}(X)$ modulo linear equivalence. Keeping this isomorphism in mind, it is clear how requirement (iv) immediately tells us that the given bilinear form descends to a pairing $\mathrm{Pic}(X) \times \mathrm{Pic}(X) \to \mathbb{Z}$, which retains all the properties above.

Now, consider two divisors $C, D \in \mathrm{Div}(X)$. We will say that C and D are *numerically equivalent* ($C \sim D$) if they yield the same intersection number when paired with every curve on the surface, i.e., $C \sim D \iff C.E = D.E$ for all $E \in \mathrm{Div}(X)$; this is obviously an equivalence relation on $\mathrm{Div}(X)$.

Definition 4.3. A divisor $D \in \mathrm{Div}(X)$ is said to be *nef* (shorthand for *numerically effective*) if $D.C \geq 0$ for any irreducible effective divisor $C \in \mathrm{Div}(X)$.

Having introduced the relation $\sim$ on $\mathrm{Div}(X)$, we can now define the *numerical Néron-Severi group* $\mathrm{Num}(X) \coloneqq \mathrm{Div}(X)/\sim$, and clearly the intersection form descends to a form over $\mathrm{Num}(X)$ as well. Observe that this is an integer-valued, symmetric bilinear form: thus, the intersection form over $\mathrm{Num}(X)$ endows it with a structure of *integral lattice*, i.e. a structure of free finitely generated abelian group equipped with a (non-degenerate) integer-valued symmetric bilinear form. A lattice is said to be *unimodular* if its *discriminant*, i.e. the determinant of the matrix associated to the lattice's bilinear form, is ± 1. This will be the case for $\mathrm{Num}(S)$, with S an Enriques surface, where the discriminant will evaluate to -1.

Before moving on to Enriques surfaces, we will define one more group associated to a given variety X. The Picard group is naturally isomorphic to $H^1(X, \mathcal{O}_X^*)$, the first cohomology group of the sheaf of non-zero holomorphic functions $\mathcal{O}_X^*$. This is thanks to the correspondence between holomorphic line bundles (i.e., elements of $\mathrm{Pic}(X)$), and their associated transition functions g_{ij} (i.e., elements of $H^1(X, \mathcal{O}_X^*)$). We want to consider the so-called *exponential* sequence of sheaves:

$$0 \longrightarrow \mathbb{Z}_X \xrightarrow{i} \mathcal{O}_X \xrightarrow{j} \mathcal{O}_X^* \longrightarrow 0.$$

Here, $\mathbb{Z}_X$ denotes the constant sheaf $\mathbb{Z}$; i is just an inclusion, while $j(s) = e^{2\pi i s}$, with s a section of $\mathcal{O}_X$. As always, this induces a long exact sequence in cohomology, which is called the *exponential cohomology sequence*:

$$\cdots \to H^1(\mathbb{Z}_X) \to H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_X^*) \xrightarrow{\delta} H^2(\mathbb{Z}_X) \to H^2(\mathcal{O}_X) \to \cdots . \tag{4.1}$$

The connecting map δ is also called the *first Chern class* of a bundle, denoted c_1.

Remark 4.4. The first Chern class $c_1 \colon \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ provides an explicit formulation for the intersection pairing defined in Theorem 4.2. Specifically, for any two divisors $C, D \in \mathrm{Div}(X)$ the intersection number coincides with applying the canonical surjection $\mathrm{Div}(X) \to \mathrm{Pic}(X)$, and then integrating the cup product on $H^2(X, \mathbb{Z})$ (or, rather, its restriction to $\mathrm{Pic}(X)$) over X:

$$C.D = \int_X c_1(\mathcal{O}_X(C)) \smile c_1(\mathcal{O}_X(D));$$

equivalently, for any two line bundles $L, P \in \mathrm{Pic}(X)$, this reads simply as:

$$L.P = \int_X c_1(L) \smile c_1(P). \tag{4.2}$$

Let us write $\mathrm{Pic}^0(X) \coloneqq \mathrm{Ker}(c_1)$.

Remark 4.5. Consider the following portion of the exponential cohomology sequence:

$$\cdots \xrightarrow{i^0} H^0(\mathcal{O}_X) \xrightarrow{j^0} H^0(\mathcal{O}_X^*) \xrightarrow{\delta} H^1(\mathbb{Z}_X) \xrightarrow{i^1} H^1(\mathcal{O}_X) \xrightarrow{j^1} H^1(\mathcal{O}_X^*) \xrightarrow{c_1} \cdots .$$

The morphism i^1 is injective. Indeed, since X is compact and connected, global holomorphic functions are constant, so $H^0(X, \mathcal{O}_X) \cong \mathbb{C}$ and $H^0(X, \mathcal{O}_X^*) \cong \mathbb{C}^*$. The global exponential map $j^0 \colon \mathbb{C} \to \mathbb{C}^*$ is surjective, which implies:

$$\mathrm{Ker}(i^1) = \mathrm{Im}(\delta) \cong H^0(\mathcal{O}_X^*)/\,\mathrm{Ker}(\delta) = H^0(\mathcal{O}_X^*)/\,\mathrm{Im}(j^0) \cong 0.$$

Consequently, $H^1(X, \mathbb{Z}_X)$ can be regarded as a discrete subgroup of the complex vector space $H^1(X, \mathcal{O}_X)$. Therefore, $\mathrm{Pic}^0(X) = \mathrm{Ker}(c_1)$, which is isomorphic to the quotient $H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z}_X)$, possesses a natural structure of a complex torus.

Definition 4.6. Let X be a surface. The *(algebraic) Néron-Severi group* associated to X is:

$$\mathrm{NS}(X) \coloneqq \mathrm{Pic}(X)/\,\mathrm{Pic}^0(X).$$

By the first isomorphism theorem, $\mathrm{NS}(X)$ is isomorphic to a subgroup of $H^2(X, \mathbb{Z}_X)$, namely to the image of the first Chern class. Of course, the quotient is meant with respect to the group structure provided to $\mathrm{Pic}(X)$ by the tensor product: two line bundles $L, M \in \mathrm{Pic}(X)$ represent the same class in $\mathrm{NS}(X)$ if $L \otimes M^\vee \equiv N$, with $N \in \mathrm{Ker}(c_1)$. In this case, L and M are said to be *algebraically equivalent*.

Remark 4.7. An alternative way to define NS is as follows. In this case, two line bundles $L_0, L_1 \in \mathrm{Pic}(X)$ are said to be algebraically equivalent if there exist a connected algebraic curve T, two points $t_0, t_1 \in T$, and a line bundle $\mathcal{L}$ over $X \times T$, such that:

$$\mathcal{L}|_{X\times\{t_0\}} \cong L_0, \quad \text{and} \quad \mathcal{L}|_{X\times\{t_1\}} \cong L_1.$$

Let us denote this new notion of algebraic equivalence as $L_0 \approx L_1$. Similarly, two divisors $D_0, D_1 \in \mathrm{Div}(X)$ are considered algebraically equivalent ($D_0 \approx D_1$) if $\mathcal{O}_X(D_0) \approx \mathcal{O}_X(D_1)$. We can now define:

$$\mathrm{NS}(X) \coloneqq \mathrm{Div}(X)/\approx .$$

Naturally, this construction can be shown to be equivalent to the one stated in Definition 4.6.

Remark 4.8. The Picard group $\mathrm{Pic}(X)$ naturally carries the structure of a complex variety, consisting of a disjoint union of connected components. The connected component of the identity is $\mathrm{Pic}^0(X)$, which parametrizes line bundles that are algebraically equivalent to the trivial bundle $\mathcal{O}_X$. Since the group action of $\mathrm{Pic}(X)$ acts by translation, every connected component is isomorphic to a coset of the form $L \otimes \mathrm{Pic}^0(X)$ for some $L \in \mathrm{Pic}(X)$.

Consequently, the Néron-Severi group parametrizes the set of connected components of $\mathrm{Pic}(X)$. By quotienting out the "algebraic deformations" contained in $\mathrm{Pic}^0(X)$, the Néron-Severi group inherits a discrete structure.

It immediately follows from (4.2) that for any $N \in \mathrm{Pic}^0(X)$ and any $L \in \mathrm{Pic}(X)$, one has $N.L = 0$. Consequently, the intersection form vanishes on $\mathrm{Pic}^0(X)$, ensuring that it descends to a well-defined integer-valued symmetric bilinear pairing on the Néron-Severi group:

$$\mathrm{NS}(X) \times \mathrm{NS}(X) \to \mathbb{Z}.$$

The algebraic Néron-Severi group $\mathrm{NS}(X)$ can be characterized in terms of the previously-introduced $\mathrm{Num}(X)$. To prove this, we need:

Definition 4.9. A line bundle $L \in \mathrm{Pic}(X)$ is said to be *very ample* if its associated evaluation morphism $\mathrm{ev}_x \colon H^0(X, L) \to L_x$ is surjective for every $x \in X$, and the associated map

$$\phi_L \colon X \to \mathbb{P}(H^0(X, L)^*)$$

is a closed embedding. A line bundle is *ample* if $L^{\otimes n}$ is very ample for some $n \geq 1$. Likewise, a divisor $D \in \mathrm{Div}(X)$ is *(very) ample* if so is its associated line bundle $\mathcal{O}_X(D)$.

4.10 Hodge Index Theorem. *Let $H \in \mathrm{Div}(X)$ be an ample divisor on the surface X. If $D \in \mathrm{NS}(X) \otimes \mathbb{R}$ is a non-zero class such that $D \cdot H = 0$, then $D^2 < 0$.*

Theorem 4.11. *If X is a surface, then* $\mathrm{Num}(X) \cong \mathrm{NS}(X)/\mathrm{Tors}(\mathrm{NS}(X))$.

Proof. We need to show that a numerically trivial element $D \in \mathrm{NS}(X)$ is necessarily torsion, i.e. $D = 0$ in $\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, since tensoring by $\mathbb{R}$ cancels any torsion element. Therefore, suppose that $D.C = 0.C = 0$ for any curve C. In particular $D^2 = 0$, and if H is an ample divisor then $D.H = 0$ as well. If D were a non-trivial element of $\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, the Hodge Index Theorem would imply that $D^2 < 0$, contradicting $D^2 = 0$. Therefore D must be torsion. □

To recap, we have introduced various structures on the divisors and the line bundles defined over a smooth projective surface X. The following commutative diagram summarizes all of them:

$$\begin{array}{ccccccc} & & \mathrm{Div}(X) & \xrightarrow{/\equiv} & \mathrm{Pic}(X) & & \\ & & \downarrow{\scriptstyle /\sim} & & \downarrow{\scriptstyle /\,\mathrm{Pic}^0(X)} & & \\ \mathrm{NS}(X)\otimes_{\mathbb{Z}}\mathbb{R} & \xleftarrow{-\otimes_{\mathbb{Z}}\mathbb{R}} & \mathrm{NS}(X) & \xrightarrow[/\,\mathrm{Tors}]{} & \mathrm{Num}(X) & \xrightarrow{-\otimes_{\mathbb{Z}}\mathbb{R}} & \mathrm{Num}(X)\otimes_{\mathbb{Z}}\mathbb{R} \end{array}$$

Observe that $\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathrm{Num}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ are isomorphic: $\mathrm{Num}(X)$ is just $\mathrm{NS}(X)$ modulo torsion, and tensoring by $\mathbb{R}$ trivializes torsion. We will be interested in the space $\mathrm{Num}(X) \otimes_{\mathbb{Z}} \mathbb{R} = \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ for the rest of this section.

