UNIVERSITÀ
DI PAVIA

**Dipartimento di Scienze Economiche e Aziendali**

**Corso di Laurea Magistrale in Finance**

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# Binomial approximation of Brownian motion and its maximum

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**Anno Accademico 2025-2026**

# Abstract

This thesis investigates the approximation of Brownian motion and its running maximum by means of a suitably rescaled simple random walk and its application to the pricing of exotic options. The main objective is to assess whether this approximation can provide a valid and competitive alternative framework to standard Brownian-motion-based methods.

The financial instruments considered in this work are path-dependent options whose payoff depends on several characteristics of the underlying asset price process, including the entire price history during the contract lifetime, the possible crossing of predetermined barrier levels, and the terminal asset price at maturity.

The study begins by introducing the probabilistic and financial concepts required throughout the thesis, with a focus on stochastic processes, option pricing theory, and the Black-Scholes framework. These tools provide the theoretical foundation for the numerical methods developed later and represent the standard benchmark for pricing path-dependent derivatives.

The core of the thesis is based on Donsker's Theorem, which establishes the weak convergence of a suitably normalized random walk to Brownian motion. This result is used to construct a numerical pricing procedure based on a binomial-tree framework, extending its traditional application to the valuation of pure European options by considering path-dependent contracts involving the running maximum, with particular attention to barrier-type contracts.

After developing the discrete approximation of the pricing function, the convergence rate towards the continuous model is investigated both analytically and numerically. The analysis shows that the approximation error decreases as the number of discretization steps increases.

Finally, numerical experiments are performed to evaluate the accuracy of the proposed methodology. Values obtained through the random-walk approximation are compared with Monte Carlo estimates and numerical integration results. The results show that the proposed approach provides accurate approximations for barrier-type contracts and represents an effective computational framework. When the discretization level remains computationally manageable, the random-walk approach provides competitive results while retaining the structural advantages of lattice methods. In particular, the backward-induction framework naturally extends to dynamic hedging applications, although at the cost of increased memory requirements.

# Riassunto

Questa tesi indaga l'approssimazione del moto browniano e del suo massimo mediante una passeggiata casuale semplice opportunamente riscalata e la sua applicazione alla determinazione del prezzo di opzioni esotiche. L'obiettivo principale è valutare se questa approssimazione possa fornire un quadro alternativo valido e competitivo ai metodi standard basati sul moto browniano.

Gli strumenti finanziari considerati in questo lavoro sono opzioni europee il cui profitto dipende da diverse caratteristiche del processo di prezzo delle attività sottostanti, tra cui l'intera cronologia dei prezzi durante la durata del contratto, il possibile superamento di livelli di barriera predeterminati e il prezzo terminale delle attività alla scadenza.

Lo studio inizia introducendo i concetti probabilistici e finanziari richiesti in tutta la tesi, con particolare attenzione ai processi stocastici, alla teoria dei prezzi delle opzioni e al modello Black-Scholes. Questi strumenti forniscono la base teorica per i metodi numerici sviluppati successivamente e rappresentano il punto di riferimento standard per la determinazione del prezzo dei derivati dipendenti dal percorso.

Il nucleo della tesi si basa sul Teorema di Donsker, che stabilisce la debole convergenza di una passeggiata casuale opportunamente riscalata al moto browniano. Questo risultato viene utilizzato per costruire una procedura di determinazione esplicita dei prezzi basata su una struttura ad albero binomiale, estendendo la sua applicazione tradizionale alla valutazione di opzioni europee pure, considerando contratti dipendenti dal percorso che coinvolgono il massimo corrente, con particolare attenzione ai contratti di tipo barriera.

Dopo aver sviluppato l'approssimazione discreta della funzione di prezzo, il tasso di convergenza verso il modello continuo viene studiato sia analiticamente che numericamente. L'analisi mostra che l'errore di approssimazione diminuisce all'aumentare del numero di passaggi di discretizzazione.

Infine, vengono eseguiti esperimenti numerici per valutare l'accuratezza della metodologia proposta. I valori ottenuti tramite l'approssimazione random-walk vengono confrontati con le stime Monte Carlo e i benchmark di integrazione numerica. I risultati mostrano che l'approccio proposto fornisce approssimazioni accurate per i contratti di tipo barriera e rappresenta un quadro computazionale efficace. Quando il livello di discretizzazione rimane computazionalmente gestibile, l'approccio random-walk fornisce risultati competitivi pur mantenendo i vantaggi strutturali dei metodi reticolari. In particolare, il framework di induzione all'indietro si estende naturalmente alle applicazioni di copertura dinamica, anche se a costo di maggiori requisiti di memoria.

# Contents

# Introduction

Brownian motion is an extremely important stochastic process in mathematics, physics, and quantitative finance. In financial mathematics, it plays a central role in modeling the random evolution of asset prices and constitutes the foundation of many modern option pricing models. In particular, it is a key component of the Black-Scholes framework and of several numerical methods used to evaluate derivative securities.

This thesis investigates the approximation of Brownian motion and its running maximum through a suitably rescaled simple random walk and studies its application to the pricing of path-dependent derivatives, with particular attention to path-dependent contracts involving the running maximum, especially barrier-type contracts. While Brownian motion naturally arises in continuous-time models, random walks provide a discrete-time framework that is often easier to analyze and implement computationally. The relationship between these two processes therefore represents an important theoretical and practical topic in quantitative finance.

The first chapter is divided into two sections. The first introduces the probabilistic concepts required throughout the thesis, including probability spaces, filtrations, the Markov property, stochastic processes, and stochastic integration. The Markov property plays a central role in the numerical implementation developed later, since it allows the pricing problem to be solved through backward induction. Particular attention is devoted to the two stochastic processes that form the basis of this work: the simple random walk, denoted by $(S_n)_{n\geq 0}$, and Brownian motion, denoted by $(B_t)_{t\geq 0}$. These concepts provide the mathematical framework necessary to study option pricing in both discrete and continuous time.

The second section introduces the financial concepts used throughout the thesis. It begins with the notions of trading strategies and the associated notation, and then presents the main classes of options considered in this work, with particular attention to barrier contracts. The chapter concludes by introducing the Cox–Ross–Rubinstein and Black–Scholes pricing models. In both frameworks, option prices can be expressed in terms of expected values. Within the Black–Scholes model, these expectations are determined by the dynamics of Brownian motion and the trajectories of the underlying asset-price process. The presentation is divided into discrete-time and continuous-time settings in order to distinguish clearly between the different models and notations. These tools provide the theoretical foundation for the numerical methods developed later and serve as reference models for assessing the validity of the proposed approximation.

The second chapter introduces a fundamental theorem underlying this thesis: Donsker's Theorem, which establishes the weak convergence of a suitably normalized simple random walk to Brownian motion. This result provides the theoretical justification for replacing a continuous-time pricing problem with a suitable discrete-time approximation. The chapter also presents the main probabilistic results required for the proof of Donsker's Theorem and for its extension to path-dependent functionals. This extension is particularly important for the class of options considered in this work, since their valuation depends on the running maximum of the underlying process and therefore on the path followed by the process rather than solely on its terminal value.

The chapter then turns to the class of path-dependent contracts considered in this work. Unlike pure European options, whose value depends only on the terminal value of the underlying

asset, these contracts also depend on path-related quantities. In particular, special attention is devoted to derivatives whose payoff depends on the running maximum of the underlying process. This additional source of path dependence requires the introduction of a second state variable and leads naturally to a two-dimensional pricing problem.

More precisely, for an option with maturity $T$, the quantity of interest is the conditional expectation

$$u : [0,T] \times \{(x,y) \in \mathbb{R}^2 : x \le y,\ y \ge 0\} \longrightarrow \mathbb{R},$$
$$u(t,x,y) = \mathbb{E}\left[ f(B_T, B_T^*) \mid B_t = x,\ B_t^* = y \right],$$

where $(B_t)_{t\ge 0}$ denotes a Brownian motion and $(B_t^*)_{t\ge 0} = \max_{0\le s\le t} B_s$ its running maximum.

This formulation highlights the central role played by the pair $(B_t, B_t^*)$, consisting of Brownian motion and its running maximum. Consequently, the analysis is no longer restricted to the terminal distribution of Brownian motion but also requires the study of the joint behaviour of the process and its maximum.

By exploiting the weak convergence established by Donsker's Theorem, the pair $(B_T, B_T^*)$ can be approximated by the rescaled random walk and its running maximum,

$$\left( \frac{S_n}{\sqrt{n}}, \frac{S_n^*}{\sqrt{n}} \right),$$

where $S_n$ denotes the simple random walk after $n$ steps and $S_n^*$ its running maximum. This allows the original continuous-time problem to be studied within a discrete framework.

The final part of the chapter develops the tools required to analyse this approximation. In particular, it derives the joint density of Brownian motion and its running maximum and presents the convergence-rate results that quantify the approximation error of the random-walk approach and justify the numerical methods implemented in the following chapter.

The third and final chapter develops the numerical implementation of the approximation introduced in the previous chapter. The proposed methodology is compared with two reference methods: Monte Carlo simulation of Brownian motion and numerical integration based on the joint density of Brownian motion and its running maximum. For the class of contracts considered in this thesis, and owing to the availability of an explicit joint density, numerical integration represents both the accuracy benchmark and the computationally most efficient method, since it avoids statistical error and requires only a moderate number of grid points.

The chapter begins by presenting the theoretical procedures and the corresponding algorithms used in each numerical method. The three approaches are then compared through a series of path-dependent examples. In order to obtain a fair comparison, the computational parameters are selected so that the different methods require approximately the same number of operations.

Particular attention is devoted to the analysis of convergence and approximation errors. For both Monte Carlo simulation and the tree method, the numerical results are used to assess the accuracy, robustness, and computational efficiency of the proposed approximation. In addition, the chapter highlights the trade-off between accuracy, computational cost, and the structural advantages provided by the lattice framework.

The objective of this thesis is therefore twofold. First, it provides both a theoretical and a numerical analysis of the approximation of Brownian motion and its running maximum through

a simple random walk. Second, it evaluates the extent to which this approximation can provide a valid and competitive numerical framework for the valuation of path-dependent contracts by comparing its performance with Monte Carlo simulation and numerical integration.

# Chapter 1

# Probability and Finance

This chapter introduces the probabilistic and financial concepts required throughout the thesis. Its objective is to provide the theoretical framework necessary to study the approximation of Brownian motion through a simple random walk and to understand the pricing methodologies employed in the subsequent chapters.

The first part of the chapter focuses on probability theory and stochastic processes. Starting from the definition of a probability space, we introduce filtrations, stochastic processes, the Markov property, stochastic integration, and Itô's Lemma. Particular attention is devoted to Brownian motion and the simple random walk, which constitute the two fundamental stochastic processes considered throughout this work.

The second part of the chapter presents the financial concepts and pricing models required for the valuation of derivative securities. We introduce contingent claims, trading strategies, arbitrage-free pricing, and market completeness before presenting the Cox-Ross-Rubinstein model and the Black-Scholes framework. The latter provides the continuous-time framework from which the benchmark values used in the numerical analysis are derived.

## 1.1 Elements of Probability

**Definition 1.1.1.** *Let $\Omega$ be a non-empty set. A $\sigma$-field $\mathcal{F}$ on $\Omega$ is a family of subsets of $\Omega$ such that*

1) the empty set $\emptyset$ belongs to $\mathcal{F}$,

2) if $A$ belongs to $\mathcal{F}$, then so does the complement $\Omega \backslash A$,

3) if $A_1, A_2, \ldots$ is a sequence of sets in $\mathcal{F}$, then their union $A_1 \cup A_2 \cup \ldots$ also belongs to $\mathcal{F}$.

The sets belonging to $\mathcal{F}$ are called events and $\Omega$ is the set of all possible outcomes. The pair $(\Omega, \mathcal{F})$ is called a measurable space.

**Definition 1.1.2.** *Let $\mathcal{F}$ be a $\sigma$-field on $\Omega$. A probability measure $P$ on the measurable space $(\Omega, \mathcal{F})$ is a function*

$$P : \mathcal{F} \to [0, 1],$$

*such that*

*1) $P(\Omega) = 1$,*

*2) if $A_1, A_2, \ldots$ are pairwise disjoint sets belonging to $\mathcal{F}$, then*

$$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots.$$

*The triplet $(\Omega, \mathcal{F}, P)$ is called a probability space.*

**Definition 1.1.3.** *A family of $\sigma$-fields $(\mathcal{F}_t)_{t\in T}$ on $\Omega$ is called a filtration if*

$$\mathcal{F}_s \subseteq \mathcal{F}_t, \qquad s \leq t,$$

*for every $s, t \in T$ such that $s \leq t$.*

In general, $\mathcal{F}_t$ represents our knowledge at time $t$. It contains all events $A$ such that, at time $t$, it is possible to decide whether $A$ has occurred or not.

**Definition 1.1.4.** *A stochastic process is a family of random variables $(X(t))_{t\in T}$. When $T$ is a discrete set, such as: $T = \{1, 2, \ldots\}$ we shall say that $X(t)$ is a stochastic process in discrete time. When $T$ is a continuous set, (for instance $T = [0, \infty)$ or a finite interval), we shall say that $X(t)$ is a stochastic process in continuous time.*

Sometimes the stochastic process can be seen as a function of both elementary events and time by the relation

$$X(t, \omega) = X(t)(\omega), \quad \text{for any } t \in T \text{ and } \omega \in \Omega.$$

**Definition 1.1.5.** *For any fixed $\omega \in \Omega$ the function*

$$t \mapsto X(t, \omega), \quad t \in T$$

*is called a sample path (or trajectory) of the stochastic process $X$.*

The concept of path is particularly relevant in this thesis, since the path-dependent contracts considered in this thesis depend on the entire evolution of the underlying process rather than only on its terminal value. Moreover, the Brownian motion simulation algorithm developed in Chapter 3 is based entirely on the simulation of sample paths.

**Definition 1.1.6.** *Let $X = (X(t))_{t\in T}$ be a stochastic process. The filtration*

$$\mathcal{F}_t = \sigma\{X(s) : \ s \leq t\}$$

*is called the natural filtration generated by $X$.*

The natural filtration represents the information generated by the process up to time $t$. Throughout this thesis, whenever no further specification is given, stochastic processes are considered with respect to their natural filtration.

**Definition 1.1.7.** *A family of random variables $(X(t))_{t\in T}$ is said to be adapted to the filtration $(\mathcal{F}_t)_{t\in T}$ if $X(t)$ is $\mathcal{F}_t$-measurable for every $t \in T$.*

**Definition 1.1.8.** *Let $\mathcal{F}_n,\ n \geq 1$, be a filtration. A sequence $X_n,\ n \geq 1$ is said to be a predictable sequence if $X_0$ is $\mathcal{F}_0$ -measurable and $X_n$ is $\mathcal{F}_{n-1}$ -measurable for every $n \geq 1$.*

In other words, the value of $X_n$ may be predicted from the information available at time $n - 1$.

In this thesis we are particularly concentrated on two stochastic processes that play a central role in probability theory and quantitative finance: the simple random walk and Brownian motion. The former provides a discrete-time framework, while the latter represents its

continuous-time counterpart and constitutes the fundamental building block of the Black-Scholes model.

**Definition 1.1.9.** *Let $p \in (0,1)$, given a sequence of i.i.d. random variables $\{X_i\}_{i\in\mathbb{N}}$ with*

$$P(X_i = +1) = p, \quad P(X_i = -1) = 1 - p =: q.$$

*We define the random walk $S = (S_n)_{n\in\mathbb{N}}$ as the process:*

$$S_0 = s_0, \quad S_n = s_0 + X_1 + X_2 + \ldots + X_n \quad \text{for } n \geq 1,$$

*for a given initial value $s_0 \in \mathbb{Z}$.*

The running maximum associated with the simple random walk is defined as

$$S_n^* = \max_{0\leq k\leq n} S_k, \qquad n \geq 0.$$

The second stochastic process considered throughout this thesis is Brownian motion. It was originally introduced to describe the random motion of microscopic particles suspended in a fluid. Over time, it has found applications in several fields, including physics, mathematics, finance, and engineering.

Brownian motion is the fundamental stochastic process of continuous-time financial modelling. Its importance stems from the fact that it constitutes the source of randomness in the Black-Scholes model and in many other continuous-time models used in quantitative finance. Moreover, Donsker's Theorem establishes that Brownian motion can be approximated by a suitably rescaled simple random walk, providing the theoretical foundation for the numerical methodology developed in the subsequent chapters.

**Definition 1.1.10.** *A stochastic process $(B(t))_{t\geq 0}$ with values in $\mathbb{R}$ defined for $t \in [0,\infty)$, is called a Brownian motion if and only if the following conditions hold:*

1) *$B(0) = 0$ a.s.;*
2) *the sample paths $t \mapsto B(t)$ are a.s. continuous;*
3) *$B(t)$ has stationary independent increments;*
4) *the increment $B(t) - B(s)$ has the normal distribution with mean $0$ and variance $t - s$ for any $0 \leq s < t$.*

The combination of independent increments, stationarity, and continuity makes Brownian motion particularly suitable for modelling the evolution of financial assets.

