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Interazioni non lineari onda particella in fisica dei plasmi
Non Linear wave particle interactions in plasma physics

Tesi di Laurea Magistrale in Matematica

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Abstract

In questa tesi, studiamo un modello cinetico ridotto che descrive la beam-plasma instability (BPI) in un plasma omogeneo non magnetizzato. La fisica di base della BPI può essere riassunta come un'onda di Langmuir che è resa instabile dalla non uniformità nello spazio delle fasi di una popolazione di elettroni energetici. Dimostriamo esistenza globale e unicità di una soluzione debole con dato iniziale non negativo in $L^1 \cap L^\infty$, a supporto compatto in velocità. Mostriamo conservazione di massa, norme L^p, positività e propagazione della compattezza del supporto del dato iniziale. Studiamo inoltre gli stati di equilibrio e analizziamo numericamente il caso unidimensionale. La BPI è usata come modello ridotto per lo studio delle instabilità nei plasmi dei tokamak. In particolare, viene messa a confronto con gli EGAM (Energetic particle induced geodesic acoustic modes) che sono perturbazioni simmetriche dell'onda sonora (associata al campo elettrico radiale) nei plasma nei tokamak. Sono resi instabili dalla non uniformità di una famiglia di particelle energetice (EP), in questo caso ioni. Le oscillazioni non lineari degli EGAMs, dopo la saturazione, sono studiate utilizzando il codice girocinetico ORB5. ORB5 è stato originariamente sviluppato in Svizzera per studiare le turbolenze nel caso elettrostatico, e oggi è usato in molti Paesi Europei e in India per modellizzare la fisica dei tokamak e di futuri reattori. Uno scaling simile è stato trovato tra frequenza delle oscillazioni non lineari e livello di saturazione per EGAMs e BPI, a conferma del fatto che la loro dinamica non lineare è determinata dallo stesso meccanismo fisico: l'interazione con le EP. Come prodotto di questo studio, una nuova diagnostica è proposta per la valutazione dell'intensità degli EGAM nei tokamak.

Abstract

In this thesis, we study a reduced kinetic model describing the beam-plasma instability (BPI) in a homogeneous unmagnetized plasma. The basic physics of the BPI can be summarized as a Langmuir wave which is driven unstable by phase space nonuniformity of a population of energetic electrons. We prove global existence and uniqueness of weak solutions for nonnegative initial data in $L^1 \cap L^\infty$ with compact velocity support. We establish conservation of mass, preservation of L^p norms, positivity, and propagation of compact velocity support. We also study the existence of steady states and numerically study the problem in the one-dimensional setting. The reduced model of the BPI is used to shed light on instabilities in magnetized tokamak plasmas. In particular, similarities are discussed with Energetic Particle (EP) induced geodesic acoustic modes (EGAMs) that are axisymmetric perturbations of the sound wave (associated to a radial electric field) in tokamak plasmas. They are driven unstable by the phase space nonuniformity of a population of EPs (in this case, ions). The nonlinear oscillation in the amplitude of the EGAMs after their saturation, is studied by means of the gyrokinetic particle-in-cell code ORB5. ORB5 was originally developed in Switzerland for electrostatic turbulence studies, and is nowadays used in many countries of Europe and India, for the modeling of present day tokamaks and future reactors. A similar scaling of the nonlinear oscillation frequency as a function of the mode amplitude is found for the EGAMs and for the BPI, confirming that their nonlinear dynamics is strongly determined by the same physical mechanisms: the interaction with EPs. As a product of this study, a novel diagnostics is proposed for the evaluation of the EGAM intensity in tokamak plasmas.

Contents

Chapter 1

Introduction

Nuclear fusion is one of the possible solutions to the growing need of a clean sustainable energy. A fundamental nuclear fusion reaction takes two light nuclei (like deuterium and tritium) and has a heavier atom (like helium) as a product, and a neutron, and energy. Magnetic confinement fusion devices aim at heating and stabilizing the fusion fuel, by means of a strong magnetic field. At ignition temperatures, the fuel is in a state of plasma - an ionized gas.
The kinetic description of collisionless plasmas is classically based on the Vlasov–Poisson or Vlasov–Maxwell systems. In the electrostatic regime, the self-consistent electric field is usually determined by Poisson's equation, which couples the field to the charge density. In the present work we consider a different reduced kinetic model, motivated by the nonlinear theory of beam–plasma interaction developed by Levin, Lyubarskii, Onishchenko, Shapiro and Shevchenko [1].

The model proposed in [1] describes the interaction between resonant beam particles and a nearly monochromatic plasma wave. The spatial profile of the wave is prescribed, while its scalar amplitude evolves in time according to an energy-balance closure law. In particular, the particles are transported by a fixed electrostatic wave profile, whereas the wave amplitude is coupled back to the particle distribution through the current projected onto the wave mode. Thus the self-consistent field is not determined by Poisson's equation.

This type of reduced model is natural in regimes where the wave remains close to monochromatic and where the main nonlinear feedback comes from wave–particle interaction near the chosen mode. The non-resonant background plasma is then understood to be encoded in the linear dispersion relation and in the constants entering the reduced amplitude equation.

Although such reduced descriptions are common in the plasma physics literature, especially in beam–plasma interaction and wave–particle dynamics relevant to fusion plasmas, they appear to have received comparatively little systematic mathematical attention.
Plasmas in magnetic confinement fusion devices are expected to have a large population of energetic (i.e. suprathermal) particles (EP) due to the nuclear fusion reactions, and to the external heating mechanisms. A large variety of instabilities can take energy out of the EP population, via inverse Landau damping [2], for example Alfvén waves [3]. We are interested here in the formation

of unstable branches of the geodesic acoustic mode (GAM) [4], due to inverse Landau damping with EPs, giving rise to EP induced GAMs (EGAMs) [5]. GAMs are a type of zonal (i.e. axisymmetric) flows (ZF), associated to a perturbation of the radial electric field. GAMs are often observed in the presence of turbulence in tokamaks. EGAMs can redistribute the EP population in phase space, thus modifying their distribution function and consequently affecting the heating mechanism. The BPI comes from the interaction of a beam of energetic electrons with a Langmuir wave [1] (see also Ref. [6] for a recent study).
The purpose of this thesis is to provide a first well-posedness result, at the level of global weak solutions, for a generalization of the Vlasov–wave amplitude model derived in [1]. In addition, we focus on the EGAM behavior just after the nonlinear saturation. Our goal is to give insights on the amplitude evolution in time, and show how the physics of the BPI can help understanding it.
The work is organized as follows. In chapter 2 we give a general description of plasma physics and his application to nuclear fusion. We move then to chapter 3 to present the mathematical models that describe wave-particle interactions in plasmas. In section 3.5.1 we study the BPI following Levin's model, (see [1]). In Sec. 3.5.2 we study the model numerically, implementing a particle-in-cell code. This code is used to investigate the evolution of the amplitude of the BPI. We recover the results of Levin, and in particular we investigate in detail the oscillations during the nonlinear phase.
Chapter 4 is devoted to the mathematical study of the Vlasov wave-amplitude model. In Section 4.1 we give a proof of existence and uniqueness of the solution of the model of Levin. In Section 4.2 we introduce the generalized model and describe its relation with the moving frame of the wave. In Section 4.3 we prove global weak well-posedness. The proof is based on a local contraction argument for the scalar amplitude and on a priori estimates preventing finite-time blow-up. Section 4.4 is devoted to the study of equilibrium states and related qualitative properties. In Sec. 4.5 a numerical simulation is performed using a PIC code.
After having discussed the nonlinear oscillations of the BPI in chapter 3, in chapter 5 we perform simulations of EGAMs considering the case of Ref. [7, 8], where the EGAM had already been studied in detail for its linear phase and nonlinear saturation. Our numerical simulations are performed with the ORB5 particle-in-cell code. ORB5 is a multispecies global gyrokinetic code [9]. This code is described in Sec. 5.2, where the tokamak case is also shown, and the linear dynamics is given. A scan with the EP concentration is performed, allowing us to investigate the dynamics of the EGAMs for different drive intensities. In Sec. 5.3 we discuss the behavior of the maximum of the radial electric field, and we show that during the nonlinear phase, some oscillations of the EGAM radial electric field are present. The scaling of this nonlinear frequency with the mode amplitude is studied. Sec. 5.4, is devoted to a comparison with BPI and the EGAM. To this aim, we use the theoretical result of Ref. [10], that links the bounce frequency of EP trapped in the wave, with the saturation level. Finally, a discussion and outlook is given in Sec. 5.4, where a novel diagnostics is also proposed for the measurement of the EGAM amplitude in tokamak plasmas.

The investigation of the nonlinear dynamics of EGAMs discussed in this thesis is the subject of a dedicated paper [11].

Chapter 2

Plasma Physics

2.1 Definition of plasma

Plasma can be defined as an ionized gas in which, due to the high temperature, electrons are stripped from atoms. As a result, the plasma consists of a mixture of charged particles, namely ions and electrons. Since these particles are electrically charged, they generate electromagnetic fields that, in turn, influence the motion of other particles. This is because the Coulomb force goes as $\frac{1}{r^2}$, so it also acts at a long range. This gives rise to what is known as *collective behavior*, whereby particles interact through electromagnetic forces even when they are far apart. Plasma is considered to be the fourth state of matter. It is not naturally present on earth, but we can find it for example in the stars.
One of the main features of plasmas is *quasineutrality.* This is linked to the Debye length λ_D, that is defined as:

$$\lambda_D = \left(\frac{\epsilon_0 k_B T_e}{n e^2}\right)^{\frac{1}{2}}$$

Here ϵ_0 is the vacuum permittivity, k_B is the Boltzmann constant for gasses, T_e is the electron temperature, n the density. It represents the characteristic scale over which the electrostatic potential generated by a charged particle is shielded . The requirement for plasmas is that

$$\lambda_D << L$$

, that means having a high density of particles.
We can also have *non neutral* plasmas, such as electron beams for example.
The range of temperatures and densities on which we can find plasmas is quite wide. Temperatures can go from 10^3 to $10^8 \quad K$, densities from 10^{10} to $10^{30} \quad m^{-3}$. In order to achieve nuclear fusion we need a temperature of the order of 10^8 and a density of the order of 10^{20}.

For more details we refer to the book of Chen on Ref. [12].

2.2 Nuclear fusion

Artificial plasmas can be used in various fields, such as medicine, industry, space and aerospace, and lighting. Among these applications, one of the most interesting is nuclear fusion. It is the process in which, at high temperatures, light nuclei form heavier elements. This is a natural process that occurs in stars, but it does not occur naturally on Earth. Research is being conducted to achieve controlled thermonuclear fusion on Earth because of the massive amount of energy produced by these nuclear reactions. The best candidates are reactions involving Deuterium (D) and Tritium (T), which are:

$$\mathrm{D} + \mathrm{D} \to {}^{3}\mathrm{He} + \mathrm{n} + 3.2, \mathrm{MeV}, \tag{2.2.1}$$
$$\mathrm{D} + \mathrm{D} \to \mathrm{T} + \mathrm{p} + 4.0, \mathrm{MeV}, \tag{2.2.2}$$
$$\mathrm{D} + \mathrm{T} \to {}^{4}\mathrm{He} + \mathrm{n} + 17.6, \mathrm{MeV}. \tag{2.2.3}$$

This can happen at temperatures of the order of $10^8, \mathrm{K}$, where we have a totally ionized plasma.
Another important element is the Q factor, given by the ratio between the energy produced by the reaction and the energy spent to reach the plasma state.
Currently, this is performed in tokamaks, which are toroidal devices where the plasma is confined by a strong magnetic field.
An example is the ITER project, co-funded by the European Union, Japan, China, Russia, South Korea, and India. It is being constructed in Cadarache, in the south of France, and its goal is to reach a Q value of 10, which could lead to the production of energy from fusion reactions for commercial purposes. In Fig. 2.1 we can see the configuration of a tokamak.

2.3 Wave-particle interactions in plasmas

In plasmas, several instabilities can arise. One of the main causes is the *wave-particle* interaction. Particles exchange energy with the wave under resonance conditions, causing the mode to grow.
In this section we are going to present two cases: the beam-plasma instability problem (BPI) and Energetic-Particle-Driven Geodesic Acoustic Modes (EGAMs).

2.3.1 BPI

This is an instability that arises from the interaction of a plasma wave with a beam of electrons.
The particles are initialized with a bump-on-tail distribution function. When the phase velocity of the wave is found where the slope of the bump is positive, the effect is the so-called *Inverse Landau Damping*. In this case, we have more particles moving faster than the resonant velocity, so the particles transfer more energy to the wave. This causes the amplitude of the wave to grow exponentially until it reaches saturation. See the work of Landau in Ref. [2] for more details.

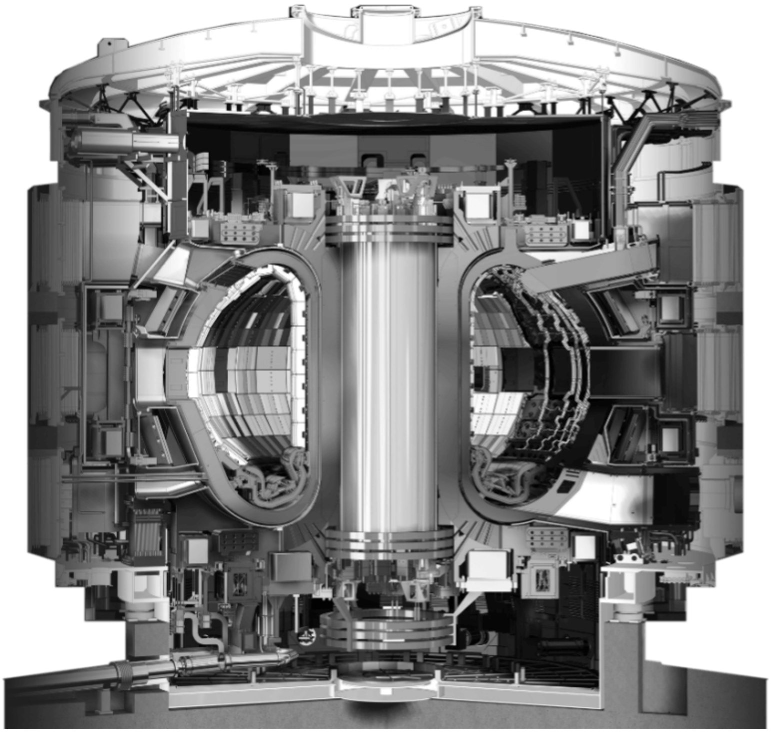


Figure 2.1: Design of ITER. © ITER Organization, http://www.iter.org/.

The beam-plasma instability can be described by a simple one-dimensional model, since we have a monochromatic wave and a beam of electrons. It is still of interest, though, because it can be used as a reduced model for the study of more complex problems, such as instabilities that arise in tokamaks.

2.3.2 EGAMs

Energetic-Particle-Driven Geodesic Acoustic Modes (EGAMs) are instabilities that can be found in a tokamak due to the interaction of a plasma wave with a beam of energetic particles (EPs). This is different from the BPI since the geometry is more complex. In fact, tokamaks have a toroidal shape, so this becomes a three-dimensional problem.
The redistribution of EPs in phase space caused by EGAMs can affect the plasma heating mechanism.
EGAMs have been studied theoretically, see Refs. [5, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23], and experimentally, see Ref. [24].
Similarities between the physics of EGAMs and the Beam-Plasma Instability (BPI) have been proposed, for example, in Ref. [10]. Following the analytical study presented in Ref. [10], numerical simulations have also been performed, and their nonlinear saturation levels due to wave-particle interaction have been discussed and compared with those of the BPI [7, 8].

Chapter 3

Mathematical models for plasmas

In this chapter we present a mathematical description of plasmas, following the lecture notes of E. Sonnendrücker on ref. [25].

Since plasma can be seen both as a continuum and as a set of charged particles, we will present both kinetic models and a continuous description of the problem.

3.1 The Vlasov-Maxwell system

Let us define $f = f(t, \mathbf{x}, \mathbf{v})$ as the distribution function that gives the density of particles with position $\mathbf{x} \in \mathbb{R}^3$, velocity $\mathbf{v} \in \mathbb{R}^3$ at time $t \in \mathbb{R}^+$. The evolution in time and space of the distribution function f is governed by the Vlasov equation that for particles with mass m and charge q, reads:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f = 0. \tag{3.1.1}$$

The electromagnetic field is self-induced by the charged particles, therefore is determined as solution of Maxwell's equations:

$$\begin{cases} -\dfrac{1}{c^2}\dfrac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \\ \dfrac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0, \\ \nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0}, \\ \nabla \cdot \mathbf{B} = 0, \end{cases} \tag{3.1.2}$$

where

$$\rho(t, \mathbf{x}) = q \int f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v}, \quad \mathbf{J}(t, \mathbf{x}) = q \int f(t, \mathbf{x}, \mathbf{v}) \mathbf{v} d\mathbf{v}.$$

Remark 3.1.1. In the relativistic case the Vlasov equation is defined as follows:

$$\frac{\partial f}{\partial t} + \mathbf{v}(\mathbf{p}) \cdot \nabla_x f + q(\mathbf{E} + \mathbf{v}(\mathbf{p}) \times \mathbf{B}) \cdot \nabla_p f = 0, \tag{3.1.3}$$

where $\mathbf{v}(\mathbf{p}) = \frac{\mathbf{p}}{m\gamma}$ and the Lorentz factor that is: $\gamma = \sqrt{1 + \frac{|p|^2}{m^2c^2}}$.

3.1.1 Kinetic models: the Vlasov-Boltzmann equation

Plasma can be considered as a set of particles that interact with each other. This can be modeled with kinetic theory. Having defined $f_s = f_s(t, \mathbf{x}, \mathbf{v})$ as the distribution function of species s, we find f_s as the solution of the Vlasov-Boltzmann equation that reads:

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla_x f_s + \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B})\nabla_v f_s = \sum_{s'} Q(f_s, f_{s'}). \tag{3.1.4}$$

$Q(f_s, f_{s'})$ is called the Boltzmann operator and is defined as:

$$Q(f, f_s)(\mathbf{v}) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(|\mathbf{v} - \mathbf{v}_1|, \theta)[f(\mathbf{v}')f_s(\mathbf{v}_1') - f(\mathbf{v})f_s(\mathbf{v}_1)]\, d\mathbf{v}_1\, d\mathbf{n} \tag{3.1.5}$$

Here $\mathbf{v}'$ and $\mathbf{v}_1'$ are the post-collisional velocities. They are obtained using the conservation of momentum and energy as follows:

$$\begin{cases} m\mathbf{v} + m_s\mathbf{v}_1 = m\mathbf{v}' + m_s\mathbf{v}_1', \\ \frac{m}{2}|\mathbf{v}|^2 + \frac{m_s}{2}|\mathbf{v}_1|^2 = \frac{m}{2}|\mathbf{v}'|^2 + \frac{m_s}{2}|\mathbf{v}_1'|^2. \end{cases}$$

This leads to the relation:

$$\begin{aligned} \mathbf{v}' &= \mathbf{v} - \frac{2\mu}{m}\big[(\mathbf{v} - \mathbf{v}_1) \cdot \mathbf{n}\big]\mathbf{n}, \\ \mathbf{v}_1' &= \mathbf{v}_1 + \frac{2\mu}{m_s}\big[(\mathbf{v} - \mathbf{v}_1) \cdot \mathbf{n}\big]\mathbf{n}. \end{aligned} \tag{3.1.6}$$

Here $\mu = \frac{mm_s}{m+m_s}$ and $\mathbf{n} \in \mathbb{S}^2$.
The kernel $B(|\mathbf{v} - \mathbf{v}_1|, \theta)$ is known from the physics behind the collisions, where θ denotes the scattering angle, typically defined by

$$\cos\theta = \frac{(\mathbf{v} - \mathbf{v}_1) \cdot \mathbf{n}}{|\mathbf{v} - \mathbf{v}_1|}.$$

Proposition 3.1.2 (Some properties of the Boltzmann operator)**.** *Let $Q(f, f)$ be the Boltzmann operator defined as in 3.1.5. Then the following proprieties hold:*

- *Let $\varphi \in C(\mathbb{R}^3)$ be an arbitrary test function. Then:*

$$\begin{aligned}\int_{\mathbb{R}^3} Q(f,f)\,\varphi(\mathbf{v})\,d\mathbf{v} = \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\\ [\varphi(\mathbf{v})+\varphi(\mathbf{v}_1)-\varphi(\mathbf{v}')-\varphi(\mathbf{v}_1')]B(|\mathbf{v}-\mathbf{v}_1|,\theta)\\ \times[f(\mathbf{v}')f(\mathbf{v}_1')-f(\mathbf{v})f(\mathbf{v}_1)]\,d\mathbf{n}\,d\mathbf{v}_1\,d\mathbf{v}.\end{aligned} \tag{3.1.7}$$

- *The Boltzmann inequality holds. Suppose that $B(|\mathbf{v}-\mathbf{v}_1|,\theta) \geq 0$. Then:*

$$\int_{\mathbb{R}^3} Q(f,f)\ln f\,d\mathbf{v} \leq 0 \quad \forall f \geq 0.$$

- *If f is a solution of 3.1.4 then, having defined :*

$$H(t) = \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} f\ln f\,d\mathbf{x}\,d\mathbf{v},$$

we have

$$\frac{dH}{dt} \leq 0.$$

See the works of Cergignani and Glassey on ref. [26, 27] for the details of the proof.