Definition 4.12. Let X be a surface. The *nef cone* of X is the region of the real vector space $\mathrm{Num}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ defined by:

$$\mathrm{Nef}(X) \coloneqq \{D \in \mathrm{Num}(X) \otimes_{\mathbb{Z}} \mathbb{R} \text{ such that } D.C \geq 0 \text{ for all curves } C \subset X\}.$$

By *cone* we mean a subset C of a real vector space V such that, for any $x \in C$ and any $\lambda \in \mathbb{R}_{>0}$, we have $\lambda x \in C$: thus, the nef cone is obviously a cone. We can also define the *ample cone* and the *effective cone*:

$$\mathrm{Amp}(X) \coloneqq \{D \in \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R} \text{ such that } D \text{ is ample}\} \subset \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R},$$
$$\mathrm{Eff}(X) \coloneqq \{D \in \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R} \text{ such that } D \text{ is effective}\} \subset \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}.$$

The closure $\overline{\mathrm{Eff}}(X)$ is called the *pseudo-effective cone* of X. Before relating these newly-introduced notions to one another, let us introduce one more cone:

$$\mathrm{NE}(X) \coloneqq \left\{\sum a_i[C_i] \text{ s.t. } a_i \geq 0 \text{ and } C_i \text{ is an irreducible curve over } X, \text{for all } i\right\},$$

where the equivalence class $[C_i]$ is meant with respect to numerical equivalence. Formally, the *Mori cone* $\mathrm{NE}(X)$ studies classes of dimension 1 objects, while $\mathrm{Num}(X)$ focuses on subvarieties of codimension 1. It is clear that, when working with surfaces, these two structures coincide.

The intersection form on divisors realizes the following duality:

4.13 Kleiman Duality. *Let X be a surface. Then the nef cone and the closure of the Mori cone satisfy:*

$$\mathrm{Nef}(X) = \left(\overline{\mathrm{NE}}(X)\right)^{\vee}.$$

The nef cone can also be described in terms of the ample cone, thanks to:

4.14 Nakai–Moishezon Criterion. *Let $D \in \mathrm{Div}(X)$ be a divisor over X. Then:*

$$D \text{ is ample} \iff D^2 > 0, D.C > 0 \text{ for every curve } C \subset X.$$

Corollary 4.15. *Let X be a surface. Then the nef cone and the closure of the ample cone coincide:*

$$\mathrm{Nef}(X) = \overline{\mathrm{Amp}}(X).$$

Remark 4.16. In the Nakai–Moishezon criterion, it suffices to check $D.C > 0$ for every *integral* curve $C \subset S$: if $C = \sum_i m_i C_i$ is an arbitrary effective divisor with integral components C_i, then by additivity of the intersection pairing (Theorem 4.2(iii)), $D.C = \sum_i m_i (D.C_i)$, which is positive whenever each $D.C_i$ is.

Example 4.17. As an illustration of how the cones $\mathrm{Eff}(X)$, $\mathrm{Nef}(X)$ and $\mathrm{Amp}(X)$ can be computed in practice, consider the blow-up $\pi\colon X \to \mathbb{P}^2$ of $\mathbb{P}^2$ at a point p. Let us set:

$$H \coloneqq \pi^*[\text{line in } \mathbb{P}^2] \in \mathrm{Div}(X), \qquad E \coloneqq [\text{exceptional divisor of } \pi] \in \mathrm{Div}(X).$$

It is a standard fact about blow-ups that $\mathrm{Num}(X) = \mathbb{Z}H \oplus \mathbb{Z}E$, with intersection numbers:

$$H^2 = 1, \qquad H.E = 0, \qquad E^2 = -1,$$

so that $\mathrm{Num}(X) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^2$.

We first describe which classes in $\mathrm{Num}(X) \otimes \mathbb{R}$ are effective, by listing the irreducible curves on X:

- the exceptional divisor E itself;
- the strict transform $\widetilde{C}$ of an irreducible plane curve $C \subset \mathbb{P}^2$ of degree d passing through p with multiplicity m (where necessarily $0 \le m \le d$), whose class is:

$$[\widetilde{C}] = dH - mE.$$

Since $0 \le m \le d$, we can rewrite this class as:

$$[\widetilde{C}] = d(H - E) + (d - m)E,$$

a combination of E and $H - E$ with non-negative coefficients. As every irreducible curve on X is either E or one such strict transform, every effective divisor on X is a non-negative combination of E and $H - E$. Hence, $\mathrm{Eff}(X)$ is the cone generated by these two divisors:

$$\mathrm{Eff}(X) = \mathrm{Cone}(E,\, H - E).$$

We will now study the nef cone. Let $D = aH - bE$ be a generic class in $\mathrm{Num}(X) \otimes \mathbb{R}$. Since $\mathrm{Eff}(X)$ is generated by E and $H - E$, by Definition 4.3 it suffices to check the sign of D paired against these two generators:

$$D.E = (aH - bE).E = -bE^2 = b, \qquad D.(H - E) = (aH - bE).(H - E) = a - b.$$

Therefore $D \in \mathrm{Nef}(X)$ if and only if $b \ge 0$ and $a - b \ge 0$, i.e.

$$\mathrm{Nef}(X) = \{aH - bE \;:\; a \ge b \ge 0\} = \mathrm{Cone}(H - E,\, H).$$

Remark 4.18. Note that $H \in \mathrm{Nef}(X)$, but $E \notin \mathrm{Nef}(X)$: indeed $E^2 = -1 < 0$, and since nef classes must pair non-negatively with every effective divisor, in particular with themselves when effective, no curve of negative self-intersection can ever be nef. This is a useful general remark, which we will use when discussing Enriques surfaces in Section 4.3.

Remark 4.19. By the Nakai–Moishezon criterion and Corollary 4.15, $\mathrm{Amp}(X)$ is the interior of $\mathrm{Nef}(X)$, namely:

$$\mathrm{Amp}(X) = \{aH - bE \;:\; a > b > 0\}.$$

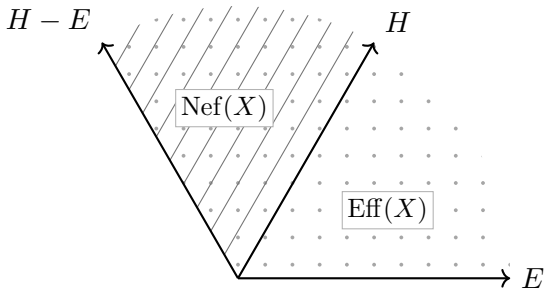


Figure 4.1: The cones $\mathrm{Eff}(X)$ and $\mathrm{Nef}(X)$ inside $\mathrm{Num}(X)\otimes\mathbb{R}$. The effective cone $\mathrm{Eff}(X) = \mathrm{Cone}(E, H-E)$ is generated by E and $H-E$; the nef cone $\mathrm{Nef}(X) = \mathrm{Cone}(H-E, H)$ is the narrower sub-cone generated by $H-E$ and H. Since $H = E + (H-E)$, the ray H lies strictly inside $\mathrm{Eff}(X)$, making the inclusion $\mathrm{Nef}(X) \subset \mathrm{Eff}(X)$ visually immediate.

4.2 K3 and Enriques Surfaces: Generalities

The intent of this section is to introduce some basic facts about Enriques surfaces, in order to set the stage for the study of their semiorthogonal decomposition. In this regard, we will first explore a situation where semiorthogonal decompositions do not exist at all, before finally moving on to Enriques surfaces. For the general theory of K3 and Enriques surfaces, their lattice theory, and elliptic fibrations, we refer to [BHPVdV04, Chapter VIII] and [CDL25].

Definition 4.20. A *K3 surface* X is a smooth projective surface with no non-trivial global holomorphic 1-forms, and with a trivial canonical bundle. In symbols:

(i) $H^1(X, \mathcal{O}_X) = 0$;
(ii) $K_X \cong \mathcal{O}_X$.

The name for K3 surfaces was given by André Weil (1906–1998) in [Wei58], in honor of three mathematicians: Ernst Kummer (1810–1893), Kunihiko Kodaira (1915–1997), and Erich Kähler (1906–2000), whose pioneering work on singular surfaces, complex surface classification, and Kähler geometry laid the groundwork for their study, as well as in homage to the mountain K2.

While we will not introduce the concepts necessary to properly prove this fact, a classical example of K3 surface is the class of smooth quartics in $\mathbb{P}^3$; here, a particularly interesting special case is the *Fermat quartic*, defined by the equation $x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$.

By definition, any K3 surface X is connected. This implies, by Proposition 3.49, that its bounded derived category $D^b(X)$ is indecomposable, i.e., $D^b(X)$ admits no non-trivial fully orthogonal decomposition. In order to prove that the same holds true even for semiorthogonal decompositions, the idea is to show that a semiorthogonal decomposition is also orthogonal, and conclude its triviality by the previous observation.

Remark 4.21. Let X be a K3 surface. Then its Serre functor, given by Proposition 3.62, is just the shift functor $[\dim(X)] = [2]$, since the canonical bundle is trivial by definition.

A quick consequence of this is the non-existence of exceptional objects in $D^b(X)$, with X a K3 surface. Indeed, suppose that $\mathcal{E}^\bullet \in D^b(X)$ is exceptional. Recall that this implies:

$$\mathrm{Hom}(\mathcal{E}^\bullet, \mathcal{E}^\bullet[i]) = \begin{cases} k & \text{if } i = 0 \\ 0 & \text{if } i \neq 0, \end{cases}$$

therefore $\mathrm{Hom}(\mathcal{E}^\bullet, \mathcal{E}^\bullet) = k$. We can now use the definition of Serre functor to get:

$$\mathrm{Hom}(\mathcal{E}^\bullet, \mathcal{E}^\bullet[2]) \cong \mathrm{Hom}(\mathcal{E}^\bullet, \mathcal{E}^\bullet)^\vee,$$

which yields $\mathrm{Hom}(\mathcal{E}^\bullet, \mathcal{E}^\bullet[2]) = k^\vee \neq 0$, against the definition of exceptional object. The proof of the fact that semiorthogonal decompositions are orthogonal follows a similar argument.