### 1.1.1 Markov Property

The Markov property is a fundamental concept that enables the development of algorithms with reduced computational complexity. In this thesis, its relevance also comes from the fact that both processes considered here, namely the simple random walk and Brownian motion, are Markov processes. The definition of the Markov property depends on the underlying time framework. In the following, we first introduce the discrete- and continuous-time definitions and

then verify that the two stochastic processes considered throughout this thesis, namely the simple random walk and Brownian motion, satisfy the Markov property.

**Definition 1.1.11.** *Suppose that $D$ is a finite or a countable set. Suppose also that a probability space $(\Omega, \mathcal{F}, P)$ is given. A $D$-valued sequence of random variables $(X_n)_{n\in\mathbb{N}}$ is called a $D$-valued Markov chain if for all $n \in \mathbb{N}$*

$$P(X_{n+1} = s_{n+1}|X_0 = s_0, \ldots, X_n = s_n) = P(X_{n+1} = s_{n+1}|X_n = s_n).$$

We now verify that the simple random walk $(S_n)_{n\geq 0}$ satisfies the Markov property. From Definition 1.1.9 we have

$$S_n = S_{n-1} + X_n$$

and therefore

$$S_{n+1} = S_n + X_{n+1}$$

where the increments $(X_n)_{n\geq 1}$ are i.i.d.

Since the increment $X_{n+1}$ is independent of the previous increments $X_1, \ldots, X_n$, and each $S_k,\ k \leq n$ is a function of $X_1, \ldots, X_n$, it follows that $X_{n+1}$ is independent of $S_0, S_1, \ldots$.

Since we condition on the event $S_n = s_n$, the value of $S_n$ is fixed.

$$\begin{aligned} P(S_{n+1} = s_{n+1}|S_0 = s_0, \ldots, S_n = s_n) &= P(S_n + X_{n+1} = s_{n+1}|S_0 = s_0, \ldots, S_n = s_n) \\ &= P(s_n + X_{n+1} = s_{n+1}) \\ &= P(X_{n+1} = s_{n+1} - s_n). \end{aligned}$$

Similarly

$$\begin{aligned} P(S_{n+1} = s_{n+1}|S_n = s_n) &= P(S_n + X_{n+1} = s_{n+1}|S_n = s_n) \\ &= P(s_n + X_{n+1} = s_{n+1}) \\ &= P(X_{n+1} = s_{n+1} - s_n). \end{aligned}$$

Comparing the two expressions, we obtain

$$P(S_{n+1} = s_{n+1}|S_0 = s_0, \ldots, S_n = s_n) = P(S_{n+1} = s_{n+1}|S_n = s_n).$$

The Markov property plays a fundamental role in option pricing and stochastic modelling. It implies that the future evolution of a process depends only on its current state and not on its entire past history. This property considerably simplifies both theoretical analysis and numerical implementation, since conditional expectations can be expressed in terms of the current state variable alone.

In continuous time instead we have the following definition

**Definition 1.1.12.** *A stochastic process $(X_t)_{t\geq 0}$ adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfies the Markov property if for any bounded measurable function $f$ and for any $s$ and $t$ such that $s \leq t$, we have*

$$\mathbb{E}[f(X_t)|\mathcal{F}_s] = \mathbb{E}[f(X_t)|\sigma(X_s)].$$

We now verify that Brownian motion satisfies the Markov property.

Let $(B_t)_{t\geq 0}$ be a Brownian motion. Using the properties of the Brownian motion we are allowed to write $B_t = B_s + (B_t - B_s)$. Taking the conditional expectations with respect to $\mathcal{F}_s$

$$\mathbb{E}[f(B_t)|\mathcal{F}_s] = \mathbb{E}[f(B_s + (B_t - B_s))|\mathcal{F}_s].$$

Since $B_t - B_s$ is independent of $\mathcal{F}_s$ and $B_s$ is $\mathcal{F}_s$-measurable, we define

$$g(x) := \mathbb{E}[f(x + (B_t - B_s))].$$

Then $g(B_s)$ is $\mathcal{F}_s$-measurable and satisfies

$$g(B_s) = \mathbb{E}[f(B_s + (B_t - B_s))|\mathcal{F}_s].$$

By definition of conditional expectation with respect to $\sigma(B_s)$,

$$g(B_s) = \mathbb{E}[f(B_t)|\sigma(B_s)].$$

Therefore,

$$\mathbb{E}[f(B_t) \mid \mathcal{F}_s] = \mathbb{E}[f(B_t)|\sigma(B_s)].$$

Hence Brownian motion satisfies the Markov property.

**Definition 1.1.13.** *A family of random variables $(X_t)_{t\geq 0}$ is called a martingale with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ if*

*1) $X(t)$ is integrable for each $t$;*

*2) $X(t)_{t\geq 0}$ is adapted to $(\mathcal{F}_t)_{t\geq 0}$;*

*3) $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$ a.s for every $0 \leq s \leq t$.*

Brownian motion is a martingale with respect to its natural filtration. Likewise, a simple random walk is a martingale whenever it is symmetric

$$P(X_i = +1) = P(X_i = -1) = \frac{1}{2}.$$

Indeed, under this condition the increments have zero mean.

### 1.1.2 Stochastic Integral

The classical Riemann integral is defined as the limit of finite sums of the form

$$\sum_{i=1}^{n} f(\xi_i)\Delta x_i,$$

where $\Delta x_i = x_i - x_{i-1}$ and $\xi_i \in [x_{i-1}, x_i]$. As the mesh of the partition tends to zero, these sums converge to the integral

$$\int_a^b f(x)\, dx.$$

The Itô integral follows a similar construction, replacing deterministic increments $\Delta x_i$ with increments of a Brownian motion.

**Definition 1.1.14.** *A stochastic process $(f(t))_{t\geq 0}$ is called a random step process if there is a finite sequence of numbers $0 = t_0 < t_1 < \ldots < t_n$ and square integrable random variables $\eta_0, \eta_1, \ldots, \eta_{n-1}$ such that*

$$f(t) = \sum_{j=0}^{n-1} \eta_j \cdot \mathbb{1}_{[t_j, t_{j+1})}(t),$$

*where $\eta_j$ is $\mathcal{F}_{t_j}$-measurable for $j = 0, 1, \ldots, n-1$. The set of random step processes will be denoted by $\mathcal{M}^2_{steps}$.*

**Definition 1.1.15.** *Let $f$ be a stochastic process introduced in Definition 1.1.14. We denote by $\mathcal{M}^2$ the class of stochastic processes $(f_t)_{t\geq 0}$ such that*

$$\mathbb{E}\left[\int_0^\infty |f(t)|^2\, dt\right] < \infty,$$

*and there is a sequence $f_1, f_2, \ldots \in \mathcal{M}^2_{steps}$ of random step processes such that*

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^\infty |f(t) - f_n(t)|^2\, dt\right] = 0.$$

*In this case we shall say that the sequence of random step processes $f_1, f_2, \ldots$ approximates $f$ in $\mathcal{M}^2$.*

The space $\mathcal{M}^2$ contains all square-integrable stochastic processes and constitutes the natural domain for the construction of the Itô integral.

The Itô integral is first defined on the class of step processes $\mathcal{M}^2_{\text{steps}}$. The extension to the whole space $\mathcal{M}^2$ is then obtained by density arguments. More precisely, every process in $\mathcal{M}^2$ can be approximated by a sequence of step processes, while Itô's isometry guarantees that the corresponding sequence of stochastic integrals converges in $L^2$.

**Definition 1.1.16.** *Let $f \in \mathcal{M}^2_{\text{steps}}$ be a random step process of the form*

$$f(t) = \sum_{j=0}^{n-1} \eta_j \mathbb{1}_{[t_j, t_{j+1})}(t), \qquad 0 \leq t \leq t_n.$$

*The stochastic integral of $f$ on $[0, t_n]$ is defined by*

$$I(f) = \sum_{j=0}^{n-1} \eta_j \left(B(t_{j+1}) - B(t_j)\right).$$

*By the density of $\mathcal{M}^2_{\text{steps}}$ in $\mathcal{M}^2$, this construction extends to every process $f \in \mathcal{M}^2_T$.*
*The resulting stochastic integral satisfies the following fundamental properties, which hold for any $f, g \in \mathcal{M}^2_T$, any $\alpha, \beta \in \mathbb{R}$, and any $0 \leq s < t \leq T$:*

*1) **Linearity.** For $f, g \in \mathcal{M}^2_T$ and $\alpha, \beta \in \mathbb{R}$,*

$$\int_0^t (\alpha \cdot f(r) + \beta \cdot g(r)) dB(r) = \alpha \cdot \int_0^t f(r) dB(r) + \beta \cdot \int_0^t g(r) dB(r).$$

*2)* ***Itô isometry.*** *For $f \in \mathcal{M}_T^2$,*

$$\mathbb{E}\Big(\Big|\int_0^t f(r)dB(r)\Big|^2\Big) = \mathbb{E}\Big(\int_0^t |f(r)|^2 dr\Big).$$

*3)* ***Martingale property.*** *For $f$ progressively measurable and in $\mathcal{M}_T^2$, the process*

$$I_t = \int_0^t f(r)\, dB(r), \qquad 0 \le t \le T,$$

*is a martingale. Equivalently, for every $0 \le s < t \le T$,*

$$\mathbb{E}\left[\int_0^t f(r)\, dB(r)|\mathcal{F}_s\right] = \int_0^s f(r)\, dB(r).$$

While Brownian motion provides the fundamental source of randomness, most financial quantities cannot be described by a pure Brownian motion alone. In practice, stochastic models combine a deterministic component and a random component. This naturally leads to the notion of an Itô process, which constitutes the basic building block of continuous-time financial models.

**Definition 1.1.17.** *Let $(\Omega, \mathcal{F}, (\mathcal{F})_{t\ge0}, P)$ be a filtered probability space and let $(B_t)_{t\ge0}$ be an $\mathcal{F}_t$-Brownian motion. $(X_t)_{t\ge0}$ is an $\mathbb{R}$-valued Itô process if it can be represented as*

$$P \text{ a.s. for all } t \le T \quad X(T) = X(0) + \int_0^T a(t)dt + \int_0^T b(t)dB_t.$$

The first integral represents the deterministic component of the process, usually referred to as the drift, while the second integral represents the random fluctuations generated by Brownian motion. For this reason, Itô processes provide a natural framework for modelling asset prices in continuous time.

### 1.1.3 Itô's Lemma

**Theorem 1.1.1.** *Suppose that $F(t,x)$ is a real-valued function with continuous partial derivatives $F_t'(t,x), F_x'(t,x)$ and $F_{xx}''(t,x)$ for all $t > 0$ and $x \in \mathbb{R}$. We also assume that the process $F_x'(t, X(t))$ belongs to $M_T^2$, for all $T \ge 0$. Then $F(t, X(t))$ is an Itô process such that*

$$\begin{aligned} F(T,X(T)) - F(0,X(0)) = &\int_0^T F_t'(t,X(t))dt + \int_0^T F_x'(t,X(t))dX(t) + \\ &+ \frac{1}{2}\int_0^T F_{xx}''(t,X(t))d\langle X,X\rangle_t, \end{aligned}$$

*with: $dX(t) = adt + bdB(t),\ d\langle X,X\rangle_t = b^2dt$*

$$\begin{aligned} F(T,X(T)) - F(0,X(0)) = &\int_0^T F_t'(t,X(t))\, dt + \int_0^T aF_x'(t,X(t))\, dt + \\ &\int_0^T bF_x'(t,X(t))\, dB(t) + \frac{1}{2}\int_0^T F_{xx}''(t,X(t))b^2\, dt \\ = &\int_0^T (F_t'(t,X(t)) + aF_x'(t,X(t)) + \frac{1}{2}F_{xx}''(t,X(t))b^2)\, dt \end{aligned}$$

$$+ \int_0^T bF'_x(t, X(t))\, dB(t).$$

*In the form of a stochastic differential equation*

$$dF(t, X(t)) = (F'_t(t, X(t)) + a \cdot F'_x(t, X(t)) + \frac{1}{2}b^2 F''_{xx}(t, X(t)))dt + bF'_x(t, X(t))dB(t).$$

## 1.2 Financial Elements

The financial concepts introduced in this section are presented in both a discrete-time and a continuous-time framework.

In the discrete-time setting, we consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and denote by

$$S_n = (S_n^0, S_n^1, \ldots, S_n^d), \qquad 0 \le n \le N,$$

the vector of asset prices at time $n$, where $S_n^0$ represents the risk-free asset and $S_n^1, \ldots, S_n^d$ denote the risky assets.

The corresponding continuous-time process is denoted by $(S_t)_{t\ge 0}$ and will be used throughout the Black-Scholes model.

### 1.2.1 Strategies

One potential advantage of using a random-walk-based framework for pricing exotic options, compared with standard Brownian motion simulations, is the possibility of naturally constructing dynamic hedging strategies. Before introducing this point, we define the main concepts related to trading strategies, replication, and arbitrage-free pricing. As in the case of a Markov property, the definitions of a trading strategy and self-financing strategy depend on the underlying framework. In a discrete-time setting, they are defined as follows.

**Definition 1.2.1.** *A strategy (or trading strategy) is a predictable stochastic process*

$$\phi = \left((\phi_n^0, \phi_n^1, \ldots, \phi_n^d)\right)_{0\le n\le N}$$

*taking values in $\mathbb{R}^{d+1}$, where $\phi_n^i$ denotes the number of units of asset i held in the portfolio at time n. The process is predictable if*

$$\forall i \in 0, 1, \ldots, d, \begin{cases} \phi_0^i \text{ is } \mathcal{F}_0\text{-measurable}, \\ \phi_n^i \text{ is } \mathcal{F}_{n-1}\text{-measurable}, \quad \text{for } n \ge 1. \end{cases}$$

This means that portfolio positions at time $n$ are determined using only the information available at time $n-1$ and are held unchanged during the following period.

The value of the portfolio associated with the strategy $\phi$ at time $n$ is defined as

$$V_n(\phi) = \phi_n \cdot S_n = \sum_{i=0}^{d} \phi_n^i S_n^i, \qquad \forall n \in \{0, \ldots, N\}.$$

**Definition 1.2.2.** *A strategy is called self-financing if the following equation is satisfied*

$$\phi_n \cdot S_n = \phi_{n+1} \cdot S_n, \qquad \forall n \in \{0, \ldots, N-1\}.$$

A self-financing strategy allows the investors to rebalance the portfolio without injecting additional capital or withdrawing wealth. The portfolio adjustments are financed exclusively through the assets already held in the portfolio. This condition is fundamental in derivative pricing since replicating portfolios must be self-financing.

**Definition 1.2.3.** *A strategy $\phi$ is admissible if it is self-financing and satisfies*

$$V_n(\phi) \geq 0, \qquad \forall n \in \{0, \ldots, N\}.$$

**Definition 1.2.4.** *A contingent claim is a financial contract whose payoff depends on the realization of one or more underlying random variables. Formally, a contingent claim with maturity $N$ is represented by an $\mathcal{F}_N$-measurable random variable $h$. A European option with maturity $N$ is a non-negative contingent claim represented by an $\mathcal{F}_N$-measurable random variable $h \geq 0$.*

The same idea extends naturally to continuous-time models such as Black-Scholes, where the maturity is denoted by $T$ and the payoff is represented by a non-negative $\mathcal{F}_T$-measurable random variable.

Options constitute one of the most important classes of contingent claims and are the main financial instruments considered throughout this thesis.

**Definition 1.2.5.** *Let $h$ be an $\mathcal{F}_N$-measurable, non-negative random variable representing a contingent claim. An admissible strategy $\phi$ is said to replicate $h$ if its terminal portfolio value satisfies*

$$V_N(\phi) = h.$$

*In this case, $\phi$ is called a replicating strategy for the contingent claim $h$.*

The existence of replicating strategies is one of the fundamental ideas underlying modern option pricing theory. If a contingent claim can be replicated, then its arbitrage-free price must coincide with the initial value of the replicating portfolio.

**Definition 1.2.6.** *An arbitrage strategy is an admissible strategy with zero initial value and non-zero final value.*

$$V_0 = 0,$$
$$P(V_N > 0) > 0,$$

*A market is said to be viable if no arbitrage opportunities exist.*

**Theorem 1.2.1.** *The market is viable if and only if there exists a probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ such that the discounted prices of assets are $\mathbb{P}^*$-martingales.*

Two probability measures $\mathbb{P}_1$ and $\mathbb{P}_2$ are equivalent if and only if for any event $A$, $\mathbb{P}_1(A) = 0 \iff \mathbb{P}_2(A) = 0$.

**Definition 1.2.7.** *The market is complete if every contingent claim is attainable.*

In a complete and arbitrage-free market, every contingent claim admits a unique arbitrage-free price. This principle is the basis behind both the Cox-Ross-Rubinstein model and the Black-Scholes model. Furthermore, in complete markets, such as the CRR and Black–Scholes models, a replicating strategy provides a perfect hedge against the contingent claim. Therefore,

the notions of replicating strategy and hedging strategy coincide. In incomplete markets, however, perfect replication may not be possible and hedging strategies generally aim at reducing rather than eliminating risk.

The CRR model naturally supports the construction of dynamic hedging strategies. Since the binomial tree explicitly describes all possible future states of the underlying asset, backward induction provides both the option value and the corresponding replicating portfolio at every node.