Remark 3.1.3. Using eq. 3.1.7 we can derive that:

$$\int_{\mathbb{R}^3} Q(f,f)\,\varphi(\mathbf{v})\,d\mathbf{v} = 0 \iff \varphi(\mathbf{v}) = a + bv_x + cv_y + dv_z + e|v|^2$$

Remark 3.1.4. From the H theorem it follows that the distribution function tends to a Maxwellian.

Remark 3.1.5. The Vlasov equation presented on the previous section is obtained as a kinetic model where the effects of the collisions are not considered due to the high temperatures of plasmas.

3.1.2 Derivation of a fluid model

Near thermodynamic equilibrium, a plasma can be regarded as a fluid. Therefore, the kinetic description can be reduced to a system of equations for macroscopic quantities. First of all we give the definition of the following macroscopic quantities.

Definition 3.1.6. Consider $f(t,\mathbf{x},\mathbf{v})$ as the distribution function of the particles. Then we define:

1. The particle density $n(\mathbf{x},t) = \displaystyle\int_{\mathbb{R}^3} f(t,\mathbf{x},\mathbf{v})\,d\mathbf{v}$;
2. The mean velocity $n(\mathbf{x},t)\mathbf{u}(\mathbf{x},t) = \displaystyle\int_{\mathbb{R}^3} f(t,\mathbf{x},\mathbf{v})\mathbf{v}\,d\mathbf{v}$;
3. The pressure tensor $\mathbb{P}(\mathbf{x},\approx) = m\displaystyle\int_{\mathbb{R}^3} f(t,\mathbf{x},\mathbf{v})\,(\mathbf{v}-\mathbf{u}(\mathbf{x},t))\otimes(\mathbf{v}-\mathbf{u}(\mathbf{x},t))d\mathbf{v}$;

4. The scalar pressure $p(\mathbf{x},t) = \dfrac{m}{3}\displaystyle\int_{\mathbb{R}^3} f(t,\mathbf{x},\mathbf{v})\,|\mathbf{v}-\mathbf{u}(\mathbf{x},t)|^2\,d\mathbf{v}$;

5. The temperature $T(\mathbf{x},t) = \dfrac{p(\mathbf{x},t)}{k_B n(\mathbf{x},t)}$, where k_B is the Boltzmann constant;

6. The energy flux $\mathbf{Q}(\mathbf{x},t) = \frac{m}{2}\int_{\mathbb{R}^3} f(t,\mathbf{x},\mathbf{v})\,|\mathbf{v}|^2\,\mathbf{v}\,d\mathbf{v}$.

To get the equations of the fluid description we start from the Vlasov–Boltzmann equation:

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla_x f + \frac{q}{m}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_v f = Q(f,f). \tag{3.1.8}$$

If we integrate respect to velocity we have:

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f(t,\mathbf{x},\mathbf{v})\,d\mathbf{v} + \nabla_x\cdot\int_{\mathbb{R}^3} f(t,\mathbf{x},\mathbf{v})\,\mathbf{v}\,d\mathbf{v}&\\ + \frac{q}{m}\int_{\mathbb{R}^3}\nabla_v\cdot\Big[f(t,\mathbf{x},\mathbf{v})(\mathbf{E}+\mathbf{v}\times\mathbf{B})\Big]\,d\mathbf{v}&\\ = \int_{\mathbb{R}^3} Q(f,f)\,d\mathbf{v}.&\end{aligned}$$

We suppose that the distribution function goes to zero at infinity, we have that:

$$\int_{\mathbb{R}^3}\nabla_v[f(t,\mathbf{x},\mathbf{v})(\mathbf{E}+\mathbf{v}\times\mathbf{B})]d\mathbf{v} = 0.$$

Moreover, from 3.1.3 we also have:

$$\int_{\mathbb{R}^3} Q(f,f)d\mathbf{v} = 0.$$

Then we get:

$$\frac{\partial n}{\partial t} + \nabla_x\cdot(n\mathbf{u}) = 0. \tag{3.1.9}$$

Similarly, by calculating the moments of f we have:

$$\begin{aligned}&m\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f(t,\mathbf{x},\mathbf{v})\,\mathbf{v}\,d\mathbf{v} + m\nabla_{\mathbf{x}}\cdot\int_{\mathbb{R}^3}(\mathbf{v}\otimes\mathbf{v})f(t,\mathbf{x},\mathbf{v})\,d\mathbf{v}\\ &-q\left(\mathbf{E}(\mathbf{x},t)\int_{\mathbb{R}^3} f(t,\mathbf{x},\mathbf{v})\,d\mathbf{v} + \int_{\mathbb{R}^3} f(t,\mathbf{x},\mathbf{v})\,\mathbf{v}\,d\mathbf{v}\times\mathbf{B}(\mathbf{x},t)\right) = 0,\end{aligned} \tag{3.1.10}$$

where we obtained the last term by integrating by parts. To substitute the second term with the macroscopic quantities we defined we can write:

$$\int_{\mathbb{R}^3}\mathbf{v}\otimes\mathbf{v}\,f(t,\mathbf{x},\mathbf{v})\,d\mathbf{v} = \int_{\mathbb{R}^3}(\mathbf{v}-\mathbf{u})\otimes(\mathbf{v}-\mathbf{u})\,f(t,\mathbf{x},\mathbf{v})\,d\mathbf{v} + n\,\mathbf{u}\otimes\mathbf{u}.$$

This leads us to:

$$m\frac{\partial}{\partial t}(n\mathbf{u}) + m\nabla \cdot (n\,\mathbf{u} \otimes \mathbf{u}) + \nabla \cdot \mathbb{P} = qn\,(\mathbf{E} + \mathbf{u} \times \mathbf{B})\,. \tag{3.1.11}$$

Lastly, we integrate eq. 3.1.8 and multiply by $\frac{1}{2}m|v|^2$ and we derive with a similar computation:

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p + \frac{1}{2}mn|\mathbf{u}|^2\right) + \nabla \cdot \mathbf{Q} = \mathbf{E} \cdot (qn\mathbf{u}). \tag{3.1.12}$$

To close the system we need to rewrite the energy flux as function of density, temperature and mean velocity. To do so, we use the hypothesis that the fluid model is valid at thermodynamic equilibrium. This means, using 3.1.2, that the distribution function is a Maxwellian, defined as follows:

$$f_M(t, \mathbf{x}, \mathbf{v}) = \frac{n(\mathbf{x}, t)}{(2\pi k_B T(\mathbf{x}, t)/m)^{3/2}} \exp\left(-\frac{m\,|\mathbf{v} - \mathbf{u}(\mathbf{x}, t)|^2}{2\,k_B T(\mathbf{x}, t)}\right).$$

Then we substitute in the pressure tensor:

$$\mathbb{P} = m\frac{n}{(2\pi k_B T/m)^{3/2}} \int_{\mathbb{R}^3} \exp\left(-\frac{|\mathbf{v} - \mathbf{u}|^2}{2k_B T/m}\right)(\mathbf{v} - \mathbf{u}) \otimes (\mathbf{v} - \mathbf{u})\,d\mathbf{v} =$$

$$= m\frac{n}{(2\pi v_{th}^2)^{3/2}} \int \exp\left(-\frac{|\mathbf{w}|^2}{2}\right) v_{th}^2 \mathbf{w} \otimes \mathbf{w} v_{th}^3\,d\mathbf{w}.$$

Here we defined $v_{th} = \sqrt{\frac{k_B T}{m}}$ and the change of variables $\mathbf{w} = \frac{\mathbf{v}-\mathbf{u}}{v_{th}}$. To conclude we use the following:

$$\int_{\mathbb{R}^3} w_i w_j\, e^{-\frac{|w|^2}{2}}\,d\mathbf{w} = \delta_{ij} \int_{\mathbb{R}^3} e^{-\frac{|w|^2}{2}}\,d\mathbf{w}. \tag{3.1.13}$$

This leads us to:

$$\begin{cases} \mathbb{P}_{ij} = 0, \\ \mathbb{P}_{ii} = nk_B T\dfrac{1}{(2\pi)^{\frac{3}{2}}} \displaystyle\int \exp\left(-\frac{|\mathbf{w}|^2}{2}\right) w_i^2\,d\mathbf{w} = nk_B T. \end{cases}$$

This can be written in compact form as :

$$\mathbb{P} = nT\mathbb{I} = p\mathbb{I}, \tag{3.1.14}$$

using the definition of temperature.
With the same change of variables in the definition we compute:

$$\mathbf{Q} = \frac{5}{2}\,nk_B T\,\mathbf{u} + \frac{m}{2}\,n\,|\mathbf{u}|^2\,\mathbf{u} = \frac{5}{2}\,p\,\mathbf{u} + \frac{m}{2}\,n\,|\mathbf{u}|^2\,\mathbf{u}. \tag{3.1.15}$$

If we substitute (3.1.14)-(3.1.15) into (3.1.9)-(3.1.11)-(3.1.12) we obtain the following fluid model:

$$
\begin{cases}
\dfrac{\partial n}{\partial t} + \nabla_{\mathbf{x}} \cdot (n\mathbf{u}) = 0, \\[2ex]
m\dfrac{\partial}{\partial t}(n\mathbf{u}) + m\nabla_{\mathbf{x}} \cdot (n\mathbf{u} \otimes \mathbf{u}) + \nabla_{\mathbf{x}} p = q\, n(\mathbf{E} + \mathbf{u} \times \mathbf{B}), \\[2ex]
\dfrac{\partial}{\partial t}\left(\dfrac{3}{2}p + \dfrac{1}{2}mn|\mathbf{u}|^2\right) + \nabla_{\mathbf{x}} \cdot \left(\dfrac{5}{2}p\,\mathbf{u} + \dfrac{m}{2}n|\mathbf{u}|^2\mathbf{u}\right) = \mathbf{E} \cdot (qn\mathbf{u}).
\end{cases}
\tag{3.1.16}
$$

3.2 Theory on the Vlasov equation

In this section, suppose that the elctromagnetic field is given, so that $\mathbf{E}(\mathbf{t}, \mathbf{x})$ and $\mathbf{B}(t, \mathbf{x})$ are two known fields.
If we define:

$$
\mathbf{A}(t, \mathbf{x}, \mathbf{v}) = \begin{pmatrix} \mathbf{v} \\ \mathbf{E} + \mathbf{v} \times \mathbf{B} \end{pmatrix},
$$

we can rewrite the Vlasov equation as a linear advection equation of the form:

$$
\frac{\partial f}{\partial t} + A \cdot \nabla_{(x,v)} f = 0. \tag{3.2.1}
$$

To show the well posedness of the problem we need to define the concept of characteristic of a linear advection equation and recall their main properties.

Definition 3.2.1 (Characteristics)**.** Let us consider equation 3.2.1. Then, we call characteristics the curve $\mathbf{X}(\mathbf{t}; \mathbf{s}, \mathbf{x})$, the solution of the following system:

$$
\begin{cases}
\dfrac{dX}{dt} = A(X, t), \\
X(s) = x.
\end{cases}
\tag{3.2.2}
$$

To find a proof of existence of the solution of this system we need to refer to ODE theory, as in the work of Amann on ref. [28].

The characteristics are used to proof the following theorem of existence of the solution of equation 3.2.1.

Theorem 3.2.2. *Let us consider* $\mathbf{A}\,, \nabla\mathbf{A} \in C^{k-1}([0, T] \times \mathbb{R}^{\ntriangleleft})$ *with* $k \geq 1$ *and*

$$
|\mathbf{A}(t, \mathbf{x})| \leq k(1 + |\mathbf{x}|)\, \forall \mathbf{x} \in \mathbb{R}^6.
$$

We consider the following system:

$$
\begin{cases}
\dfrac{\partial f}{\partial t} + \mathbf{A} \cdot \nabla f = 0, \\
\qquad f(0, \mathbf{x}) = f_0(\mathbf{x}),
\end{cases}
\tag{3.2.3}
$$

where we suppose that $f_0 \in C^1(\mathbb{R}^6)$*. Then* $f(t, \mathbf{x}) = f_0(\mathbf{X}(0; t, \mathbf{x}))$ *is the unique* *solution of the system. Here* $\mathbf{X}$ *represent the characteristic of the linear equation.*

For the proof of this theorem we refer to the lecture notes on ref. [25].

3.3 The Vlasov-Poisson system

In the electrostatic case, where we neglect the contribution of the magnetic field the model becomes simpler. The self-induced electric field $\mathbf{E}(t, \mathbf{x})$ can be found from the Maxwell equations :

$$\begin{cases} \nabla \times \mathbf{E} = 0, \\ \nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0}. \end{cases} \tag{3.3.1}$$

It follows that it exists a scalar potential ϕ such that $\mathbf{E} = -\nabla\phi$ and $-\Delta\phi = \rho$. This leads to the Vlasov-Poisson model:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_x f + \frac{q}{m}\mathbf{E} \cdot \nabla_v f = 0. \tag{3.3.2}$$

$$-\Delta\phi = \rho,\ \mathbf{E} = -\nabla\phi,\ \rho(t, \mathbf{x}) = \int f(t, \mathbf{x}, \mathbf{v})\, d\mathbf{v}. \tag{3.3.3}$$

Solving the system following the characteristics we can derive the *maximum principle*:

$$0 \leq f(t, \mathbf{x}, \mathbf{v}) \leq \max_{(\mathbf{x},\mathbf{v})} f_0(\mathbf{x}, \mathbf{v}).$$

Proposition 3.3.1 (Conservation of L^p norms)**.** *Let* $f : \mathbb{R}^+ \times \mathbb{R}^3_x \times \mathbb{R}^3_v \to \mathbb{R}^+$ *be a solution of 3.3.2-3.3.3. Then* $\forall 1 \leq p \leq \infty$*:*

$$\frac{d}{dt}\left(\int (f(t, \mathbf{x}, \mathbf{v}))^p\, d\mathbf{v}\right) = 0. \tag{3.3.4}$$

Proof. If we multiply 3.3.2 by f^{p-1} and we integrate in phase space we get:

$$\int_{\mathbb{R}^3_x}\int_{\mathbb{R}^3_v} \frac{\partial f}{\partial t} f^{p-1}\, d\mathbf{x}\, d\mathbf{v} + \int_{\mathbb{R}^3_x}\int_{\mathbb{R}^3_v} \mathbf{v}\nabla_x f f^{p-1}\, d\mathbf{x}\, d\mathbf{v} +$$
$$\frac{q}{m}\int_{\mathbb{R}^3_x}\int_{\mathbb{R}^3_v} \mathbf{E}\nabla_v f f^{p-1}\, d\mathbf{x}\, d\mathbf{v} = 0.$$
$$\iff \quad \frac{d}{dt}\left(\int_{\mathbb{R}^3_x}\int_{\mathbb{R}^3_v} f^p\, d\mathbf{x}\, d\mathbf{v}\right) + \int_{\mathbb{R}^3_v} \mathbf{v}\left(\int_{\mathbb{R}^3_x} \nabla_x f^p\, d\mathbf{x}\right) d\mathbf{v}$$
$$+ \frac{q}{m}\int_{\mathbb{R}^3_x} \mathbf{E}\left(\int_{\mathbb{R}^3_v} \nabla_v f^p\, d\mathbf{v}\right) d\mathbf{x} = 0.$$
$$\iff \frac{d}{dt}\left(||f||_{L^p(\mathbb{R}^3_x \times \mathbb{R}^3_v)}\right) = 0,$$

where in the last step we used that f vanishes at infinity. □

Proposition 3.3.2 (Conservation of momentum and energy)**.** *For a density f that is a solution of system* (3.3.2)-(3.3.3) *the momentum and the total energy od the system are preserved. That means:*

- $\frac{d}{dt}\left(\int \mathbf{v} f \, d\mathbf{v} \, d\mathbf{x}\right) = 0$
- $\frac{d}{dt}\left(\frac{1}{2}\int v^2 f \, dx \, dv + \frac{1}{2}\int E^2 \, dx\right) = 0.$

Proof. This conservation proprierties follow directly from multiplying Vlasov equation by $\mathbf{v}$ for the momentum and by $\frac{1}{2}|v|^2$ for the energy, and then integrating in phase space, see [25] for details. □

3.4 Numerical methods

In this section we are going to give a description of the numerical methods used to solve the Vlasov Poisson equation presented in the previous sections. Different techniques are used, but the most present in literature is the Particle in cell (PIC) method. The idea is to follow the particle trajectories and couple a particle solver with a mesh solver for the Electric field.
This is widely described in literature, see as reference [29, 30, 31, 32, 33, 34].

3.4.1 The PIC method

We consider a finite number N of macro-particles, each one with a given weight. Then we define the distribution function as follows:

$$f_N(x, v, t) = \sum_{k=1}^{N} w_k \, \delta(x - x_k(t)) \, \delta(v - v_k(t)). \tag{3.4.1}$$

The particles are initialized following the initial distribution function $f_0(\mathbf{x}, \mathbf{v})$ that is given in the Vlasov system.
Then the evolution of the particles is given by the characteristics:

$$\begin{cases} \dfrac{dx_k}{dt} = v_k, \\ \dfrac{dv_k}{dt} = \dfrac{q}{m} E(x_k, t), \\ x_k(0) = x_k^0, \qquad v_k(0) = v_k^0. \end{cases} \qquad k = 1, \dots, N.$$

Proposition 3.4.1. *Let us consider the distribution function f_N defined in 3.4.1 with a given initial condition*

$$f_N^0(x, v) = \sum_{k=1}^{N} w_k \, \delta(x - x_k^0) \, \delta(v - v_k^0).$$

Then f_N solves the Vlasov equation in the sense of distributions, meaning that:

$$\left\langle \frac{\partial f_N}{\partial t}, \varphi \right\rangle = \left\langle -v \cdot \nabla_x f_N - \frac{q}{m} E(x,t) \cdot \nabla_v f_N, \varphi \right\rangle \ \forall \varphi \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R}^3 \times (0, +\infty)). \tag{3.4.2}$$

Proof. Consider an arbitrary function $\varphi \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R}^3 \times (0, +\infty))$. Then:

$$\left\langle \frac{\partial f_N}{\partial t}, \varphi \right\rangle = -\left\langle f_N, \frac{\partial \varphi}{\partial t} \right\rangle = -\sum_{k=1}^{N} w_k \int_0^T \frac{\partial \varphi}{\partial t}(x_k(t), v_k(t), t)\, dt.$$

We now use that the particles are evolving following the characteristic, so that:

$$\begin{aligned} \int_0^T \frac{\partial \varphi}{\partial t}(x_k(t), v_k(t), t)\, dt = & \int_0^T \frac{d}{dt}\Big(\varphi(\mathbf{x}_k(t), \mathbf{v}_k(t), t)\Big)\, dt \\ & - \int_0^T \mathbf{v}_k \cdot \nabla_x \varphi \, dt - \int_0^T \frac{q}{m}\, \mathbf{E}(\mathbf{x}_k, t) \cdot \nabla_v \varphi \, dt. \end{aligned}$$

We then use that the support of φ is compact, so consequently:

$$\int_0^T \frac{d}{dt}\left(\varphi(\mathbf{x}_k(t), \mathbf{v}_k(t), t)\right) dt = 0$$

$$\Rightarrow \left\langle \frac{\partial f_N}{\partial t}, \varphi \right\rangle = \sum_{k=1}^{N} w_k \int_0^T \left(\mathbf{v}_k \cdot \nabla_x \varphi + \frac{q}{m} \mathbf{E}(\mathbf{x}_k, t) \cdot \nabla_v \varphi \right) dt \tag{3.4.3}$$

and our thesis follows. □

3.4.2 The PIC algorithm

To develop an algorithm, we need the following steps:

1. Compute the initial distribution function;
2. Use a time scheme for the evolution of the particles;
3. Choose a solver to let evolve the electromagnetic field.