Theorem 4.22. *Let X be a K3 surface. Then $D^b(X)$ admits no non-trivial semiorthogonal decomposition.*

Proof. Let us suppose that there exists a proper semiorthogonal decomposition: $D^b(X) = \langle \mathcal{A}, \mathcal{B} \rangle$. Then we have $\mathrm{Hom}(B, A) = 0$ for all $A \in \mathcal{A}$ and all $B \in \mathcal{B}$, since $\mathcal{B} \subset \mathcal{A}^\perp$. By the observations above, it suffices to prove that morphisms in the opposite direction are also 0. By applying Remark 4.21 we obtain:

$$\mathrm{Hom}(A, B) \cong \mathrm{Hom}(B, S_X(A))^\vee \cong \mathrm{Hom}(B, A[2])^\vee.$$

Now, $A[2] \in \mathcal{A}$, therefore the semiorthogonality condition implies $\mathrm{Hom}(B, A[2]) = 0$; hence, $\mathrm{Hom}(A, B) = 0$ as well. □

To bypass this obstruction, one can slightly relax the condition $K_X \cong \mathcal{O}_X$ by considering the case where the canonical bundle is instead 2-torsion. We are now ready to define the central object of this chapter. Let us reiterate that we are only going to work in characteristic 0.

Definition 4.23. An *Enriques surface* S is a smooth projective surface with no non-trivial global holomorphic 1-forms, and with a non-trivial, 2-torsion canonical bundle. In symbols:

(i) $H^1(S, \mathcal{O}_S) = 0$;
(ii) $K_S \ncong \mathcal{O}_S$, but $K_S^{\otimes 2} \cong \mathcal{O}_S$.

We will also use the standard fact (which we do not prove) that an Enriques surface has $H^2(S, \mathcal{O}_S) \cong H^0(S, K_S)^* = 0$, i.e. $p_g(S) = 0$; together with $H^1(S, \mathcal{O}_S) = 0$ this gives $\chi(\mathcal{O}_S) = 1$.

Enriques surfaces are closely linked to K3 surfaces by the following fact.

Proposition 4.24. *Every Enriques surface S admits a universal unramified cover $p\colon X \to S$ of degree 2, with X a K3 surface. Equivalently, every Enriques surface is the quotient of a K3 surface by a fixed-point-free involution ι, i.e., $S \cong X/\iota$.*

The quotient turns the trivial canonical bundle of the K3 surface into the 2-torsion canonical bundle of the Enriques, thus making the Serre functor S_S more than just a shift.

The proof of Theorem 4.22 cannot work anymore, and indeed, Enriques surfaces do admit non-trivial semiorthogonal decompositions.

The converse of this result also holds (see [Kum]): the quotient of a K3 surface by a fixed-point free involution is an Enriques surface. This is very useful when building examples:

Example 4.25. Consider the smooth complete intersection of 3 quadrics in $\mathbb{P}^5$. Take:

$$f_i = Q_i(x_0, x_1, x_2) + Q'_i(x_3, x_4, x_5), \quad \text{for } i = 1, 2, 3,$$

where Q_i and Q'_i are general homogeneous quadratic forms. Let us denote by X' the variety given by $X' = \{f_1 = f_2 = f_3 = 0\}$. By the adjunction formula [Har77, Exercise II.8.4]:

$$K_{X'} \cong \mathcal{O}_{X'}(-(n+1) + \sum_{i=1}^{k} d_i) = \mathcal{O}_{X'},$$

since here $n = 5$, $k = 3$, and $d_i = 2$ for $i = 1, 2, 3$. Furthermore, since X' is a complete intersection of dimension 2 in $\mathbb{P}^5$, by [Har77, Exercise III.5.5] we have $H^i(X', \mathcal{O}_{X'}(m)) = 0$ for $0 < i < 2$ and for all $m \in \mathbb{Z}$. Taking $i = 1$ and $m = 0$, we obtain $h^1(X', \mathcal{O}_{X'}) = 0$. This implies that X' is indeed K3. Now, consider:

$$\sigma \colon \mathbb{P}^5 \to \mathbb{P}^5, \quad \sigma(x_0 \colon \cdots \colon x_5) = (x_0 \colon x_1 \colon x_2 \colon -x_3 \colon -x_4 \colon -x_5),$$

which is clearly an involution. Observe that $\sigma(X') = X'$. The fixed locus of σ in $\mathbb{P}^5$ is the union of the planes $\Pi_1 = \{x_3 = x_4 = x_5 = 0\}$ and $\Pi_2 = \{x_0 = x_1 = x_2 = 0\}$. Now, $X' \cap \Pi_1$ is given by $Q_1 = Q_2 = Q_3 = 0$ in $\Pi_1 \cong \mathbb{P}^2$, and three general quadrics in $\mathbb{P}^2$ have no common point; similarly $X' \cap \Pi_2 = \varnothing$ for general Q'_i. Therefore $\sigma|_{X'}$ is a fixed-point free involution of the K3 surface X', and the quotient X'/σ is an Enriques surface.

A fundamental consequence of the definition of Enriques surface is the relationship between the Picard group $\mathrm{Pic}(S)$ and $\mathrm{NS}(S)$. Recall the exponential cohomology sequence (4.1). We defined $\mathrm{Pic}^0(S)$ as the kernel of the connecting morphism δ. In the case of an Enriques surface, however, the hypothesis $H^1(S, \mathcal{O}_S) = 0$ makes δ injective. This means that any two line bundles in $\mathrm{Pic}(S)$ are algebraically equivalent if and only if they are linearly equivalent. Therefore, for any Enriques surface S:

$$\mathrm{NS}(S) = \mathrm{Pic}(S).$$

This is not the case for $\mathrm{NS}(S)$ and $\mathrm{Num}(S)$. Specifically, it can be proved that the torsion subgroup of $\mathrm{NS}(S)$ is generated by the class of the canonical bundle, therefore Theorem 4.11 implies that:

$$\mathrm{Num}(S) \cong \mathrm{NS}(S)/\operatorname{Tors}(\mathrm{NS}(S)) \cong \mathrm{NS}(S)/\langle K_S \rangle. \tag{4.3}$$

Another way to convince ourselves that the class of K_S in $\mathrm{Num}(S)$ is 0 is the following: for any divisor $D \in \mathrm{Div}(S)$, the additivity of the intersection form implies that $2(K_S.D) = (2K_S).D = \mathcal{O}_S.D = 0$, which forces $K_S.D = 0$ since the intersection number must be an integer. Consequently, the canonical class is numerically equivalent to zero.

The geometric properties of Enriques surfaces uniquely determine the structure of $\mathrm{Num}(S)$ as a lattice, as stated by the following fundamental theorem:

Theorem 4.26. *Let S be an Enriques surface. Then the numerical Néron-Severi group* $\mathrm{Num}(S)$ *equipped with the intersection pairing is a unimodular lattice of rank* 10 *and signature* $(1, 9)$*. In particular, it is isometric to the unique even unimodular hyperbolic lattice of rank 10:*

$$\mathrm{Num}(S) \cong U \oplus E_8(-1) \tag{4.4}$$

where U is the hyperbolic lattice $\left(\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$*, and $E_8(-1)$ is the unique negative definite even unimodular lattice of rank 8.*

The fact that the signature is $(1, 9)$ is a direct consequence of the *Hodge Index Theorem* for surfaces, while the computation of the rank (the *Picard number* $\rho = 10$) follows from another application of the exponential cohomology sequence (4.1).

One more pivotal property of Enriques surfaces is the following.

Proposition 4.27. *Every Enriques surface S admits a* genus 1 fibration *over the projective line, i.e., a map $\pi\colon S \to \mathbb{P}^1$ whose general fiber is a smooth curve of genus* 1*.*

This important result is complemented by the following proposition, which enables us to identify a specific class of divisors of an Enriques surface:

Proposition 4.28. *Every genus 1 fibration $X \to \mathbb{P}^1$ of an Enriques surfaces has exactly two* multiple fibers $2F$ *and* $2F'$*.*

To understand what a multiple fiber is, we can reason in terms of pullbacks. Given the genus 1 fibration $\pi\colon S \to \mathbb{P}^1$, the fiber $\pi^{-1}(p)$ over a general point $p \in \mathbb{P}^1$ is a smooth connected curve of genus 1 on S; it coincides with $\pi^*(p)$, the pullback of p as a divisor.

Recall that a curve $C = \sum_i n_i C_i$ on S is said to be *reduced* if $n_i = 1$ for every i, and *non-reduced* otherwise. Given any curve $C = \sum_i n_i C_i$, its *reduction* is the reduced curve $C_{\mathrm{red}} := \sum_i C_i$, obtained by setting all coefficients equal to 1.

The second proposition states that there exist exactly two special points $q, q' \in \mathbb{P}^1$ whose associated fibers are *multiple*, meaning that their pullback as divisors yields a multiple of a curve:

$$\pi^*(q) = 2F \quad \text{and} \quad \pi^*(q') = 2F',$$

where F and F' are the reductions of the multiple fibers (called, in this case, *half-fibers*). We are now ready to define an important invariant of Enriques surfaces. First, observe that a half-fiber F is also a divisor, and we can therefore define its class $[F]$ in $\mathrm{Num}(S)$.

Definition 4.29. Let S be an Enriques surface. Its *non-degeneracy invariant* [MRS25], denoted $\mathrm{nd}(S)$, is defined as the maximum number of half-fibers $F_1, \ldots, F_m$ which satisfy:

$$[F_i].[F_j] = 1 - \delta_{ij} \tag{4.5}$$

for all $1 \leq i, j \leq m$.

A sequence of primitive vectors $\{F_1, \ldots, F_m\} \subset \mathrm{Num}(S)$ whose elements satisfy (4.5) is said to be *isotropic*. From the point of view of linear algebra, the isotropy of a sequence

of half-fibers $\{F_1, \dots, F_m\}$ implies that the sublattice generated by $F_1, \dots, F_m$ has rank m (or, equivalently, that the F_i are linearly independent). Since $\mathrm{Num}(S)$ itself has rank 10, it is clear that, for any Enriques surface S:

$$\mathrm{nd}(S) \leq 10.$$

4.3 The Weyl group and (-2)-curves

The lattice-theoretic material of this section (roots, reflections, and Weyl groups) is standard, see [CDL25, Sections 0.8 and 2.3]; the results on isotropic sequences, their canonical form and the non-degeneracy invariant follow [DK25, Section 6.1] and [MRS25]. Before proceeding, we introduce some terminology on curves that will be needed below.

Definition 4.30. A curve $C = \sum_i n_i C_i$ on S is called *irreducible* if its support has only one irreducible component, i.e. $C = n_1 C_1$. A curve which is both reduced and irreducible, i.e. $C = C_1$, is called *integral*.

Since the numerical Néron-Severi group $\mathrm{Num}(S)$ of an Enriques surface is endowed with a lattice structure, we can naturally define its group of symmetries using notions from lattice theory. Let L be an even lattice equipped with a symmetric bilinear form denoted by $\alpha \cdot \beta$.