Monte Carlo simulation, by contrast, is fundamentally a forward-looking methodology. Rather than constructing the state space of the process, it generates a collection of sample paths. Consequently, standard Monte Carlo methods do not directly provide dynamic hedging strategies and require additional techniques to estimate hedge ratios.

The possibility of constructing dynamic hedging strategies in a natural way represents one of the main practical advantages of the CRR framework.

### 1.2.2 Options

Options are the financial instruments considered throughout this thesis and constitute one of the most important classes of contingent claims.

An option gives its holder the right, but not the obligation, to buy or sell an underlying asset at a predetermined price on, or in some cases before, a specified maturity date.

The two most popular categories of options are calls and puts.

1) A call option gives the holder the right to buy the underlying asset. Its payoff at maturity is

$$h = (S_T - K)_+ = \max(S_T - K, 0).$$

2) A put option gives the holder the right to sell the underlying asset. Its payoff at maturity is

$$h = (K - S_T)_+ = \max(K - S_T, 0).$$

Here $S_T$ denotes the price of the underlying asset at maturity $T$, and $K$ is the strike price.

Options can be classified into several categories, including European, American, Asian, and barrier options. In this thesis, we focus on path-dependent options, with particular attention to barrier contracts.

For the class of barrier contracts considered in this thesis, the payoff is conveniently expressed in terms of the running maximum

$$M_T = \max_{0 \le t \le T} S_t.$$

Barrier options depend on whether the underlying asset crosses a predetermined level during the life of the contract. Barriers can be either soft or hard. A soft barrier modifies the payoff by introducing a penalty or a reward when the barrier is crossed, while a hard barrier can activate or deactivate the option.

Hard barrier options are divided into two main categories:

1) Knock-out options: the option expires if the barrier is reached.

2) Knock-in options: the option becomes active only if the barrier is reached.

They can be further classified as follows:

1) Down-and-out: the option expires if the underlying asset decreases below the barrier.

2) Up-and-out: the option expires if the underlying asset increases above the barrier.

3) Down-and-in: the option becomes active if the underlying asset decreases below the barrier.

4) Up-and-in: the option becomes active if the underlying asset increases above the barrier.

In this thesis, we consider in particular path-dependent contracts involving the running maximum, with particular attention to barrier-type contracts. The numerical analysis focuses on three path-dependent contracts involving the running maximum: a standard floating-strike lookback option, a contract with a soft barrier, and an up-and-out barrier contract. The corresponding payoff functions will be introduced explicitly in the final chapter.

### 1.2.3 Cox-Ross-Rubinstein Model

The Cox-Ross-Rubinstein model is a discrete-time model for option pricing. It consists of one risk-free asset with value

$$S_n^0 = (1+r)^n,$$

where $r$ is the one-period risk-free rate, and one risky asset with price $S_n$, $0 \le n \le N$.

Between two consecutive periods, the risky asset price changes by a multiplicative factor $u$ or $d$. The no-arbitrage condition is

$$0 < d < 1 + r < u.$$

Let

$$Y_n = \begin{cases} u, & \text{with probability } p, \\ d, & \text{with probability } 1-p. \end{cases}$$

The asset price evolves according to

$$S_n = S_{n-1} Y_n,$$

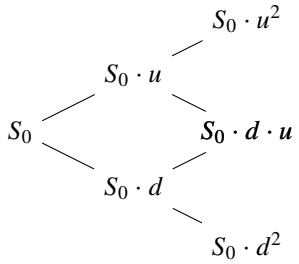
and therefore

$$S_n = S_0 \prod_{j=1}^{n} Y_j.$$

If $k \in 0, \ldots, n$ denotes the number of upward movements up to time $n$, then

$$S_n = S_0 u^k d^{n-k}.$$

The model allows one to price and replicate European contingent claims by backward induction on a binomial tree.




Thus, at each node, the asset price is fully determined by the number of upward movements, regardless of the order in which they occurred.

In the CRR model, the risk-neutral probability of an upward movement is

$$p = \frac{(1+r) - d}{u - d}.$$

Under the risk-neutral measure $\mathbb{P}^*$, the probability of reaching the node $S_0 u^k d^{n-k}$ at time $n$ is

$$\mathbb{P}^*(S_n = S_0 u^k d^{n-k}) = \binom{n}{k} (p)^k (1-p)^{n-k}.$$

The CRR model is both arbitrage-free and complete. Consequently, every European contingent claim admits a unique arbitrage-free price that can be obtained through replication arguments.

Let $h = h(S_N)$ be a European contingent claim. Its value is obtained by backward induction through the recursive equation

$$V(n, S_n) = \frac{p\, V(n+1, S_n u) + (1-p)\, V(n+1, S_n d)}{1+r}, \tag{1.1}$$

with terminal condition

$$V(N, S_N) = h(S_N).$$

A fundamental property explaining the wide use of the CRR model is the recombining structure of the tree. In a non-recombining tree, the number of final nodes grows exponentially and equals $2^N$. In the CRR tree, instead, there are only $N + 1$ final nodes at time $N$. The total number of nodes is

$$\frac{(N+1)(N+2)}{2},$$

so the computational cost is of order $O(N^2)$, rather than $O(2^N)$. This makes backward induction computationally efficient for standard European options.

The model is especially important in this work because it adapts a random-walk structure to the dynamics of a risky asset. In the final chapter, this discrete framework will be extended to price path-dependent options, including barrier contracts.

### 1.2.4 Black-Scholes Model

The Black-Scholes model is the standard continuous-time model for option pricing. Like the CRR model, it consists of a risky asset and a risk-free asset. Since it is formulated in continuous time, the model is defined on an interval $[0, T]$.

In this section, we present its main assumptions, properties, limitations, and its relevance for this thesis. The analytical pricing formula will be derived in the following section through Girsanov's theorem.

The model assumes that asset prices follow stochastic differential equations. The risk-free asset satisfies

$$dS_t^0 = rS_t^0\,dt,$$

where $r$ is the instantaneous risk-free interest rate. Hence,

$$S_t^0 = e^{rt}.$$

The risky asset follows the stochastic differential equation

$$dS_t = S_t(\mu\,dt + \sigma\,dB_t),$$

where $\mu$ is the drift, $\sigma$ is the volatility, and $B_t$ is a standard Brownian motion.

By Itô's Lemma, the solution is

$$S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right].$$

To verify this, define

$$X_t = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t.$$

Then

$$dX_t = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma\,dB_t.$$

Applying the Itô formula with

$$F(t,X) = f(t,x) = S_0e^x.$$

Its derivatives are

$$\begin{aligned}\partial_t f(t,x) &= 0,\\ \partial_x f(t,x) &= S_0e^x,\\ \partial_x^2 f(t,x) &= S_0e^x.\end{aligned}$$

We obtain

$$\begin{aligned}dS_t &= 0 + S_0e^x\,dX_t + \frac{1}{2}S_0e^x\,d\langle X,X\rangle_t,\\ dS_t &= S_0e^x\left(dX_t + \frac{1}{2}\,d\langle X,X\rangle_t\right).\end{aligned}$$

Since

$$d\langle X,X\rangle_t = \sigma^2dt,$$

we have

$$dS_t = S_t\left[\left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma\,dB_t + \frac{1}{2}\sigma^2dt\right],$$

and therefore

$$dS_t = S_t(\mu\, dt + \sigma\, dB_t).$$

The risky asset has several important properties:

1) It has continuous sample paths.

2) It follows a lognormal distribution. Equivalently,

$$\ln\left(\frac{S_t}{S_0}\right) \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right).$$

3) Its logarithmic increments are independent.

4) Its logarithmic increments are stationary.

Having specified the dynamics of the assets, we now introduce trading strategies and the notion of a self-financing portfolio.

**Definition 1.2.8.** *A strategy is a process $\phi = (\phi_t)_{0\le t\le T} = (H_t^0, H_t)_{0\le t\le T}$ with values in $\mathbb{R}^2$, adapted to the natural filtration $(\mathcal{F}_t)_{0\le t\le T}$ of the Brownian motion.*

$H_t^0$ and $H_t$ are respectively the quantities of riskless and risky asset held in portfolio at time $t$. The value of the portfolio at time $t$ is given by

$$V_t(\phi) = H_t^0\, S_t^0 + H_t\, S_t.$$

**Definition 1.2.9.** *A self-financing strategy is defined by a pair $\phi$ of adapted process $(H_t^0)_{0\le t\le T}$ and $(H_t)_{0\le t\le T}$ satisfying:*

*1)* $\int_0^T |H_t^0|\, dt + \int_0^T H_t^2\, dt < \infty \quad a.s.$

*2)* $H_t^0\, S_t^0 + H_t\, S_t = H_0^0\, S_0^0 + H_0\, S_0 + \int_0^t H_u^0\, dS_u^0 + \int_0^t H_u\, dS_u \quad a.s. \quad \text{for all } t \in [0,T].$

The relevance of the Black-Scholes model is not limited to these properties. Its main importance comes from the existence of an equivalent martingale measure $\mathbb{P}^*$, under which discounted asset prices become martingales. This allows one to derive arbitrage-free prices through risk-neutral valuation.

Despite its importance, the model has limitations. One of the main critical aspects is the assumption that logarithmic returns are normally distributed. Empirical financial returns often display skewness and excess kurtosis, which are not captured by the Gaussian distribution. For this reason, alternative models or modified distributions may be more appropriate in some applications.

Nevertheless, the Black-Scholes model remains fundamental for this thesis. First, it provides the continuous-time benchmark for the CRR model. Indeed, the CRR binomial tree converges to the Black-Scholes model as the number of time steps $N \to \infty$. This convergence is obtained by setting, for $\Delta t = \frac{T}{N}$,

1) $u = e^{\sigma\sqrt{\Delta t}}$,

2) $d = e^{-\sigma\sqrt{\Delta t}} = \dfrac{1}{u}$,

3) $p = \dfrac{e^{r\Delta t} - d}{u - d}$.

Under this parametrization, as $\Delta t \to 0$, the log-returns of the binomial tree converge in distribution to the Gaussian increments of the geometric Brownian motion. The CRR model therefore provides a natural discrete approximation of the Black-Scholes model, and its backward-induction formula converges to the Black-Scholes price.

Second, the analytical Black-Scholes formula will be used in the final chapter as a benchmark to assess the accuracy of the numerical results.

### 1.2.5 Girsanov's Theorem

**Definition 1.2.10.** *Let $(\theta_t)_{0\le t\le T}$ be an adapted process satisfying $\int_0^T \theta_s^2 ds < \infty$ a.s. and such that the process $(L_t)_{0\le t\le T}$ defined by:*

$$L_t = \exp\Big( - \int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t \theta_s^2 ds\Big)$$

*is a martingale. Then, under the probability $\mathbb{P}^{(L)}$ with density $L_T$ with respect to $\mathbb{P}$, the process $(W_t)_{0\le t\le T}$ defined by $W_t = B_t + \int_0^t \theta_s ds$ is an $(\mathcal{F}_t)$-standard Brownian motion.*

Girsanov's theorem allows the drift of Brownian motion to be modified through a change of probability measure while preserving the volatility structure. This result plays a fundamental role in mathematical finance because it permits the construction of an equivalent martingale measure under which discounted asset prices become martingales.

**Theorem 1.2.2** (Novikov condition)**.** *A sufficient condition for $(L_t)_{0\le t\le T}$ to be a martingale is:*

$$\mathbb{E}\Big[\exp\Big(\frac{1}{2}\cdot\int_0^T \theta_t^2 dt\Big)\Big] < \infty.$$

We show that there exists a probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ under which the discounted price process is a martingale.

*Proof.* The discounted risky asset is

$$\tilde{S}_t = e^{-rt} S_t.$$

Using the Black-Scholes dynamics, we obtain

$$d\tilde{S}_t = \tilde{S}_t\left((\mu - r)dt + \sigma\, dB_t\right).$$

By Girsanov's theorem,

$$W_t = B_t + \int_0^t \theta_s\, ds,$$

and therefore

$$dB_t = dW_t - \theta_t\, dt.$$

Substituting this expression into the dynamics of $\tilde{S}_t$, we obtain

$$d\tilde{S}_t = \tilde{S}_t\left((\mu - r - \sigma\theta_t)dt + \sigma\, dW_t\right).$$

The process $\tilde{S}_t$ is a martingale if its drift is equal to zero. Hence,

$$\mu - r - \sigma\theta_t = 0,$$

which gives

$$\theta_t = \frac{\mu - r}{\sigma}.$$

With this choice of $\theta_t$,

$$d\tilde{S}_t = \sigma\tilde{S}_t \, dW_t.$$

Therefore, under the equivalent probability measure $\mathbb{P}^*$, the discounted price process $\tilde{S}_t$ is a martingale. □

Under the risk-neutral probability measure $\mathbb{P}^*$, the risky asset follows

$$S_t = S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right].$$

This follows from the relation $B_t = W_t - \theta t$, with $\theta = \frac{(\mu - r)}{\sigma}$, applied to the original Black-Scholes solution. The drift $\mu$ is replaced by the risk-free rate $r$, while the volatility $\sigma$ remains unchanged.

Girsanov's theorem is therefore crucial for the derivation of the Black-Scholes pricing formula, since it allows the valuation problem to be written under the risk-neutral probability measure.

**Theorem 1.2.3.** *In the Black-Scholes model, any option defined by a non-negative, $\mathcal{F}_T$-measurable random variable h, square integrable under $\mathbb{P}^*$, is replicable. The value at time t of any replicating portfolio is*

$$V_t = \mathbb{E}^*\left[e^{-r(T-t)}h|\mathcal{F}_t\right].$$

If $h = f(S_T)$, the option value at time $t$ can be written as a function of $t$ and $S_t$. Hence,

$$V_t = \mathbb{E}^*\left[e^{-r(T-t)}f(S_T)|\mathcal{F}_t\right].$$

Under $\mathbb{P}^*$,

$$S_T = S_t \exp\left[\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma(W_T - W_t)\right].$$

Since $S_t$ is $\mathcal{F}_t$-measurable and $W_T - W_t$ is independent of $\mathcal{F}_t$, we can write $V_t = F(t, S_t)$, where

$$F(t,x) = e^{-r(T-t)}\int_{-\infty}^{+\infty} f\left(x\exp\left[\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma y\sqrt{T-t}\right]\right)\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}dy.$$

For a European call option,

$$F(t,x) = xN(d_1) - Ke^{-r\theta}N(d_2),$$

while for a European put option,

$$F(t,x) = Ke^{-r\theta}N(-d_2) - xN(-d_1),$$

where:

1) $x = S_t$,

2) $\theta = T - t$,

3) $d_1 = \dfrac{\log\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\theta}{\sigma\sqrt{\theta}}$,

4) $d_2 = d_1 - \sigma\sqrt{\theta}$,

5) $N(d) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{d} e^{\frac{-x^2}{2}}\, dx$.

The concepts introduced in this chapter provide the probabilistic and financial framework required for the remainder of the thesis. In particular, stochastic processes, Brownian motion, Itô calculus, trading strategies, replication, and equivalent martingale measures form the theoretical basis of the pricing methods considered later.

The CRR and Black-Scholes models play a central role because they represent the standard discrete-time and continuous-time approaches to option pricing. The following chapter establishes the weak convergence result that allows the continuous Brownian framework to be approximated by a normalized simple random walk. This connection will make it possible to replace pricing quantities involving $(B_t, B_t^*)$ with corresponding quantities involving the rescaled random walk and its running maximum.

# Chapter 2

# Brownian Motion and Random Walk

The objective of this chapter is to approximate path-dependent option payoffs by means of a normalized simple random walk converging to Brownian motion. A natural way to represent the class of options considered in this thesis is through functions of the form $f(B_t, B_t^*)$, where $B_t$ is a Brownian motion and $B_t^*$ denotes its running maximum.

Before studying this function, it is necessary to introduce the theoretical framework that justifies the convergence between the simple random walk and Brownian motion. For this reason, the first part of the chapter recalls the main notions of weak convergence, convergence in distribution, tightness, the continuous mapping theorem, and the space $C[0,1]$ of continuous functions.

These tools are required to state and prove Donsker's Theorem, which is the central theoretical result of this chapter. The theorem establishes that a suitably normalized and linearly interpolated random walk converges weakly to Brownian motion. This result provides the mathematical justification for replacing quantities involving Brownian motion with corresponding quantities involving a rescaled random walk.

The second part of the chapter focuses on the function used in this work, its density, and its convergence rate. Finally, we present some results from the literature that extend or refine this type of approximation in the context of option pricing.

**Notation.** Throughout this chapter we use the following convergence arrows consistently:

- $\Rightarrow$ denotes weak convergence of probability measures, or of stochastic processes in a metric space, in the sense of Definition 2.1.1. This is the notion used in Donsker's theorem and in the functional limit theorems of Section 2.1.
- $\xrightarrow{d}$ denotes convergence in distribution of real-valued or finite-dimensional random variables, namely convergence of the associated probability measures on $\mathbb{R}^n$. For random variables taking values in $\mathbb{R}$, the two notions coincide; the distinction becomes relevant when the limit belongs to an infinite-dimensional space such as $C[0,1]$.
- $\xrightarrow{P}$ denotes convergence in probability.