Starting with the initial distribution function, we can use both a *Deterministic method* or a *Monte-Carlo method*.
The first consists in fixing a phase space mesh $(\mathbf{x}_k^0, \mathbf{v}_k^0)$ and obtain the weights as

$$w_k = \int_{V_k} f_0(\mathbf{x}, \mathbf{v})\, d\mathbf{x} d\mathbf{v}$$

.
On the other hand, with the Monte-Carlo method, the particles are extracted pseudo-randomly from the initial density f_0.
With this description we don't have a point-wise distribution in all the phase-space so a convolutional kernel is needed to compute the density that has to be use to calculate the electric field solving the Poisson equation. The most

common way is to use B-splines. They are defined as follows:

$$S_0(x) = \begin{cases} \dfrac{1}{\Delta x}, & -\dfrac{\Delta x}{2} \leq x < \dfrac{\Delta x}{2}, \\ 0, & \text{elsewhere.} \end{cases}$$

$$S^m(x) = \frac{1}{\Delta x} \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} S^{m-1}(u)\, du \quad \forall m \in \mathbb{N}^*.$$

Once the kernel S is chosen we can calculate ρ_h as folows.

$$\rho_N(x) = q \sum_k w_k\, \delta(x - x_k).$$

$$\rho_h(x,t) = \int S(x-y)\, \rho_N(y)\, dy = q \sum_{k=1}^{N} w_k\, S(x - x_k). \tag{3.4.4}$$

Having defined ρ point-wise we can solve the Poisson equation with one of the known solvers, like finite elements or finite differences.
For the particle's time evolution we know give an example of a numerichal scheme commonly used in literature, the Verlet scheme. We denote with $\mathbf{x}_k^n, \mathbf{v}_k^n, \mathbf{E}_k^n$, the position, velocity and value of electric field at time t_n. Then we evolve the system from t_n to t_{n+1} in the following way:

$$\mathbf{v}_k^{n+\frac{1}{2}} = \mathbf{v}_k^n + \frac{q\Delta t}{2m} \mathbf{E}_k^n(\mathbf{x}_k^n),$$

$$\mathbf{x}_k^{n+1} = \mathbf{x}_k^n + \Delta t\, \mathbf{v}_k^{n+\frac{1}{2}},$$

$$\mathbf{v}_k^{n+1} = \mathbf{v}_k^{n+\frac{1}{2}} + \frac{q\Delta t}{2m} \mathbf{E}_k^{n+1}(\mathbf{x}_k^{n+1}).$$

Here, the electric field at step $n+1$ is calculated using the chosen method.

3.5 Beam Plasma Instability

In the following section we present a simplified case that models the physics of the Beam Plasma Instability. This will serve as the basis for the analysis carried out in the following chapters.

3.5.1 The Levin model

We adopt the model for the dynamics of the beam-plasma instability (BPI) provided by Levin in Ref. [1]. The BPI comes from the interaction of a beam of electrons and a Langmuir wave. A homogeneous plasma is considered, and the dynamics is studied in the electrostatic limit. Thus, the problem reduces to a one dimension in real space (with coordinate x), and one dimension in velocity space (with coordinate v). Consistently with Ref. [1], we study the amplitude of the Langmuir wave, but we are not interested in the evolution of its frequency and radial structure. Therefore in this model the wave has constant frequency and radial structure (a sinusoidal shape with wavelength λ). On the other hand, the evolution of the wave amplitude is studied self-consistently with the electron beam nonlinear dynamics. The dynamics of these energetic particles is studied in a self-consistent way, by following their trajectories in the wave field.

The model equations that we use are those of Levin [1]. The electric field has the form: $E(x,t) = E(t)\sin(kx - \omega t)$, where k and ω are supposed to be known. The system of equations that rules the dynamics of particles and the evolution in time of the amplitude of the wave is the following:

$$\begin{cases} \dfrac{d\xi}{dt} = v', \\ \dfrac{dv'}{dt} = -\dfrac{e}{m}E(t)\sin(k\xi), \\ \dfrac{1}{4\pi}E(t)\dfrac{dE}{dt} = -eE(t)\dfrac{1}{\lambda}\displaystyle\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}}\int_{-v_0^m}^{v_0^m} \sin(k\xi)\left(v' + \frac{\omega}{k}\right) f\left(v' + \frac{\omega}{k}\right) d\xi\, dv. \end{cases} \tag{3.5.1}$$

Here ξ and v' are, respectively, the position and velocity of particles in the wave frame: $\xi = x - \frac{\omega}{k}t$ and $v' = v - \frac{\omega}{k}$.
To derive the equation for the variation of the power of the field over time the author used a result of O'Niel, (on Ref. [35]) stating that the only particles exchanging energy with the wave were the ones whose velocity is close to the resonant velocity of the wave. Therefore, we don't need to integrate in all the real space, but it is sufficient to do the integration in a small interval around the resonance, with boundaries $-v_0^m$ and v_0^m. Note that these boundaries are free to choose: the only condition is to take the interval sufficiently large, so that the main physics is all inside, and the result does not depend anymore on their position. For more detail of this model, the reader is referred to Ref. [1].

3.5.2 Numerical Simulations of the BPI system

As a first step we recovered Levin's numerical results by solving the dimensionless system [1]:

$$\begin{cases} \dfrac{d\nu}{d\tau} = -\dfrac{\mathcal{E}}{2\pi} sin(2\pi\zeta), \\ \dfrac{d\zeta}{d\tau} = \nu, \\ \dfrac{d\mathcal{E}}{d\tau} = 16\pi \displaystyle\int_0^{\frac{1}{2}}\int_{-v_0^m}^{v_0^m} \nu_0 \sin(2\pi\zeta) d\nu_0 d\zeta_0. \end{cases} \tag{3.5.2}$$

This was obtained from system (3.5.1) by defining the dimensionless variables:

$$\zeta = \frac{1}{2\pi}k\xi, \quad \nu = \frac{1}{2\pi}\frac{kv'}{\gamma_L}, \quad \mathcal{E} = \frac{eEk}{m\gamma_L^2}, \quad \tau = \gamma_L t.$$

After the change of variables, we used Liouville's theorem that states that the distribution function f remains constant following the particles trajectories. The evolution in time of the dimensionless amplitude is obtained by linearizing the system and taking a Taylor expansion of the first order around $v_\phi = \frac{\omega}{k}$. Then, assuming that ν and ξ are odd functions, we get:

$$\frac{d\mathcal{E}}{d\tau} = (\frac{8\pi^2 e^2}{m\gamma_L^2})(\frac{2\pi\gamma_L}{k})(2v_\phi \frac{\partial f_0}{\partial v_\phi}) \int_0^{\frac{1}{2}} \int_{-\nu_0^m}^{\nu_0^m} \sin(2\pi\zeta)\nu_0 =$$

$$(\frac{16\pi}{\gamma_L}) \underbrace{(\frac{2\pi^2 e^2}{mk} v_\phi \frac{\partial f_0}{\partial v_\phi})}_{\gamma_L} \int_{-\nu_0^m}^{\nu_0^m} \sin(2\pi\zeta)\nu_0$$

$$\Rightarrow \frac{d\mathcal{E}}{d\tau} = 16\pi \int_{-\nu_0^m}^{\nu_0^m} \sin(2\pi\zeta)\nu_0$$

with f_0 being the equilibrium distribution function. Here, we have used the expression for the linear growth rate for the inverse Landau Damping: $\gamma_L = \frac{2\pi^2 e^2}{mk} v_\phi \frac{\partial f_0}{\partial v_\phi}$. The result is a ODE system that can be solved numerically, with a Runge-Kutta method.

Following Levin's steps we performed a simulation with 4000 particles with $\nu_0 \in (-2, 2), \quad E_0 = 0.01$. We remark that the system does not depend on the initial distribution anymore, since the dependence on the initial distribution function is contained in the linear growth rate, so we can initialize the particles with a uniform distribution. A linear growth of this instability can be seen in the initial phase of the simulation (see Fig. 3.2). The growth rate measured matches well the theoretical growth rate: in this case, $\gamma = 1$, as the system is normalized to the linear growth rate. A nonlinear saturation of the electric field amplitude at $E \simeq 10$ is observed around $\tau = 7.5$. Then, the field amplitude oscillates around this value.

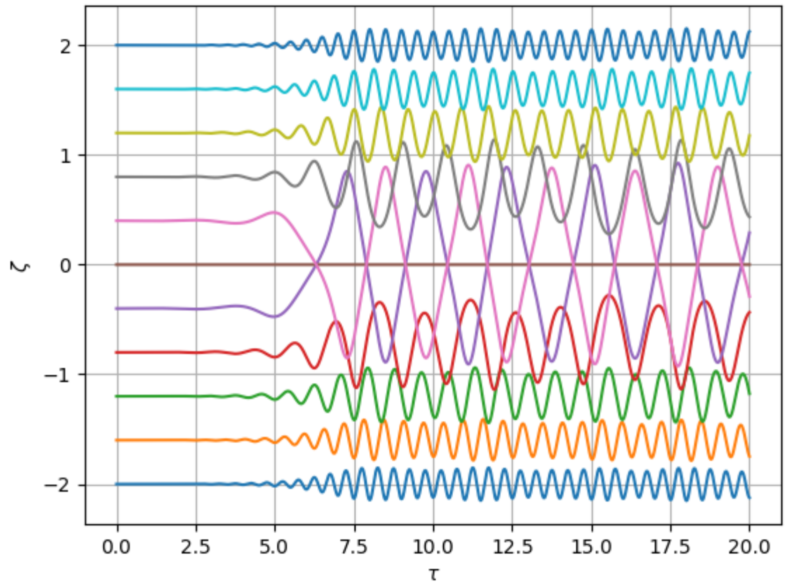


Figure 3.1: Velocity of particles as a function of τ. The linear and nonlinear phase are clearly visible, and the effect of the field on both passing and trapped particles can be observed in their velocity ζ.

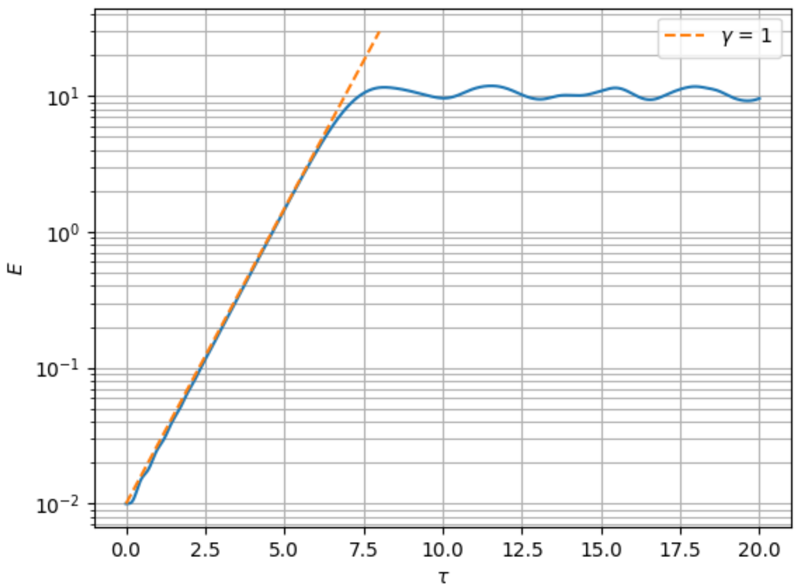
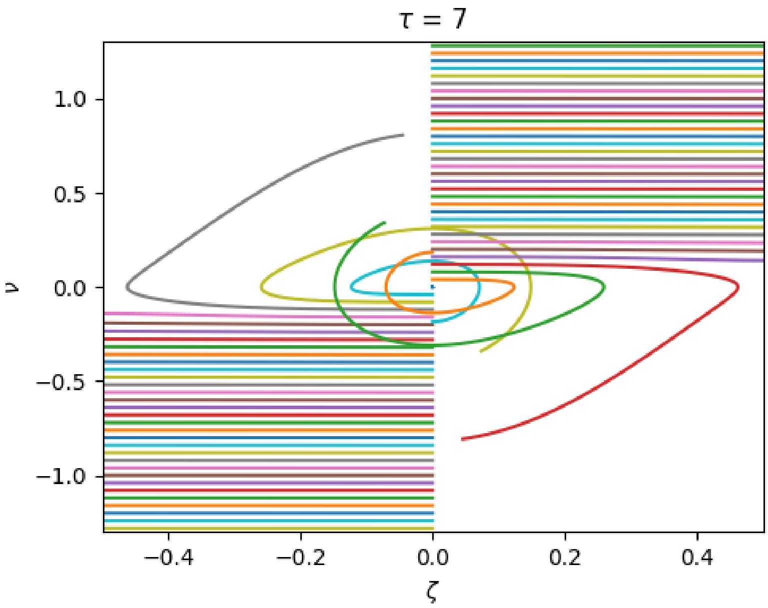
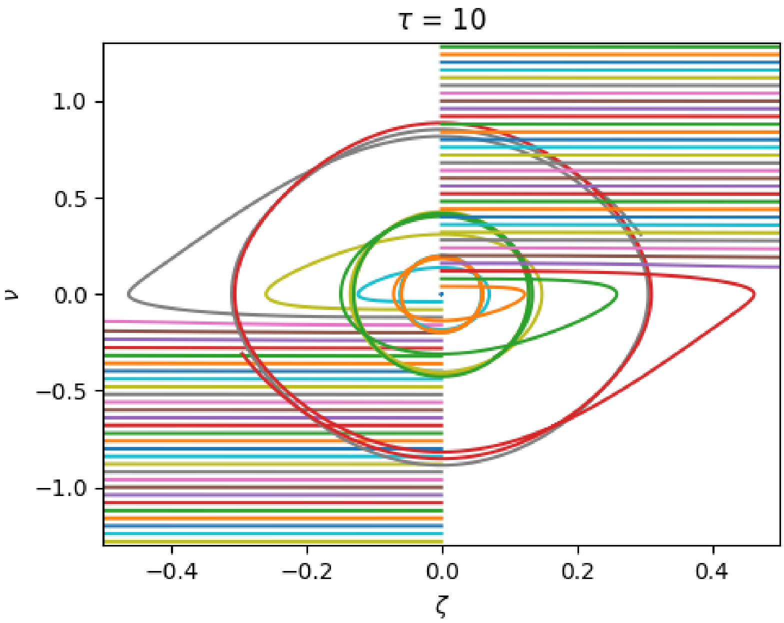


Figure 3.2: Dimensionless amplitude of the BPI electric field, as a function of τ. After an initial linear phase, the wave saturates. The oscillations that we see during the nonlinear phase have the same frequency of the bounce of the trapped particles, as predicted by Levin. The orange line is obtained with the predicted dimensionless growth rate.

It is interesting to study the dynamics of the electron beam during the linear and nonlinear phase. This is shown in Fig. 3.1. All particles follow a straight line in the very beginning of the simulation, when the BPI field is negligible. After a while, some particles remain passing, and some become trapped. It is clear the distinction between trapped particles and passing particles: the particles near the resonance velocity are trapped, and in fact their velocity oscillates around zero in ζ. We can also see as the particles in green after an initial phase where they are passing, as the amplitude grows become trapped. On the contrary in violet we see particles that during the saturation phase become passing particle. Then for those whose initial velocity is greater than 1 in module we see that for the entire simulation they aren't trapped.
The dynamics of the particles in phase space can also be visualized in Fig. **??**, 3.4. Note that, for larger and larger times, more and more passing particles become trapped, forming closed trajectories around the origin of the phase space. The frequency of this bounce motion is identified as the same frequency of the nonlinear oscillations of the BPI amplitude of Fig. 3.2. This result serves as a basis for the study of the next sections, where a beam of energetic ions nonlinearly interacts with the electric field of an EGAM instability in magnetized tokamak plasmas.


τ = 7
1.0
0.5
0.0
−0.5
−1.0
ν
−0.4
−0.2
0.0
0.2
0.4
ζ


τ = 10
1.0
0.5
0.0
−0.5
−1.0
ν
−0.4
−0.2
0.0
0.2
0.4
ζ

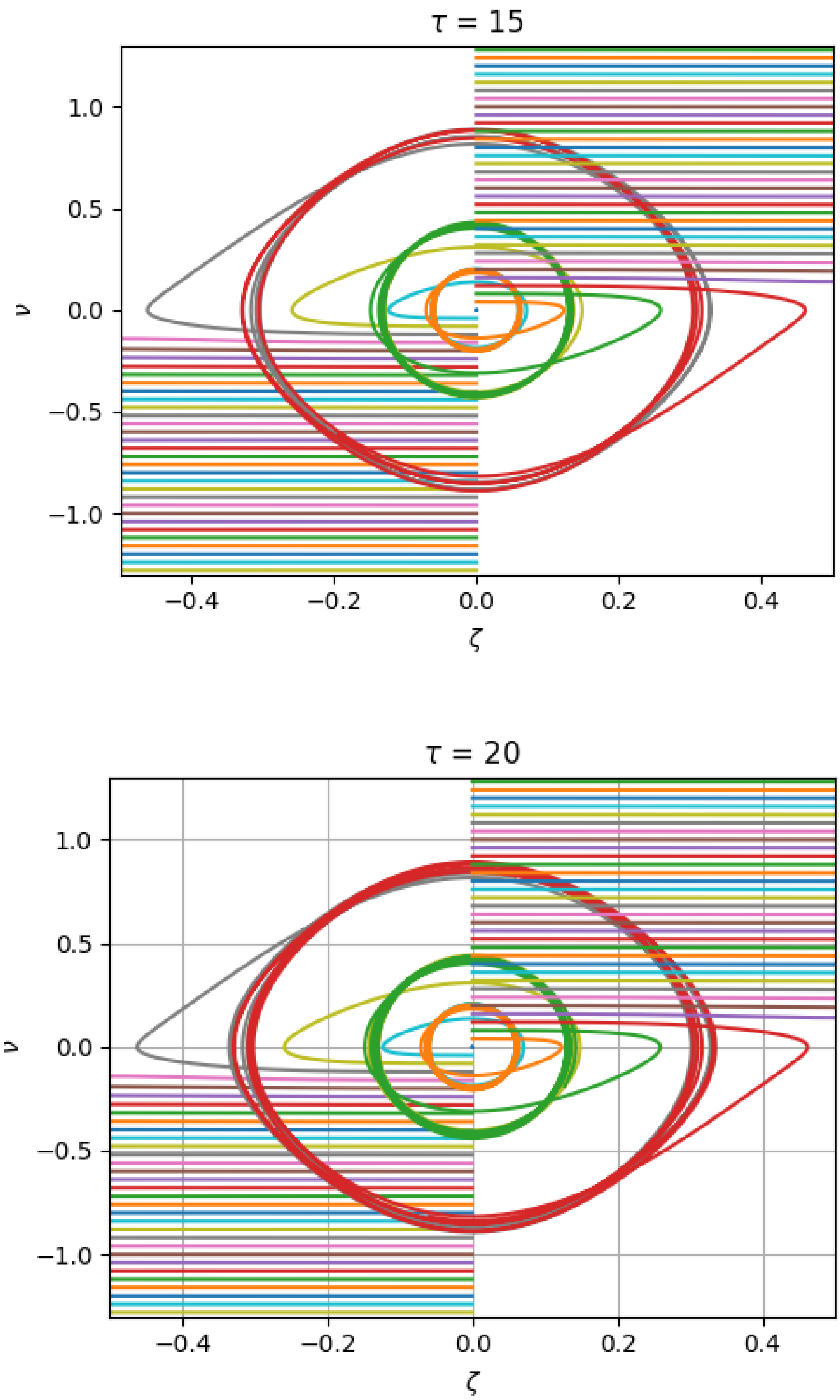


Figure 3.4: Formation of the island of trapped particles

Chapter 4

Two mathematical models for the Beam Plasma Instability

This chapter is devoted to the analysis of two closely related models describing the nonlinear interaction between a population of resonant electrons and a monochromatic plasma wave.

We first consider a one-dimensional model inspired from the article by Levin et al. in [1], formulated in the reference frame moving with the phase velocity of the wave. In this model, the electron distribution evolves according to a Vlasov equation, while the scalar wave amplitude is coupled to the current generated by the resonant-particle population.

We then introduce a multidimensional Eulerian extension of the model proposed in [1], in which the particle distribution is defined on $\mathbb{T}^d \times \mathbb{R}^d$ and the wave propagates along the direction of a prescribed wave vector $k \in \mathbb{Z}^d \setminus \{0\}$.

Although presented separately, the two models share the same underlying wave–particle coupling mechanism. More precisely, in the one-dimensional setting, the multidimensional system reduces to the coupling structure of the model of Levin et al. [1], when the physical parameters are suitably identified and the distribution function is interpreted as describing the initially selected resonant-particle population.

4.1 The mono-dimensional case

Let us consider the $d = 1$ case that is inspired by the BPI model shown by Levin et al.. Following [1], we can simplify the system by integrating just on an interval of the phase velocity of the wave. This was justified by the author using a previous result of O'Niel of 1965, see [35], stating that the only particle interacting with the wave were those initialized near v_{ph}.
Throughout this section, we assume that $k\lambda \in 2\pi\mathbb{Z}$, so that the function $\xi \mapsto \sin(k\xi)$ is well-defined on the torus $\mathbb{T}_\xi := \mathbb{R}/\lambda\mathbb{Z}$.