Definition 4.31. An element $\alpha \in L$ is called a *root* of the lattice if $\alpha^2 = -2$. For each root α, we define the *reflection* $s_\alpha \colon L \to L$ associated with α as the linear isometry given by:

$$s_\alpha(x) = x + (x \cdot \alpha)\alpha \quad \text{for all } x \in L.$$

The *Weyl group* of the lattice L, denoted by $W(L)$, is the subgroup of the isometry group $\mathrm{O}(L)$ generated by all reflections s_α as α varies among all the roots of L.

The fact that $\mathrm{Num}(S) \cong U \oplus E_8(-1)$ lets us study the Weyl group $W(S) \coloneqq W(\mathrm{Num}(S))$, which acts on the numerical classes of divisors. The geometric meaning of this action on an Enriques surface S relies on the properties of its curves. This reliance is made precise by the following characterization of smooth rational curves in terms of their numerical class:

Lemma 4.32. *Let $C \subset S$ be an integral curve on S. Then, C is in particular a subvariety of codimension 1, therefore a divisor on S. Let us denote its associated numerical class by $[C] \in \mathrm{Num}(S)$. Then:*

i) if $[C]^2 < 0$, then $[C]^2 = -2$ and C is a smooth rational curve;
ii) if C is a smooth rational curve, then $[C]^2 = -2$.

In particular, C is a smooth rational curve if and only if $[C]^2 = -2$.

A curve with self-intersection -2 will be called a (-2)-*curve*. Now, the Lemma immediately tells us that the numerical class of any smooth rational curve on an Enriques surface is a root of the lattice $\mathrm{Num}(S)$. Therefore, for every smooth rational curve $R \subset S$ the reflection

$$s_R(x) = x + (x \cdot R)R, \qquad x \in \mathrm{Num}(S),$$

belongs to $W(S)$. The subgroup of $W(S)$ generated by these reflections is called the *nodal Weyl group* of S:

$$W^{\mathrm{nod}}(S) := \langle s_R \ : \ R \subset S \text{ a } (-2)\text{-curve} \rangle \subset W(S).$$

In general $W^{\mathrm{nod}}(S)$ is strictly smaller than $W(S)$: not every root of $\mathrm{Num}(S)$ is the class of a (-2)-curve (for instance, a general Enriques surface has no (-2)-curves at all, see Proposition 4.38, so that $W^{\mathrm{nod}}(S)$ is trivial). Since reflections are isometries, both groups preserve the intersection product, and hence act on sequences of isotropic vectors. This is the setup for Lemmas 4.34 and 4.35: $W(S)$ is used to extend sequences, while $W^{\mathrm{nod}}(S)$ is used to put them in canonical form. This algebraic tool allows us to manipulate configurations of divisors via successive reflections, which is the foundational setup for Lemma 4.35, regarding the standard reduction of isotropic sequences. First, a sequence $\{v_1, \dots, v_n\} \subset \mathrm{Num}(S)$ is said to be *non-degenerate* if every v_i is (numerically) equivalent to the class of a half-fiber of some genus 1 fibration on S.

Remark 4.33. Recall that an effective, isotropic divisor class D on S is the class of a half-fiber precisely when D is *primitive* (i.e., not a multiple of another effective class) and nef. More precisely, any term D of an isotropic sequence falls into exactly one of three cases:

- D is not primitive: then $D = mF$ for some half-fiber F and some integer $m \geq 2$;
- D is primitive but not nef: then $D = F + R_1 + \cdots + R_k$ for some half-fiber F and smooth rational curves $R_1, \dots, R_k$;
- D is primitive and nef: then, as we said above, $D = F$ is itself the class of a half-fiber.

The terms of the canonical sequence in Lemma 4.35 are always instances of the second or third case.

Lemma 4.34. *Let $\{v_1, \dots, v_k\} \subset \mathrm{Num}(S)$, with $1 \leq k \leq 10$ and $k \neq 9$ be an isotropic sequence. There exist vectors $w_1, \dots, w_{10-k}$ which extend the sequence to one with length 10, which is still isotropic.*

Proof. A key fact we will not prove is that $W(S)$ acts transitively on isotropic sequences of length $k \neq 9$ (see [DK25, Section 6.1]). Thus, $\{v_1, \dots, v_k\}$ is the image under some $w \in W(S)$ of the first k vectors of a fixed isotropic sequence of length 10, and we can extend it by applying w to the remaining vectors. □

In general, the extension is not unique. Note that there is no guarantee about the non-degeneracy of the extended sequence. Once the sequence has been extended to length 10, the nodal Weyl group can intervene in order to bring it into a special form.

Lemma 4.35. *Let S be an Enriques surface and let $\{v_1, \dots, v_k\} \subset \mathrm{Num}(S)$ be an isotropic sequence with $k \neq 9$. Let $\{v_1, \dots, v_{10}\}$ be an isotropic extension of it, as in Lemma 4.34. Then there exists a unique $w \in W^{\mathrm{nod}}(S)$ such that, up to reordering, the sequence $\{w(v_1), \dots, w(v_{10})\}$ has the* canonical *form*

$$\{f_1,\ f_1 + R_{1,1},\ \dots,\ f_1 + R_{1,1} + \cdots + R_{1,k_1},\ f_2,\ f_2 + R_{2,1},\ \dots,\\ f_n,\ \dots,\ f_n + R_{n,1} + \cdots + R_{n,k_n}\}.$$

Moreover, the classes $f_1, \dots, f_n$ are nef, while the classes $f_i + R_{i,1} + \cdots + R_{i,j}$ with $j \geq 1$ are not; in particular n is the number of nef classes in the sequence.

Remark 4.36. Due to its reliance on Lemma 4.34, the sequence obtained by Lemma 4.35 has length 10, i.e. $k_1 + \cdots + k_n + n = 10$. The f_i represent classes of half-fibers of the Enriques, while the $R_{i,j}$ are classes of smooth rational curves. The requirement $k \neq 9$ is also a consequence of the hypotheses of Lemma 4.34.

Remark 4.37. The Weyl group $W(S)$ acts on the set of isotropic sequences of $\mathrm{Num}(S)$ by acting on each element; reflections are isometries, so isotropy is preserved. The hypothesis $k \neq 9$ in Lemma 4.34 reflects the structure of the orbits of this action. For $k \neq 9$ the action is transitive, which is why every isotropic k-sequence can be extended to one of length 10. For $k = 9$ there are exactly two orbits:

- O_1, the sequences of length 9 that can be extended to an isotropic sequence of length 10;
- O_2, the sequences of length 9 that cannot be extended any further.

Consider a length-9 sequence $\{f_1, \dots f_9\}$. The two cases are distinguished by the sum of the components $f_1 + \cdots + f_9 \in \mathrm{Num}(S)$: the sequence is in O_1 if and only if the sum is *primitive* (i.e., it is not a multiple of any element of the lattice), and in O_2 if and only if the sum is divisible by 2. In particular, Lemmas 4.34 and 4.35 also hold for $k = 9$ when the sequence lies in O_1.

Moreover, extending a sequence never decreases its *non-degeneracy* (i.e., the number of nef classes in the sequence). More precisely, let $\{f_1, \dots, f_m\}$ be a sequence in canonical form with n nef classes. If $m \leq 8$, it can be extended both to a canonical sequence of length 10 and to a canonical sequence of length 9 in O_2, in each case with at least n nef classes. If $m = 9$ and the sequence lies in O_1, it can be extended to a canonical sequence of length 10 with at least n nef classes, whereas if it lies in O_2 (or if $m = 10$) no further extension is possible. As a consequence,

$$\mathrm{nd}(S) = \max\{\mathrm{Fnd}(S), \mathrm{Mnd}(S)\},$$

where the *Fano non-degeneracy* $\mathrm{Fnd}(S)$ (resp. the *Mukai non-degeneracy* $\mathrm{Mnd}(S)$) is the maximal number of nef classes in a canonical sequence of length 10 (resp. of length 9 in O_2). Indeed, a non-degenerate sequence of length $\mathrm{nd}(S)$ extends to one of these sequences, which cannot have more than $\mathrm{nd}(S)$ nef classes. All of this is discussed in [MRS25].

For almost every Enriques, the canonical sequence described in Lemma 4.35 has a very special form, thanks to:

Proposition 4.38. *A general Enriques surface S admits no (-2)-curve, and thus no smooth rational curve either. Therefore, $W^{\mathrm{nod}}(S)$ is trivial, while $W(S)$ is not.*

Remark 4.39. Note that, outside the generality hypothesis, the canonical form above does not have to consist of half-fiber classes alone: there exist Enriques surfaces with non-degeneracy invariant strictly smaller than 10, whose canonical isotropic 10-sequences necessarily contain degenerate blocks $f_i + R_{i,1} + \cdots$.

Proposition 4.38 implies that, given a maximal sequence $\{v_1, \dots, v_{10}\} \subset \mathrm{Num}(S)$, its associated canonical form is $\{f_1, \dots, f_{10}\}$, where the f_i are classes of half-fibers. In this thesis we will not deal with moduli spaces, however, when we specify *general* Enriques, we mean that the Proposition holds true for any Enriques outside of a certain closed subset of the moduli space of Enriques surfaces, $\mathcal{M}_{\mathrm{Enr}}$. In other words, smooth rational curves

do exist on some Enriques surfaces, but such examples are sporadic: in some sense, the result is true for "almost every" Enriques (i.e., outside of a subset of codimension at least 1 in $\mathcal{M}_{\text{Enr}}$).

The previous results tell us that a canonical sequence in the form of $\{f_1, \dots f_{10}\}$ exists for a general Enriques S: indeed, recall that, thanks to 4.28, any Enriques surface admits a half-fiber f_1. Thus, we can apply Lemma 4.34 to the length-1 sequence $\{f_1\}$, extending it to a sequence of the form:

$$\{f_1, v_2, \dots, v_{10}\}.$$

By Proposition 4.38 the nodal Weyl group is trivial, therefore the sequence is already in canonical form. To highlight that all of its elements are classes of half-fibers, we can rewrite the sequence as:

$$\{f_1, \dots, f_{10}\}, \tag{4.6}$$

thanks to Lemma 4.35, and to the previous observation regarding Proposition 4.38. The existence of a sequence like (4.6) implies that, for a general Enriques, $\text{nd}(S) = 10$.

Remark 4.40. Proposition 4.28 actually implies the existence of two half-fibers for every Enriques. However, one could show that, if the half-fibers f and f' come from the same genus 1 fibration, then $[f].[f'] = 0$, making the sequence $\{f, f'\} \subset \text{Num}(S)$ not isotropic, which is one of the hypotheses of Lemma 4.35.