## 2.1 Brownian Motion Approximation

### 2.1.1 Convergence Criteria

Let $\mathcal{X}$ be a metric space and let $\mathcal{B}(\mathcal{X})$ denote its Borel $\sigma$-field. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. The distribution of $X_n$ is denoted by $P_n$, also called the law of $X_n$, and written as $\mathcal{L}(X_n)$.

**Definition 2.1.1.** *A sequence of probability measures $(P_n)_{n\in\mathbb{N}}$ converges weakly to a probability measure $P$ as $n \to \infty$ if, for every bounded and continuous real-valued function $f$ on $\mathcal{X}$,*

$$\int_{\mathcal{X}} f\, dP_n \xrightarrow[n\to\infty]{} \int_{\mathcal{X}} f\, dP.$$

**Definition 2.1.2.** *A sequence $\{X_n\}_{n\in\mathbb{N}}$ of random variables is said to converge in distribution to a random variable $X$ if*

$$P_n \Rightarrow P,$$

*where $P_n$ and $P$ are the distributions of $X_n$ and $X$, respectively.*

### 2.1.2 Central Limit Theorem

**Theorem 2.1.1.** *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables with*

$$\mathbb{E}[X_i] = \mu, \qquad \mathrm{Var}(X_i) = \sigma^2 < \infty.$$

*Let*

$$S_n = \sum_{i=1}^{n} X_i.$$

*be the sum of these variables, then as $n \to \infty$ the normalized variables converges in distribution to a standard normal variable $Z \sim N(0,1)$,*

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1).$$

### 2.1.3 Tightness

**Definition 2.1.3.** *A probability measure $P$ on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ is tight if, for every $\epsilon > 0$, there exists a compact set $K \subseteq \mathcal{X}$ such that*

$$P(K) > 1 - \epsilon.$$

**Theorem 2.1.2.** *If $\mathcal{X}$ is separable and complete, then every probability measure on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ is tight.*

### 2.1.4 Mapping

Let

$$h : \mathcal{X} \to \mathcal{X}'$$

be a measurable function, with metric $\rho'$ and Borel $\sigma$-field $\mathcal{B}(\mathcal{X}')$. If $h$ is measurable, then each probability measure $P$ on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ induces a probability measure $Ph^{-1}$ on $(\mathcal{X}', \mathcal{B}(\mathcal{X}'))$, defined by

$$Ph^{-1}(A) = P(h^{-1}(A)).$$

**Theorem 2.1.3.** *If $P_n \Rightarrow P$ and $P(D_h) = 0$, then*

$$P_n h^{-1} \Rightarrow Ph^{-1}.$$

where $D_h = \{x \in \mathcal{X} : h \text{ is discontinuous at } x\}$,

**Corollary 2.1.1.** *Suppose $h : \mathcal{X} \to \mathcal{X}'$ is measurable and let $D_h$ be the set of its discontinuity points. If $X$ has distribution $P$, then $h(X)$ has distribution $Ph^{-1}$. Therefore,*

$$X_n \Rightarrow X \quad \Longrightarrow \quad h(X_n) \Rightarrow h(X),$$

*provided that*

$$P[X \in D_h] = 0.$$

Suppose that $(X_n, Y_n)$ is an $(\mathcal{X} \times \mathcal{X})$-valued random variable. Then $X_n$ and $Y_n$ are random elements of $\mathcal{X}$. If $\rho$ is the metric on $\mathcal{X}$, then $\rho(x, y)$ maps $\mathcal{X} \times \mathcal{X}$ continuously into $\mathbb{R}$. Hence, it is meaningful to consider the distance $\rho(X_n, Y_n)$, which is the random variable taking value $\rho(X_n(\omega), Y_n(\omega))$ at $\omega$.

**Theorem 2.1.4.** *Suppose that $(X_n, Y_n)$ are $(\mathcal{X} \times \mathcal{X})$-valued random variables for $n \geq 0$. If $X_n \Rightarrow X$ and $\rho(X_n, Y_n) \xrightarrow{P} 0$, then*

$$Y_n \Rightarrow X.$$

### 2.1.5 The Space $C[0, 1]$ of Continuous Functions

Let $C = C[0, 1]$ denote the space of real-valued continuous functions on the unit interval. We endow $C$ with the uniform topology, defining the distance between two functions $x$ and $y$ as

$$\rho(x, y) = |x - y| = \sup_{t \in [0,1]} |x(t) - y(t)|.$$

**Theorem 2.1.5.** *Let $P_n$ and $P$ be probability measures on $(C, C)$. If the finite-dimensional distributions of $P_n$ converge weakly to those of $P$, and if $P_n$ is tight, then*

$$P_n \Rightarrow P.$$

**Theorem 2.1.6.** *The sequence $\{P_n\}$ is tight if and only if the following conditions hold:*

1) *For each positive real number $\eta$, there exist a positive real number $a$ and an integer $\eta_0$ such that*

$$P_n[x : |x(0)| \geq a] \leq \eta, \qquad n \geq \eta_0.$$

2) *For each positive real number $\epsilon$ and each positive real number $\eta$, there exist a real number $\delta \in (0, 1)$ and an integer $\eta_0$ such that*

$$P_n[x : w_x(\delta) \geq \epsilon] \leq \eta, \qquad n \geq \eta_0.$$

Here $w_x(\delta)$ denotes the modulus of continuity of an arbitrary function $x : [0, 1] \to \mathbb{R}$, defined by

$$w_x(\delta) = w(x, \delta) = \sup_{|s-t| \leq \delta} |x(s) - x(t)|, \qquad 0 < \delta \leq 1.$$

### 2.1.6 Donsker's Theorem

At this point, we have introduced the theoretical elements required to establish the weak convergence of a normalized simple random walk to Brownian motion. Before stating Donsker's theorem, we fix the notation.

Let $B = (B_t)_{t\in[0,1]}$ be a Brownian motion and let $(X_i)_{i\in\mathbb{N}}$ be the sequence of random variables used to define the random walk. We consider the symmetric simple random walk, namely

$$P(X_i = +1) = \frac{1}{2}, \qquad P(X_i = -1) = \frac{1}{2},$$

with initial value $S_0 = 0$.

Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, with

$$\begin{aligned} \mathbb{E}[X_i] &= 0, \qquad i \in \mathbb{N}, \\ \mathrm{Var}(X_i) &= \sigma^2, \qquad i \in \mathbb{N}. \end{aligned}$$

We define the process $B^{(n)} = (B_t^{(n)})_{t\in[0,1]}$ by

$$B_t^{(n)}(\omega) = \frac{S_{\lfloor nt\rfloor}(\omega) + (nt - \lfloor nt\rfloor)X_{\lfloor nt\rfloor+1}(\omega)}{\sigma\sqrt{n}}.$$

Thus, for each $\omega$, the function $t \mapsto B_t^{(n)}(\omega)$ is obtained by linear interpolation between the values

$$B_{\frac{i}{n}}^{(n)}(\omega) = \frac{S_i(\omega)}{\sigma\sqrt{n}}, \qquad \forall\, i \in \{0, \dots, n\}.$$

For a fixed $t \in [0, 1]$, let $i = \lfloor nt\rfloor$, so that

$$\frac{i}{n} \le t < \frac{i+1}{n}.$$

The value $B_t^{(n)}$ is the linear interpolation between $B_{\frac{i}{n}}^{(n)}$ and $B_{\frac{i+1}{n}}^{(n)}$:

$$\begin{aligned} &\frac{B_t^{(n)} - B_{\frac{i}{n}}^{(n)}}{B_{\frac{i+1}{n}}^{(n)} - B_{\frac{i}{n}}^{(n)}} = \frac{t - \frac{i}{n}}{\frac{1}{n}}, \\ \iff &\frac{B_t^{(n)}}{B_{\frac{i+1}{n}}^{(n)} - B_{\frac{i}{n}}^{(n)}} = \frac{B_{\frac{i}{n}}^{(n)}}{B_{\frac{i+1}{n}}^{(n)} - B_{\frac{i}{n}}^{(n)}} + \frac{t - \frac{i}{n}}{\frac{1}{n}}, \\ \iff &B_t^{(n)} = B_{\frac{i}{n}}^{(n)} + \frac{t - \frac{i}{n}}{\frac{1}{n}}\left(B_{\frac{i+1}{n}}^{(n)} - B_{\frac{i}{n}}^{(n)}\right). \end{aligned}$$

Since

$$B_{\frac{i}{n}}^{(n)}(\omega) = \frac{S_i(\omega)}{\sigma\sqrt{n}}, \qquad B_{\frac{i+1}{n}}^{(n)}(\omega) = \frac{S_{i+1}(\omega)}{\sigma\sqrt{n}},$$

we deduce that

$$\begin{aligned} B_t^{(n)} &= \frac{S_{\lfloor nt\rfloor}(\omega)}{\sigma\sqrt{n}} + (nt - i)\left(\frac{S_{i+1}(\omega)}{\sigma\sqrt{n}} - \frac{S_i(\omega)}{\sigma\sqrt{n}}\right) \\ &= \frac{S_{\lfloor nt\rfloor}(\omega)}{\sigma\sqrt{n}} + (nt - \lfloor nt\rfloor)\left(\frac{S_{\lfloor nt\rfloor}(\omega) + X_{\lfloor nt\rfloor+1}(\omega)}{\sigma\sqrt{n}} - \frac{S_{\lfloor nt\rfloor}(\omega)}{\sigma\sqrt{n}}\right) \\ &= \frac{S_{\lfloor nt\rfloor}(\omega) + (nt - \lfloor nt\rfloor)X_{\lfloor nt\rfloor+1}(\omega)}{\sigma\sqrt{n}}. \end{aligned} \tag{2.1}$$

We can now state and prove Donsker's theorem.

**Theorem 2.1.7.** *If $(X_1, X_2, \ldots)$ are independent and identically distributed random variables with mean 0 and variance $\sigma^2$, and if $B^{(n)}$ is the random function defined in* (2.1)*, then*

$$B^{(n)} \Rightarrow W,$$

*where $W = (W_t)_{t\in[0,1]}$ is a Brownian motion.*

*Proof.* We verify the convergence of the finite-dimensional distributions at one and two time points. The extension to any finite number of time points follows analogously.

1) $B_t^{(n)} \Rightarrow W_t$.

2) $(B_t^{(n)}, B_s^{(n)}) \Rightarrow (W_t, W_s)$, for $0 \le s < t \le 1$.

1) We first prove convergence at a fixed time $t \in [0, 1]$. Starting from the definition of $B_t^{(n)}$, we have

$$B_t^{(n)} = \frac{S_{\lfloor nt \rfloor}(\omega) + (nt - \lfloor nt \rfloor) X_{\lfloor nt \rfloor + 1}(\omega)}{\sigma\sqrt{n}},$$
$$B_t^{(n)} - \frac{S_{\lfloor nt \rfloor}(\omega)}{\sigma\sqrt{n}} = \frac{(nt - \lfloor nt \rfloor) X_{\lfloor nt \rfloor + 1}(\omega)}{\sigma\sqrt{n}}.$$

By the Central Limit Theorem,

$$\frac{S_{\lfloor nt \rfloor}}{\sigma\sqrt{n}} = \frac{S_{\lfloor nt \rfloor}}{\sigma\sqrt{\lfloor nt \rfloor}} \sqrt{\frac{\lfloor nt \rfloor}{n}}.$$

Since

$$\sqrt{\frac{\lfloor nt \rfloor}{n}} \to \sqrt{t},$$

it follows that

$$\frac{S_{\lfloor nt \rfloor}}{\sigma\sqrt{n}} \xrightarrow{d} N(0, t), \qquad n \to \infty.$$

Since $W_t \sim N(0, t)$, it remains to show that the interpolation term converges to zero in probability:

$$\frac{(nt - \lfloor nt \rfloor) X_{\lfloor nt \rfloor + 1}(\omega)}{\sigma\sqrt{n}} \xrightarrow{P} 0.$$

By definition of the integer part,

$$\lfloor nt \rfloor \le nt \le \lfloor nt \rfloor + 1,$$
$$0 \le nt - \lfloor nt \rfloor \le 1.$$

Thus,

$$\left| \frac{(nt - \lfloor nt \rfloor) X_{\lfloor nt \rfloor + 1}(\omega)}{\sigma\sqrt{n}} \right| \le \left| \frac{X_{\lfloor nt \rfloor + 1}(\omega)}{\sigma\sqrt{n}} \right|.$$

By Markov's inequality,

$$P\left( \left| \frac{X_{\lfloor nt \rfloor + 1}(\omega)}{\sigma\sqrt{n}} \right| \ge \epsilon \right) \le \frac{\mathbb{E}\left[ \left| \frac{X_{\lfloor nt \rfloor + 1}(\omega)}{\sigma\sqrt{n}} \right|^2 \right]}{\epsilon^2}$$

$$= \frac{\mathbb{E}\left[X_{\lfloor nt \rfloor+1}(\omega)^2\right]}{\epsilon^2\sigma^2 n}$$
$$= \frac{1}{\epsilon^2 n}.$$

Therefore,

$$P\left(\left|\frac{X_{\lfloor nt \rfloor+1}(\omega)}{\sigma\sqrt{n}}\right| \geq \epsilon\right) \longrightarrow 0, \qquad n \to \infty.$$

It follows that

$$\frac{(nt - \lfloor nt \rfloor)X_{\lfloor nt \rfloor+1}(\omega)}{\sigma\sqrt{n}} \xrightarrow{P} 0.$$

By Theorem 2.1.4, we conclude that

$$B_t^{(n)} \Rightarrow W_t.$$

2) We now verify the convergence of increments. Let $0 \leq s < t \leq 1$. The increment of the random walk can be written as

$$S_{\lfloor nt \rfloor} - S_{\lfloor ns \rfloor} = \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} X_i.$$

This is a sum of independent and identically distributed random variables. Let

$$k_n = \lfloor nt \rfloor - \lfloor ns \rfloor.$$

Then

$$\frac{k_n}{n} \longrightarrow t - s, \qquad n \to \infty.$$

By the Central Limit Theorem,

$$\frac{S_{\lfloor nt \rfloor} - S_{\lfloor ns \rfloor}}{\sigma\sqrt{k_n}} \Rightarrow N(0,1).$$

Since $\frac{k_n}{n} \to t - s$, we obtain

$$\frac{S_{\lfloor nt \rfloor} - S_{\lfloor ns \rfloor}}{\sigma\sqrt{n}} \Rightarrow N(0, t-s).$$

Therefore, the increment of the rescaled process satisfies

$$B_t^{(n)} - B_s^{(n)} \Rightarrow N(0, t-s).$$

Moreover, increments over disjoint intervals are independent by construction of the random walk. Hence, for any finite collection of times $0 \leq t_1 < \cdots < t_n \leq 1$, the vector

$$\left(B_{t_1}^{(n)}, \ldots, B_{t_n}^{(n)}\right),$$

converges in distribution to a centered Gaussian vector with covariance function

$$\mathrm{Cov}(W_s, W_t) = \min(s,t).$$

In particular,

$$(B_t^{(n)}, B_s^{(n)}) \Rightarrow (W_t, W_s),$$

where $W = (W_t)_{t\in[0,1]}$ is a Brownian motion.

The same argument extends to any finite number of time instants. Together with the tightness criterion stated in Theorem 2.1.5, this proves the weak convergence of $B^{(n)}$ to Brownian motion in $C[0,1]$. □

## 2.2 Brownian Motion, Random Walk and Their Running Maximum

In order to introduce the fundamental equation used in this work, let $B = (B_t)_{t\geq 0}$ be a Brownian motion and define its running maximum by

$$B_t^* = \sup_{0\leq s\leq t} B_s, \qquad t \geq 0.$$

The core function of this thesis is

$$\begin{aligned} u : [0,T] \times \{(x,y) \in \mathbb{R}^2 : x \leq y,\ y \geq 0\} &\longrightarrow \mathbb{R}, \\ u(t,x,y) = \mathbb{E}\left[f(B_T, B_T^*)|B_t = x,\ B_t^* = y\right], & \end{aligned} \tag{2.2}$$

where $T$ is fixed and $f$ is a two-variable function.

The objective is to approximate (2.2) by replacing the continuous-time process $(B_t, B_t^*)_{t\in[0,T]}$ with the discrete pair $(S_n, S_n^*)_{n\in\mathbb{N}}$. The approximation is based on the binomial structure induced by the simple random walk and on the convergence result established in the previous section.

### 2.2.1 Joint Density of Brownian Motion and Its Maximum

Throughout this section, x and y denote generic values of $B_t$ and $B_t^*$, respectively. We start by observing that by the independent and stationary increments of Brownian motion

$$\begin{aligned} B_T^* &= \sup_{s\in[0,T]} B_s \\ &= \left(\sup_{s\in[0,t]} B_s\right) \vee \left(\sup_{s\in[t,T]} B_s\right) \\ &= B_t^* \vee \left(\sup_{s\in[t,T]} (B_s - B_t + B_t)\right) \\ &= y \vee \left(x + \sup_{s\in[t,T]} (B_s - B_t)\right) \\ &= y \vee (x + B_{T-t}^*). \end{aligned}$$

Therefore, (2.2) can be written as

$$\begin{aligned} u(t,x,y) &= \mathbb{E}\left[f(B_T, B_T^*)|B_t = x,\ B_t^* = y\right] \\ &= \mathbb{E}\left[f(B_T - B_t + B_t,\ y \vee (x + B_{T-t}^*))\right] \end{aligned}$$

$$= \mathbb{E}\left[f(B_{T-t} + x,\ y \vee (x + B^*_{T-t}))\right]$$
$$= \mathbb{E}\left[f_0(B_{T-t}, B^*_{T-t})\right],$$

where

$$f_0(x', y') = f(x + x',\ y \vee (x + y')).$$

Using the scaling property of Brownian motion, we obtain

$$\begin{aligned} u(t,x,y) &= \mathbb{E}\left[f_0(B_{T-t}, B^*_{T-t})\right] \\ &= \mathbb{E}\left[f_0\left(\sqrt{T-t}\, B_1, \sqrt{T-t}\, B^*_1\right)\right] \\ &= \mathbb{E}\left[\widetilde{f}(B_1, B^*_1)\right], \end{aligned}$$

where

$$\widetilde{f}(x,y) = f_0\left(\sqrt{T-t}\, x, \sqrt{T-t}\, y\right).$$

From now on, without loss of generality, we work on the time interval $[0,1]$.

Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables such that

$$\mathbb{P}(X_n = -1) = \frac{1}{2}, \qquad \mathbb{P}(X_n = +1) = \frac{1}{2}.$$

The associated simple random walk is

$$S_0 = 0, \qquad S_n = X_1 + \cdots + X_n, \qquad n \geq 1.$$

From Section 2.1.6, the linearly interpolated and normalized random walk converges weakly to Brownian motion:

$$B^{(n)}_t = \frac{S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor) X_{\lfloor nt \rfloor + 1}}{\sqrt{n}}.$$

In the present symmetric case, $\sigma = 1$. We are interested in the convergence

$$\frac{1}{\sqrt{n}}(S_n, S^*_n) \Rightarrow (B_1, B^*_1).$$

The state space of the pair $(S_n, S^*_n)$ is

$$E = \{(l,m) \in \mathbb{Z}^2 : l \leq m,\ m \geq 0\}.$$

Its transition probabilities are

$$\begin{aligned} &\mathbb{P}\{(S_{n+1}, S^*_{n+1}) = (l', m') \mid (S_n, S^*_n) = (l,m)\} \\ &= \begin{cases} \frac{1}{2} & \text{if } l < m,\ m' = m,\ l' \in \{l-1, l+1\}, \\ \frac{1}{2} & \text{if } l = m,\ (l', m') \in \{(l-1, l), (l+1, l+1)\}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We now determine the law of the discrete vector $(S_n, S^*_n)$.

Let

$$\tau_\lambda = \inf\{n \geq 0 : S_n \geq \lambda\}, \qquad \lambda \in \mathbb{N}.$$

For any Borel function $h$, by the strong Markov property and the symmetry of the simple random walk we have

$$\begin{aligned}\mathbb{E}\left[h(S_n)\mathbb{1}_{\tau_\lambda \le n}\right] &= \mathbb{E}\left[\mathbb{1}_{\{\tau\lambda \le n\}} h(S_{n-\tau_\lambda} + \lambda)\right] \\ &= \mathbb{E}\left[\mathbb{1}_{\{\tau_\lambda \le n\}} h(\lambda - S_{n-\tau_\lambda})\right] \\ &= \mathbb{E}\left[\mathbb{1}_{\{\tau_\lambda \le n\}} h(2\lambda - S_n)\right].\end{aligned}$$

Choosing $h = \mathbb{1}_{(-\infty,\lambda]}$, we obtain

$$\begin{aligned}\mathbb{P}\{S_n \ge \lambda, \tau_\lambda \le n\} &= \mathbb{P}\{S_n \le \lambda, S_n^* \ge \lambda\} \\ &= \mathbb{P}\{S_n \ge \lambda, S_n^* \ge \lambda\} \\ &= \mathbb{P}\{S_n \ge \lambda\}.\end{aligned}$$

Consequently,

$$\begin{aligned}\mathbb{P}\{S_n^* \ge \lambda\} &= \mathbb{P}\{S_n \le \lambda,\ S_n^* \ge \lambda\} + \mathbb{P}\{S_n > \lambda,\ S_n^* \ge \lambda\} \\ &= \mathbb{P}\{S_n \ge \lambda\} + \mathbb{P}\{S_n > \lambda\}.\end{aligned}$$

More precisely,

$$\mathbb{P}\{S_n^* = \lambda\} = \mathbb{P}\{S_n = \lambda\} + \mathbb{P}\{S_n = \lambda + 1\}.$$

Taking $h = \mathbb{1}(-\infty, \mu]$, two cases arise.
**Case 1:** $\mu \le \lambda$**.**

$$\begin{aligned}\mathbb{P}\{S_n \le \mu,\ S_n^* \ge \lambda\} &= \mathbb{P}\{S_n \ge 2\lambda - \mu,\ S_n^* \ge \lambda\} \\ &= \mathbb{P}\{S_n \ge 2\lambda - \mu\}.\end{aligned}$$

**Case 2:** $\mu \ge \lambda$**.**

$$\begin{aligned}\mathbb{P}\{S_n \le \mu,\ S_n^* \ge \lambda\} &= \mathbb{P}\{S_n \le \mu\} - \mathbb{P}\{S_n^* < \lambda\} \\ &= \mathbb{P}\{S_n \ge \lambda\} + \mathbb{P}\{\lambda < S_n \le \mu\}.\end{aligned}$$

From these identities, we deduce that the joint distribution of $(S_n, S_n^*)$ is

$$\mathbb{P}\{S_n = \mu\,, S_n^* = \lambda\} = \mathbb{P}\{S_n = 2\lambda - \mu\} - \mathbb{P}\{S_n = 2\lambda - \mu + 2\}. \tag{2.3}$$

### 2.2.2 Rate of Convergence

To determine the convergence rate, we use the following theorem.

**Theorem 2.2.1.** *Let $f$ be either a function of the form*

1. $f(x, y) = f_1(x)\mathbb{1}_{(M_0, M_1)}(y)$,
2. *$f$ is a Lipschitz continuous function of two variables.*

*where $f_1$ is Lipschitz and $0 \le M_0 \le M_1 \le +\infty$. Then there exists a constant $C_0 > 0$ such that, for*

*all n,*

$$\left|\mathbb{E}[f(B_1, B_1^*)] - \mathbb{E}\left[f\left(\frac{S_n}{\sqrt{n}}, \frac{S_n^*}{\sqrt{n}}\right)\right]\right| \leq \frac{C_0}{\sqrt{n}}.$$

**Definition 2.2.1.** *A function $f : \mathbb{R}^2 \to \mathbb{R}$ is called Lipschitz if there exists $L > 0$ such that*

$$|f(x) - f(y)| \leq L|x - y|, \qquad \forall x, y \in \mathbb{R}^2.$$

*The smallest such constant L is called the Lipschitz constant of $f$.*

We divide the proof of Theorem 2.2.1 into three steps. First, we introduce condition $(LL)$.

We say that a function $\varphi : \mathbb{R} \to \mathbb{R}$ satisfies condition $(LL)$ if there exists a constant $C > 0$ and $r \geq 0$ such that, for all $x \in \mathbb{R}$ and all $w \in [0, 1]$,

$$|\varphi(x + w) - \varphi(x)| \leq Cw(1 + |x|)^r.$$

A Lipschitz function satisfies this condition. Moreover, such a function $\varphi$ increases not faster than a polynomial, given that for $x > 0$

$$|\varphi(x)| = \left|\varphi(0) + \sum_{j=0}^{[x]-1} (\varphi(j + 1) - \varphi(j) + \varphi(x) - \varphi([x])\right| \leq |\varphi(0)| + C(1 + x)^{r+1},$$

this is valid also for $x < 0$.

*STEP 1*

In this step, we express the quantities

$$\mathbb{E}[f(B_1, B_1^*)] \qquad \text{and} \qquad \mathbb{E}\left[f\left(\frac{S_n}{\sqrt{n}}, \frac{S_n^*}{\sqrt{n}}\right)\right],$$

using only the densities of $B_1$ and $S_n$.

Starting from the random walk, by equation (2.3) we can write

$$\begin{aligned}
\mathbb{E}\left[f\left(\frac{S_n}{\sqrt{n}}, \frac{S_n^*}{\sqrt{n}}\right)\right] &= \sum_{l \geq 0,\ j \leq l} f\left(\frac{j}{\sqrt{n}}, \frac{l}{\sqrt{n}}\right) (\mathbb{P}\{S_n = 2l - j\} - \mathbb{P}\{S_n = 2l - j + 2\}) \\
&= \sum_{l \geq 0,\ m \geq l} f\left(\frac{2l - m}{\sqrt{n}}, \frac{l}{\sqrt{n}}\right) (\mathbb{P}\{S_n = m\} - \mathbb{P}\{S_n = m + 2\}) \\
&= \sum_{m \geq 0} \frac{m + 1}{n + 1} (\mathbb{P}\{S_n = m\} + \mathbb{P}\{S_n = m + 2\}) \sum_{l=0}^{m} f\left(\frac{2l - m}{\sqrt{n}}, \frac{l}{\sqrt{n}}\right). \qquad (2.4)
\end{aligned}$$

The last equality follows from a rearrangement of the summation indices together with the following identity for the distribution of the simple symmetric random walk. Indeed, for $m \geq 0$,

$$\mathbb{P}\{S_n = m\} = \binom{n}{\frac{n+m}{2}} 2^{-n},$$

whenever $n + m$ is even; otherwise is 0.

Setting $k = \frac{n+m}{2}$, we obtain

$$\mathbb{P}\{S_n = m+2\} = \binom{n}{k+1} 2^{-n}.$$

Using the relation between consecutive binomial coefficients,

$$\binom{n}{k+1} = \binom{n}{k} \frac{n-k}{k+1},$$

and observing that

$$\frac{n-k}{k+1} = \frac{n - \frac{n+m}{2}}{\frac{n+m}{2} + 1} = \frac{n-m}{n+m+2}.$$

Hence,

$$\mathbb{P}\{S_n = m+2\} = \mathbb{P}\{S_n = m\} \frac{n-m}{n+m+2}.$$

Consequently,

$$\begin{aligned}
\mathbb{P}\{S_n = m\} + \mathbb{P}\{S_n = m+2\} &= \mathbb{P}\{S_n = m\} + \mathbb{P}\{S_n = m\} \frac{n-m}{n+m+2} \\
&= \mathbb{P}\{S_n = m\} \left(1 + \frac{n-m}{n+m+2}\right) \\
&= \mathbb{P}\{S_n = m\} \frac{2n+2}{n+m+2} = \mathbb{P}\{S_n = m\} \frac{2(n+1)}{n+m+2},
\end{aligned}$$

while

$$\mathbb{P}\{S_n = m\} - \mathbb{P}\{S_n = m+2\} = \mathbb{P}\{S_n = m\} \frac{2(m+1)}{n+m+2}.$$

Taking the ratio of the last two identities yields

$$\frac{\mathbb{P}\{S_n = m\} - \mathbb{P}\{S_n = m+2\}}{\mathbb{P}\{S_n = m\} + \mathbb{P}\{S_n = m+2\}} = \frac{m+1}{n+1},$$

and therefore

$$\mathbb{P}\{S_n = m\} - \mathbb{P}\{S_n = m+2\} = \frac{m+1}{n+1} \left(\mathbb{P}\{S_n = m\} + \mathbb{P}\{S_n = m+2\}\right).$$

Substituting this identity into the previous sum yields the representation in the last line of (2.4). For Brownian motion, the joint density of the vector $(B_t, B_t^*)$ with respect to the Lebesgue measure is

$$p_t(x,y) = \mathbb{1}_{\{y \geq 0\}} \mathbb{1}_{\{x \leq y\}} \frac{2(2y-x)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2y-x)^2}{2t}\right).$$

see [6, Equation (8.2)].

By a change of variables and with $t = 1$, we obtain

$$\mathbb{E}[f(B_1, B_1^*)] = \int_0^{+\infty} \int_y^{+\infty} \frac{2z}{\sqrt{2\pi}} e^{-z^2/2} f(2y-z, y) dz\, dy = 2\mathbb{E}[g(B_1)], \tag{2.5}$$

where

$$g(x) = x \int_0^x f(2y-x, y)\, dy\, \mathbb{1}_{(0,+\infty)}(x).$$

The function $g$ is not necessarily Lipschitz, but it satisfies a condition of type $(LL)$. Indeed, for

$w \in [0,1]$ and $x > 0$,

$$\begin{aligned} g(x+w) &= (x+w)\int_0^x f(2y-x-w,y)\,dy\,\mathbb{1}_{(0,+\infty)}(x+w) \\ &= x\int_0^x f(2y-x-w,y)\,dy\,\mathbb{1}_{(0,+\infty)}(x+w) \\ &\quad + w\int_0^x f(2y-x-w,y)\,dy\,\mathbb{1}_{(0,+\infty)}(x+w), \end{aligned}$$

and by hypothesis $(LL)$

$$\begin{aligned} &|g(x)-g(x+w)| \leq \\ &\leq \Big| x\int_0^x f(2y-x,y)dy\,\mathbb{1}_{(0,+\infty)}(x) - x\int_0^x f(2y-x-w,y)\,dy\,\mathbb{1}_{(0,+\infty)}(x+w) + \\ &\quad + w\int_0^x f(2y-x-w,y)\,dy\,\mathbb{1}_{(0,+\infty)}(x+w)\Big| \\ &\leq \Big| x\int_0^x f(2y-x,y)dy - x\int_0^x f(2y-x-w,y)\,dy + w\int_0^x f(2y-x-w,y)\,dy\Big| \\ &\leq \Big| x\int_0^x f(2y-x,y)dy - x\int_0^x f(2y-x-w,y)\,dy + w\int_0^x f(2y-x-w,y)\,dy + \\ &\quad + x\int_0^x f(2y-x-w,y)\,dy - x\int_0^x f(2y-x-w,y)\,dy\Big| \\ &\leq x\int_0^x |f(2y-x,y)-f(2y-x-w,y)|\,dy + w\int_0^{x+w} |f(2y-x-w,y)|\,dy + \\ &\quad + x\int_x^{x+w} |f(2y-x-w,y)|\,dy \\ &\leq wLx^2 + w(2x+w)[L\sqrt{2}(x+w) + |f(0,0)|] \\ &\leq w(|f(0,0)|+4L)(1+x)^2 \qquad \text{for } x>0 \text{ and } w \in [0,1]. \end{aligned}$$

*STEP 2*

After proving equation (2.5), we approximate the term $\mathbb{E}[g(B_1)]$ through the random walk. To do this, we use the following non-uniform version of the Berry-Esseen inequality.

**Theorem 2.2.2.** *Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with*

$$\mathbb{E}[X_1] = 0, \qquad \mathbb{E}[X_1^2] = 1, \qquad \mathbb{E}[|X_1|^b] < \infty$$

*for some integer $b \geq 3$, and define*

$$S_n = X_1 + \cdots + X_n.$$

*Let $F_n$ and $\phi$ be the cumulative distribution functions of $\frac{S_n}{\sqrt{n}}$ and of the standard normal distribution, respectively. Then, for any $b \geq 3$, there exists a constant $c(b)$ such that, for any $x \in \mathbb{R}$ and $n \in \mathbb{N}^*$,*

$$|F_n(x) - \phi(x)| \leq \frac{c(b)}{\sqrt{n}(1+|x|^b)}.$$

This theorem refines the Central Limit Theorem by providing a quantitative bound on the approximation error between the distribution of the normalized random walk and the standard normal distribution. In the independent case, the convergence rate is of order $n^{-\frac{1}{2}}$. The

non-uniform version gives a pointwise control of the error in $x$, which is the form needed in the present proof.