For a fixed $k > 0$, we consider

$$\begin{cases} \dfrac{\partial f}{\partial t} + v' \dfrac{\partial f}{\partial \xi} - \dfrac{e}{m} E(t) \sin(k\xi) \dfrac{\partial f}{\partial v'} = 0 \\ \dfrac{dE}{dt} = -e \dfrac{4\pi}{\lambda} \displaystyle\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} \int_{-v_0^m}^{v_0^m} \sin(k\xi)(v' + \frac{\omega}{k}) f(t, \xi, v') d\xi dv' \\ f(0, \xi, v') = f_0(\xi, v') \\ E(0) = E_0. \end{cases} \tag{4.1.1}$$

In this system, v' denotes the velocity relative to the wave, whereas $v' + \omega/k$ is the corresponding velocity in the laboratory frame. The constant $v_0^m > 0$ is a fixed cutoff in the current relative velocity.

Definition 4.1.1. A couple (f, E) is said to be a *weak solution* of system 4.1.1 if:

1. $f \in C([0, T]; L^2(\mathbb{T}_\xi \times \mathbb{R})) \cap L^\infty(0, T; H^1(\mathbb{T}_\xi \times \mathbb{R}))$;

2. f solves the Vlasov equation on its weak form:

$$\begin{aligned} &\int_0^T \int_{\mathbb{T}_\xi} \int_{\mathbb{R}_{v'}} f(t, \xi, v')\, \partial_t \phi \, dt\, d\xi\, dv' \\ &+ \int_0^T \int_{\mathbb{T}_\xi} \int_{\mathbb{R}_{v'}} f(t, \xi, v')\, v'\, \partial_\xi \phi \, dt\, d\xi\, dv' \\ &- \frac{e}{m} \int_0^T \int_{\mathbb{T}_\xi} \int_{\mathbb{R}_{v'}} E(t) \sin(k\xi)\, f(t, \xi, v')\, \partial_{v'} \phi \, dt\, d\xi\, dv' \\ &+ \int_{\mathbb{T}_\xi} \int_{\mathbb{R}_{v'}} f_0(\xi, v')\, \phi(0, \xi, v')\, d\xi\, dv' = 0. \end{aligned} \tag{4.1.2}$$

   $\forall\, \phi \in C_c^\infty([-T, T] \times \mathbb{T}_\xi \times \mathbb{R}_{v'})$.

3. $E \in W^{1,\infty}(0, T)$

4. E solves the amplitude equation in its integral form:

$$E(t) = E_0 - e \frac{4\pi}{\lambda} \int_0^t \int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} \int_{-v_0^m}^{v_0^m} \sin(k\xi)(v' + \frac{\omega}{k}) f(s, \xi, v')\, ds\, d\xi\, dv' \quad \forall t \in [0, T]. \tag{4.1.3}$$

The H^1 regularity in the preceding definition is used to justify the stability estimate required in the uniqueness argument. Uniqueness in the larger class $L^\infty(0, T; L^2)$ alone would require an additional renormalization argument for the transport equation.

To show existence and uniqueness of the solution in its weak form we need to show some preliminary results.

4.1.1 The sequence of approximated solutions

Definition 4.1.2 (Sequence of approximated solutions)**.** Let us consider $E^0 \in \mathbb{R}$ and $f^0 : \mathbb{T}_\xi \times \mathbb{R}_{v'} \to \mathbb{R}$.

Then the system of approximated solution of system (4.1.1) is a couple $(f_n, E_n)_{n\in\mathbb{N}}$ such that:

$$\begin{cases} E_0 = E^0, \\ f_0(\xi, v') = f^0(\xi, v') \quad \forall(\xi, v') \in \mathbb{T}_\xi \times \mathbb{R}_{v'}. \end{cases}$$

And, $\forall n \geq 1$,

$$E_n(t) = E^0 - -e\frac{4\pi}{\lambda}\int_0^t \int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} \int_{-v_0^m}^{v_0^m} \sin(k\xi)(v' + \frac{\omega}{k})f_{n-1}(s,\xi,v')dv'd\xi ds. \quad (4.1.4)$$

Then f_n is the solution of the Vlasov system:

$$\begin{cases} \dfrac{\partial f_n}{\partial t} + v'\dfrac{\partial f_n}{\partial \xi} - \dfrac{e}{m}E_n(t)\sin(k\xi)\dfrac{\partial f_n}{\partial v'} = 0, \\ f_n(0, \xi, v') = f_0(\xi, v'). \end{cases} \quad (4.1.5)$$

We now prove some proprieties that will be useful for the proof of the main result of this section.

Lemma 4.1.3. *Let $f_0 \in H^1(\mathbb{T}_\xi \times \mathbb{R}_{v'})$, $E \in C([0,T])$.*

Then there exists a weak solution

$$f \in C([0,T]; H^1(\mathbb{T}_\xi \times \mathbb{R}_{v'}))$$

of the Vlasov system:

$$\begin{cases} \dfrac{\partial f}{\partial t} + v'\dfrac{\partial f}{\partial \xi} - \dfrac{e}{m}E(t)\sin(k\xi)\dfrac{\partial f}{\partial v'} = 0, \\ f(0, \xi, v') = f_0(\xi, v'), \end{cases} \quad (4.1.6)$$

such that

$$\|f(t,\cdot)\|_{L^2(\mathbb{T}_\xi\times\mathbb{R}_{v'})} = \|f_0\|_{L^2(\mathbb{T}_\xi\times\mathbb{R}_{v'})} \qquad \forall t \in [0,T].$$

Moreover, there exists a constant $C > 0$, depending only on e/m and k, such that

$$\|f(t)\|_{H^1} \leq \|f_0\|_{H^1} \exp\left(C\int_0^t (1 + |E(s)|)ds\right), \quad (4.1.7)$$

for all $t \in [0,T]$.

Proof. We prove this statement following the characteristics that for this system are defined as follows. $\forall s, t \in [0,T], (\xi, v) \in \mathbb{T}_\xi \times \mathbb{R}_{v'}$ the characteristic flux

$X(s,t,(\xi,v)) = (\xi(t),v(t))$ is defined as the solution of:

$$\begin{cases} \dot{\xi} = v(t), \\ \dot{v} = -\frac{e}{m}E(t)\sin(k\xi(t)), \end{cases} \tag{4.1.8}$$

with $X(t,t,(\xi,v)) = (\xi,v)$. The associated vector field

$$b_E = (v(t), -\frac{e}{m}E(t)\sin(k\xi(t))$$

is continuous for hypothesis and globally Lipschitz with respect to (ξ,v), and $\mathrm{div}_{(\xi,v)}(b_E) = 0$. Then $X \in C^1(0,T)$ and, using Liouville's theorem:

$$\frac{d}{dt}(\det(J)) = (div_{(\xi,v)}(b))\det J = 0, \tag{4.1.9}$$

where $J = D_{(\xi,v)}X$. Using that $X(t,t,(\xi,v)) = (\xi,v)$ we have that:

$$|\det J| = 1. \tag{4.1.10}$$

We now define:

$$f(t,\xi,v) = f_0(X(0,t,(\xi,v))). \tag{4.1.11}$$

Using (4.1.10) we deduce that

$$||f(t,\cdot)||_{L^2(\mathbb{T}_\xi\times\mathbb{R}_{v'})} = ||f_0||_{L^2(\mathbb{T}_\xi\times\mathbb{R}_{v'})} \quad \forall t \in [0,T].$$

It remains to derive the H^1 estimate. Differentiating (4.1.6) with respect to ξ gives

$$\partial_t(\partial_\xi f) + v'\partial_\xi(\partial_\xi f) - \frac{e}{m}E(t)\sin(k\xi)\partial_{v'}(\partial_\xi f) = \frac{ek}{m}E(t)\cos(k\xi)\partial_{v'} f.$$

Differentiating with respect to v' gives

$$\partial_t(\partial_{v'} f) + v'\partial_\xi(\partial_{v'} f) - \frac{e}{m}E(t)\sin(k\xi)\partial_{v'}(\partial_{v'} f) = -\partial_\xi f.$$

Multiplying the first equation by $\partial_\xi f$, the second one by $\partial_{v'} f$, integrating over $\mathbb{T}_\xi \times \mathbb{R}$, and using the divergence-free property of the field b_E, we find

$$\frac{d}{dt}\left(\|\partial_\xi f(t)\|_2^2 + \|\partial_{v'} f(t)\|_2^2\right) \le C(1+|E(t)|)\left(\|\partial_\xi f(t)\|_2^2 + \|\partial_{v'} f(t)\|_2^2\right).$$

The conclusion follows from Grönwall's lemma and the bounds for the L^2-norm. □

Lemma 4.1.4 (Well-posedness of the approximated sequence)**.** *Let* $E^0 \in \mathbb{R}$ *and* $f^0 \in H^1(\mathbb{T}_\xi \times \mathbb{R})$*. Then the approximated sequence* $(f_n, E_n)_{n\in\mathbb{N}}$ *defined in* (4.1.2) *is well defined* $\forall n \in \mathbb{N}$*. We also have that:*

$$E_n \in W^{1,\infty}(0,T), \quad f_n(\cdot,t) \in H^1(\mathbb{T}_\xi \times \mathbb{R}),$$

and

$$||f_n(\cdot,t)||_{L^2(\mathbb{T}_\xi\times\mathbb{R})} = ||f^0||_{L^2(\mathbb{T}_\xi\times\mathbb{R})} \quad \forall t\in[0,T].$$

For every $T>0$, there exists a constant $M_T>0$, independent of n, such that

$$\sup_{n\in\mathbb{N}} \|f_n\|_{L^\infty(0,T;H^1)} \leq M_T. \tag{4.1.12}$$

Moreover, there exists $T^*>0$ such that $(f_n)_n$ is a Cauchy sequence in $C([0,T^*];L^2(\mathbb{T}_\xi\times\mathbb{R}))$.

Proof. We prove this statement by induction on n.
Base case.
The $n=0$ case follows directly from the hypothesis.
Inductive step.
Assume that at step n, the statement is valid, we will now show that it is valid for $n+1$.

Firs of all, we start by showing that E_{n+1} is well defined and continuous. To simplify the notation we denote $\Omega=(-\frac{\lambda}{2},\frac{\lambda}{2})\times(-v_o^m,v_0^m)$ and we define

$$g(\xi,v') = \sin(k\xi)(v'+\frac{\omega}{k})$$

. Then we can rewrite E_{n+1} as follows:

$$E_{n+1}(t) = E^0 - \frac{e4\pi}{\lambda}\int_0^t\int_\Omega g(\xi,v')f_n(s,\xi,v')\,ds\,d\xi\,dv'.$$

Since Ω is bounded and $f_n(t)\in L^2(\mathbb{T}_\xi\times\mathbb{R})$, the amplitude E is well defined. To show the continuity we consider $t_1,t_2\in(0,T)$. Then we have:

$$|E_{n+1}(t_1)-E_{n+1}(t_2)|\leq \frac{4\pi e}{\lambda}\left|\int_{t_1}^{t_2}\int_\Omega g(\xi,v')f_n(s,\xi,v')\,ds\,d\xi\,dv'\right|.$$

Using the Cauchy-Schwartz inequality we get:

$$|E_{n+1}(t_1)-E_{n+1}(t_2)| \leq \frac{4\pi e}{\lambda}\int_{t_1}^{t_2} ||g||_{L^2(\Omega)}||f_n(s,\cdot)||_{L^2(\Omega)}ds$$

$$\leq \frac{4\pi e}{\lambda}||g||_{L^2(\Omega)}||f^0||_{L^2(\Omega)}|t_2-t_1|,$$

where we use the inductive hypotheses on f_n. We showed that E_{n+1} is Lipschitz, therefore $E_{n+1}\in W^{1,\infty}(0,T)\subset C[0,T]$.

Furthermore

$$\|E_{n+1}\|_{L^\infty(0,T)} \leq |E^0| + \frac{4\pi e}{\lambda}T\|g\|_{L^2(\Omega)}\|f^0\|_{L^2}. \tag{4.1.13}$$

The estimate (4.1.7), together with (4.1.13), also gives a constant $M_T>0$, independent of n, such that

$$\|f_n\|_{L^\infty(0,T;H^1)} \leq M_T.$$

The existence of f_{n+1}, and its properties follow from Lemma 4.1.3.
To conclude we need to show that :

$$||f_n(t,\cdot) - f_{n-1}(t,\cdot)||_{L^2} \leq C||f_{n-1} - f_{n-2}(t,\cdot)||_{L^2}.$$

Since both f_n and f_{n-1} are solutions to the Vlasov equation, their difference solve:

$$\frac{\partial}{\partial t}(f_n - f_{n-1}) + v\frac{\partial}{\partial \xi}(f_n - f_{n-1}) - \frac{e}{m}\sin(k\xi)[E_n\frac{\partial}{\partial v}f_n - E_{n-1}\frac{\partial}{\partial v}f_{n-1}] = 0 \tag{4.1.14}$$

If we multiply by $f_n - f_{n-1}$ and we integrate in space and velocity we get:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|f_n - f_{n-1}\|^2_{L^2} &= \frac{e}{m}\int \sin(k\xi)\, E_n\, \frac{\partial}{\partial v}\left(\frac{(f_n - f_{n-1})^2}{2}\right) \\ &\quad + \frac{e}{m}\int \sin(k\xi)\, \frac{\partial f_{n-1}}{\partial v}(E_n - E_{n-1}) \\ &= \frac{e}{m}\int \sin(k\xi)\, \frac{\partial f_{n-1}}{\partial v}(E_n - E_{n-1}).\end{aligned}$$

To do an estimate on $E_n - E_{n-1}$, we use def. 4.1.2:

$$|E_n - E_{n-1}| = |\int_0^t \int_\Omega g(\xi, v)(f_{n-1} - f_{n-2})| \leq cT||f_{n-1} - f_{n-2}||^2_{L^2}.$$

Substituting and again using Cauchy Schwartz we get:

$$\frac{d}{dt}\|f_n - f_{n-1}\|^2_{L^2} \leq \tilde{C}T^2||f_{n-1} - f_{n-2}||^2_{L^2},$$

and our thesis follows choosing T^* sufficiently small.

Therefore, $(f_n)_n$ is a Cauchy sequence in $C([0,T^*]; L^2)$.

□

4.1.2 Well Posedness of the 1-dimensional BPI system

We are now ready to give the statement of the theorem of existence and uniqueness of the solution of system 4.1.1.

Theorem 4.1.5. *Let $f_0 : \mathbb{T}_\xi \times \mathbb{R}_{v'} \to \mathbb{R}+$ such that $f_0 \in H^1(\mathbb{T}_\xi \times \mathbb{R}_{v'})$, let $E_0 \in \mathbb{R}$. Then, $\forall\, T < \infty$, there exists a unique weak solution (f, E), with*

$$f \in C([0,T]; L^2(\mathbb{T}_\xi \times \mathbb{R}_{v'})) \cap L^\infty(0,T; H^1(\mathbb{T}_\xi \times \mathbb{R}_{v'}))$$

and

$$E \in W^{1,\infty}(0,T)$$

of the system:

$$\begin{cases} \dfrac{\partial f}{\partial t} + v' \cdot \dfrac{\partial f}{\partial \xi} - \dfrac{e}{m} E(t) \sin(k\xi) \dfrac{\partial f}{\partial v'} = 0 \\ \dfrac{dE}{dt} = -e \dfrac{4\pi}{\lambda} \displaystyle\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} \int_{-v_0^m}^{v_0^m} \sin(k\xi)(v' + \frac{\omega}{k}) f(t, \xi, v') d\xi dv' \\ f(0, \xi, v') = f_0(\xi, v') \\ E(0) = E_0 \end{cases}$$

Proof. Let us start by showing the existence of this solution on a sufficiently small interval $[0, T^*]$.

Consider a sequence $(f_n, E_n)_{n\in\mathbb{N}}$ of approximated solutions as defined in 4.1.2. Using Lemma 4.1.4, we have that $\forall n \in \mathbb{N}$, the couple is well defined and there exists $T^* \in [0, T]$ such that $(f_n)_n$ is a a Cauchy sequence in $C([0, T^*]; L^2(\mathbb{T}_\xi \times \mathbb{R}_{v'}))$.

Therefore, there exists

$$f \in C([0, T^*]; L^2(\mathbb{T}_\xi \times \mathbb{R}_{v'}))$$

such that

$$f_n \longrightarrow f \qquad \text{strongly in } C([0, T^*]; L^2). \tag{4.1.15}$$

We now prove the strong convergence of E_n. For every n, m,

$$\begin{aligned} |E_n(t) - E_m(t)| &\leq \frac{4\pi e}{\lambda} \int_0^t \left| \int_\Omega g(\xi, v')(f_{n-1} - f_{m-1})(s, \xi, v') d\xi \, dv' \right| ds \\ &\leq \frac{4\pi e}{\lambda} \|g\|_{L^2(\Omega)} \int_0^t \|f_{n-1}(s) - f_{m-1}(s)\|_{L^2} ds. \end{aligned}$$

Thus,

$$\|E_n - E_m\|_{L^\infty(0,T^*)} \leq \frac{4\pi e}{\lambda} T^* \|g\|_{L^2(\Omega)} \|f_{n-1} - f_{m-1}\|_{C([0,T^*];L^2)}.$$

It follows that $(E_n)_n$ is a Cauchy sequence in $C([0, T^*])$. Therefore, there exists $E \in C([0, T^*])$ such that

$$E_n \longrightarrow E \qquad \text{strongly in } C([0, T^*]). \tag{4.1.16}$$

Our goal is now to prove that f is the weak solution of our problem in the sense that we defined earlier. Let $\phi \in C_c^\infty([-T^*, T^*] \times \mathbb{T}_\xi \times \mathbb{R}_{v'}))$, then by definition of weak limit we have that:

1.
$$\int_0^{T^*} \int_{\mathbb{T}_\xi} \int_{\mathbb{R}_v'} f_n \partial_t \phi dt \, d\xi \, dv' \to \int_0^{T^*} \int_{\mathbb{T}_\xi} \int_{\mathbb{R}_v'} f \partial_t \phi dt \, d\xi \, dv'$$

2.
$$\int_0^{T^*} \int_{\mathbb{T}_\xi} \int_{\mathbb{R}_v'} f_n v \cdot \partial_\xi \phi dt \, d\xi \, dv' \to \int_0^{T^*} \int_{\mathbb{T}_\xi} \int_{\mathbb{R}_v'} f v \cdot \partial_\xi \phi dt \, d\xi \, dv'$$

For the term containing E_n, we write

$$E_n f_n - Ef = (E_n - E)f_n + E(f_n - f).$$

Therefore,

$$\begin{aligned}\|E_n f_n - Ef\|_{L^1(0,T^*;L^2)} \le T^*\|E_n - E\|_{L^\infty(0,T^*)}\|f_n\|_{L^\infty(0,T^*;L^2)} \\ + T^*\|E\|_{L^\infty(0,T^*)}\|f_n - f\|_{L^\infty(0,T^*;L^2)} \to 0.\end{aligned}$$

Hence f satisfies the Vlasov equation in its weak form.

To conclude the proof of existence, we need to verify that $E \in W^{1,\infty}(0,T^*)$.

Following Lemma 4.1.4 $(E_n)_{n\in\mathbb{N}} \in C([0,T^*])$; since it is continuous in a compact set it is also limited, therefore $E \in L^\infty(0,T^*)$, being the strong limit of a sequence in $L^\infty(0,T)$.
The weak derivative of E is:

$$E'(t) = \int_\Omega g(\xi, v')f(t,\xi,v')\,d\xi\,dv'.$$

Using the Cauchy-Schwartz inequality we get:

$$|E'(t)| \le ||g||_{L^2(\Omega)}||f(t,\cdot)||_{L^2(\Omega)}.$$

We hence deduce that $E \in W^{1,\infty}(0,T^*)$.

The previous uniform estimates and weak lower semicontinuity imply that $f \in L^\infty(0,T^*;H^1)$. Since f solves the linear Vlasov equation associated with E, the characteristic representation of Lemma 4.1.3 implies that

$$\|f(t)\|_{L^2} = \|f_0\|_{L^2} \qquad \forall t \in [0,T^*].$$

This concludes the proof of existence.

We now prove uniqueness. Suppose that (f_1, E_1) and (f_2, E_2) are two weak solutions in the class stated in the theorem, with the same initial data. Set

$$h := f_1 - f_2, \qquad F := E_1 - E_2.$$

Subtracting the two Vlasov equations, we obtain

$$\partial_t h + v'\partial_\xi h - \frac{e}{m}E_1(t)\sin(k\xi)\partial_{v'}h = \frac{e}{m}F(t)\sin(k\xi)\partial_{v'}f_2, \tag{4.1.17}$$

with $h(0) = 0$.