The canonical sequence $\{f_1, \dots, f_{10}\} \subset \text{Num}(S)$ of numerical classes of half-fibers provides the first step in the construction of a decomposition of the bounded derived category $D^b(S)$. Our goal is to lift these numerical classes to line bundles on S that can generate a semiorthogonal decomposition. However, passing from the numerical lattice to the derived category requires some care.

Recall that for a classical Enriques surface S, the Picard group $\text{Pic}(S)$ is isomorphic to the Néron-Severi group $\text{NS}(S)$. On the other hand, the canonical map $\text{NS}(S) \to \text{Num}(S)$ is not an isomorphism: as noted in (4.3), its kernel is generated by the canonical class K_S, which is a non-trivial 2-torsion element in $\text{Pic}(S)$. Thus, the projection to $\text{Num}(S)$ cancels the torsion given by the canonical bundle, mapping K_S to zero. This means that:

$$[F] = [F] + [K_S] \in \text{Num}(S).$$

Consequently, given the canonical sequence of numerical classes $\{f_1, \dots, f_{10}\}$, lifting each class f_i to a line bundle in $\text{Pic}(S)$ (by lifting f_i to a divisor class $[D_i] \in \text{NS}(S)$ first) involves a binary choice. For each numerical class f_i, there are exactly two available line bundles in $\text{Pic}(S)$ whose numerical class is f_i, namely $\mathcal{O}_S(D_i)$ and $\mathcal{O}_S(D_i) \otimes K_S$.

Crucially, these two lifts are mutually exclusive within the context of our construction. Indeed, the two line bundles associated with the same numerical class can never belong to the same semiorthogonal decomposition, due to the existence of non-trivial morphisms between them.

By choosing (in a correct way) a unique line bundle $\mathcal{O}_S(F_i)$ for each class f_i in the sequence, we ensure that the components satisfy the semiorthogonality hypothesis (we will

prove this for some important cases at the end of this section). Viewing these line bundles as complexes of locally free sheaves concentrated in degree zero, we obtain a collection of objects $\{\mathcal{O}_S(F_1), \dots, \mathcal{O}_S(F_{10})\} \subset D^b(S)$. We will show that this lifted sequence constitutes an exceptional collection, thereby generating a semiorthogonal decomposition of the bounded derived category of the Enriques surface.

Remark 4.41. As previously noted, for a general Enriques surface S (i.e., when no (-2)-curves exist), the non-degeneracy invariant satisfies $\mathrm{nd}(S) = 10$. In terms of the derived category $D^b(S)$, this corresponds to a semiorthogonal decomposition consisting of 10 fully orthogonal blocks.

Conversely, when (-2)-curves do exist, the invariant can drop strictly below 10. In this non-general setting, the value of nd geometrically measures the maximum number of *fully* orthogonal blocks appearing in any decomposition of the category. More precisely, if we start with a length-10 sequence (or use Lemma 4.34 to extend a shorter one to length 10) with non-degeneracy d (see Remark 4.37), the derived category still decomposes into a total of 10 exceptional components, but these are now partitioned into exactly d blocks, plus an initial special component:

$$\begin{aligned} \mathcal{L} &\coloneqq \langle \mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_d \rangle; \\ D^b(S) &= \langle \mathcal{K}u(S, \mathcal{L}), \mathcal{L} \rangle. \end{aligned} \tag{4.7}$$

The subcategory $\mathcal{K}u(S, \mathcal{L})$ is called *residual component* (or, in the case of Fano varieties and in particular cubic fourfolds [Kuz10], *Kuznetsov component*, in honor of Alexander Kuznetsov (b. 1973)), and it is just the orthogonal complement of $\mathcal{L}$ (see Definition 3.35). Its presence ensures that the resulting sequence $\{\mathcal{K}u(S, \mathcal{L}), \mathcal{L}\}$ is full, by Example 3.39. The structural behavior of decomposition (4.7) is characterized as follows (see [LNSZ21, Proposition 3.5]):

- **Internal Semiorthogonality:** Within each individual block $\mathcal{B}_i$, the components are only semiorthogonal to one another.
- **Mutual Full Orthogonality:** Between distinct blocks, the orthogonality is complete and mutual, meaning that for any $i \neq j$, we have $\mathrm{Hom}(B_i, B_j) = 0$ and $\mathrm{Hom}(B_j, B_i) = 0$ for all $B_i \in \mathcal{B}_i, B_j \in \mathcal{B}_j$.

Therefore, the invariant $\mathrm{nd}(S)$ detects the failure of full orthogonality among the components of the sequence, grouping them into exactly $\mathrm{nd}(S)$ mutually orthogonal blocks.

We will now focus our attention on showing why these (semi)orthogonality results hold in this context. Beforehand, we need to recall some important facts.

Remark 4.42. Let X be a smooth projective variety, with D a divisor on X. Recall that there is a natural identification between the linear system $|D|$, i.e. the set of effective divisors linearly equivalent to D, and the projectivization of the global sections of the sheaf $\mathcal{O}_X(D)$. In symbols:

$$|D| \cong \mathbb{P}(H^0(X, \mathcal{O}_X(D))).$$

In particular, the existence of a non-zero global section $s \in H^0(X, \mathcal{O}_X(D))$ corresponds to the existence of an effective divisor in the class of linear equivalence of D.

Recall that the *Euler-Poincaré characteristic* of a line bundle L over a surface X is defined as:

$$\chi(L) = h^0(X, L) - h^1(X, L) + h^2(X, L),$$

where $h^i(X, L) \coloneqq \dim(H^i(X, L))$.

4.43 Riemann-Roch Theorem. *Let X be a smooth projective surface, with $D \in \mathrm{Div}(X)$ a divisor on X. Then the following equality holds:*

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{2}D.(D - K_X).$$

We are now ready to study the (semi)orthogonality facts that we mentioned earlier. First we will focus on the case of a general Enriques S. Consider the canonical non-degenerate sequence $\{f_1, \dots, f_{10}\} \subset \mathrm{Num}(S)$, which lifts to the sequence of half-fibers:

$$\{F_1, \dots, F_{10}\} \subset \mathrm{NS}(S). \tag{4.8}$$

Our intention is to show the full orthogonality of the collection of subcategories of $D^b(S)$ generated by the bundles $\mathcal{O}_S(F_1), \dots, \mathcal{O}_S(F_{10}) \in \mathrm{Pic}(S)$:

$$\langle \mathcal{O}_S(F_1) \rangle, \dots, \langle \mathcal{O}_S(F_{10}) \rangle \subset D^b(S). \tag{4.9}$$

Let us start by checking that $\mathrm{Hom}(\mathcal{O}_S(F_k), \mathcal{O}_S(F_h))$ is 0 for all $1 \leq k, h \leq 10$, with $k \neq h$. By Lemma 3.51 and Remark 3.52, we have:

$$\mathrm{Hom}(\mathcal{O}_S(F_k), \mathcal{O}_S(F_h)) \cong \mathrm{Hom}(\mathcal{O}_S, \mathcal{O}_S(F_h - F_k)) \cong H^0(S, \mathcal{O}_S(F_h - F_k)).$$

Lemma 4.44. *Let $\pi \colon S \to \mathbb{P}^1$ be a genus 1 fibration, whose existence is guaranteed by Proposition 4.27. Then, for every $p \in \mathbb{P}^1$, the divisor $\pi^*(p)$ is nef.*

Corollary 4.45. *Every half-fiber F of a genus 1 fibration on an Enriques surface S is nef.*

Proof. By Lemma 4.44, $2F = \pi^*(q)$ is nef. For any divisor D and any $\lambda \in \mathbb{R}_{>0}$, D is nef if and only if λD is nef: this obviously follows from the defining inequality. Therefore, $F = \frac{1}{2}(2F)$ is nef as well. □

Since (4.8) is isotropic, $(F_h - F_k).F_h = -1$. Now, F_h is a fiber of a genus 1 fibration, therefore it is nef by Corollary 4.45: this implies that $F_h - F_k$ cannot be effective. Then, Remark 4.42 tells us that $H^0(S, \mathcal{O}_S(F_h - F_k))$ must be 0: indeed, if it contained any non-zero element, we would be able to construct an effective divisor which is linearly equivalent to $F_h - F_k$.

This observation, however, is not sufficient to prove the orthogonality of (4.9), as we also need to prove that morphisms towards shifts are zero. Theorem 3.32 guarantees that:

$$\mathrm{Hom}(\mathcal{O}_S(F_k), \mathcal{O}_S(F_h)[i]) \cong \mathrm{Ext}^i(\mathcal{O}_S(F_k), \mathcal{O}_S(F_h)),$$

and the right-hand side is isomorphic to $H^i(S, \mathcal{O}_S(F_h - F_k))$ by Remark 3.52. From now on, let us denote $D \coloneqq F_h - F_k$. We already know that $H^i(S, \mathcal{O}_S(D))$ is 0 for all $i \neq 1, 2$:

we just need to prove triviality in these two degrees. The case $i = 2$ follows from Serre duality:

$$H^2(S, \mathcal{O}_S(D)) \cong H^0(\mathcal{O}_S(K_S - D))^*,$$

which is 0 by the same argument as above: $(K_S - D).F_k = -1$, and F_k is nef, therefore $K_S - D$ cannot be effective and, again, $H^0(\mathcal{O}_S(K_S - D)) = 0$ by Remark 4.42. Finally, Riemann-Roch on $\mathcal{O}_S(D)$ yields:

$$\chi(\mathcal{O}_S(D)) = \chi(\mathcal{O}_S) + \frac{1}{2}D.(D - K_S),$$

which evaluates to 0, since $D^2 = -2$ and $\chi(\mathcal{O}_S) = 1$. Thus $h^1(S, \mathcal{O}_S(D)) = 0$, which proves that $H^1(S, \mathcal{O}_S(D)) = 0$ as well.

The vanishing of these Ext groups for the line bundles $\mathcal{O}_S(F_1), \ldots, \mathcal{O}_S(F_{10})$ on a general Enriques surface was also stated in [Zub94, Section 2.6].