**Lemma 2.2.1.** *Let $\varphi$ be a real-valued function satisfying condition $(LL)$. Under the assumptions of Theorem 2.2.2, with $b = r + 2$, the random variable $\varphi\left(\frac{S_n}{\sqrt{n}}\right)$ has finite expectation for every n, and*

$$\left|\mathbb{E}[\varphi(B_1)] - \mathbb{E}\left[\varphi\left(\frac{S_n}{\sqrt{n}}\right)\right]\right| \leq 2Cc(r+2)n^{-\frac{1}{2}},$$

*where $c(r+2)$ is the constant appearing in Theorem 2.2.2.*

*Proof.* Since $\varphi$ satisfies condition $(LL)$, it has at most polynomial growth. Hence, the expectations considered below are finite. We can write

$$\begin{aligned}
&\left|\mathbb{E}[\varphi(B_1)] - \mathbb{E}\Big[\varphi\Big(\frac{S_n}{\sqrt{n}}\Big)\Big]\right| = \Big|\int_{\mathbb{R}} \varphi(x)d(\phi - F_n)(x)\Big| \\
&= \left|\lim_{M\to+\infty}\lim_{w\to 0^+}\sum_{j=-\left[\frac{M}{w}\right]+1}^{\left[\frac{M}{w}\right]} \varphi(jw)\cdot(\phi(jw) - F_n(jw) - \phi(jw-w) + F_n(jw-w))\right| \\
&\leq \lim_{M\to+\infty}\left|\lim_{w\to 0^+}\sum_j(\phi(jw) - F_n(jw)(\varphi(jw+w))\right| + \lim_{M\to+\infty}\Big|((\phi - F_n)\Big(\Big[\frac{M}{w}\Big]w\Big) - \\
&\quad - ((\phi - F_n)\Big(-\Big[\frac{M}{w}\Big]w\Big))\varphi(\Big(1 - \Big[\frac{M}{w}\Big]\Big)w\Big| \\
&\leq \lim_{M\to+\infty}\left(\lim_{w\to 0^+}\sum_j \frac{c(r+2)}{\sqrt{n}(1+|jw|)^{r+2}}Cw(1+|j|w)^r \; + 2\frac{c(r+2)(|\varphi(0)| + C(1+M)^{r+1})}{\sqrt{n}\Big(1 + w\Big[\frac{M}{w}\Big]\Big)^{r+2}}\right) \\
&= \frac{c(r+2)C}{\sqrt{n}}\int_{\mathbb{R}}\frac{1}{(1+|x|)^2}dx \\
&= \frac{2Cc(r+2)}{\sqrt{n}}.
\end{aligned}$$

Applying Lemma 2.2.1 with $r = 2$ to the function $g$ defined in (2.5), we get

$$\left|\mathbb{E}\left[g\left(\frac{S_n}{\sqrt{n}}\right)\right] - \mathbb{E}[g(B_1)]\right| \leq \frac{2c(4)}{\sqrt{n}}\left(|f(0,0)| + 4L\right),$$

where $c(4)$ is the constant introduced in Theorem 2.2.2. □

*STEP 3*

From Step 1, using equation (2.5), we can write

$$\begin{aligned}
\left|\mathbb{E}[f(B_1, B_1^*)] - \mathbb{E}\left[f\left(\frac{S_n}{\sqrt{n}}, \frac{S_n^*}{\sqrt{n}}\right)\right]\right| &= \left|2\mathbb{E}[g(B_1)] - \mathbb{E}\left[f\left(\frac{S_n}{\sqrt{n}}, \frac{S_n^*}{\sqrt{n}}\right)\right]\right| \\
&= \left|2\mathbb{E}[g(B_1)] - 2\mathbb{E}\left[g\left(\frac{S_n}{\sqrt{n}}\right)\right] + 2\mathbb{E}\left[g\left(\frac{S_n}{\sqrt{n}}\right)\right] - \mathbb{E}\left[f\left(\frac{S_n}{\sqrt{n}}, \frac{S_n^*}{\sqrt{n}}\right)\right]\right|.
\end{aligned}$$

By the triangle inequality,

$$\left|\mathbb{E}[f(B_1, B_1^*)] - \mathbb{E}\left[f\left(\frac{S_n}{\sqrt{n}}, \frac{S_n^*}{\sqrt{n}}\right)\right]\right| \leq 2\left|\mathbb{E}[g(B_1)] - \mathbb{E}\left[g\left(\frac{S_n}{\sqrt{n}}\right)\right]\right| + \left|2\mathbb{E}\left[g\left(\frac{S_n}{\sqrt{n}}\right)\right] - \mathbb{E}\left[f\left(\frac{S_n}{\sqrt{n}}, \frac{S_n^*}{\sqrt{n}}\right)\right]\right|.$$

The first term has already been treated in Step 2. To complete the proof of Theorem 2.2.1, it remains to prove the following proposition.

**Proposition 2.2.1.** *There exists a constant $C > 0$ such that, for all n,*

$$\left|2\mathbb{E}\left[g\left(\frac{S_n}{\sqrt{n}}\right)\right] - \mathbb{E}\left[f\left(\frac{S_n}{\sqrt{n}}, \frac{S_n^*}{\sqrt{n}}\right)\right]\right| \leq \frac{C}{\sqrt{n}}.$$

*Proof.* To simplify notation, let

$$p_n(m) = \mathbb{P}\{S_n = m\}.$$

Using equation (2.4),

$$\begin{aligned}
&\left|2\mathbb{E}\left[g\left(\frac{S_n}{\sqrt{n}}\right)\right] - \mathbb{E}\left[f\left(\frac{S_n}{\sqrt{n}}, \frac{S_n^*}{\sqrt{n}}\right)\right]\right| = \\
&\qquad\qquad \left|2\sum_{m\geq 1} p_n(m)\frac{m}{\sqrt{n}}\int_0^{\frac{m}{\sqrt{n}}} f\left(2y - \frac{m}{\sqrt{n}}, y\right)dy - \mathbb{E}\left[f\left(\frac{S_n}{\sqrt{n}}, \frac{S_n^*}{\sqrt{n}}\right)\right]\right| \\
&\qquad\qquad \leq \left|2\sum_{m\geq 1} p_n(m)\frac{m}{\sqrt{n}}\sum_{l=0}^{m-1}\left(\int_{\frac{l}{\sqrt{n}}}^{\frac{l+1}{\sqrt{n}}} f\left(2y - \frac{m}{\sqrt{n}}, y\right)dy - \frac{1}{\sqrt{n}}f\left(\frac{2l-m}{\sqrt{n}}, \frac{l}{\sqrt{n}}\right)\right)\right| + \\
&\quad + \left|\sum_{m\geq 1} p_n(m)\frac{m}{n}\sum_{l=0}^{m-1} f\left(\frac{2l-m}{\sqrt{n}}, \frac{l}{\sqrt{n}}\right) - \sum_{m\geq 0} p_n(m)\frac{m+1}{n+1}\sum_{l=0}^{m} f\left(\frac{2l-m+2}{\sqrt{n}}, \frac{l}{\sqrt{n}}\right)\right| + \\
&\quad + \left|\sum_{m\geq 1} p_n(m)\frac{m}{n}\sum_{l=0}^{m-1} f\left(\frac{2l-m}{\sqrt{n}}, \frac{l}{\sqrt{n}}\right) - \sum_{m\geq 0} p_n(m)\frac{m+1}{n+1}\sum_{l=0}^{m-2} f\left(\frac{2l-m+2}{\sqrt{n}}, \frac{l}{\sqrt{n}}\right)\right|.
\end{aligned}$$

To make the notation more compact, we denote by $A_n$, $B_n$, and $C_n$ the first, second, and third absolute values, respectively. The goal is to obtain an upper bound of order $n^{-\frac{1}{2}}$ for each of them.

For the first term, using the Lipschitz property of $f$, we obtain

$$\begin{aligned}
A_n &\leq \sum_{m\geq 1} p_n(m)\frac{m}{\sqrt{n}}\sum_{l=0}^{m-1}\int_{\frac{l}{\sqrt{n}}}^{\frac{(l+1)}{\sqrt{n}}}\left|f\left(2y - \frac{m}{\sqrt{n}}, y\right) - f\left(\frac{2l-m}{\sqrt{n}}, \frac{l}{\sqrt{n}}\right)\right|dy \\
&\leq \sum_{m\geq 1} p_n(m)\frac{m}{\sqrt{n}}\sum_{l=0}^{m-1}\frac{\sqrt{3}L}{2n} \\
&= \frac{\sqrt{3}L}{2\sqrt{n}}\mathbb{E}\left[\frac{S_n^2}{n}\mathbb{1}_{\{S_n>0\}}\right] = \frac{\sqrt{3}L}{2\sqrt{n}}.
\end{aligned}$$

The estimate for the second and third terms is analogous. Therefore, for simplicity, we report the one made for $C_n$

$$\begin{aligned}
C_n &\leq \sum_{m\geq 2} p_n(m)\frac{m-1}{n+1}\sum_{l=0}^{m-2}\left|f\left(\frac{2l-m}{\sqrt{n}}, \frac{l}{\sqrt{n}}\right) - f\left(\frac{2l-m+2}{\sqrt{n}}, \frac{l}{\sqrt{n}}\right)\right| + \\
&\quad + \sum_{m\leq 1} p_n(m)\frac{m}{n}\left|f\left(\frac{m-2}{\sqrt{n}}, \frac{m-1}{\sqrt{n}}\right)\right| + \sum_{m\geq 2} p_n(m)\left|\frac{m-1}{n+1} - \frac{m}{n}\right|\sum_{l=0}^{m-2}\left|f\left(\frac{2l-m}{\sqrt{n}}, \frac{l}{\sqrt{n}}\right)\right| \\
&< p_n(m)\frac{(m-1)^2}{n+1}\frac{2L}{\sqrt{n}} + \sum_{m\geq 1} p_n(m)\frac{m}{n}\left(|f(0,0)| + \sqrt{2}L\frac{m}{n}\right) + \\
&\quad + \sum_{m\geq 2} p_n(m)\frac{m+n}{n(n+1)}\sum_{l=0}^{m-2}\left(|f(0,0)| + L\frac{m\sqrt{2}}{\sqrt{n}}\right)
\end{aligned}$$

$$
\begin{aligned}
&< \frac{L(2+\sqrt{2})}{2\sqrt{2}}\mathbb{E}\left[\frac{S_n^2}{n}\right] + \sum_{m\geq 1} p_n(m)\left(\frac{|f(0,0)|}{n+1}\left(2m+1+\frac{m^2}{n}\right) + \frac{L\sqrt{2}}{\sqrt{n}}\frac{m(m-1)(n+m)}{n(n+1)}\right)\\
&\leq \frac{L(2+\sqrt{2})}{2\sqrt{n}} + \frac{|f(0,0)|}{\sqrt{n}}\mathbb{E}\left[\frac{|S_n|}{\sqrt{n}}\right] + \frac{L\sqrt{2}}{\sqrt{n}}\sum_{m\geq 2}\frac{2m^2}{n}p_n(m)\\
&< \frac{|f(0,0)|+4L}{\sqrt{n}}.
\end{aligned}
$$

The same type of estimate applies to $B_n$. Hence, there exists a constant $C > 0$ such that

$$A_n + B_n + C_n \leq \frac{C}{\sqrt{n}}.$$

This proves the proposition and completes the proof of Theorem 2.2.1. □

**The case of barrier options.** The proof of Theorem 2.2.1 can also be adapted to the case of barrier options.

Consider a function of the form

$$f(x,y) = f_1(x)\mathbf{1}_{(M_0,M_1)}(y),$$

where $M_0 < M_1 \leq +\infty$ and $f_1$ is Lipschitz.

In this setting, the auxiliary function $g$ introduced in the proof can be written as

$$g(x) = 2x\int_{M_0}^{x\wedge M_1} f_1(2y-x)\,dy\,\mathbf{1}_{(M_0,+\infty)}(x) = 2x\int_{2M_0-x}^{x\wedge(2M_1-x)} f_1(z)\,dz\,\mathbf{1}_{(M_0,+\infty)}(x).$$

This function satisfies condition $(LL)$. Hence, the arguments used in Steps 1 and 2 remain valid. The same ideas used in Step 3 then allow one to extend the convergence result to this class of payoff functions.

Possible choices of $f_1$ can be

1 $f_1(x) = (K-x)_+$,

2 $f_1(x) = (x-K)_+$,

3 $f_1(x) = (K-e^{\sigma x})_+$,

4 $f_1(x) = (e^{\sigma x}-K)_+$.

These functions correspond to European option payoffs with barriers.

## 2.3 Related Research

Further developments in this field have been provided by Leduc, Palmer, and Lin. More specifically, we consider three contributions that are particularly relevant to the present work.

The first contribution consists of two independent studies carried out by Leduc [8] and by Palmer and Lin [11]. Both studies analyze binomial approximations for option pricing and obtain similar convergence results, although they focus on different classes of contracts.

Leduc derived a general first-order error formula for European options. Palmer and Lin, on the other hand, studied the CRR approximation for barrier options. They considered two main classes of contracts:

1) Up-and-out and up-and-in options, where the barrier is located above the initial asset price.

2) Down-and-out and down-and-in options, where the barrier is located below the initial asset price.

For example, an up-and-out option with barrier level $b$ and terminal payoff $f(S_T)$ can be represented as

$$h = f(S_T)\mathbb{1}_{\{M_T \leq b\}},$$

while the corresponding up-and-in option is

$$h = f(S_T)\mathbb{1}_{\{M_T > b\}}.$$

Similarly, a down-and-out option with barrier level $a$ is given by

$$h = f(S_T)\mathbb{1}_{\{m_T \geq a\}},$$

whereas the corresponding down-and-in option is

$$h = f(S_T)\mathbb{1}_{\{m_T < a\}},$$

where

$$m_t = \min_{0 \leq s \leq t} S_s, \qquad 0 \leq t \leq T,$$

denotes the running minimum of the price process.

For an up-and-out option, the contract is knocked out and becomes worthless as soon as the underlying asset price exceeds the barrier before maturity. Analogously, a down-and-out option is knocked out when the asset price falls below the barrier.

They showed that the Cox-Ross-Rubinstein framework converges to the corresponding Black-Scholes barrier option price with the following first-order asymptotic error expansion, analogous to that obtained by Leduc for European options:

$$C_n = C_{BS} + \frac{A_n}{\sqrt{n}} + \frac{B_n}{n} + O(n^{-\frac{3}{2}}).$$

where $n$ denotes the number of time steps, $C_n$ denotes the binomial price, $C_{BS}$ the corresponding Black-Scholes price, and $A_n$ and $B_n$ are coefficients depending on the position of the barrier and the strike in the binomial lattice. Moreover, $O(\cdot)$ denotes the asymptotic order of the pricing error as the number of time steps tends to infinity.

Since their work focuses on barrier options, they also proved that if the barrier lies on a node of the binomial tree, the convergence rate improves from $O(n^{-\frac{1}{2}})$ to $O(n^{-1})$.

Starting from this initial study, Leduc [9] generalized the previous formula to a broader class of frameworks that discretize the Black-Scholes model. He proved that the convergence rate of order $O(n^{-\frac{1}{2}})$ is not a specific feature of the CRR model, but rather a consequence of discretizing a continuous-time process. He also showed that modifying the model parameters alone does not improve the asymptotic convergence rate.

A further contribution was made by Leduc and Palmer [10]. Their work develops new methods for pricing lookback options. The approach is based on transforming path-dependent options into equivalent non-path-dependent claims, allowing the derivation of more accurate

binomial approximations. As a result, the pricing error is reduced to order $O(n^{-1})$. Further corrections to the framework improve the convergence rate to $O(n^{-\frac{3}{2}})$.

These studies show that the convergence properties of binomial models depend primarily on the way a continuous-time process is discretized. They also demonstrate that suitable modifications of the lattice structure or of the option representation can significantly improve the accuracy of pricing and hedging procedures.

These results are particularly relevant to the present thesis. They support the idea that lattice methods can remain competitive for the valuation of exotic derivatives when appropriate corrections are introduced. They also show that binomial models can provide accurate approximations not only for pure European options, but also for exotic derivatives such as barrier options. Nevertheless, the convergence rate of the pricing error is not the only factor affecting practical implementation. Nevertheless, convergence rate is not the only aspect determining the practical usefulness of a numerical method. Computational efficiency is equally important. For this reason, the final chapter presents the numerical algorithm adopted in this work and discusses its computational consistency and performance.

# Chapter 3

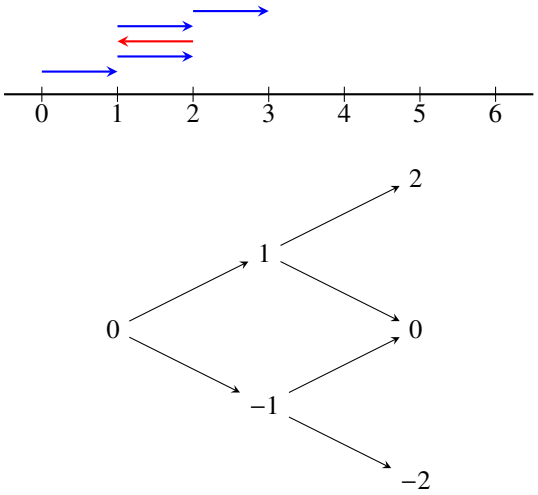
# Algorithms and Simulations

This chapter presents the numerical algorithms used to assess the theoretical results developed in the previous chapters. The analysis compares three different methodologies: the random-walk tree algorithm, Monte Carlo simulation of Brownian motion, and numerical integration used as a benchmark.

The comparison is performed on three path-dependent test functions. The first corresponds to a standard lookback payoff, while the remaining examples are chosen to illustrate more general path-dependent structures. In particular, the third function introduces a hard barrier through the running maximum. The objective is to evaluate the accuracy, computational cost, and practical limitations of the proposed random-walk approximation.

## 3.1 Random Walk Algorithm

In this section, we use the Markov property and the tree structure to construct an efficient backward algorithm. A simple random walk can be represented not only as a path evolving on the integer line, but also as a recombining binomial tree.




At each period, the random walk changes its position by $+1$ or $-1$ with probabilities $p$ and $1-p$, respectively. In the symmetric case considered in this thesis, $p = \frac{1}{2}$. This structure is analogous to the Cox-Ross-Rubinstein binomial model. In both cases, the process evolves on a recombining binomial tree where each node has two possible successors. In the CRR framework, the branches represent upward and downward movements of the underlying asset price, whereas in the present setting they correspond to increments of the random walk.

As introduced in Chapter 1, $S_n$ denotes the simple random walk after $n$ steps and $S_n^*$ its running maximum. The Markov property allows the pricing problem to be solved by backward induction over the state space.