Since h is not necessarily a smooth test function, the following energy identity is justified by regularizing (4.1.17) in (ξ, v') and then passing to the limit. The H^1 regularity of f_1 and f_2 is sufficient for this standard argument.

Multiplying the regularized equation by the regularized difference and passing to the limit, we obtain

$$\frac{1}{2}\frac{d}{dt}\|h(t)\|_{L^2}^2 = \frac{e}{m}F(t)\int_{\mathbb{T}_\xi\times\mathbb{R}} \sin(k\xi)\frac{\partial f_2}{\partial v'}(t,\xi,v')h(t,\xi,v')d\xi\,dv'.$$

Thus,
$$\frac{d}{dt}\|h(t)\|_{L^2} \leq \frac{e}{m}|F(t)| \left\|\frac{\partial f_2}{\partial v'}(t)\right\|_{L^2}. \tag{4.1.18}$$
On the other hand, subtracting the two amplitude equations gives
$$F(t) = -\frac{4\pi e}{\lambda}\int_0^t \int_\Omega g(\xi, v')h(s,\xi,v')d\xi\, dv'\, ds.$$
Therefore,
$$|F(t)| \leq \frac{4\pi e}{\lambda}\|g\|_{L^2(\Omega)} \int_0^t \|h(s)\|_{L^2} ds. \tag{4.1.19}$$
Let
$$M = \left\|\frac{\partial f_2}{\partial v'}\right\|_{L^\infty(0,T^*;L^2)}.$$
Combining (4.4.8) and (4.1.19), we obtain
$$\begin{aligned}\|h(t)\|_{L^2} &\leq \frac{e}{m}M\int_0^t |F(s)|\, ds\\ &\leq CM\int_0^t\int_0^s \|h(\tau)\|_{L^2} d\tau\, ds,\end{aligned}$$
where
$$C = \frac{4\pi e^2}{m\lambda}\|g\|_{L^2(\Omega)}.$$
If
$$H(t) = \sup_{0\leq s\leq t}\|h(s)\|_{L^2},$$
then
$$H(t) \leq CMt^2 H(t).$$
For $t > 0$ sufficiently small so that $CMt^2 < 1$, it follows that $H(t) = 0$. Thus
$$f_1 = f_2 \qquad \text{and} \qquad E_1 = E_2$$
on a sufficiently small time interval. Repeating the same argument on successive time intervals proves uniqueness on the whole interval of existence.

Finally, the solution can be extended to every finite time. Indeed,
$$\begin{aligned}|E(t)| &\leq |E_0| + \frac{4\pi e}{\lambda}\int_0^t \left|\int_\Omega g(\xi,v')f(s,\xi,v')d\xi\, dv'\right| ds\\ &\leq |E_0| + \frac{4\pi e}{\lambda}t\|g\|_{L^2(\Omega)}\|f_0\|_{L^2}.\end{aligned}$$
Thus E remains bounded on every finite interval. By Lemma 4.1.3, the H^1 norm also remains finite on every finite interval. Therefore, the local construction can be repeated until the whole interval $[0,T]$ is covered.

□

4.2 The Vlasov-wave amplitude system

We now study the multidimensional Eulerian extension of the model proposed in [1].

We use the convention $\mathbb{T}^d := \mathbb{R}^d/(2\pi\mathbb{Z})^d$. With this convention, if $k \in \mathbb{Z}^d$, the function $x \mapsto \sin(k \cdot x)$ is well-defined on $\mathbb{T}^d$.

We consider an electron distribution function

$$f = f(t, x, v) \geq 0, \qquad (t, x, v) \in \mathbb{R}_+ \times \mathbb{T}^d \times \mathbb{R}^d,$$

where x is the position variable and v the velocity variable in a frame moving with the phase velocity of the wave. The space dimension is $d \in \mathbb{N}^*$.

Let $k \in \mathbb{Z}^d \setminus \{0\}$ be the wave vector and set

$$b := \frac{k}{|k|}.$$

In laboratory variables, denoted here by (X_ℓ, V_ℓ), the electrostatic wave is formally of the form

$$\mathcal{E}_{\text{lab}}(t, X_\ell) = E(t) \sin(k \cdot X_\ell - \omega t)\, b.$$

We introduce the phase velocity

$$v_{\text{ph}} := \frac{\omega}{|k|}$$

and pass to the frame moving with velocity $v_{\text{ph}} b$ by setting

$$x = X_\ell - v_{\text{ph}} t\, b, \qquad v = V_\ell - v_{\text{ph}} b.$$

Hence, $k \cdot X_\ell - \omega t = k \cdot x$, and the field becomes stationary in space up to its scalar amplitude, i.e.

$$\mathcal{E}(t, x) = E(t) \sin(k \cdot x)\, b.$$

The electron charge is $-e$, with $e > 0$, and the electron mass is $m > 0$. Writing

$$\alpha := \frac{e}{m} > 0,$$

the force acting on the electrons in the moving frame is $F(t, x) = -\alpha E(t) \sin(k \cdot x)\, b$. Thus the particle characteristics are

$$\dot{X} = V, \qquad \dot{V} = -\alpha E(t) \sin(k \cdot X)\, b,$$

where (X, V) denotes the variables in the moving frame. Equivalently, the distribution function satisfies

$$\partial_t f + v \cdot \nabla_x f - \alpha E(t) \sin(k \cdot x)\, b \cdot \nabla_v f = 0. \tag{4.2.1}$$

The remaining ingredient is the scalar evolution equation for the amplitude $E(t)$.

Since v is the velocity in the moving frame, the physical velocity in the direction of propagation is

$$V_\ell \cdot b = v \cdot b + v_{\mathrm{ph}}.$$

Following [1], the relevant feedback onto the wave amplitude is the current projected onto the chosen sinusoidal mode; the equation linking the amplitude with the modal current is

$$\frac{dE}{dt}(t) = -\beta \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \sin(k \cdot x) \, (v \cdot b + v_{\mathrm{ph}}) \, f(t, x, v) \, dv \, dx, \tag{4.2.2}$$

where $\beta > 0$ is a positive coupling constant.

Combining (4.2.1) and (4.2.2), the Vlasov–wave amplitude model reads

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \alpha E(t) \sin(k \cdot x) \, b \cdot \nabla_v f = 0, \\ \dfrac{dE}{dt}(t) = -\beta \displaystyle\int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \sin(k \cdot x) \, (v \cdot b + v_{\mathrm{ph}}) \, f(t, x, v) \, dv \, dx, \\ f(0, x, v) = f_0(x, v), \\ E(0) = E_0. \end{cases} \tag{4.2.3}$$

4.3 Weak well posedness in the general case

In this section we prove global weak well-posedness for (4.2.3). The argument is based on the fact that, once the scalar amplitude $E(t)$ is prescribed, the kinetic equation is a linear transport equation with a globally defined characteristic flow. The nonlinear coupling is then reduced to a fixed point problem for E.

We introduce the notation

$$\Psi(x, v) := \sin(k \cdot x) \, (v \cdot b + v_{\mathrm{ph}}) \, . \tag{4.3.1}$$

Throughout the section we assume that

$$f_0 \in L^1(\mathbb{T}^d \times \mathbb{R}^d) \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d), \qquad f_0 \geq 0, \tag{4.3.2}$$

and that f_0 has compact velocity support. More precisely, there exists $R_0 > 0$ such that

$$f_0(x, v) = 0 \quad \text{for a.e. } (x, v) \in \mathbb{T}^d \times \mathbb{R}^d \text{ with } |v| > R_0. \tag{4.3.3}$$

Equivalently,

$$\operatorname{ess\,supp} f_0 \subset \mathbb{T}^d \times B_{R_0}.$$

Here and below, B_R denotes the Euclidean ball of $\mathbb{R}^d$ centered at the origin with radius R.

The total mass is denoted by

$$M_0 := \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_0(x, v) \, dv \, dx.$$

4.3.1 Definition of weak solution

The following definition formalizes the concept of weak solution.

Definition 4.3.1. Let $T > 0$. A pair (f, E) is called a weak solution of (4.2.3) on $[0, T]$ if the following conditions hold:

- $f \geq 0$ and $f \in L^\infty(0, T; L^1(\mathbb{T}^d \times \mathbb{R}^d) \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d))$;
- f admits a strongly continuous representative in time, namely
$$f \in C([0, T]; L^1(\mathbb{T}^d \times \mathbb{R}^d));$$
- f has essentially bounded velocity support on $[0, T]$, i.e., there exists $R_T > 0$ such that, for every $t \in [0, T]$ after choosing the L^1-continuous representative,
$$\operatorname{ess\,supp} f(t) \subset \mathbb{T}^d \times B_{R_T};$$
- $E \in W^{1,\infty}(0, T)$;
- for every smooth test function φ, periodic in x, compactly supported in v, and satisfying $\varphi(T, \cdot, \cdot) = 0$, one has
$$\int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f(t, x, v) \left(\partial_t \varphi + v \cdot \nabla_x \varphi - \alpha E(t) \sin(k \cdot x) b \cdot \nabla_v \varphi\right) dv\, dx\, dt + \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f_0(x, v) \varphi(0, x, v)\, dv\, dx = 0; \tag{4.3.4}$$
- the amplitude equation holds in integral form:
$$E(t) = E_0 - \beta \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \Psi(x, v) f(s, x, v)\, dv\, dx\, ds \tag{4.3.5}$$
for every $t \in [0, T]$.

4.3.2 The linear transport problem

Let $T > 0$ and let $E \in C([0, T])$ be prescribed. Consider
$$\partial_t f + v \cdot \nabla_x f - \alpha E(t) \sin(k \cdot x) b \cdot \nabla_v f = 0. \tag{4.3.6}$$

The associated characteristic system is
$$\begin{cases} \dot{X}(t) = V(t), \\ \dot{V}(t) = -\alpha E(t) \sin(k \cdot X(t))\, b. \end{cases} \tag{4.3.7}$$

For $0 \leq s, t \leq T$ and $z = (x, v) \in \mathbb{T}^d \times \mathbb{R}^d$, we denote by
$$\Phi_E(t, s, z) := (X_E(t; s, z), V_E(t; s, z))$$

the value at time t of the unique solution of (4.3.7) satisfying

$$X_E(s;s,z) = x, \qquad V_E(s;s,z) = v.$$

In particular, we write

$$\Phi_E(t,z) := \Phi_E(t,0,z).$$

The corresponding phase-space vector field is

$$B_E(t,x,v) := (v, -\alpha E(t)\sin(k\cdot x)b)\,.$$

For every solution of (4.3.7), we have

$$\frac{d}{dt}|V_E(t;s,z)| \le \alpha|E(t)|.$$

Hence

$$|V_E(t;s,z)| \le |v| + \alpha\int_s^t |E(\tau)|\,d\tau,$$

with the usual interpretation after exchanging s and t if $t < s$. This estimate excludes finite-time blow-up in velocity. Consequently, the characteristic flow is globally defined on $[0,T]$.

Moreover, since B_E is continuous in t and C^1 in (x,v), standard ODE theory gives that the map

$$\Phi_E(t,s,\cdot) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d \times \mathbb{R}^d$$

is a C^1 diffeomorphism for every $s,t \in [0,T]$, and the flow property holds:

$$\Phi_E(t,r,\Phi_E(r,s,z)) = \Phi_E(t,s,z)$$

for all $0 \le s,r,t \le T$ and all $z \in \mathbb{T}^d \times \mathbb{R}^d$. In particular,

$$\Phi_E(s,t,\cdot) = \Phi_E(t,s,\cdot)^{-1}.$$

The vector field is divergence-free:

$$\operatorname{div}_{x,v} B_E(t,x,v) = 0.$$

Indeed,

$$\nabla_x \cdot v = 0, \qquad \nabla_v \cdot (-\alpha E(t)\sin(k\cdot x)b) = 0.$$

Consequently, by Liouville's formula,

$$\frac{d}{dt}\det D_z\Phi_E(t,s,z) = (\operatorname{div}_{x,v} B_E)\,(t,\Phi_E(t,s,z))\det D_z\Phi_E(t,s,z) = 0,$$

where $D_z\Phi_E(t,s,z)$ is the Jacobian matrix of the flow with respect to the initial phase-space variable $z = (x,v)$, namely

$$D_z\Phi_E(t,s,z) = \begin{pmatrix} D_xX_E(t;s,x,v) & D_vX_E(t;s,x,v) \\ D_xV_E(t;s,x,v) & D_vV_E(t;s,x,v) \end{pmatrix}, \qquad z = (x,v).$$

Since
$$\det D_z\Phi_E(s,s,z) = 1,$$
we obtain
$$\det D_z\Phi_E(t,s,z) = 1$$
for all $0 \le s,t \le T$ and all $z \in \mathbb{T}^d \times \mathbb{R}^d$. Thus $\Phi_E(t,s,\cdot)$ preserves the Lebesgue measure on $\mathbb{T}^d \times \mathbb{R}^d$.

We define the transported density f_E as the push-forward of the initial measure $f_0(z)\,dz$ under the characteristic flow:
$$f_E(t,\cdot)\,dz := \Phi_E(t,0,\cdot)_{\#}\left(f_0(z)\,dz\right).$$
Equivalently, for every bounded Borel function χ,
$$\int_{\mathbb{T}^d\times\mathbb{R}^d} \chi(z)\, f_E(t,z)\,dz = \int_{\mathbb{T}^d\times\mathbb{R}^d} \chi(\Phi_E(t,0,z))\, f_0(z)\,dz.$$
Since $\Phi_E(t,0,\cdot)$ is a measure-preserving C^1 diffeomorphism, this is equivalent to
$$f_E(t,z) = f_0(\Phi_E(0,t,z)). \tag{4.3.8}$$

The following lemma shows that f_E is the unique weak solution of the linear transport equation associated with the field B_E.

Lemma 4.3.2. *Let $E \in C([0,T])$ and assume* (4.3.2)–(4.3.3). *Then the function f_E defined by* (4.3.8) *is the unique weak solution of* (4.3.6) *in the class*
$$L^\infty(0,T;L^1\cap L^\infty(\mathbb{T}^d\times\mathbb{R}^d)) \cap C([0,T];L^1(\mathbb{T}^d\times\mathbb{R}^d))$$
with essentially bounded velocity support. Moreover, $f_E \ge 0$ and, for every $1 \le p \le \infty$,
$$\|f_E(t)\|_{L^p(\mathbb{T}^d\times\mathbb{R}^d)} = \|f_0\|_{L^p(\mathbb{T}^d\times\mathbb{R}^d)}$$
for every $t \in [0,T]$. Finally, if $\|E\|_{L^\infty(0,T)} \le R$, then
$$\operatorname{ess\,supp} f_E(t) \subset \mathbb{T}^d \times B_{R_0+\alpha RT} \tag{4.3.9}$$
for every $t \in [0,T]$.

Proof. We first assume that $f_0 \in C_c^\infty(\mathbb{T}^d \times \mathbb{R}^d)$. Define
$$f_E(t,z) := f_0(\Phi_E(0,t,z)),$$
i.e.,
$$f_E(t,\Phi_E(t,0,z)) = f_0(z).$$
Since the flow is C^1, differentiating along characteristics gives
$$\partial_t f_E + v\cdot\nabla_x f_E - \alpha E(t)\sin(k\cdot x) b\cdot\nabla_v f_E = 0,$$
which is exactly (4.3.6). Thus f_E is a classical solution in the smooth case.

Moreover, since the flow depends continuously on time, one has

$$f_E \in C([0,T]; L^1)$$

.

We now consider a general datum satisfying (4.3.2)–(4.3.3). Choose a sequence $f_0^n \in C_c^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ such that $f_0^n \geq 0$,

$$f_0^n \to f_0 \quad \text{strongly in } L^1(\mathbb{T}^d \times \mathbb{R}^d),$$

$$\sup_{n\in\mathbb{N}} \|f_0^n\|_{L^\infty} < +\infty, \qquad \operatorname{ess\,supp} f_0^n \subset \mathbb{T}^d \times B_{R_0+1}.$$

For each n, define

$$f_E^n(t,z) = f_0^n(\Phi_E(0,t,z)).$$

By the smooth case, each f_E^n is a classical solution of (4.3.6) and belongs to $C([0,T]; L^1(\mathbb{T}^d \times \mathbb{R}^d))$.

Since $\Phi_E(t,0,\cdot)$ preserves the Lebesgue measure, we have

$$\|f_E^n(t) - f_E^m(t)\|_{L^1(\mathbb{T}^d\times\mathbb{R}^d)} = \|f_0^n - f_0^m\|_{L^1(\mathbb{T}^d\times\mathbb{R}^d)}$$

for every $t \in [0,T]$. Hence $(f_E^n)_n$ is a Cauchy sequence in $C([0,T]; L^1(\mathbb{T}^d \times \mathbb{R}^d))$ and converges to a limit, still denoted by f_E. In particular,

$$f_E \in C([0,T]; L^1(\mathbb{T}^d \times \mathbb{R}^d)).$$

Moreover,

$$\|f_E^n(t) - f_E(t)\|_{L^1(\mathbb{T}^d\times\mathbb{R}^d)} = \|f_0^n - f_0\|_{L^1(\mathbb{T}^d\times\mathbb{R}^d)}$$

for every $t \in [0,T]$, so that

$$f_E^n \to f_E \quad \text{strongly in } L^\infty(0,T; L^1(\mathbb{T}^d \times \mathbb{R}^d)).$$

Passing to the limit in the weak formulation satisfied by f_E^n yields that f_E is a weak solution of (4.3.6).

The representation formula and the measure-preserving property imply positivity. They also give, for every $1 \leq p < \infty$,

$$\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} |f_E(t,x,v)|^p \, dv \, dx = \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} |f_0(x,v)|^p \, dv \, dx.$$

For $p = \infty$, the identity follows directly from $f_E(t,z) = f_0(\Phi_E(0,t,z))$ and the bijectivity of the flow.

The velocity support estimate follows from the characteristic bound

$$|V_E(t,0,x,v)| \leq |v| + \alpha \int_0^t |E(s)| \, ds.$$

If $|v| \leq R_0$ and $\|E\|_{L^\infty(0,T)} \leq R$, then

$$|V_E(t,0,x,v)| \leq R_0 + \alpha RT,$$

which proves (4.3.9). Therefore f_E has essentially bounded velocity support on $[0,T]$.

It remains to justify uniqueness. Let f_E^* be another weak solution of (4.3.6) with the same initial datum, strong time continuity in L^1, and essentially bounded velocity support on $[0,T]$. We set

$$h := f_E^* - f_E.$$

Then h satisfies the homogeneous weak transport equation with zero initial datum, namely

$$\int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} h(t,x,v) \left(\partial_t \varphi + v \cdot \nabla_x \varphi - \alpha E(t) \sin(k \cdot x) b \cdot \nabla_v \varphi\right) \, dv \, dx \, dt = 0$$

for every admissible smooth test function φ.

For brevity, set

$$\mathcal{L}_E \varphi := \partial_t \varphi + v \cdot \nabla_x \varphi - \alpha E(t) \sin(k \cdot x) b \cdot \nabla_v \varphi.$$

Let now $\zeta \in C_c^\infty((0,T) \times \mathbb{T}^d \times \mathbb{R}^d)$ be arbitrary. Choose $\tau \in (0,T)$ such that

$$\operatorname{supp} \zeta \subset (0,\tau) \times \mathbb{T}^d \times \mathbb{R}^d.$$

We define, for $0 \le t \le \tau$,

$$\varphi(t,z) = -\int_t^\tau \zeta\left(s, \Phi_E(s,t,z)\right) \, ds, \qquad z = (x,v),$$

and extend φ by zero for $t \ge \tau$.

The next step consists in proving that $\mathcal{L}_E \varphi = \zeta$ on $[0,\tau) \times \mathbb{T}^d \times \mathbb{R}^d$. Indeed, fix (t,z) and set $Z(r) := \Phi_E(r,t,z)$. By the flow property, for $r \in [t,\tau]$ one has

$$\varphi(r, Z(r)) = -\int_r^\tau \zeta\left(s, \Phi_E(s,t,z)\right) \, ds.$$

Differentiating with respect to r gives

$$\frac{d}{dr} \varphi(r, Z(r)) = \zeta(r, Z(r)).$$

Since

$$\frac{d}{dr} \varphi(r, Z(r)) = \left(\mathcal{L}_E \varphi\right)(r, Z(r)),$$

and since the flow is a diffeomorphism, the desired identity follows.

Let us record the regularity and support properties of φ.