Example 4.46. We can abandon the generality hypothesis to consider the case of an Enriques surface S which admits at least one rational curve R: specifically, we are now going to study the sequence $\{f_1, f_1+R\} \subset \mathrm{Num}(S)$, and show that its associated collection of subcategories:

$$\langle \mathcal{O}_S(F_1) \rangle, \langle \mathcal{O}_S(F_1 + R) \rangle \subset D^b(S) \tag{4.10}$$

is strictly semiorthogonal. It is clear that

$$\mathrm{Hom}(\mathcal{O}_S(F_1 + R), \mathcal{O}_S(F_1)) \cong \mathrm{Hom}(\mathcal{O}_S, \mathcal{O}_S(-R))$$

by Lemma 3.51; the latter is 0, since $-R$ is obviously not effective. Again, this does not suffice, as we need to check the morphisms towards shifts of $\mathcal{O}_S(F_1)$. Indeed, the argument is the same as before. Theorem 3.32, Lemma 3.51 and Remark 3.52 imply that:

$$\begin{aligned}\mathrm{Hom}(\mathcal{O}_S(F_1 + R), \mathcal{O}_S(F_1)[i]) \cong \mathrm{Ext^i}(\mathcal{O}_S(F_1 + R), \mathcal{O}_S(F_1)) &\cong \mathrm{Ext}^i(\mathcal{O}_S, \mathcal{O}_S(-R)) \\ &\cong H^i(S, \mathcal{O}_S(-R)).\end{aligned}$$

Since S is a surface, this is obviously 0 for all $i \notin \{0, 1, 2\}$. Keeping in mind the previous computations, the case $i = 0$ follows from the fact that $-R$ is not effective (much like $F_h - F_k$ above). For $i = 2$, Serre duality gives:

$$H^2(S, \mathcal{O}_S(-R)) \cong H^0(S, \mathcal{O}_S(K_S + R))^*, \tag{4.11}$$

so we have to show that $H^0(S, \mathcal{O}_S(K_S + R)) = 0$. Tensoring the exact sequence:

$$0 \to \mathcal{O}_S(-R) \to \mathcal{O}_S \to \mathcal{O}_R \to 0$$

by the line bundle $\mathcal{O}_S(K_S+R)$ preserves exactness by Proposition 2.13, since $\mathcal{O}_S(K_S+R)$ is a locally free sheaf. Thus, we obtain:

$$0 \to \mathcal{O}_S(K_S) \to \mathcal{O}_S(K_S + R) \to \mathcal{O}_R(K_S + R) \to 0,$$

where $\mathcal{O}_R(K_S + R) \coloneqq \mathcal{O}_S(K_S + R)|_R$. Passing to cohomology, we get the exact sequence

$$0 \to H^0(S, \mathcal{O}_S(K_S)) \to H^0(S, \mathcal{O}_S(K_S + R)) \to H^0(R, \mathcal{O}_R(K_S + R)).$$

The first H^0 is 0, since $p_g(S) = 0$. As for the last term, we use two standard facts, which we do not prove: for a smooth curve $R \subset S$ and a divisor D one has $\deg(\mathcal{O}_S(D)|_R) = D \cdot R$, and

a line bundle of negative degree on $\mathbb{P}^1$ has no non-zero global sections. Here, $R \cong \mathbb{P}^1$, and $\mathcal{O}_R(K_S+R)$ is a line bundle on R of degree $(K_S+R)\cdot R = R^2 = -2$, since $K_S \cdot R = 0$ (K_S is numerically trivial). By the two standard facts, we conclude that $H^0(R, \mathcal{O}_R(K_S+R)) = 0$. Hence $H^0(S, \mathcal{O}_S(K_S+R)) = 0$, and (4.11) implies $H^2(S, \mathcal{O}_S(-R)) = 0$. Finally, Riemann–Roch yields:

$$\chi(\mathcal{O}_S(-R)) = \chi(\mathcal{O}_S) + \tfrac{1}{2}(-R)\cdot(-R-K_S) = 1 + \tfrac{1}{2}(R^2 + K_S \cdot R) = 0,$$

and since $h^0 = h^2 = 0$ we get $h^1(S, \mathcal{O}_S(-R)) = 0$ as well.

Therefore, semiorthogonality is satisfied. On the other hand, Lemma 3.51 (as usual, along with Remark 3.52) also tells us that:

$$\mathrm{Hom}(\mathcal{O}_S(F_1), \mathcal{O}_S(F_1+R)) \cong \mathrm{Hom}(\mathcal{O}_S, \mathcal{O}_S(R)) \cong H^0(S, \mathcal{O}_S(R)).$$

Since R is clearly an effective divisor, it follows from Remark 4.42 that $H^0(S, \mathcal{O}_S(R)) \neq 0$. Therefore, orthogonality does not hold: collection (4.10) is semiorthogonal, but not fully orthogonal.

Example 4.47. To conclude this set of examples, we will consider an Enriques surface S which has at least two rational curves R_1 and R_2, and admits the isotropic sequence:

$$\{f_1, f_1+R_1, f_2, f_2+R_2\} \subset \mathrm{Num}(S); \tag{4.12}$$

we will show that $\langle \mathcal{O}_S(F_1+R_1)\rangle, \langle \mathcal{O}_S(F_2+R_2)\rangle \subset D^b(S)$ are fully orthogonal. Note that f_1 and f_2 represent two distinct half-fiber classes, forming different blocks of the canonical isotropic sequence (4.12). Therefore, their respective lifts F_1 and F_2 must be distinct.

Let us denote $A \coloneqq F_1 + R_1$ and $B \coloneqq F_2 + R_2$. Once again, we have that:

$$\mathrm{Ext}^i(\mathcal{O}_S(B), \mathcal{O}_S(A)) \cong H^i(S, \mathcal{O}_S(A-B)).$$

If we are able to prove that $A - B$ cannot be effective, then we can apply the same argument of Example 4.46 to obtain that $H^0(S, \mathcal{O}_S(A-B)) = 0$. The cases $i = 2$ and $i = 1$ again follow from Serre duality and Riemann-Roch, respectively. At that point, a perfectly symmetrical argument conducted on:

$$\mathrm{Ext}^i(\mathcal{O}_S(A), \mathcal{O}_S(B)) \cong H^i(S, \mathcal{O}_S(B-A))$$

guarantees full orthogonality.

Hence, we will show that $D \coloneqq A - B$ is not effective. By contradiction, we assume that it is. Recall that the elements v_i of an isotropic sequence satisfy $v_i.v_j = 1 - \delta_{ij}$. This yields $F_1^2 = F_2^2 = 0$, $F_1.F_2 = 1$, and the following useful computations:

$$\begin{gathered}
1 = F_1.(F_1+R_1) = F_1^2 + F_1.R_1 \implies F_1.R_1 = 1;\\
1 = F_1.(F_2+R_2) = F_1.F_2 + F_1.R_2 \implies F_1.R_2 = 0;\\
1 = (F_1+R_1).F_2 = F_1.F_2 + R_1.F_2 \implies F_2.R_1 = 0;\\
1 = (F_1+R_1).(F_2+R_2) = F_1.F_2 + F_1.R_2 + F_2.R_1 + R_1.R_2 \implies R_1.R_2 = 0.
\end{gathered}$$

Since $R_1^2 = -2$, we also obtain:

$$\begin{aligned}
D.F_1 &= (F_1 + R_1 - F_2 - R_2).F_1 = 0 + 1 - 1 - 0 = 0;\\
D.R_1 &= F_1.R_1 + R_1^2 - F_2.R_1 - R_2.R_1 = 1 - 2 - 0 - 0 = -1.
\end{aligned}$$

Now write $D = \sum_i a_i C_i$, with C_i distinct irreducible curves and $a_i > 0$, since D is effective by assumption. If R_1 were not among the C_i, then every C_i would be an irreducible curve different from R_1, hence without common components with R_1. Thus, we would get:

$$C_i.R_1 = \sum_{P \in C_i \cap R_1} (C_i.R_1)_P \geq 0$$

for all i, where $(C_i.R_1)_P$ denotes the local intersection multiplicity at P. This would imply:

$$D.R_1 = \sum_i a_i (C_i.R_1) \geq 0,$$

contradicting $D \cdot R_1 = -1$. Therefore R_1 is a component of D, i.e. $D = R_1 + E$ with E effective. Since F_1 is nef (by Corollary 4.45, since it is a half-fiber), $E \cdot F_1 \geq 0$, and:

$$0 = D \cdot F_1 = R_1 \cdot F_1 + E \cdot F_1 = 1 + E \cdot F_1 \geq 1,$$

a contradiction. Hence $A - B$ is not effective.

Since the computations above are symmetric, by exchanging the indices 1 and 2 the same argument shows that $B - A$ is not effective: indeed $(B - A) \cdot F_2 = 0$ and $(B - A) \cdot R_2 = -1$, and F_2 is nef as well.

Before concluding this section, we briefly step back from the categorical setting to explore the geometric consequences of the objects we just described. Specifically, we want to illustrate how the existence (or absence) of (-2)-curves and the action of the nodal Weyl group determine the shape of the nef and ample cones of an Enriques surface. Consider the following result:

Lemma 4.48. *Let S be an Enriques surface, with $F \in \mathrm{Num}(S)$ nef. If $w \in W^{\mathrm{nod}}(S)$ is an element of the nodal Weyl group such that $w(F)$ is nef as well, then $F = w(F)$.*

We will not prove this Lemma thoroughly, but a geometric intuition of why it must hold will be explained by the following Example (see Remark 4.54). Very briefly, the nef cone corresponds to a fundamental chamber for the action of the nodal Weyl group $W^{\mathrm{nod}}(S)$, whose walls are defined by the reflections associated with (-2)-curves. Consequently, any non-trivial element $w \in W^{\mathrm{nod}}(S)$ maps the interior of the nef cone outside of itself, unless w fixes the given class.

Example 4.49. If S is an Enriques surface, then the cone $\mathrm{Nef}(S)$ admits an explicit description in terms of (-2)-curves. By Theorem 4.26, $\mathrm{Num}(S)$ is a lattice of rank 10 and signature $(1, 9)$, isometric to $U \oplus E_8(-1)$. Consequently, the *positive cone*:

$$\mathcal{P} \coloneqq \{x \in \mathrm{Num}(S) \otimes \mathbb{R} : \ x^2 > 0\}$$

has exactly two connected components, exchanged by $x \mapsto -x$. We let $\mathcal{P}^+$ denote the component containing the class of an ample divisor, and by $\overline{\mathcal{P}^+}$ its closure. We need two preliminary lemmas.

Lemma 4.50. *If D, H are ample divisors on a smooth projective surface X, then $D.H > 0$.*

For the following Lemma, we refer to [BHPVdV04, Section IV.7].

Lemma 4.51. *Let V be a real vector space with a non-degenerate bilinear form of signature $(1, \dim V - 1)$, and let C be one of the two connected components of $\{v \in V : v^2 > 0\}$ (the other being $-C$). Then:*

1. *For all $v, w \in \overline{C}$, $v.w \geq 0$, with strict inequality whenever $v \in C$ or $w \in C$, and $v, w \neq 0$.*
2. *Consequently, if $v, w \in C \cup (-C)$ and $v.w > 0$, then v and w lie in the same one of $C, -C$.*
3. *For any fixed $H \in C$, $\overline{C} = \{x \in V : x^2 \geq 0,\ x.H \geq 0\}$.*

Proof. We leave out the proof of (1) and (3). For part (2), suppose $v \in C$ and, for contradiction, that $w \in -C$, i.e. $-w \in C$. Applying part (1) to $v, -w \in C$ (both in the open cone, guaranteeing the strict inequality) gives $v.(-w) > 0$, i.e. $v.w < 0$, contradicting the hypothesis $v.w > 0$. Hence $w \in C$. The case $v \in -C$ is symmetric. □

Proposition 4.52. *Let S be an Enriques surface. Then:*

$$\mathrm{Nef}(S) = \overline{\mathcal{P}^+} \cap \bigcap_{\substack{R \subset S \\ R^2 = -2}} \{D : D.R \geq 0\}, \qquad \mathrm{Amp}(S) = \mathcal{P}^+ \cap \bigcap_{\substack{R \subset S \\ R^2 = -2}} \{D : D.R > 0\},$$

where the intersections range over all smooth rational curves $R \subset S$, which are classes of roots of $\mathrm{Num}(S)$.