Let $m$ denote the current time step and let $\mathbb{E}_m$ be the set of states attainable by $(S_n, S_n^*)$ in $m$ steps starting from $(0,0)$. The value function is defined recursively by

$$v(m,j,l) = \mathbb{E}\left[ f\left(\frac{S_n}{\sqrt{n}}, \frac{S_n^*}{\sqrt{n}}\right) | S_m = j,\ S_m^* = l \right].$$

From the state $(j,l)$ the process can move only to two possible states:

$$(j', l') = (j+1, l \vee (j+1)),$$

or

$$(j'', l'') = (j-1, l).$$

Each transition has probability $\frac{1}{2}$. Therefore, by the Markov property,

$$\begin{aligned} v(m,j,l) &= \mathbb{E}\left[ f\left(\frac{j + S_{n-m}}{\sqrt{n}}, \frac{l \vee (j + S_{n-m}^*)}{\sqrt{n}}\right)\right] \\ &= \frac{1}{2}\left(v(m+1, j', l') + v(m+1, j'', l'')\right). \end{aligned} \tag{3.1}$$

The terminal condition is

$$v(n,j,l) = f\left(\frac{j}{\sqrt{n}}, \frac{l}{\sqrt{n}}\right).$$

Starting from the terminal nodes and applying (3.1) backward, the required expectation is obtained as

$$\mathbb{E}\left[ f\left(\frac{S_n}{\sqrt{n}}, \frac{S_n^*}{\sqrt{n}}\right)\right] = v(0,0,0).$$

The algorithm can be summarized as follows:

for $(j,l)$ in $\mathbb{E}_n$, $v(n,j,l) := f\left(\frac{j}{\sqrt{n}}, \frac{l}{\sqrt{n}}\right)$,

for $m = n-1$ down to 0,

for $(j,l)$ in $\mathbb{E}_m$, $v(m,j,l) := \frac{1}{2}(v(m+1,j',l') + v(m+1,j'',l''))$.

The procedure consists of two phases. First, all attainable terminal states are identified and the function $f$ is evaluated at those states. Second, the value function is computed recursively by backward induction.

The function $f$ is evaluated approximately $\frac{n^2}{4}$ times at the terminal level. The backward induction requires approximately $\frac{n^3}{12}$ operations, since equation (3.1) is applied a number of times asymptotically equal to

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (k+1)^2 + \sum_{k=0}^{\lfloor \frac{(n-1)}{2} \rfloor} (k+1)(k+2) \sim \frac{n^3}{12}. \tag{3.2}$$

This follows from

$$\sum_{k=0}^{n} k^2 = \frac{2n^3 + 3n^2 + n}{6}.$$

*So the total number of operations is generally of order* $\frac{n^3}{12}$.

Listing 3.1: Simple random walk algorithm

```
clear; clc; close all;
%% TERMINAL LAYER

% V_next stores the value function at a fixed time layer.
% Rows correspond to the random walk position.
% Columns correspond to the running maximum.
% Npath is set equal to 1000 in order to achieve a computational cost
    comparable to the other methods and to allow a fair comparison
    between the different numerical procedures.

N = 1000;
V_next = NaN(2*N + 1, N + 1);
K = 1;
M = 1;

% i generate all possible values of terminal nodes.
% j is the row index associated with the terminal random walk position.
% l is the column index associated with the running maximum.
% m represents the possible values of the running maximum.
% Some states may not be attainable, but this does not affect the
    algorithm since backward induction visits only attainable states.

% The terminal value is computed using one of the illustrative
    functions below.
% Different choices of f can be implemented by modifying this terminal
    condition.

for i = -N : 2 : N
    j = i + N + 1;
    for m = max(0,i) : N
        l = m + 1;

        x = i / sqrt (N);
        y = m / sqrt (N);

        % function 1: floating-strike lookback call
        terminal_value = y - x;

        % function 2: lookback-type function with soft penalty
        % terminal_value = (y - x) * exp(-y);

        % function 3: up-and-out call-type function
        % terminal_value = max(exp(x) - K, 0) * (y <= M);

        V_next(j, l) = terminal_value;
    end
```

```
end

%% BACKWARD INDUCTION

% Since we have already computed the terminal nodes, we start from N - 1.
% i is the time index.
% k is the random walk position at time i.
% j is the row index associated with the random walk position.
% l is the column index associated with the running maximum.
% m is the running maximum reached along the path.
% j_up and j_down are the row indices of the next possible nodes.
% l_up and l_down are the corresponding column indices of the running maximum.

for i = N-1 : -1 : 0
    V_curr = NaN(2*i + 1, i + 1);
    for k = -i : 2 : i
        j = k + i + 1;
        for m = max(0,k) : i
            l = m + 1;
            k_up   = k + 1;
            k_down = k - 1;
            m_up   = max(m, k_up);
            m_down = m;
            j_up   = k_up   + (i + 1) + 1;
            j_down = k_down + (i + 1) + 1;
            l_up   = m_up + 1;
            l_down = m_down + 1;
            V_curr(j, l) = 0.5 * (V_next(j_up, l_up) + V_next(j_down, l_down));
        end
    end
    V_next = V_curr;
end

result = V_next(1,1);
```

In this section and in the following ones, the code listings provide the skeleton of the implemented algorithms. The objective is not to present fully optimized code, but to describe the computational logic used to obtain the numerical results. In practice, both the random-walk algorithm and the Brownian motion simulation can be optimized further to improve computational speed.

Furthermore, the functions used in the code listings are only illustrative examples. Different functions can be implemented by modifying only the terminal condition.

Although this thesis focuses on European path-dependent contracts, the algorithm presented in the previous section can be adapted naturally to price an American put option.

Let $\mathcal{P}_n$ denote the value of an American put option at time $n$, with maturity $N$, strike

price $K$, and underlying asset price $x$. The price $\mathcal{P}_n$ can be written as

$$\mathcal{P}_n = P_{\text{am}}(n, x),$$

where $P_{\text{am}}(n, x)$ is defined by

$$P_{\text{am}}(N, x) = (K - x)_+.$$

Unlike a European option, which can only be exercised at maturity, an American option can be exercised at any time before expiration. Therefore, at each node the option value is given by the maximum between the immediate exercise value and the continuation value:

$$P_{\text{am}}(n, x) = \max\left((K - x)_+, \frac{f(n+1, x)}{1 + r}\right) \quad \text{for } n \leq N - 1.$$

where

$$f(n + 1, x) = pP_{\text{am}}(n + 1, xu) + (1 - p)P_{\text{am}}(n + 1, xd)$$

The formula is very similar to equation 1.1 for European options. The pricing procedure is therefore based on the same backward-induction mechanism used in Cox-Ross-Rubinstein model with the same assumptions. However, unlike the path-dependent algorithm presented previously, no additional state variable is required. As a consequence, the recombining property of the binomial tree is preserved and the computational complexity is significantly reduced.

Listing 3.2: American option algorithm

```
clear; clc; close all;
%% TERMINAL LAYER

% Input parameters

S0 = 100;
K = 100;
T = 1;
r = 0.05;
sigma = 0.2;
N = 500;

% CRR parameters obtained from the standard continuous-time
    parametrization
% to ensure consistency with the Black-Scholes framework

dt = T / N;
u = exp(sigma * sqrt(dt));
d = exp(-sigma * sqrt(dt));
p = (exp(r * dt) - d) / (u - d);
discount_rate = exp(-r * dt);

% The Matrix containing the option values for all nodes and time steps

Values = NaN(N + 1, N + 1);

```

```
% i compute the option payoff at all terminal nodes

for i = 0 : N
    S = S0 * u^i * d^(N-i);
    Values(N + 1, i + 1) = max(K - S, 0);
end

%% Backward induction

% Since we have already computed the terminal nodes, we start from N -
    1
% i denotes the time step
% j denotes the number of upward moves at time step i

for i = N-1:-1:0
    for j = 0:i
        continuation = discount_rate * (p * Values(i+2, j+2) + (1-p) *
            Values(i+2, j+1));
        S = S0 * u^j * d^(i-j);
        exercise = max(K - S, 0);
        Values(i+1, j+1) = max(exercise, continuation);

    end
end

% The option price is obtained from the root node of the tree

price = Values(1,1);
fprintf('American put price = %.6f\n', price);
```

For the parameters $S_0 = 100$, $K = 100$, $r = 5\%$, $\sigma = 20\%$, and $T = 1$, the algorithm yields an American put price equal to 6.089595, illustrating the implementation of the pricing procedure described above.

The American-option algorithm has been included for completeness. Although it is not used in the numerical experiments of this work, it illustrates how the backward-induction framework can be extended naturally from European to American-style contracts.

## 3.2 Brownian Motion Simulations

Since this thesis studies the approximation of Brownian motion by a random walk, a natural benchmark is given by Monte Carlo simulation of Brownian paths. The Brownian motion is simulated by discretizing the interval $[0, T]$ into time steps of length $\Delta t$. The resulting increments are independent and normally distributed with mean zero and variance $\Delta t$. The discretized dynamics are

$$B_{t+1} = B_t + \sqrt{\Delta t}\, Z_t,$$

where $Z_t \sim N(0, 1)$ are independent standard normal random variables.

For a payoff $f(B_T, B_T^*)$, the Monte Carlo estimator is

$$\widehat{V} = \frac{1}{N_{\text{path}}} \sum_{i=1}^{N_{\text{path}}} f(B_T^{(i)}, (B_T^*)^{(i)}),$$

where $N_{\text{path}}$ is the number of simulated paths.

Monte Carlo simulation presents a clear trade-off. On the one hand, it is one of the most flexible and intuitive methods for pricing exotic options. On the other hand, since it relies on random sampling, it introduces a statistical error. Reducing this error requires a large number of paths, which may lead to high computational cost.

The total error can be decomposed qualitatively as

$$\text{Total Error} = \underbrace{\text{Statistical Error}}_{\sim \frac{1}{\sqrt{N_{\text{path}}}}} + \underbrace{\text{Discretization Bias}}_{\sim \Delta t}. \tag{3.3}$$

The first component is the statistical error. It is measured through the standard error

$$\text{SE} = \frac{\sigma_{\text{payoff}}}{\sqrt{N_{\text{path}}}},$$

where $\sigma_{\text{payoff}}$ is the standard deviation of the simulated payoffs. Halving the statistical error requires increasing the number of simulated paths by a factor of four. Therefore, a high level of accuracy can become computationally expensive.

To reduce the variance, we use the antithetic variates method. Since the standard normal distribution is symmetric, for every simulated increment $Z$ we also consider the opposite increment $-Z$. This generates two negatively correlated paths. The final estimator is obtained by averaging the payoffs of the two paths.

$$\widehat{V} = \frac{1}{N} \sum_{i=1}^{N} \frac{f(B^{+,i}) + f(B^{-,i})}{2}$$

The effectiveness of the method is based on the identity

$$\text{Var}\left(\frac{B^+ + B^-}{2}\right) = \frac{\text{Var}(B^+) + \text{Var}(B^-) + 2\text{Cov}(B^+, B^-)}{4}.$$

When the covariance term is negative, the variance of the average estimator is reduced. This is the reason why antithetic variables can improve the efficiency of Monte Carlo simulation.

This technique has limitations, especially for complex path-dependent options, American-style options, and multidimensional problems. In this thesis, however, the options considered are European-style path-dependent options, so antithetic variates can be effectively combined with a large number of simulations.

Figure 3.1 compares the convergence of standard Monte Carlo and antithetic Monte Carlo.

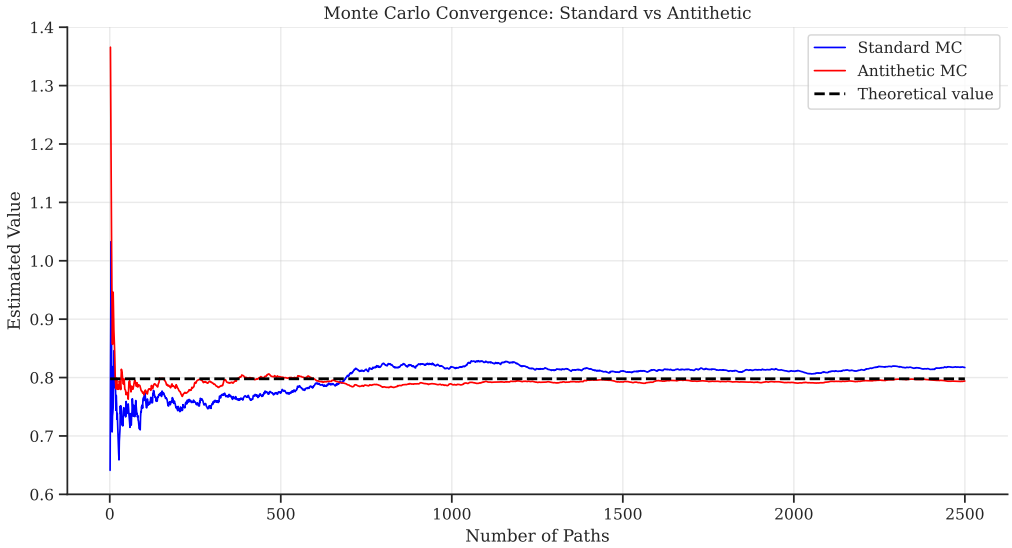



Figure 3.1: Monte Carlo convergence: standard estimator versus antithetic-variates estimator.

For the payoff depending on the maximum of Brownian motion, the theoretical value is

$$\mathbb{E}[M_T] = \sqrt{\frac{2T}{\pi}}.$$

Since in this work $T = 1$, this becomes

$$\mathbb{E}[M_T] = \sqrt{\frac{2}{\pi}}.$$

The two remaining lines in the graph represent the estimates obtained using the standard estimator and the Monte Carlo estimator obtained through the antithetic variates method. This one, displays a more stable convergence pattern than the standard Monte Carlo estimator. In particular, the antithetic method approaches the theoretical value after roughly 1200 paths, while the standard estimator remains less stable even with a larger number of simulated paths.

The main advantages of the antithetic method can be summarized as follows:

1) For the same number of simulations, it achieves a lower variance and therefore a lower standard error than standard Monte Carlo.

2) For a fixed level of accuracy, it requires fewer simulated paths.

The second component of the error in (3.3) is the discretization bias. It can be reduced by increasing the number of time steps. In standard financial applications, the time step is often chosen according to the financial calendar. For exotic options, however, this choice may not be sufficient. If the payoff depends on the maximum of the process, the maximum may occur between two consecutive discretization dates and may therefore be missed by the simulation.

To address this issue, we use a Brownian bridge correction between consecutive discretization points. The Brownian bridge allows us to account for the maximum reached between two simulated values. This technique is particularly useful for path-dependent options and barrier options.

Let $t_i$ and $t_{i+1}$ be two consecutive discretization times, with

$$\Delta t = t_{i+1} - t_i.$$

Conditionally on the values

$$B(t_i) = a,$$
$$B(t_{i+1}) = b,$$

the process between $t_i$ and $t_{i+1}$ is a Brownian bridge. The conditional distribution of its maximum $M$ is given by

$$\mathbb{P}(M \geq x \mid B(t_i) = a,\ B(t_{i+1}) = b) = \exp\left(-\frac{2(x-a)(x-b)}{\Delta t}\right).$$

By setting this probability equal to a uniform random variable $u \sim U(0,1)$, one obtains the sampled maximum

$$M = \frac{a + b + \sqrt{(b-a)^2 - 2\Delta t \ln(u)}}{2}.$$

Without this correction, increasing the number of paths reduces the variance but does not remove the bias caused by missing the true maximum between discretization points.

The error trade-off in Monte Carlo simulation is complementary to that of the random-walk algorithm. In the random-walk framework, the computation is more complex, but the approximation error is deterministic and controlled by the convergence rate. In Monte Carlo simulation, the method is more scalable, but the estimator is affected by both sampling error and discretization bias.

To validate the simulations, we compute the confidence interval

$$\bar{X} \pm z_{0.975} \times \text{SE},$$

where $\bar{X}$ is the sample mean, SE is the standard error, and $z_{0.975} = 1.96$ for a 95% confidence interval.

A confidence interval does not mean that the true value belongs to that specific interval with probability 95%. Rather, it means that if the simulation were repeated many times, approximately 95% of the resulting confidence intervals would contain the true value. Therefore, confidence intervals are useful for comparing the efficiency of different estimators, provided that they estimate the same quantity and are constructed with the same confidence level.