Since B_E is continuous in time and C^1 in the phase variable, with locally bounded first derivatives on bounded velocity sets, standard ODE theory implies that the flow is C^1 with respect to the phase variable and absolutely continuous with respect to the time variables. The function φ is therefore continuous, has continuous phase derivatives, and its derivative along the vector field is continuous. In particular, the identity $\mathcal{L}_E \varphi = \zeta$ holds pointwise.

Moreover, φ is periodic in x. If $\operatorname{supp}_v \zeta \subset B_{R^0_\zeta}$ and $\zeta(s, \Phi_E(s,t,z)) \neq 0$, then $|V_E(s;t,z)| \leq R^0_\zeta$.

Using the velocity estimate for the inverse characteristic, we obtain

$$|v| \leq R^0_\zeta + \alpha \int_t^s |E(\sigma)|\, d\sigma \leq R^0_\zeta + \alpha T \|E\|_{L^\infty(0,T)}.$$

Hence φ is compactly supported in velocity. More precisely, there exists $R_\zeta > 0$ such that $\operatorname{supp} \varphi \subset [0,\tau] \times \mathbb{T}^d \times B_{R_\zeta}$.

Finally, since ζ vanishes in a neighbourhood of $t = \tau$, the extension of φ by zero for $t \geq \tau$ is still of class C^1 across $t = \tau$. Hence φ is an admissible C^1 test function, compactly supported in velocity, but it is not necessarily smooth in time because the amplitude E is only continuous.

We now justify that such a C^1 test function can be used in the weak formulation. Let K be a compact subset of $\mathbb{R}^d$ containing the velocity supports of φ and $\mathcal{L}_E\varphi$. By standard regularization, using convolution in time and velocity and periodic convolution in the spatial variable, together with a fixed cut-off in velocity, there exists a sequence of admissible smooth test functions $\varphi_n \in C_c^\infty([0,T) \times \mathbb{T}^d \times \mathbb{R}^d)$, periodic in x, such that

$$\operatorname{supp} \varphi_n \subset [0,T) \times \mathbb{T}^d \times K$$

for all n, and

$$\varphi_n \to \varphi \quad \text{in } C^1([0,T] \times \mathbb{T}^d \times K).$$

Since the coefficients of $\mathcal{L}_E$ are continuous and bounded on $[0,T] \times \mathbb{T}^d \times K$, this implies

$$\mathcal{L}_E\varphi_n \to \mathcal{L}_E\varphi \quad \text{uniformly on } [0,T] \times \mathbb{T}^d \times K.$$

Applying the weak formulation to φ_n and passing to the limit, using $h \in L^\infty(0,T; L^1(\mathbb{T}^d \times \mathbb{R}^d))$, we obtain

$$\int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} h(t,x,v)\, \mathcal{L}_E\varphi(t,x,v)\, dv\, dx\, dt = 0.$$

Since $\mathcal{L}_E\varphi = \zeta$, it follows that

$$\int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} h(t,x,v)\zeta(t,x,v)\, dv\, dx\, dt = 0.$$

Since ζ is arbitrary in $C_c^\infty((0,T) \times \mathbb{T}^d \times \mathbb{R}^d)$, we conclude that

$$h = 0 \qquad \text{a.e. in } (0,T) \times \mathbb{T}^d \times \mathbb{R}^d.$$

By the strong continuity in L^1, the equality holds for every time in the sense of L^1. Therefore $f_E^* = f_E$ on $[0,T]$, and uniqueness for the linear transport problem follows. $\square$

4.3.3 The fixed-point map

Let $R > 0$ and $T > 0$. Define

$$\mathscr{B}_R(T) := \left\{ E \in C([0,T]) \,:\, \|E\|_{L^\infty(0,T)} \le R \right\}.$$

For $E \in \mathscr{B}_R(T)$, let f_E be the unique solution of the linear problem constructed in Lemma 4.3.2. We define the modal current

$$\mathscr{J}_E(t) := \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \Psi(x,v) f_E(t,x,v)\, dv\, dx. \tag{4.3.10}$$

Using the push-forward identity, this may be written as

$$\mathscr{J}_E(t) = \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \Psi(\Phi_E(t,0,z)) f_0(z)\, dz. \tag{4.3.11}$$

We now define the fixed-point map

$$\mathscr{T}(E)(t) := E_0 - \beta \int_0^t \mathscr{J}_E(s)\, ds. \tag{4.3.12}$$

A fixed point of $\mathscr{T}$ is precisely an amplitude for which the associated transported distribution solves the full system (4.2.3).

Assume that $E \in \mathscr{B}_R(T)$. By Lemma 4.3.2,

$$\operatorname{ess\,supp} f_E(t) \subset \mathbb{T}^d \times B_{R_T}, \qquad R_T := R_0 + \alpha R T.$$

Therefore,

$$\begin{aligned} |\mathscr{T}(E)(t)| &\le |E_0| + \beta \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |\Psi(x,v)| f_E(s,x,v)\, dv\, dx\, ds \\ &\le |E_0| + \beta T M_0 \left(R_0 + \alpha R T + |v_{\mathrm{ph}}| \right). \end{aligned}$$

Hence, if $R > |E_0|$, there exists $T_1 > 0$ such that

$$|E_0| + \beta T M_0 \left(R_0 + \alpha R T + |v_{\mathrm{ph}}| \right) \le R \tag{4.3.13}$$

for every $T \in (0, T_1]$. For such T, the map $\mathscr{T}$ sends $\mathscr{B}_R(T)$ into itself.

Lemma 4.3.3. *Let $R > |E_0|$. There exists $T_2 > 0$, depending only on R, R_0, M_0, α, β, $|k|$ and $|v_{\mathrm{ph}}|$, such that for every $T \in (0, T_2]$ the map $\mathscr{T}$ is a contraction on $\mathscr{B}_R(T)$.*

Proof. Let $E_1, E_2 \in \mathscr{B}_R(T)$ and denote the corresponding flows by

$$\Phi_{E_i}(t,0,z) = (X_i(t;z), V_i(t;z)), \qquad i = 1,2.$$

We estimate the distance between the two flows on the essential support of f_0. Since the spatial variable lives on the torus $\mathbb{T}^d$, the difference $X_1(t;z) - X_2(t;z)$ is not intrinsically defined. However, for each initial point $z = (x,v) \in \mathbb{T}^d \times \mathbb{R}^d$, we choose one lift $\tilde{x} \in \mathbb{R}^d$ of x and consider the lifted trajectories $\tilde{X}_i(\cdot; z)$,

$i = 1, 2$, starting from the same initial lift $\tilde{x}$. In what follows we write simply X_i for these lifted trajectories. This convention is harmless: if a different lift of x is chosen, both lifted trajectories are translated by the same element of $2\pi\mathbb{Z}^d$, and therefore their difference is unchanged.
For the projected points on the torus one has

$$d_{\mathbb{T}^d}(X_1(t;z), X_2(t;z)) \leq |X_1(t;z) - X_2(t;z)|_{\mathbb{R}^d},$$

where $d_{\mathbb{T}^d}$ denotes the canonical distance induced by the quotient

$$\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$$

. Define

$$D(t) := \sup_{x\in\mathbb{T}^d,\ |v|\leq R_0} \left(|X_1(t;x,v) - X_2(t;x,v)| + |V_1(t;x,v) - V_2(t;x,v)|\right).$$

For every $z = (x, v)$ with $|v| \leq R_0$, the two characteristics have the same initial data at time 0, hence

$$X_1(t;z) - X_2(t;z) = \int_0^t \big(V_1(s;z) - V_2(s;z)\big)\, ds.$$

Therefore

$$|X_1(t;z) - X_2(t;z)| \leq \int_0^t |V_1(s;z) - V_2(s;z)|\, ds. \tag{4.3.14}$$

Similarly,

$$V_1(t;z) - V_2(t;z) = -\alpha \int_0^t [E_1(s)\sin(k\cdot X_1(s;z)) - E_2(s)\sin(k\cdot X_2(s;z))]\, b\ ds.$$

Since $|b| = 1$, $|\sin| \leq 1$, $|E_2(s)| \leq R$, and

$$|\sin(k\cdot x) - \sin(k\cdot y)| \leq |k|\,|x-y| \qquad \text{for } x, y \in \mathbb{R}^d,$$

we obtain

$$\begin{aligned}
&|V_1(t;z) - V_2(t;z)| \leq \alpha \int_0^t |E_1(s)\sin(k\cdot X_1(s;z)) - E_2(s)\sin(k\cdot X_2(s;z))|\ ds \\
&\leq \alpha \int_0^t |E_1(s) - E_2(s)|\, ds + \alpha \sup_{s\in[0,t]} |E_2(s)|\,|k| \int_0^t |X_1(s;z) - X_2(s;z)|\, ds \\
&\leq \alpha \int_0^t |E_1(s) - E_2(s)|\, ds + \alpha R|k| \int_0^t |X_1(s;z) - X_2(s;z)|\, ds.
\end{aligned} \tag{4.3.15}$$

Adding (4.3.14) and (4.3.15) gives

$$
\begin{aligned}
|X_1(t;z) - X_2(t;z)| + |V_1(t;z) - V_2(t;z)| & \\
\leq \int_0^t |V_1(s;z) - V_2(s;z)|\, ds + \alpha R|k| \int_0^t |X_1(s;z) - X_2(s;z)|\, ds & \\
+ \alpha \int_0^t |E_1(s) - E_2(s)|\, ds. &
\end{aligned}
$$

$$
\begin{aligned}
\leq (1+\alpha R|k|) \int_0^t \left(|X_1(s;z) - X_2(s;z)| + |V_1(s;z) - V_2(s;z)|\right) ds + & \\
\alpha \int_0^t |E_1(s) - E_2(s)|\, ds. &
\end{aligned}
$$

Taking the supremum over $x \in \mathbb{T}^d$ and $|v| \leq R_0$, we obtain

$$D(t) \leq (1+\alpha R|k|) \int_0^t D(s)\, ds + \alpha \int_0^t |E_1(s) - E_2(s)|\, ds.$$

By Gronwall's lemma,

$$D(t) \leq \alpha e^{(1+\alpha R|k|)t} \int_0^t |E_1(s) - E_2(s)|\, ds.$$

In particular, since $t \leq T$, if $T \leq 1$ then

$$D(t) \leq C_R T \|E_1 - E_2\|_{L^\infty(0,T)}, \qquad C_R := \alpha e^{1+\alpha R|k|}. \tag{4.3.16}$$

On $\mathbb{T}^d \times B_{R_T}$, with $R_T = R_0 + \alpha R T$, the function Ψ defined in (4.3.1) is Lipschitz. Indeed, since $|b| = 1$,

$$|\nabla_v \Psi(x,v)| \leq 1$$

and

$$|\nabla_x \Psi(x,v)| \leq |k| \left(|v| + |v_{\mathrm{ph}}|\right).$$

For $T \leq 1$ we may use the Lipschitz constant

$$L_R := 1 + |k| \left(R_0 + \alpha R + |v_{\mathrm{ph}}|\right).$$

Therefore,

$$|\Psi(x_1, v_1) - \Psi(x_2, v_2)| \leq L_R \left(d_{\mathbb{T}^d}(x_1, x_2) + |v_1 - v_2|\right)$$

for all $(x_i, v_i) \in \mathbb{T}^d \times B_{R_T}$.

Using (4.3.11), we obtain

$$
\begin{aligned}
|\mathscr{J}_{E_1}(t) - \mathscr{J}_{E_2}(t)| &= \left| \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \left[\Psi(\Phi_{E_1}(t,0,z)) - \Psi(\Phi_{E_2}(t,0,z))\right] f_0(z)\, dz \right| \\
&\leq L_R M_0 D(t).
\end{aligned}
$$

Combining this with (4.3.16), we find

$$|\mathscr{J}_{E_1}(t) - \mathscr{J}_{E_2}(t)| \leq L_R M_0 C_R T \|E_1 - E_2\|_{L^\infty(0,T)}.$$

Consequently,

$$\|\mathscr{T}(E_1) - \mathscr{T}(E_2)\|_{L^\infty(0,T)} \leq \beta L_R M_0 C_R T^2 \|E_1 - E_2\|_{L^\infty(0,T)}.$$

Choosing $T_2 > 0$ so that $T_2 \leq 1$ and

$$\beta L_R M_0 C_R T_2^2 < 1,$$

we obtain the desired contraction property. □

4.3.4 Local weak well-posedness

Proposition 4.3.4. *Assume* (4.3.2) *and* (4.3.3). *Let* $R > |E_0|$. *Then there exists* $T_{\mathrm{loc}} > 0$ *such that, for every* $T \in (0, T_{\mathrm{loc}}]$, (4.2.3) *admits a weak solution* (f, E) *on* $[0, T]$ *satisfying* $E \in \mathscr{B}_R(T)$.

This solution is unique among weak solutions whose amplitude belongs to $\mathscr{B}_R(T)$.

Moreover, after choosing the representative given by the characteristic formula,

$$f(t, z) = f_0(\Phi_E(0, t, z))$$

for a.e. z *and every* $t \in [0, T]$.

Proof. Let $T_{\mathrm{loc}} := \min\{T_1, T_2\}$, where T_1 is chosen so that the self-map condition (4.3.13) holds and T_2 is given by Lemma 4.3.3. Then, for every $T \in (0, T_{\mathrm{loc}}]$, the map $\mathscr{T}$ maps $\mathscr{B}_R(T)$ into itself and is a contraction there. Since $\mathscr{B}_R(T)$ is a closed subset of the Banach space $C([0, T])$, Banach's fixed point theorem gives a unique $E \in \mathscr{B}_R(T)$ satisfying $\mathscr{T}(E) = E$.

Let $f = f_E$ be the corresponding solution of the linear Vlasov equation. By Lemma 4.3.2, f satisfies the weak Vlasov formulation. The fixed point identity is exactly the amplitude equation.

Furthermore, since f_E has velocity support contained in $B_{R_0+\alpha RT}$, the modal current $\mathscr{J}_E$ belongs to $L^\infty(0, T)$. Hence the fixed point identity implies $E \in W^{1,\infty}(0, T)$. Therefore (f, E) is a weak solution in the sense of the definition.

Let $(\tilde{f}, \tilde{E})$ be another weak solution on $[0, T]$ with $\tilde{E} \in \mathscr{B}_R(T)$. By Lemma 4.3.2, applied with the prescribed amplitude $\tilde{E}$, one has $\tilde{f} = f_{\tilde{E}}$. The amplitude equation then implies $\tilde{E} = \mathscr{T}(\tilde{E})$. Since $\mathscr{T}$ has a unique fixed point in $\mathscr{B}_R(T)$, we get $\tilde{E} = E$. The uniqueness of the linear transport equation then yields $\tilde{f} = f$. □

4.3.5 A priori estimates

Let (f, E) be a weak solution on $[0, T]$ such that

$$f(t, z) = f_0(\Phi_E(0, t, z)).$$

We define its velocity radius by

$$S(t) := \operatorname{ess\,sup}\left\{|v| :\ (x,v) \in \operatorname{ess\,supp} f(t)\right\}.$$

Since the initial datum is supported in $\mathbb{T}^d \times B_{R_0}$, one has $S(0) \le R_0$. Moreover, using the characteristic representation, for every $t \in [0,T]$ we obtain

$$S(t) \le R_0 + \alpha \int_0^t |E(s)|\,ds. \tag{4.3.17}$$

Indeed, along characteristics,

$$|V_E(t,0,x,v)| \le |v| + \alpha \int_0^t |E(s)|\,ds,$$

and $|v| \le R_0$ on the initial velocity support.

On the other hand, the amplitude equation gives, for a.e. $t \in [0,T]$,

$$\begin{aligned} |E'(t)| &\le \beta \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} |\Psi(x,v)| f(t,x,v)\,dv\,dx \\ &\le \beta M_0 \left(S(t) + |v_{\mathrm{ph}}|\right). \end{aligned} \tag{4.3.18}$$

Therefore, since E is absolutely continuous,

$$|E(t)| \le |E_0| + \beta M_0 \int_0^t \left(S(s) + |v_{\mathrm{ph}}|\right)\,ds. \tag{4.3.19}$$

We now set

$$Y(t) := 1 + |E(t)| + S(t).$$

The role of the additional constant 1 is to absorb the time integral of the constant term $|v_{\mathrm{ph}}|$. Combining (4.3.17) and (4.3.19), we obtain

$$\begin{aligned} Y(t) &= 1 + |E(t)| + S(t) \\ &\le 1 + |E_0| + R_0 + \alpha \int_0^t |E(s)|\,ds + \beta M_0 \int_0^t \left(S(s) + |v_{\mathrm{ph}}|\right)\,ds \\ &\le 1 + |E_0| + R_0 + (\alpha + \beta M_0(1 + |v_{\mathrm{ph}}|)) \int_0^t Y(s)\,ds. \end{aligned} \tag{4.3.20}$$

Define

$$Y_0 := 1 + |E_0| + R_0$$

and

$$C := \alpha + \beta M_0(1 + |v_{\mathrm{ph}}|).$$

Then (4.3.20) becomes

$$Y(t) \le Y_0 + C \int_0^t Y(s)\,ds.$$

By Gronwall's lemma, $Y(t) \le Y_0 e^{Ct}$ for every $t \in [0,T]$. Since $|E(t)| + S(t) \le$

$Y(t)$, we finally obtain

$$|E(t)| + S(t) \leq (1 + |E_0| + R_0)\, e^{Ct} \tag{4.3.21}$$

for every $t \in [0, T]$.

In particular, neither the amplitude nor the velocity support can blow up in finite time.

4.3.6 Global weak well-posedness

We conclude by extending the time interval in which the solution is defined. The following result holds.

Theorem 4.3.5. *Assume* (4.3.2) *and* (4.3.3). *Then, for every* $T > 0$, *the system* (4.2.3) *admits a unique weak solution* (f, E) *on* $[0, T]$ *in the class*

$$f \in L^\infty(0, T; L^1(\mathbb{T}^d \times \mathbb{R}^d) \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d)) \cap C([0, T]; L^1(\mathbb{T}^d \times \mathbb{R}^d)),$$

with essentially bounded velocity support, and $E \in W^{1,\infty}(0, T)$.

Moreover, for every $T > 0$, $f(t, x, v) \geq 0$ *for a.e.* (t, x, v),

$$\int_{\mathbb{T}^d} \int_{\mathbb{R}^d} f(t, x, v)\, dv\, dx = M_0$$

for every $t \in [0, T]$, *and, for every* $1 \leq p \leq \infty$,

$$\|f(t)\|_{L^p(\mathbb{T}^d \times \mathbb{R}^d)} = \|f_0\|_{L^p(\mathbb{T}^d \times \mathbb{R}^d)} \tag{4.3.22}$$

for every $t \in [0, T]$. *Finally, there exists* $R_T > 0$ *such that*

$$\operatorname{ess\,supp} f(t) \subset \mathbb{T}^d \times B_{R_T} \tag{4.3.23}$$

for every $t \in [0, T]$.

Proof. We first prove existence by continuation. Proposition 4.3.4 gives a solution on a nontrivial interval. By the local uniqueness result, local solutions coincide on common intervals; hence they can be uniquely glued together. Let $[0, T_{\max})$ be the maximal interval on which the solution can be constructed by the local fixed point argument.

Assume, by contradiction, that $T_{\max} < +\infty$. By the a priori estimate (4.3.21), there exist constants R_E and R_V such that

$$|E(t)| \leq R_E, \qquad S(t) \leq R_V, \qquad 0 \leq t < T_{\max}.$$

Let $t_0 < T_{\max}$. The data at time t_0 are $f(t_0)$ and $E(t_0)$.

By the characteristic representation, $f(t_0)$ is nonnegative, belongs to $L^1(\mathbb{T}^d \times \mathbb{R}^d) \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$, has mass M_0, and satisfies

$$\operatorname{ess\,supp} f(t_0) \subset \mathbb{T}^d \times B_{R_V}.$$

Moreover, $|E(t_0)| \leq R_E$.

We restart the fixed point argument at time t_0, after the change of variable $s = t - t_0$. We take the ball radius $R := R_E + 1$. The new initial velocity radius is bounded by R_V, and the mass is still M_0. Therefore the constants in the self-map and contraction estimates are bounded uniformly with respect to $t_0 < T_{\max}$. Hence there exists $\tau > 0$, depending only on R_E, R_V, M_0, α, β, $|k|$ and $|v_{\mathrm{ph}}|$, such that the solution can be extended on $[t_0, t_0 + \tau]$ for every $t_0 < T_{\max}$. Choosing t_0 sufficiently close to $T_{\max}$ yields $t_0 + \tau > T_{\max}$, contradicting the maximality of $T_{\max}$. Therefore $T_{\max} = +\infty$.
The conservation of positivity and of the L^p norms follows from Lemma 4.3.2, since the global solution is represented by

$$f(t, z) = f_0(\Phi_E(0, t, z)).$$

Taking $p = 1$ gives mass conservation. The boundedness of the velocity support on every finite interval follows from (4.3.17) and (4.3.21). Finally, (4.3.18) gives $E' \in L^\infty(0, T)$ for every finite $T > 0$, hence

$$E \in W^{1,\infty}(0, T).$$

It remains to prove uniqueness. Let (f_1, E_1) and (f_2, E_2) be two weak solutions on $[0, T]$ with the same initial datum. On every finite interval, both amplitudes are bounded and both densities have essentially bounded velocity support. Therefore the constants entering the local contraction argument are finite.