Proof. We prove the statement for $\mathrm{Amp}(S)$; the claim for $\mathrm{Nef}(S)$ then follows by taking closures, using Corollary 4.15.

[$\subseteq$] Let D be ample. By the Nakai–Moishezon criterion (see 4.14), $D^2 > 0$ and $D.C > 0$ for every curve $C \subset S$; in particular this holds for every (-2)-curve R, giving $D.R > 0$. Moreover D lies in $\mathcal{P}^+$ rather than its opposite component, since $D.H > 0$ for the ample class H defining $\mathcal{P}^+$ by Lemma 4.50, since both are ample classes. This forces D and H to lie in the same component of $\mathcal{P}$, by point (2) of Lemma 4.51.

($\supseteq$) Conversely, suppose $D \in \mathcal{P}^+$ satisfies $D.R > 0$ for every (-2)-curve R. By the Nakai–Moishezon criterion, it suffices to show that $D^2 > 0$ (which is clear, since $D \in \mathcal{P}^+ \subset \mathcal{P}$) and $D.C > 0$ for every integral curve $C \subset S$, due to Remark 4.16.

Fix once and for all an ample class $H \in \mathcal{P}^+$. Since H is ample, $H.C > 0$ for every integral curve C, by the Nakai–Moishezon criterion applied to H itself. Therefore, fix an integral curve C.

- If $C^2 < 0$: by Lemma 4.32, an integral curve with negative self-intersection is smooth rational with $C^2 = -2$, i.e. C is a (-2)-curve. Hence $D.C > 0$ holds by hypothesis.
- If $C^2 \geq 0$: since $H.C > 0$ and $H \in \mathcal{P}^+$, part (3) of Lemma 4.51 gives $[C] \in \overline{\mathcal{P}^+}$. Since $D \in \mathcal{P}^+$, part (1) of Lemma 4.51 then gives $D.C > 0$.

In both cases $D.C > 0$, so D is ample by the Nakai–Moishezon criterion. □

Remark 4.53. It is essential that $\mathrm{Nef}(S)$ (resp. $\mathrm{Amp}(S)$) be intersected with $\overline{\mathcal{P}^+}$ (resp. $\mathcal{P}^+$), and not just with the half-spaces cut out by the (-2)-curves. To see why the conditions $D.R \geq 0$ (resp. $D.R > 0$) alone are not sufficient, consider a general Enriques surface S. Such a surface contains no (-2)-curves (see Proposition 4.38), meaning the inequalities

with respect to (-2)-curves are vacuously satisfied by every class $D \in \mathrm{Num}(S)$. In particular, if H is an ample class, its opposite $-H$ trivially satisfies these empty conditions, yet $-H$ is clearly not nef. The role of intersecting with $\overline{\mathcal{P}^+}$ (or $\mathcal{P}^+$) is to discard classes lying in the wrong nappe of the quadric $\{x^2 \geq 0\}$.

Remark 4.54. Proposition 4.52 also clarifies the geometric role of the nodal Weyl group $W^{\mathrm{nod}}(S)$ introduced earlier in this section: each reflection s_R associated to a (-2)-curve R fixes the hyperplane $R^\perp \coloneqq \{D : D.R = 0\}$, which is exactly one of the walls bounding $\mathrm{Nef}(S)$ inside $\overline{\mathcal{P}^+}$. In this sense, $\mathrm{Nef}(S)$ is a fundamental chamber for the action of $W^{\mathrm{nod}}(S)$ on $\overline{\mathcal{P}^+}$, and reflecting across its walls moves between adjacent chambers of the arrangement cut out by all the (-2)-curves of S. This explains the intuition for why Lemma 4.48 holds.

Remark 4.55. For a general Enriques surface S, Proposition 4.38 tells us that there are no (-2)-curves at all, so the intersection in Proposition 4.52 is empty and we simply get $\mathrm{Nef}(S) = \overline{\mathcal{P}^+}$: the nef cone coincides with the full positive cone, with no walls to cross.

4.4 The Derived Torelli Theorems

We conclude this chapter and the thesis by stating the derived Torelli theorems for Enriques surfaces: the classical one, and the refined one, which regards only the residual component. We will also provide a rough sketch of the proof of the refined version, in its most simple case. We assume, like we did in the rest of the thesis, that the field where we work is algebraically closed of characteristic 0. For this exposition and the following proof sketch, we closely follow the treatment given in [PS22]. Most of the results leading to the Refined Derived Torelli Theorem, along with the theorem itself, can be found in [LNSZ21].

First, let us state the classical version of the Derived Torelli Theorem.

4.56 Derived Torelli Theorem. *Let S_1 and S_2 be smooth projective surfaces. If S_1 is an Enriques surface, and there is an exact equivalence $D^b(S_1) \cong D^b(S_2)$, then $S_1 \cong S_2$.*

The proof of this strong result, which uses the K3 covers of the surfaces and various tools from Hodge theory, will be omitted, and can be found at [BM01, Proposition 6.1]. Conversely, it is clear that isomorphic surfaces have equivalent derived categories, so the derived category of an Enriques surface determines the surface up to isomorphism.

The refined version of Torelli's Theorem replaces $D^b(S)$ with the residual component of a semiorthogonal decomposition $D^b(S) = \langle \mathcal{K}u(S, \mathcal{L}), \mathcal{L} \rangle$ as in Section 4.3, where $\mathcal{L}$ is a collection of exceptional line bundles of the type constructed there. In what follows, we will provide a very simplified sketch of the proof. First, we need some preliminary notions, beginning with:

$$^\perp\mathcal{D} \coloneqq \{C \in D^b(S) \ : \ \mathrm{Hom}(C, D) = 0 \text{ for all } D \in \mathcal{D}\},$$

which will denote the *left* orthogonal complement of a certain subcategory $\mathcal{D} \subset D^b(S)$.

The following definitions were introduced in [ST01].

Definition 4.57. Let $\mathcal{T}$ be a k-linear triangulated category with Serre functor $S_{\mathcal{T}}$. Then we say that:

- An object $E \in \mathcal{T}$ is *n-spherical* if $\mathbf{R}\mathrm{Hom}(E,E) \cong k \oplus k[-n]$ and $S_{\mathcal{T}}(E) \cong E[n]$.
- An object $E \in \mathcal{T}$ is *n-pseudoprojective* if $\mathbf{R}\mathrm{Hom}(E,E) \cong k \oplus k[-1] \oplus \cdots \oplus k[-n]$ and, once again, $S_{\mathcal{T}}(E) \cong E[n]$.

Being n-spherical is preserved by exact equivalences, and the same goes for being n-pseudoprojective. Another useful fact is:

Proposition 4.58. *The residual component $\mathcal{K}u(S,\mathcal{L})$ is an admissible subcategory of $D^b(S)$; hence, the inclusion $\kappa\colon \mathcal{K}u(S,\mathcal{L}) \to D^b(S)$ has a right adjoint, which we write as $\kappa^!$.*

The following result classifies n-spherical objects in the residual component.

Theorem 4.59. *Let S be an Enriques surface and let $\mathcal{L} = \langle L_1, \dots, L_{10}\rangle \subset D^b(S)$ be a collection of 10 mutually orthogonal exceptional line bundles as in Section 4.3, so that $D^b(S) = \langle \mathcal{K}u(S,\mathcal{L}), \mathcal{L}\rangle$. Let:*

$$\kappa\colon \mathcal{K}u(S,\mathcal{L}) \to D^b(S)$$

be the inclusion, and for $j = 1, \dots, 10$, set $S_j \coloneqq \kappa^!(L_j)$. Then an object $F \in \mathcal{K}u(S,\mathcal{L})$ is 3-spherical if and only if $F \cong S_j[k]$ for some $j \in \{1,\dots,10\}$ and some $k \in \mathbb{Z}$. Moreover, the objects $S_j[k]$ are pairwise non-isomorphic.

For general decompositions, with blocks of more than one object, a similar characterization holds for 3-pseudoprojective objects; again, this will not be discussed here.

The following result is a rather technical Lemma, which leverages the key properties of Fourier-Mukai functors in order to pose some sufficient conditions for the extension of a Fourier-Mukai equivalence between the residual components to an equivalence between the whole derived categories.

Lemma 4.60. *Let S_1 and S_2 be smooth projective varieties. Consider $\mathcal{D}_1 \subset D^b(S_1)$ an admissible subcategory, and $E \in {}^{\perp}\mathcal{D}_1$ an exceptional object. Let $\Phi\colon D^b(S_1) \to D^b(S_2)$ be a Fourier–Mukai functor which:*

- *vanishes on ${}^{\perp}\mathcal{D}_1$, i.e. $\Phi({}^{\perp}\mathcal{D}_1) \cong 0$;*
- *restricts to an equivalence of $\mathcal{D}_1$ onto an admissible subcategory $\mathcal{D}_2$.*

We will denote by α_i the embeddings $\mathcal{D}_i \hookrightarrow D^b(S_i)$, for $i = 1, 2$. Finally, assume that there exists an exceptional object $F \in {}^{\perp}\mathcal{D}_2$ such that we have an isomorphism:

$$\Phi(\alpha_1(\alpha_1^!(E))) \cong \alpha_2(\alpha_2^!(F)).$$

Then there is a Fourier–Mukai functor $\widetilde{\Phi}\colon D^b(S_1) \to D^b(S_2)$ which:

i) vanishes on ${}^{\perp}\langle \mathcal{D}_1, E\rangle$, i.e. $\widetilde{\Phi}({}^{\perp}\langle \mathcal{D}_1, E\rangle) \cong 0$;
ii) satisfies $\widetilde{\Phi}|_{\mathcal{D}_1} \cong \Phi|_{\mathcal{D}_1}$ and $\widetilde{\Phi}(E) \cong F$;
iii) restricts to an equivalence $\langle \mathcal{D}_1, E\rangle \xrightarrow{\sim} \langle \mathcal{D}_2, F\rangle$.

Finally, let us define what it means for a functor between admissible subcategories to be of Fourier-Mukai type.