Define

$$t^* := \sup\left\{ t \in [0, T] : E_1 = E_2 \text{ on } [0, t] \text{ and } f_1 = f_2 \text{ a.e. on } [0, t] \times \mathbb{T}^d \times \mathbb{R}^d \right\}.$$

Clearly $t^* \geq 0$. By the continuity of the amplitudes and the strong continuity of the densities in L^1, the two solutions have the same data at time t^*. The local well-posedness argument is invariant under time translation. Restarting the local fixed point argument at t^*, with these common data, yields a time $\tau > 0$ such that the two solutions coincide on $[t^*, t^* + \tau]$, unless $t^* = T$. This contradicts the definition of t^* if $t^* < T$. Therefore $t^* = T$, and uniqueness follows. □

Remark 4.3.6. The compact velocity support assumption is used to obtain a Lipschitz estimate for the modal current functional. The same strategy may be extended to data with unbounded velocity support under suitable moment and uniform integrability assumptions, provided that the current $\mathscr{J}_E$ remains well defined and stable under the characteristic flow.

Remark 4.3.7. The proof gives existence and uniqueness through a contraction argument for the scalar amplitude. This is simpler than the classical Vlasov–Poisson coupling, where the field is recovered from an elliptic equation depending on the spatial density. Here the self-consistent feedback acts only through the modal current associated with a prescribed wave profile.

Remark 4.3.8. Unlike the standard Vlasov–Poisson system, the present closure does not immediately yield a conserved electrostatic energy in the moving variables, because the amplitude equation involves the laboratory longitudinal velocity $v \cdot b + v_{\mathrm{ph}}$. This point is consistent with the reduced character of the model and with the fact that the wave profile is prescribed while only its scalar amplitude is evolved.

4.4 Equilibrium states

In this section we characterize the stationary solutions of (4.2.3). A stationary solution is a pair (f_*, E_*), where $E_* \in \mathbb{R}$ is constant and $f_* = f_*(x, v)$ is independent of time, such that

$$v \cdot \nabla_x f_* - \alpha E_* \sin(k \cdot x) b \cdot \nabla_v f_* = 0 \tag{4.4.1}$$

in the sense of distributions on $\mathbb{T}^d \times \mathbb{R}^d$, and

$$\int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \Psi(x, v) f_*(x, v) \, dv \, dx = 0, \tag{4.4.2}$$

where Ψ is defined in (4.3.1), namely $\Psi(x, v) = \sin(k \cdot x)(v \cdot b + v_{\mathrm{ph}})$.

Throughout this section we assume that

$$f_* \in L^1(\mathbb{T}^d \times \mathbb{R}^d) \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d), \qquad f_* \geq 0, \tag{4.4.3}$$

and that f_* has essentially compact velocity support, i.e., there exists $R > 0$ such that

$$\operatorname{ess\,supp} f_* \subset \mathbb{T}^d \times B_R. \tag{4.4.4}$$

Thus, the characteristic flow $\Phi_{E_*}(t, s, z)$, associated with the constant amplitude $E(t) \equiv E_* \in \mathbb{R}$, solves

$$\begin{cases} \dot{X}(t) = V(t), \\ \dot{V}(t) = -\alpha E_* \sin(k \cdot X(t)) b, \end{cases} \tag{4.4.5}$$

with

$$X(s) = x, \qquad V(s) = v.$$

The corresponding autonomous phase-space vector field is

$$B_{E_*}(x, v) := (v, -\alpha E_* \sin(k \cdot x) b).$$

It is divergence-free in (x, v), i.e.:

$$\operatorname{div}_{x,v} B_{E_*} = 0.$$

Therefore the flow $\Phi_{E_*}(t, s, \cdot)$ preserves the Lebesgue measure on $\mathbb{T}^d \times \mathbb{R}^d$.

The stationary equation (4.4.1) can be written as

$$B_{E_*} \cdot \nabla_{x,v} f_* = 0.$$

Since B_{E_*} is smooth in (x, v) and has at most linear growth in velocity, the characteristic flow is globally defined. Moreover, by the standard theory for linear transport equations with smooth vector fields, (4.4.1) holds in the sense of distributions if and only if $f_* (\Phi_{E_*}(t, s, z)) = f_*(z)$ for every $s, t \in \mathbb{R}$ and for a.e. $z \in \mathbb{T}^d \times \mathbb{R}^d$.

We now distinguish the cases $E_* = 0$ and $E_* \neq 0$.

4.4.1 The zero-amplitude case

If $E_* = 0$, the stationary Vlasov equation becomes

$$v \cdot \nabla_x f_* = 0. \tag{4.4.6}$$

We first show that all integrable stationary states of this equation are spatially homogeneous.

Lemma 4.4.1. *Let f_* satisfy* (4.4.3) *and* (4.4.4)*. Then* (4.4.6) *holds in the sense of distributions if and only if there exists a function $F \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, with $F \geq 0$ and compact velocity support, such that*

$$f_*(x, v) = F(v) \tag{4.4.7}$$

for a.e. (x, v).

Proof. It is immediate that every function of the form $f_*(x, v) = F(v)$ solves $v \cdot \nabla_x f_* = 0$.

Conversely, assume that f_* solves (4.4.6). For each $\lambda \in \mathbb{Z}^d$, define

$$\widehat{f}_\lambda(v) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f_*(x, v) e^{-i\lambda \cdot x} \, dx.$$

Since $f_* \in L^1(\mathbb{T}^d \times \mathbb{R}^d)$, one has $\widehat{f}_\lambda \in L^1(\mathbb{R}^d)$ for every λ. Taking the Fourier transform in the x variable in $v \cdot \nabla_x f_* = 0$ gives

$$i(\lambda \cdot v)\widehat{f}_\lambda(v) = 0$$

in the sense of distributions in v. Hence, for every $\lambda \neq 0$, $\widehat{f}_\lambda$ is supported in the hyperplane

$$\{v \in \mathbb{R}^d : \ \lambda \cdot v = 0\}.$$

Since $\widehat{f}_\lambda$ is an L^1 function and this hyperplane has zero Lebesgue measure, it follows that

$$\widehat{f}_\lambda = 0 \quad \text{for every } \lambda \neq 0.$$

Therefore only the zero Fourier mode remains, and f_* is independent of x. This proves (4.4.7). □

For a spatially homogeneous state one has

$$\int_{\mathbb{T}^d} \sin(k \cdot x) \, dx = 0.$$

Therefore the modal current condition (4.4.2) is automatically satisfied. Hence every pair of the form $(f_*, E_*) = (F(v), 0)$, with $F \geq 0$, $F \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and compactly supported in velocity, is a stationary solution.

4.4.2 The nonzero-amplitude case

Assume now that $E_* \neq 0$. The stationary dynamics is Hamiltonian. Define

$$\mathcal{H}_{E_*}(x, v) := \frac{|v|^2}{2} - \frac{\alpha E_*}{|k|} \cos(k \cdot x).$$

Then

$$\nabla_v \mathcal{H}_{E_*} = v, \qquad \nabla_x \mathcal{H}_{E_*} = \alpha E_* \sin(k \cdot x) b.$$

Therefore (4.4.5) is the Hamiltonian system

$$\dot{X} = \nabla_v \mathcal{H}_{E_*}, \qquad \dot{V} = -\nabla_x \mathcal{H}_{E_*}.$$

In particular,

$$\mathcal{H}_{E_*}\left(\Phi_{E_*}(t, s, z)\right) = \mathcal{H}_{E_*}(z)$$

for every $s, t \in \mathbb{R}$ and every $z \in \mathbb{T}^d \times \mathbb{R}^d$.

The following result gives the characterization of stationary solutions with nonzero amplitude.

Proposition 4.4.2. *Let $E_* \neq 0$ and let f_* satisfy* (4.4.3) *and* (4.4.4). *Then (f_*, E_*) is a stationary solution of* (4.2.3) *if and only if*

$$f_*\left(\Phi_{E_*}(t, s, z)\right) = f_*(z) \tag{4.4.8}$$

for every $s, t \in \mathbb{R}$ and for a.e. $z \in \mathbb{T}^d \times \mathbb{R}^d$.

Equivalently, f_* is any nonnegative density, of class $L^1(\mathbb{T}^d \times \mathbb{R}^d) \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$, with essentially compact velocity support which is constant along the orbits of the autonomous Hamiltonian flow generated by $\mathcal{H}_{E_*}$.

Proof. The equivalence between the distributional equation

$$B_{E_*} \cdot \nabla_{x,v} f_* = 0$$

and the flow invariance (4.4.8) follows from the characteristic theory recalled above. Thus it remains only to show that, when $E_* \neq 0$, the modal current condition (4.4.2) is automatically satisfied.

Set $u := v \cdot b$. Since f_* has essentially compact velocity support, choose $\chi \in C_c^\infty(\mathbb{R}^d)$ such that

$$\chi(v) = 1 \qquad \text{on a neighbourhood of } B_R,$$

where R is such that $\operatorname{ess\,supp} f_* \subset \mathbb{T}^d \times B_R$.

Because B_{E_*} is divergence-free, the distributional identity $B_{E_*} \cdot \nabla_{x,v} f_* = 0$ implies that, for every smooth compactly supported test function ϕ in velocity and periodic in x,

$$\int_{\mathbb{T}^d}\int_{\mathbb{R}^d} f_*(x,v)B_{E_*}(x,v)\cdot\nabla_{x,v}\phi(x,v)\,dv\,dx = 0.$$

We first take

$$\phi_1(x,v) := \chi(v)u.$$

On the essential support of f_* one has $\chi = 1$ and $\nabla_v\phi_1 = b$. Moreover ϕ_1 is independent of x on that support. Hence, on the essential support of f_*,

$$B_{E_*}\cdot\nabla_{x,v}\phi_1 = -\alpha E_*\sin(k\cdot x).$$

Therefore

$$0 = -\alpha E_*\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\sin(k\cdot x)f_*(x,v)\,dv\,dx.$$

Since $E_* \neq 0$, we obtain

$$\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\sin(k\cdot x)f_*(x,v)\,dv\,dx = 0. \tag{4.4.9}$$

Next we take

$$\phi_2(x,v) := \chi(v)\frac{u^2}{2}.$$

On the essential support of f_* one has $\nabla_v\phi_2 = ub$. Hence, again on the essential support of f_*,

$$B_{E_*}\cdot\nabla_{x,v}\phi_2 = -\alpha E_*\sin(k\cdot x)u.$$

Thus

$$0 = -\alpha E_*\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\sin(k\cdot x)(v\cdot b)f_*(x,v)\,dv\,dx.$$

Since $E_* \neq 0$, we get

$$\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\sin(k\cdot x)(v\cdot b)f_*(x,v)\,dv\,dx = 0. \tag{4.4.10}$$

Combining (4.4.9) and (4.4.10), we obtain

$$\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\sin(k\cdot x)(v\cdot b + v_{\mathrm{ph}})f_*(x,v)\,dv\,dx = 0.$$

This is exactly (4.4.2). Hence the amplitude equation is satisfied, and the proposition follows. □

Remark 4.4.3. The condition (4.4.8) gives the most general description of equilibria in the weak class considered here. In particular, one does not need to assume that f_* is a function only of the Hamiltonian $\mathcal{H}_{E_*}$. Any measurable density invariant along the Hamiltonian orbits is admissible, provided it belongs to $L^1\cap L^\infty$ and has essentially compact velocity support.

We may summarize the classification as follows.

Theorem 4.4.4. *The stationary solutions of* (4.2.3) *in the class* (4.4.3)–(4.4.4) *are exactly the following.*

i) Zero-amplitude equilibria:

$$E_* = 0, \qquad f_*(x, v) = F(v),$$

where $F \geq 0$, $F \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, and F has compact velocity support.

ii) Nonzero-amplitude equilibria:

$$E_* \neq 0, \qquad f_*\left(\Phi_{E_*}(t, s, z)\right) = f_*(z) \quad \textit{for every } s, t \in \mathbb{R},$$

for a.e. $z \in \mathbb{T}^d \times \mathbb{R}^d$, where f_* is nonnegative, belongs to $L^1(\mathbb{T}^d \times \mathbb{R}^d) \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ and has essentially compact velocity support. Equivalently, f_* is any admissible density constant along the orbits of the Hamiltonian flow associated with

$$\mathcal{H}_{E_*}(x, v) = \frac{|v|^2}{2} - \frac{\alpha E_*}{|k|} \cos(k \cdot x).$$

4.5 Numerical simulations

We now consider a more specific case, the one presented by Levin [1]. This is obtained by considering the case $d = 1$ of system 4.2.3. To solve it numerically we initialize the particles in a limited interval in velocity. This does not limit the validity of the system because, as shown by Oniel in 1965 [35] the only particles interacting with the wave are those that are initialised around the phase velocity of the wave, so around zero in the wave frame. Following the particle trajectories where $(x, v) \in \mathbb{T} \times (-v_0^m, v_0^m)$ the system reads:

$$\begin{cases} \dot{x}(t) = v(t), \\ \dot{v}(t) = -\alpha E(t) \sin(kx(t)), \\ \dot{E}(t) = -\beta \displaystyle\int_{\mathbb{T}} \int_{-v_0^m}^{v_0^m} \sin(kx(t))(v(t) + v_{ph}) f(t, x, v)\, dx,\, dv \\ f(0, x, v) = f_0(x, v), \\ E(0) = E_0. \end{cases} \tag{4.5.1}$$

The system can be solved by using a Particle in cell (PIC) code. We initialize the particles according the chosen distribution function letting then the system evolving in time using a leap-frog method.
Consider a system of N particles and a discretization of the time interval $t_n =$

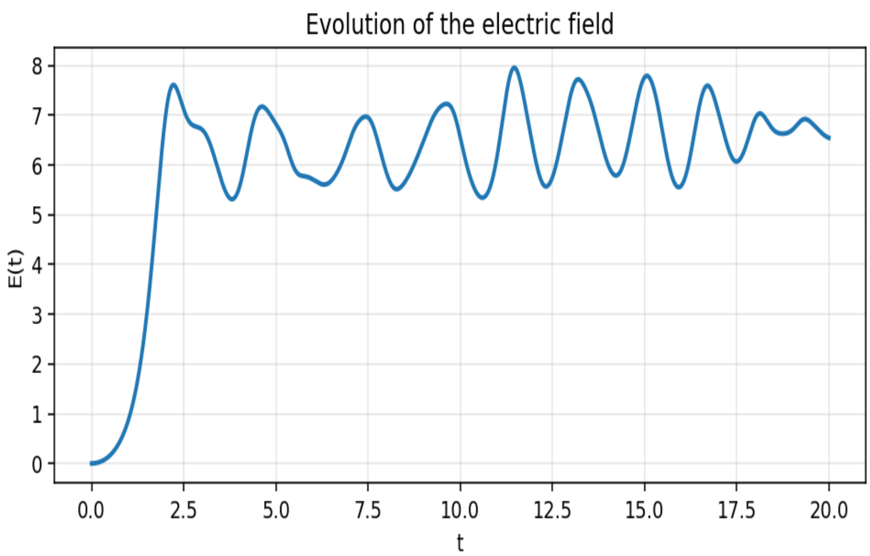


Figure 4.1: Evolution of the Amplitude of the electric field in time.

$n\Delta t$, where $\Delta t > 0$ is a chosen time step . Then $\forall i = 0, ..., N$:

$$\begin{cases} v_i^{n+\frac{1}{2}} = v_i^n + \alpha \dfrac{\Delta t}{2} E^n, \\ x_i^{n+1} = x_i^n + \Delta t v_i^{n+\frac{1}{2}}, \\ E^{n+1} = E^n - \beta \dfrac{\Delta t}{N} \displaystyle\sum_{i=0}^{N} \sin(k x_i^{n+1})(v_i^{n+\frac{1}{2}} + v_{ph}) \\ v_i^{n+1} = v_i^{n+\frac{1}{2}} + \beta \dfrac{\Delta t}{2} E^{n+1}. \end{cases} \tag{4.5.2}$$

We implemented a simulation with a total number of particles $N = 500.000$, $T = 20s$, $\Delta t = 10^{-3}$, $E_0 = 0,01$. The particles were initialized following the chosen density:

$$f_0(x, v) = F(x)G(v) = (1 - \alpha x^2)(1 + \beta v)\mathbb{I}_{v<=1}.$$

As we can see in 4.1, the amplitude of the electric field initially grows esponentially and then saturates at around $t = 2.5$. After the saturation, during the non linear phase, we see some oscillations.
Following the dynamics of the particles, we can see in Fig. 4.2 some snapshots of the phase space redistribution of particles at different fixed times during the simulation. We can see that an island of trapped particles is forming.

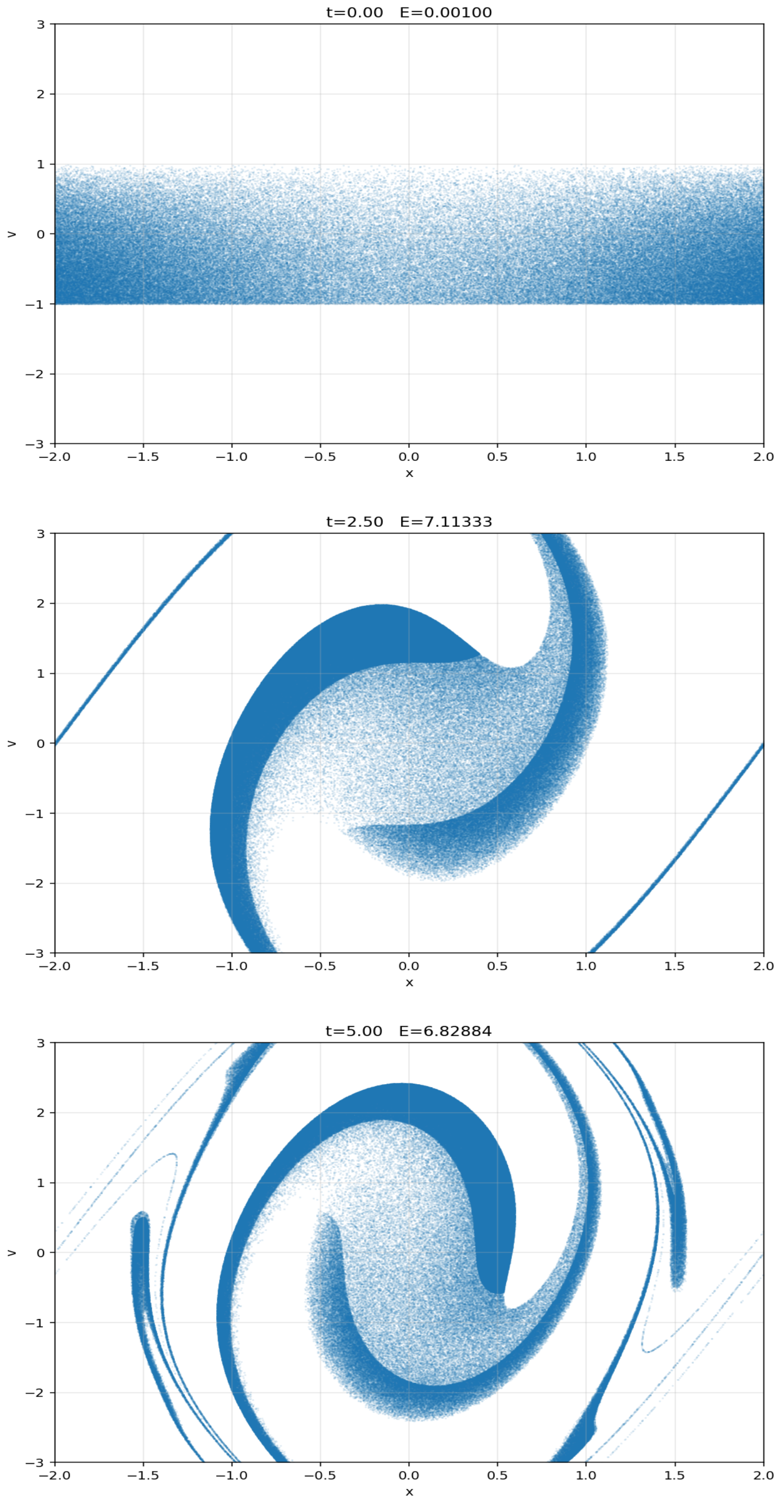


Figure 4.2: Redistribution of the particles in phase space.

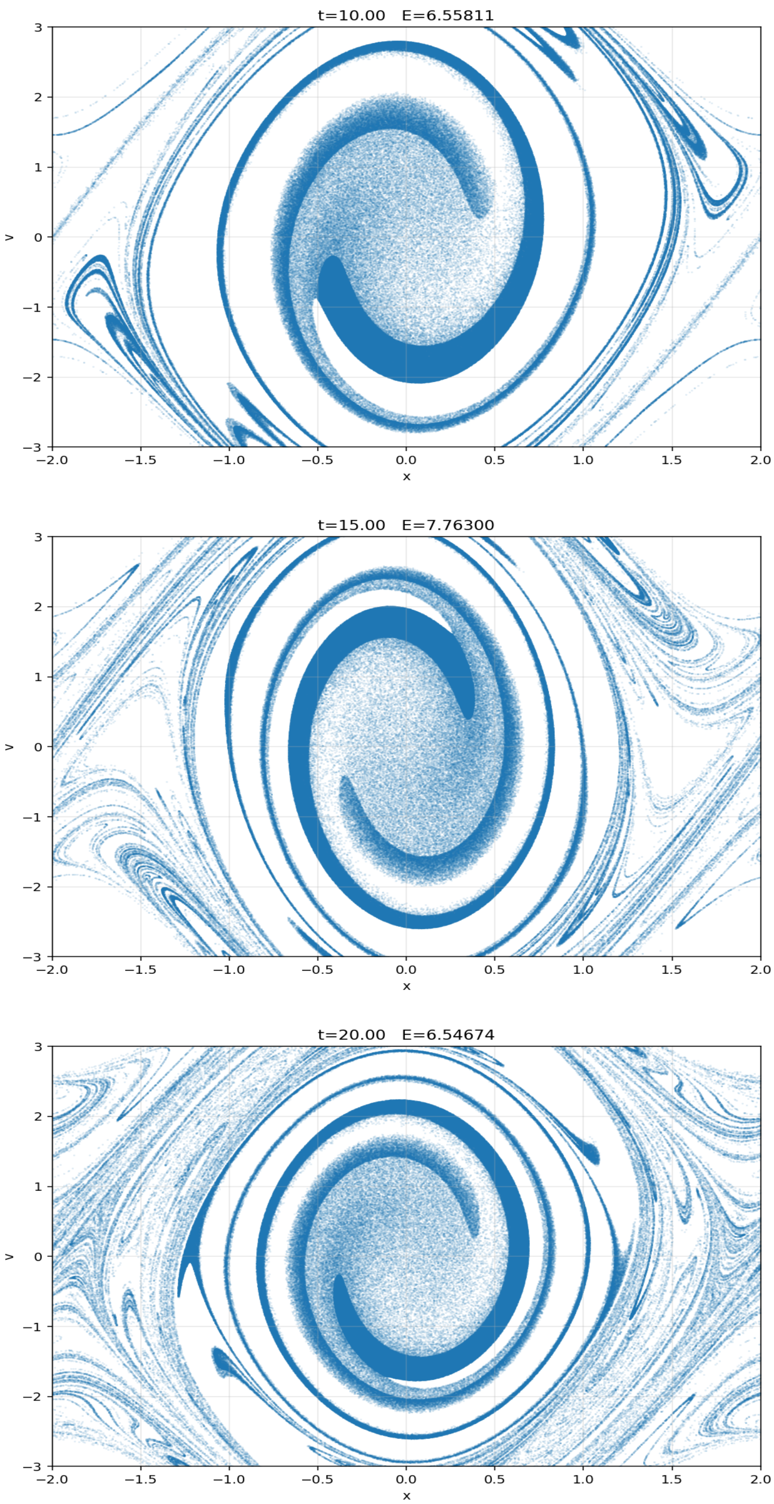


Figure 4.3: Redistribution of the particles in phase space.

Chapter 5

Comparison between Beam Plasma Instability and EGAMs

In this chapter we compare the BPI and EGAM dynamics. EGAM's simulation are different from BPI. First of all because of the different geometry of the problem, that requires a higher number of variables. To do so we refer to gyrokinetic theory, derived for example by Frieman and Chen in 1982, see [36].

5.1 ORB5 code and Gyrokinetic Model

The evolution of EGAMs is studied using ORB5, a global gyrokinetic code, see [9], with collisionless electrostatic simulations. The model equations are obtained defining the phase space variables $(\mathbf{R}, p_{||}, \mu)$, that are respectively the position, parallel momentum and magnetic momentum. The equations are the evolution in time of the markers coupled with the gyrokinetic Poisson equation, following the model given in Ref. [9] (see also Ref. [37] for a mathematical description of the electrostatic sub-model). Following the derivation of Bottino and Sonnendrücker,([37]),we get the following gyrokinetic system:

$$\dot{\mathbf{R}} = \frac{p_{\|}}{m}\frac{\mathbf{B}^*}{B_{\|}^*} - \frac{c}{eB_{\|}^*}\mathbf{b} \times \left(\mu \nabla B + e\nabla J_0 \Phi\right), \tag{5.1.1}$$

$$\dot{p}_{\|} = -\frac{\mathbf{B}^*}{B_{\|}^*} \cdot \left(\mu \nabla B + e\nabla J_0 \Phi\right), \tag{5.1.2}$$

$$\dot{\mu} = 0. \tag{5.1.3}$$

Here $\mathbf{B} = \nabla \times \mathbf{A}$ is the magnetic field, $\mathbf{b} = \frac{\mathbf{B}}{B}$ the unit vector, $p_{||} = mv_{||}$. In addition, if we define $\mathbf{A}^* = \mathbf{A} + p_{||}\frac{c}{e}\mathbf{b}$, we will have that $\mathbf{B}^* = \nabla \times \mathbf{A}^*$. Note that in the electrostatic case $\mathbf{A}$ and,consequently, $\mathbf{B}$ are at equilibrium. We also define J_0 as the gyroaverage operator, that reads:

$$(J_0\psi)(\mathbf{R}, \mu) = \frac{1}{2\pi}\int_0^{2\pi} \psi(\mathbf{R} + \boldsymbol{\rho}(\alpha))\, \mathrm{d}\alpha.$$

ϕ is the electrostatic potential that is found as a solution of the GK Poisson equation, also called polarisation equation:

$$\sum_{\rm sp}\left(\int dW\, e\, J_0^\dagger f + \nabla\cdot\frac{mn_0c^2}{B^2}\nabla_\perp\Phi\right) = 0, \tag{5.1.4}$$

denoting by $J_0^\dagger$ the adjoint operator.
Using this set of equations gives us a reduction in the dimensionality of the problem, going from 7D to 6D. This system is implemented in ORB5; see [9], which can be seen as a generalization of a PIC code. This needs to be implemented on supercomputers such as Pitagora. Pitagora is a supercomputer that counts 1176 computing nodes, hosted by CINECA. It is used by the EUROFusion community. We used it to perform the EGAMs simulations that will be presented in the following section, with ORB5 code.

5.2 Equilibrium, and linear dynamics for EGAMs

The tokamak magnetic equilibrium considered here is the same as Ref. [7] and [8]. The major radius is $R_0 = 1\,m$, the minor radius is $a = 0.3125\,m$, and we assume circular concentric flux surfaces and a magnetic field on axis of $B_0 = 1.9\,T$. The safety factor is uniform in space, equal at q = 2. The initial distribution function for the EPs is a double bump on tail.

EGAMs grow and are unstable in this configuration due to the inverse Landau damping. When a scan with the EP concentration n_{EP} is performed, one can see that this EGAM has a real frequency stemming from the GAM frequency at $n_{EP} = 0$, and its frequency decreases with increasing n_{EP}. The growth rate increases from a negative value when $n_{EP} = 0$ (corresponding to the GAM Landau damping), passing through zero (marginal stability) around $n_{EP}/n_i = 0.05$, and growing with n_{EP}. Detailed studies can be found in Ref. [7].

5.3 Study of the nonlinear oscillations of the EGAM amplitude

The EGAM linear and nonlinear dynamics can be investigated by measuring the evolution of the radial electric field in time. In Fig. 5.2a, the case of a simulation with $n_{EP}/n_i = 0.17$ is considered. One can observe a linear phase, lasting from the beginning of the simulation until $t \simeq 20000\ \Omega_i^{-1}$, a saturation at $t \simeq 22000\ \Omega_i^{-1}$, and a decrease in amplitude. The signal shows the co-existence of two scales: a high frequency component, which is the EGAM real part of the frequency, and a lower frequency component, after the nonlinear saturation. By averaging out the high-frequency part, we are left with the absolute value of the EGAM amplitude.

The evolution of the EGAM amplitude, for different values of EP concentration, is shown in Fig. 5.2b. The concentration of energetic particles has been varied from 11% to 22%. Note that a clear low-frequency oscillation of the EGAM amplitude is observed in the nonlinear phase after the saturation, for

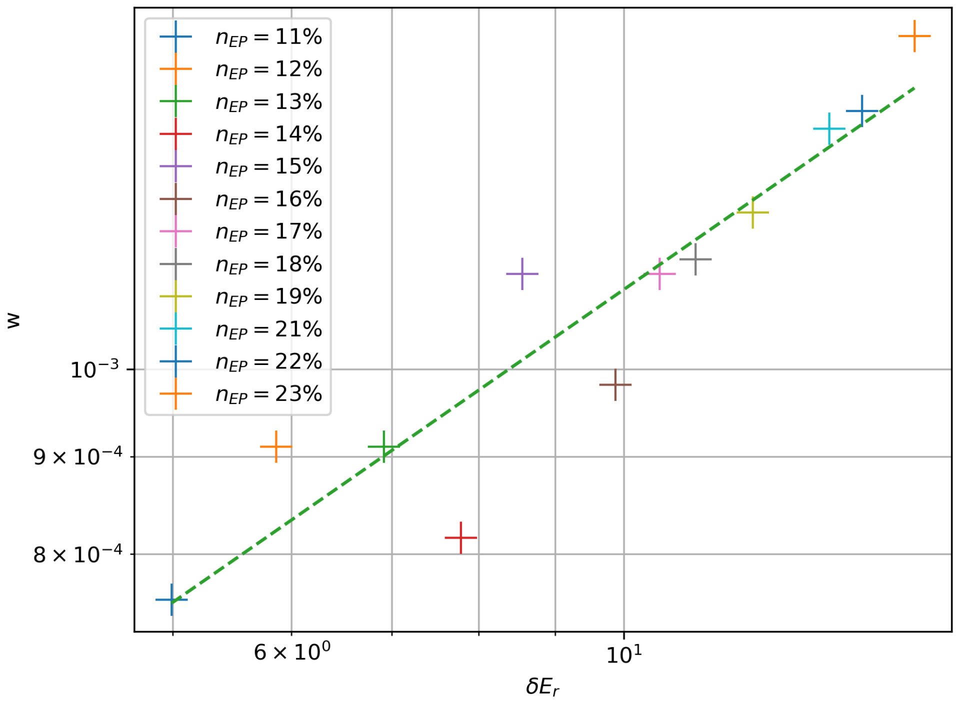


Figure 5.1: nonlinear oscillations as a function of the saturation level at different EP concentrations on a logarithmic scale.

at least one of two periods, then fading into noise. We measure this nonlinear frequency, and plot it versus the EGAM saturation level δE_r. The result can be seen in Fig. 5.1, on a logarithmic scale. The linear fitting of the points gives the scaling of the frequency of the nonlinear oscillations ω_{nl}:

$$\omega_{nl} \propto \delta E_r{}^{\alpha} \tag{5.3.1}$$

For the regime considered here, and with the scan used, we find a value of $\alpha \simeq 0.6$.

Note that the saturation level of EGAM has been described in previous papers, see, for example, [7], and its scaling with the linear properties of the EGAM has been given. Therefore, in principle, thanks to Eq. 5.3.1, we provide a way to predict the frequency of the nonlinear oscillations of EGAMs, given the linear properties. This can be used in the framework, for example, of quasilinear models, which need reduced models of the nonlinear properties, as function of the linear properties.

5.4 Comparison with BPI

As already shown, there is a one-to-one correspondence between the Beam Plasma Instability and the saturation mechanism of EGAMs. In Fig. 3.2 we can see the evolution of the amplitude of the Langmuir wave, obtained by solving the model proposed in Ref. [1]. In his model the frequency of the wave is supposed to be fixed, so that we do not see the frequency evolution we can see in the EGAMs. The saturation mechanism is similar: after a linear phase the wave saturates due to the energetic particle redistribution in phase space. We can see here the nonlinear oscillations that we analyzed in the previous section. In this case, as measured by Levin the frequency of the oscillation is of the same order of the bounce frequency of the particles.

In this section, we want to investigate if such a dependence also occurs for EGAMs. To this aim, we are interested in finding a possible scaling between the EGAM nonlinear frequency and the bounce frequency of trapped particles, as shown for BPI by Levin in 1972, see Ref. [1]. The nonlinear EGAM frequency can be measured by detecting numerically the peaks of the low-frequency oscillations. On the other end we use the result obtained by Qiu in 2011([10]), and verified by Biancalani in 2018([8]) that links the bounce frequency with the saturation value. The equation is the following:

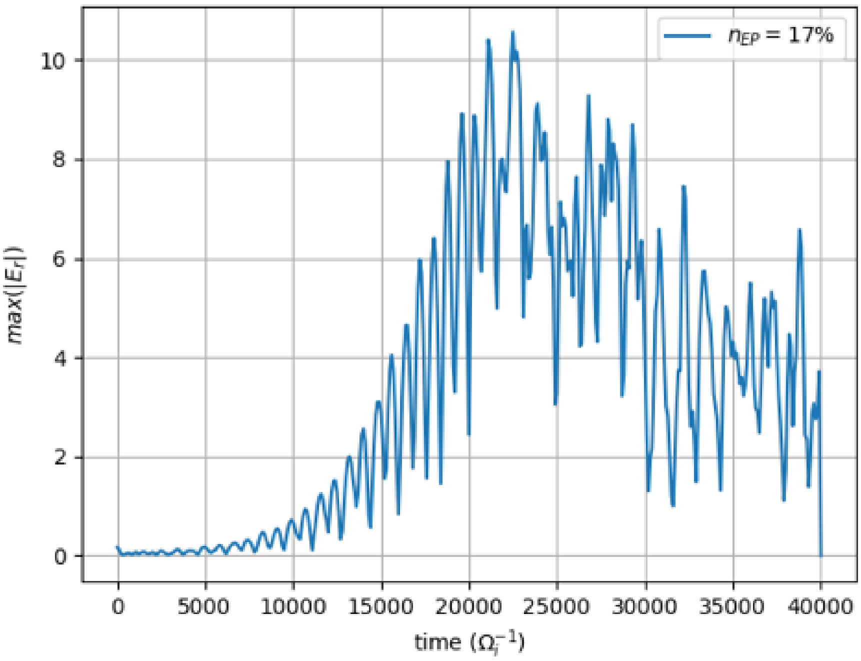
$$\omega_b^2 = \frac{e\hat{V}_{dc}}{2m_{EP}v_{||0}qR_0}\delta E_r. \tag{5.4.1}$$

The mass of energetic particles, m_{EP}, is considered to be the mass of the bulk, $v_{||0}$ is the resonant velocity and $\hat{V}_{dc} = \frac{m_{EP}v_{||0}^2}{eBR}$ is the magnetic curvature drift. So we can compare the nonlinear oscillations with the saturation level of the electric field.

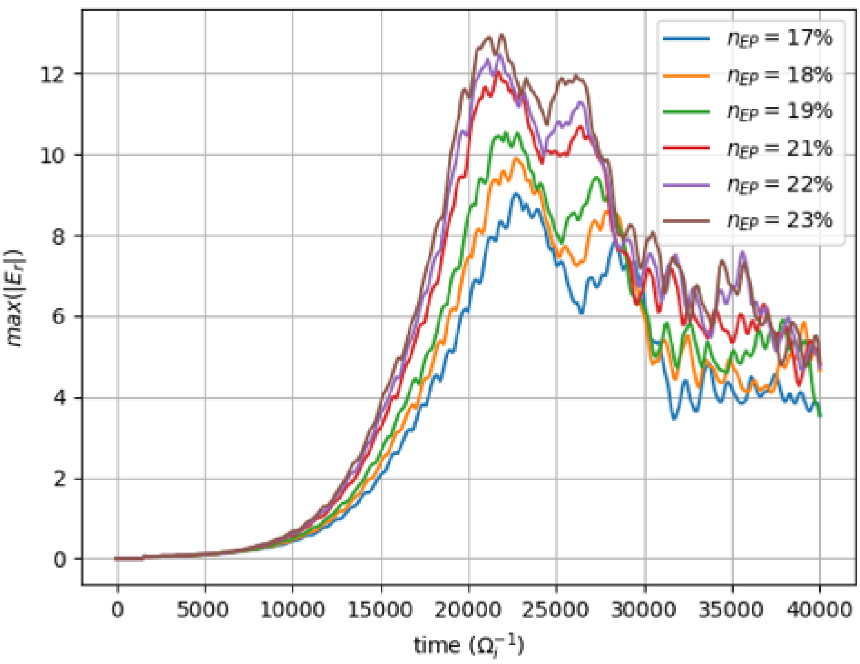
By using eq. (3) and (4) we can see that even in EGAMs the nonlinear frequency and the bounce frequency are linked by the relation:

$$\omega_b \propto \omega_{nl}^{\beta} \tag{5.4.2}$$

with $\beta = 1/2\alpha$. In the regime considered, here, we obtain $\beta_{EGAM} \simeq 0.83$, which is very similar (within the error bars of the measurements of our GK simulations) to the value of the BPI: $\beta_{BPI} \simeq 1$. This result shows one more similarity between the EGAM and the BPI. This similarity can be used to better understand the physics of the EGAM nonlinear dynamics, by using reduced models such as a solver for the BPI problem.



(a) Maximum of the radial electric field as a function of time.



(b) Mobile mean of the maximum at different EP concentration.

Figure 5.2: EGAM analysis.

Chapter 6

Summary and Outlooks

In this thesis, we have studied the Beam Plasma Instability Problem with an interest both in the theoretical study of the equations and in its applications in tokamak physics.
We adopted the model of Levin [1], where the Langmuir wave frequency and radial structure are approximated as constant, and only the amplitude is let evolve in time in a self consistent way with respect to the electron beam dynamics.
The theoretical study of this model has been performed in chapter 4. We generalized Levin's system, wich is denoted as the Vlasov-wave amplitude model. We have then prooved existence and uniqueness of the solution using a fixed point argument. We also studied the equilibrium states of the system, finding a family of equilibrium distribution. To complete the analysis we performed numerical simulations, studying the evolution of the amplitude in time and the particle dynamics with a Particle in Cell (PIC) code.
In chapter 5 we have compared the nonlinear dynamics of the beam-plasma instability (BPI) and the energetic particle induced geodesic acoustic mode (EGAM). EGAMs are zonal instabilities observed in tokamaks, driven unstable by the nonuniformity of energetic particles (EP). Note that perturbation of a zonal electric field, like the one of the EGAMs, can be studied as a renormalized nonlinear equilibrium, dubbed Zonal State [38, 39, 40].
We studied the EGAMs by means of global gyrokinetic simulations using the particle in cell code ORB5 [9]. We have found that the EGAM grows linearly, then saturates due to a similar mechanism as the BPI: a redistribution of the EPs in phase space. We observed nonlinear oscillations of the EGAM electric field, right after the saturation. The study of this nonlinear frequency, for different values of the drive, allowed us to find a similar linear scaling of the nonlinear frequency with the bounce frequency of the particles trapped in the wave. The nonlinear frequency is found to scale with a power of 0.6 as a function of the field, which is very similar (within the measurements error bars) to the value expected for the BPI (which goes like the square root of the field amplitude). This confirms that the nonlinear dynamics of forced oscillations like EGAMs in tokamak plasmas, has strong similarities with the BPI in uniform plasmas. Both saturate due to the EP redistribution in phase space.

Following this result, we propose here a novel diagnostics for the measurement of the EGAM amplitude in tokamak plasmas. This electric field is often difficult to measure due to the lack of diagnostics capable of entering the inner part of the tokamak. Here we propose to measure the frequency of the nonlinear oscillations of the field amplitude, instead. This can be found with diagnostics which are positioned outside the tokamak. Once this frequency is measured, the field amplitude can be indirectly estimated using the scaling we provide in this thesis.
This work leaves us with some possible future extensions. On the theoretical side, a further investigation of equilibrium states can be done. In fact it is interesting to see if we can find some stable and unstable equilibria, and to try and verify it with numerical simulations.

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