Definition 4.61. Let $\mathcal{A}_1 \subset D^b(S_1)$ and $\mathcal{A}_2 \subset D^b(S_2)$ be admissible subcategories, and let $\iota_i \colon \mathcal{A}_i \hookrightarrow D^b(S_i)$ denote the natural inclusion functors for $i = 1, 2$. An exact functor $F \colon \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ is said to be *of Fourier–Mukai type* if there exists an object $\mathcal{P} \in D^b(S_1 \times S_2)$ such that

$$\iota_2 \circ F \cong \Phi_{\mathcal{P}}|_{\mathcal{A}_1},$$

where $\Phi_{\mathcal{P}} \colon D^b(S_1) \longrightarrow D^b(S_2)$ is the Fourier–Mukai transform with kernel $\mathcal{P}$.

We are now ready to state the aforementioned refinement of 4.56.

4.62 Refined Derived Torelli Theorem. *Let S_1 and S_2 be Enriques surfaces. Suppose they have semiorthogonal decompositions:*

$$D^b(S_i) = \langle \mathcal{K}u(S_i, \mathcal{L}_i), \mathcal{L}_i \rangle,$$

with $i = 1, 2$ and $\mathcal{L}_i$ as above. If there is an exact equivalence of Fourier–Mukai type $F \colon \mathcal{K}u(S_1, \mathcal{L}_1) \xrightarrow{\sim} \mathcal{K}u(S_2, \mathcal{L}_2)$, then $S_1 \cong S_2$.

Proof. We treat the case in which $\mathcal{L}_i = \langle L_{i,1}, \dots, L_{i,10} \rangle$ consists of 10 completely orthogonal line bundles, so that Theorem 4.59 applies; the case with larger blocks is similar, using 3-pseudoprojective objects. We call κ_i the embeddings $\mathcal{K}u(S_i, \mathcal{L}_i) \hookrightarrow D^b(S_i)$, and we denote by $S_{i,j} \coloneqq \kappa_i^!(L_{i,j})$, the projection of $L_{i,j}$ onto $\mathcal{K}u(S_i, \mathcal{L}_i)$, which is of Fourier–Mukai type, as $\kappa_i^!$ is a Fourier–Mukai functor.

By Theorem 4.59, $S_{1,1}$ is 3-spherical. Since F is an equivalence, $F(S_{1,1})$ is 3-spherical as well, so Theorem 4.59 also implies that we have $F(S_{1,1}) \cong S_{2,j}$ for some j (up to a shift, which we absorb into F). Using orthogonality, we may reorder the $L_{2,j}$ and assume $j = 1$. One could check that the hypotheses of Lemma 4.60 are satisfied, therefore it can be applied with $\mathcal{D}_1 = \mathcal{K}u(S_1, \mathcal{L}_1)$ and $E = L_{1,1}$, in order to extend F to a Fourier–Mukai equivalence:

$$F_1 \colon \langle \mathcal{K}u(S_1, \mathcal{L}_1), L_{1,1} \rangle \xrightarrow{\sim} \langle \mathcal{K}u(S_2, \mathcal{L}_2), L_{2,1} \rangle.$$

The same argument applies to the remaining line bundles: by complete orthogonality, the projection of $L_{i,j}$ onto $\langle \mathcal{K}u(S_i, \mathcal{L}_i), L_{i,1}, \dots, L_{i,j-1} \rangle$ coincides with its projection onto $\mathcal{K}u(S_i, \mathcal{L}_i)$. After 10 steps we obtain an equivalence $D^b(S_1) \cong D^b(S_2)$, which is of Fourier–Mukai type by Corollary 3.75. Therefore, by Theorem 4.56, we can conclude that $S_1 \cong S_2$.

□

Remark 4.63. The result could be stated more generally for collections $\mathcal{L}_i$ whose blocks may contain more than one exceptional object; in that case the objects $S_{i,j}$ coming from blocks with several objects are no longer 3-spherical, but instead 3-pseudoprojective.

Remark 4.64. Theorem 4.62 has a trivial converse: an isomorphism $S_1 \cong S_2$ induces an equivalence $D^b(S_1) \cong D^b(S_2)$ of Fourier–Mukai type, which restricts to the residual components for suitable $\mathcal{L}_1, \mathcal{L}_2$. Hence, for Enriques surfaces, $S_1 \cong S_2$ if and only if there exist collections $\mathcal{L}_1, \mathcal{L}_2$ of this form with $\mathcal{K}u(S_1, \mathcal{L}_1) \cong \mathcal{K}u(S_2, \mathcal{L}_2)$, in the sense of Theorem 4.62.

Bibliography

[Bei78] Alexander A. Beilinson. Coherent sheaves on $\mathbb{P}^n$ and problems of linear algebra. *Functional Analysis and Its Applications*, 12(3):214–216, 1978.

[BHPVdV04] Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven. *Compact complex surfaces*, volume 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge*. Springer, Berlin, 2nd enlarged edition, 2004.

[BM01] Tom Bridgeland and Antony Maciocia. Complex surfaces with equivalent derived categories. *Mathematische Zeitschrift*, 236(4):677–697, 2001.

[BO01] Alexey Bondal and Dmitri O. Orlov. Reconstruction of a variety from the derived category and groups of autoequivalences. *Compositio Mathematica*, 125(3):327–344, 2001.

[BP20] Erwan Beurier and Dominique Pastor. A crash course on Category Theory, 2020. Preprint, HAL id: hal-02190027.

[Bri99] Tom Bridgeland. Equivalences of triangulated categories and Fourier–Mukai transforms. *Bulletin of the London Mathematical Society*, 31(1):25–34, 1999.

[Car50] Henri Cartan. Idéaux et modules de fonctions analytiques de variables complexes. In *Bulletin de la Société Mathématique de France*, volume 78, pages 29–64, 1950.

[CDL25] François Cossec, Igor V. Dolgachev, and Christian Liedtke. *Enriques Surfaces I*. Springer, Singapore, 2nd edition, 2025.

[CIL+26] Xiao-Wu Chen, Sergei O. Ivanov, Jian Liu, Xue-Song Lu, Vladimir Sosnilo, and Chencheng Zhang. Pretriangulated categories failing (TR4), 2026.

[DK25] Igor V. Dolgachev and Shigeyuki Kondō. *Enriques Surfaces II*. Springer, Singapore, 2025.

[GD60] Alexander Grothendieck and Jean Dieudonné. Éléments de géométrie algébrique. I. Le langage des schémas. *Publications Mathématiques de l'IHÉS*, 4:5–228, 1960.

[GM03] Sergei I. Gelfand and Yuri I. Manin. *Methods of Homological Algebra*. Springer, 2nd edition, 2003.

[Gro57] Alexander Grothendieck. Sur quelques points d'algèbre homologique. *Tôhoku Mathematical Journal*, 9(2-3):119–221, 1957.

[Har77] Robin Hartshorne. *Algebraic geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer, Berlin-Heidelberg-New York, 1977.

[Huy06] Daniel Huybrechts. *Fourier–Mukai transforms in algebraic geometry*. Oxford Mathematical Monographs. Clarendon Press, Oxford, 2006.

[Kap88] Mikhail M. Kapranov. On the derived categories of coherent sheaves on some homogeneous spaces. *Inventiones Mathematicae*, 92(3):479–508, 1988.

[KS90] Masaki Kashiwara and Pierre Schapira. *Sheaves on Manifolds*, volume 292 of *Grundlehren der mathematischen Wissenschaften*. Springer, Berlin-Heidelberg, 1990.

[Kum] Abhinav Kumar. Algebraic surfaces, lecture 17. Lecture notes, MIT OpenCourseWare, 18.727 Topics in Algebraic Geometry: Algebraic Surfaces, 2008. Notes prepared by Kartik Venkatram. Spring 2008, http://ocw.mit.edu.

[Kuz10] Alexander Kuznetsov. Derived categories of cubic fourfolds. In *Cohomological and Geometric Approaches to Rationality Problems*, volume 282 of *Progress in Mathematics*, pages 219–243. Birkhäuser, Boston, MA, 2010.

[Laz04] Robert Lazarsfeld. *Positivity in Algebraic Geometry I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge.* Springer-Verlag, Berlin, 2004.

[Ler46] Jean Leray. L'anneau d'homologie d'une représentation. *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Paris*, 222:1366–1368, 1946.

[Liu02] Qing Liu. *Algebraic geometry and arithmetic curves*, volume 6 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2002.

[LNSZ21] Chunyi Li, Howard Nuer, Paolo Stellari, and Xiaolei Zhao. A refined derived Torelli theorem for Enriques surfaces. *Mathematische Annalen*, 379(3-4):1475–1505, 2021.

[MRS25] Riccardo Moschetti, Franco Rota, and Luca Schaffler. Enriques surfaces with non-generic non-degeneracy, 2025.

[Muk81] Shigeru Mukai. Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves. *Nagoya Mathematical Journal*, 81:153–175, 1981.

[Oka50] Kiyoshi Oka. Sur les fonctions analytiques de plusieurs variables. VII. Sur quelques notions arithmétiques. *Bulletin de la Société Mathématique de France*, 78:1–27, 1950.

[Orl97] Dmitri O. Orlov. Equivalences of derived categories and K3 surfaces. *Journal of Mathematical Sciences*, 84(5):1361–1381, 1997.

[Orl03] Dmitri O. Orlov. Derived categories of coherent sheaves and equivalences between them. *Russian Mathematical Surveys*, 58(3):511–591, 2003.

[Ott88] Giorgio Ottaviani. Spinor bundles on quadrics. *Transactions of the American Mathematical Society*, 307(1):301–316, 1988.

[PS22] Laura Pertusi and Paolo Stellari. Categorical Torelli theorems: results and open problems. *arXiv preprint arXiv:2201.03899*, 2022.

[Pup62] Dieter Puppe. On the formal structure of stable homotopy theory. *Colloquium on Algebraic Topology*, pages 65–71, 1962.

[Sch12] Christian Schnell. The fundamental group is not a derived invariant. In *Derived Categories in Algebraic Geometry*, EMS Series of Congress Reports, pages 279–285. European Mathematical Society, Zürich, 2012. Preprint available on arXiv (2011).

[Ser55] Jean-Pierre Serre. Faisceaux algébriques cohérents. *Annals of Mathematics*, 61(2):197–278, 1955.

[ST01] Paul Seidel and Richard Thomas. Braid group actions on derived categories of coherent sheaves. *Duke Mathematical Journal*, 108(1):37–108, 2001.

[Ver96] Jean-Louis Verdier. Des catégories dérivées des catégories abéliennes. *Astérisque*, (239):xii+253, 1996. With a preface by Luc Illusie, edited and with a note by Georges Maltsiniotis.

[Wei58] André Weil. Final report on contract AF 18(603)-57. In *Collected Papers*, volume II, pages 390–395. Springer-Verlag, New York, 1958. Reprinted in *Collected Papers*, Vol. II, Springer-Verlag, 1979.

[Zub94] Severinas Zube. Exceptional vector bundle on Enriques surfaces. *arXiv preprint alg-geom/9410026*, 1994.