Listing 3.3: Brownian motion simulations

```
clear; clc; close all;
%%

% Divide the interval [0,1] into n time steps.
% Since the option is path-dependent, intermediate values of the
    Brownian motion path must be computed.
% Nstep is set equal to 252, corresponding approximately to the number
    of trading days in one year. This choice gives a daily
    discretization of the Brownian path over the interval [0,1].
```

```
% Npath is set equal to 700000 in order to achieve a computational cost
     comparable to the other methods and to allow a fair comparison
    between the different numerical procedures.
% T is the maturity time and t is the initial time.
% Fix a random seed to make the results reproducible.

rng(73)
Nstep = 252;
Npath = 700000;
T = 1;
t = 0;
dt = (T - t) / Nstep;
M_barrier = 1;
K = 1;

Values = zeros(Npath / 2, 1);

% For each simulation, generate one Brownian path and its antithetic
    counterpart.
% j is the Monte Carlo path index.
% k is the time-step index.
% For each subinterval, two consecutive Brownian values are used to
    sample the corresponding Brownian bridge maximum.
% a and b denote the endpoints of the interval.

% The terminal value is computed using one of the illustrative
    functions below.
% Different choices of f can be implemented by modifying this terminal
    condition.

for j = 1 : Npath/2
    dW = sqrt(dt)*randn(Nstep,1);
    dW_antithetic = -dW;
    B_1 = [0; cumsum(dW)];
    B_2 = [0; cumsum(dW_antithetic)];
    B_T_1 = B_1(end);
    B_T_2 = B_2(end);
    M_1 = -Inf;
    M_2 = -Inf;
    for k = 1:Nstep
        a_1 = B_1(k);
        b_1 = B_1(k+1);
        u_1 = rand;
        M_bridge1 = (a_1 + b_1 + sqrt((b_1-a_1)^2 - 2*dt*log(u_1))) /
            2;
        M_1 = max(M_1, M_bridge1);
        a_2 = B_2(k);
        b_2 = B_2(k+1);
        u_2 = rand;
```

```
        M_bridge2 = (a_2 + b_2 + sqrt((b_2-a_2)^2 - 2*dt*log(u_2))) /
            2;
        M_2 = max(M_2, M_bridge2);
    end

    % function 1: floating-strike lookback call
    f_1 = M_1 - B_T_1;
    f_2 = M_2 - B_T_2;

    % function 2: lookback-type function with soft penalty
    % f_1 = (M_1 - B_T_1) * exp(-M_1);
    % f_2 = (M_2 - B_T_2) * exp(-M_2);

    % function 3: up-and-out call-type function
    % f_1 = max(exp(B_T_1) - K, 0) * (M_1 <= M_barrier);
    % f_2 = max(exp(B_T_2) - K, 0) * (M_2 <= M_barrier);

    Values(j) = (f_1 + f_2)/2;
end

% Compute the estimator, standard error, and confidence interval.

E = mean(Values);
SE = std(Values) ./ sqrt(Npath/2);
CI = [E - 1.96 * SE, E + 1.96 * SE];
```

The number of time steps is set equal to 252, corresponding approximately to the number of trading days in one year. This choice provides a daily discretization of the Brownian path over the time interval $[0, 1]$. The number of simulated paths is set equal to 700 000 in order to to achieve a computational cost comparable to the other methods and to allow a fair comparison. Since the antithetic variates method is used, only $\frac{N_{\text{path}}}{2}$ independent paths are generated, each paired with its antithetic counterpart.

## 3.3 Numerical Approximation of Integrals

The third methodology used to assess the consistency of the results is numerical integration. This technique is used to approximate integrals that do not admit a closed-form solution or are computationally difficult to evaluate analytically. In this work, numerical integration is used to compute the benchmark associated with the Brownian motion and its maximum, as derived in equation (2.5).

From Chapter 1, option prices can be expressed as discounted expectations. In the case of path-dependent options, these expectations may involve multidimensional integrals. Although the theoretical integration domain is often unbounded, the rapid decay of the normal density in the tails allows the domain to be truncated in a numerically reasonable way.

Several numerical integration methods can be used. We briefly discuss three of them, although the numerical results in this thesis are computed using the trapezoidal rule.

The first method is the Finite Element Method (FEM). It is commonly used to solve partial differential equations by transforming them into systems of algebraic equations. The

method consists of dividing the domain into a mesh of finite elements, approximating the unknown function on each element by polynomial functions, assembling the equations into a global system, and solving the resulting algebraic problem. Although powerful, this approach is more complex than necessary for the benchmark computation considered in this thesis.

The second method, and the one used in this work, is the trapezoidal rule. For a single interval $[a, b]$, it approximates the integral by

$$I \approx (b - a)\frac{f(a) + f(b)}{2},$$

where $a$ and $b$ are the endpoints of the integration interval.

Dividing the interval into $n$ subintervals of equal length,

$$h = \frac{b - a}{n},$$

the composite trapezoidal rule is

$$I \approx \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right].$$

This method performs well for smooth functions and is simple to implement. Its error is of order $O(N^{-2})$, where $N$ is the number of subintervals.

The third method is Simpson's rule. As in the trapezoidal method, the interval is divided into subintervals of equal length, but the number of subintervals must be even. The integral is approximated by

$$I \approx \frac{h}{3}\left[f(x_0) + 4(f(x_1) + f(x_3) + \cdots + f(x_{n-1})) + 2(f(x_2) + f(x_4) + \cdots + f(x_n))\right].$$

Simpson's rule uses a quadratic interpolation and assign different weights to odd and even terms, and, under appropriate regularity assumptions, achieves an error of order $O(N^{-4})$. Therefore, for smooth functions, it is generally more accurate than the trapezoidal method.

However, one of the applications considered in this work is the European barrier up-and-out option. In this case, the hard barrier introduces a discontinuity through an indicator function. This violates the regularity assumptions required for Simpson's rule to achieve its theoretical convergence rate. The trapezoidal rule is more robust across all payoff specifications used in this thesis, including discontinuous functions involving indicator terms. For this reason, it was preferred.

The previous formulas are presented in the one-dimensional case for simplicity. In the numerical experiments of this work, both methods are applied to a two-dimensional integral involving the joint density of Brownian motion and its maximum. The multidimensional approximation is obtained by applying the corresponding quadrature rule successively along each integration variable.

Furthermore, since the integration domain is theoretically unbounded, finite truncation levels are required for numerical implementation. The parameters $y_{\max}$ and $z_{\max}$ denote the upper bounds used for the integration variables $y$ and $z$, respectively. Their values are selected sufficiently large so that the contribution of the neglected tail region is negligible relative to the numerical error of the quadrature method.

Listing 3.4: Numerical methods

```
clear; clc; close all;
%%

% The theoretical integration domain is unbounded.
% y_max and z_max define the truncation levels for the variables y and z.
% Their values are chosen sufficiently large so that the contribution of the omitted tail region is negligible.
% n_y and n_z are the numbers of grid points used in the two integration variables. They are set equal to 9100 in order to achieve a computational cost comparable to the other methods and to allow a fair comparison between the different numerical procedures.
% The double integral is evaluated by computing the inner integral first.

n_y = 9100;
n_z = 9100;
y_max = 4;
z_max = 8;
K = 1;
M = 1;

% Create the integration grid and store the values of the inner integral.

y = linspace(0, y_max, n_y);
I_y = zeros(1,n_y);

% Evaluate the integrand at each grid point.
% The terminal value is computed using one of the illustrative functions below.
% Different choices of f can be implemented by modifying this terminal condition.

for i = 1:n_y
    y_i = y(i);
    z = linspace(y_i, z_max, n_z);

    % function 1: floating-strike lookback call
    f = z - y_i;

    % function 2: lookback-type function with soft penalty
    % x = 2*y_i - z;
    % f = (y_i - x) .* exp(-y_i);

    % function 3: up-and-out call-type function
    % x = 2*y_i - z;
```

```
    % f = max(exp(x) - K, 0) .* (y_i <= M);

    integrand = ((2 .* z) ./ sqrt(2 * pi)) .* exp(- z.^2 /2) .* f;
    I_y(i) = trapz(z, integrand);
end
E_f = trapz(y, I_y);
```

## 3.4 Examples

We now compare the three methods using the following path-dependent test functions:

1) $f(x, y) = y - x$.

2) $f(x, y) = (y - x)e^{-y}$.

3) $f(x, y) = (e^x - 1)_+ \mathbb{1}_{\{y \le 1\}}$

The first payoff corresponds to a floating-strike lookback call option. It is the simplest case and provides a natural benchmark for comparing the three methods.

The second function introduces a soft penalty when the path reaches high values. The function depends on the gap between the running maximum and the terminal value. The exponential factor penalizes paths that attain large maxima, creating a soft-barrier effect and introducing non-linearity.

The third function is the most complex. It is discontinuous, contains a hard barrier, and depends strongly on the running maximum. While the first function corresponds to a standard financial payoff, the remaining examples are mainly included to illustrate more general path-dependent structures. In particular, the third function tests the robustness of the algorithms under a discontinuous structure. In this case, the antithetic method provides more limited benefits.

To compare the methods under comparable computational effort, the discretization parameters are chosen by fixing a common computational budget $C$. From equation (3.2), the number of operations of the tree method is approximated by $C_{\text{tree}} \approx \frac{N^3}{12}$. For Monte Carlo simulation with antithetic variates, $C_{\text{MC}} \approx (\frac{N_{\text{path}}}{2})N_{\text{step}}$, while for numerical integration, $C_{\text{int}} \approx n_y n_z$. The parameters are then selected so that these quantities are approximately equal.

The numerical results are reported below.

| Function | Method | Estimate | Standard Error | Confidence Interval |
|---|---|---|---|---|
| $y - x$ | Monte Carlo | 0.7977 | 0.0004015 | [0.79691; 0.79849] |
| | Numerical integration | 0.7979 | – | – |
| | Tree method | 0.782273 | – | – |
| $(y - x)e^{-y}$ | Monte Carlo | 0.47663 | 0.00034727 | [0.47595; 0.47731] |
| | Numerical integration | 0.4768 | – | – |
| | Tree method | 0.476067 | – | – |
| $(e^x - 1)_+ \mathbb{1}_{\{y \le 1\}}$ | Monte Carlo | 0.096878 | 0.00028039 | [0.096329; 0.097428] |
| | Numerical integration | 0.0966 | – | – |
| | Tree method | 0.099929 | – | – |

Table 3.1: Numerical approximations of $\mathbb{E}[f(B_1, B_1^*)]$.

The results show that all three methods provide estimates reasonably close to the

benchmark value. When the computational effort is approximately the same, Monte Carlo simulation achieves the highest level of accuracy in all three examples. The tree method also provides accurate approximations, particularly for the second and third functions, while the remaining discrepancy with respect to the benchmark is deterministic rather than statistical. This deterministic error can be analysed through convergence results, whereas Monte Carlo simulation provides only a statistical error quantified by confidence intervals.

We now discuss Monte Carlo simulation and the tree method separately.

*Monte Carlo.*

Using 700 000 paths, antithetic variates, and the Brownian bridge correction, the Monte Carlo estimates are very close to the numerical benchmark. The small standard errors and the narrow confidence intervals confirm the reliability of the estimator. In all three cases, the difference with respect to the benchmark remains very small.

These results confirm that Monte Carlo simulation is reliable and highly adaptable to path-dependent functions. This applies both to the standard lookback payoff and to the more general path-dependent functions considered in this work, including the discontinuous barrier-type specification. The main limitation remains the computational cost associated with the large number of simulated paths required to obtain a small statistical error.

Furthermore, the following table illustrates how the approximation improves as the number of simulations increases. This can be observed both from the stabilization of the estimated values around the benchmark and from the progressive narrowing of the confidence interval.

| Function | Steps | Estimate | Standard Error | Confidence Interval |
|---|---|---|---|---|
| $y - x$ | 1 000 | 0.79227 | 0.010393 | [0.7719; 0.81264] |
| | 10 000 | 0.79513 | 0.0033731 | [0.78852; 0.80175] |
| | 100 000 | 0.79755 | 0.0010663 | [0.79546; 0.79964] |
| | 1 000 000 | 0.79793 | 0.00033604 | [0.79727; 0.79859] |
| $(y - x)e^{-y}$ | 1 000 | 0.48005 | 0.0096875 | [0.46106; 0.49904] |
| | 10 000 | 0.47623 | 0.0029198 | [0.47051; 0.48196] |
| | 100 000 | 0.47661 | 0.00091967 | [0.47481; 0.47841] |
| | 1 000 000 | 0.47688 | 0.00029045 | [0.47631; 0.47745] |
| $(e^x - 1)_+ \mathbb{1}_{\{y \leq 1\}}$ | 1 000 | 0.10606 | 0.0077922 | [0.090791; 0.12134] |
| | 10 000 | 0.096527 | 0.0023466 | [0.091927; 0.10113] |
| | 100 000 | 0.097153 | 0.0007462 | [0.09569; 0.098615] |
| | 1 000 000 | 0.096574 | 0.00023454 | [0.096114; 0.097034] |

Table 3.2: Convergence of Monte Carlo simulations.

The convergence of the third function is less regular than in the previous cases. This is due to the discontinuity introduced by the hard barrier which increases the sensitivity of the estimator to the simulated paths. As a consequence, fluctuations can be observed in the estimated values although the confidence intervals continue to shrink.

*Tree method.*

For the tree method, the differences with respect to the benchmark are larger than those obtained with Monte Carlo simulation. The error remains small for the second function, but it becomes more significant for the first and especially for the discontinuous barrier-type function. These numerical results support the theoretical convergence results presented in Chapter 2 and

confirm that the random-walk approximation can be used to price options more complex than standard vanilla contracts, although its accuracy depends strongly on the number of tree steps.

To further assess the convergence of the tree method, the estimates are reported for different numbers of steps.

| Function | Steps | Estimate |
|---|---|---|
| $y - x$ | 500 | 0.775923 |
| | 1 000 | 0.782273 |
| | 2 500 | 0.787964 |
| | 5 000 | 0.790853 |
| $(y - x)e^{-y}$ | 500 | 0.475721 |
| | 1 000 | 0.476067 |
| | 2 500 | 0.476363 |
| | 5 000 | 0.476507 |
| $(e^x - 1)_+ \mathbb{1}_{\{y \leq 1\}}$ | 500 | 0.105684 |
| | 1 000 | 0.099929 |
| | 2 500 | 0.102937 |
| | 5 000 | 0.097904 |

Table 3.3: Convergence of $\mathbb{E}\left[f\left(\frac{S_n}{\sqrt{n}}, \frac{S_n^*}{\sqrt{n}}\right)\right]$.

The convergence for path-dependent barrier options is not necessarily monotonic and depends on the position of the barrier relative to the nodes of the tree, as discussed by Palmer and Lin [11]. Consequently, local fluctuations may occur as the number of time steps increases. The value obtained for $N = 2500$ illustrates this phenomenon and should not be interpreted as a failure of convergence.

Despite the fluctuations, the numerical results show that the lattice method provides a valid and competitive alternative for the class of barrier options considered in this thesis. Although Monte Carlo simulation provides more accurate estimates under the same computational budget, the tree method offers two important advantages: a deterministic approximation error that can be analysed theoretically and the possibility of constructing dynamic hedging strategies through backward induction.

The main limitation of the tree algorithm is computational. Since the payoff depends on the running maximum, an additional state variable must be introduced. As a consequence, the number of states grows significantly faster than in the standard Cox-Ross-Rubistein model, increasing the computational cost compared with the standard Cox–Ross–Rubinstein model. When only the option price is required, the algorithm can be implemented by storing two consecutive layers of the lattice, resulting in a memory requirement of $O(N^2)$. By contrast, when the full lattice structure is stored, the memory requirements also increase substantially. Indeed, storing the entire lattice makes it possible to construct dynamic hedging strategies, leading to an increase in the memory requirement from $O(N^2)$ to $O(N^3)$. Although more efficient implementations can reduce this memory cost, the overall computational complexity remains substantially higher than for standard recombining trees. Therefore, for sufficiently large numbers of time steps, the algorithm may become impractical.

Therefore, the tree method involves a trade-off. Increasing the number of steps improves

the approximation and provides a finer hedging structure. However, it also increases computational cost and memory usage.

## 3.5 Conclusion

This thesis has studied how stochastic processes can be used to price complex financial instruments in both continuous and discrete time. In particular, it has shown how Brownian motion can be approximated by a normalized simple random walk through Donsker's theorem and how this approximation can be implemented in the context of path-dependent option pricing.

The results have been developed both theoretically and numerically. From the theoretical point of view, the convergence of the random-walk approximation justifies the replacement of Brownian motion and its running maximum with their discrete counterparts. From the numerical point of view, the results show that the tree method can approximate the benchmark values obtained through numerical integration and Monte Carlo simulation.

Lattice methods usually express their full potential in the pricing of vanilla European and American options, where backward induction is particularly effective. In this thesis, however, we have shown that this approach can also remain competitive for path-dependent options, although with a clear computational trade-off.

To obtain results as accurate as Monte Carlo simulation, the number of tree steps must be increased. The introduction of the running maximum as an additional state variable enlarges the state space significantly compared with the standard CRR model. As a consequence, the computational cost grows rapidly with the number of time steps. When only the initial value is required, the algorithm can be implemented efficiently by storing only consecutive layers of the tree. However, recovering the full lattice structure required for dynamic hedging strategies entails a substantially larger memory cost because the running maximum introduces an additional state variable.

Monte Carlo simulation, by contrast, remains more scalable and flexible, especially for complex path-dependent structures. In this work, its main limitations concern the computational cost necessary for the reduction of the statistical error and the discretization bias, and the fact that standard Monte Carlo does not naturally provide dynamic hedging strategies.

When only the initial value is required or the number of tree steps remains manageable, the random-walk approach can represent a viable alternative to Monte Carlo simulation. Its main advantage is that it combines pricing with the possibility of constructing dynamic hedging strategies through the backward structure of the tree. This additional information is not obtained for free, however, since recovering the full hedging structure requires storing a much larger portion of the state space than is needed for pricing alone. Moreover, the approximation error is deterministic and can be analyzed theoretically through the convergence results developed in this thesis.

Numerical integration plays a different role in this work. Owing to the explicit joint density of Brownian motion and its running maximum, it provides highly accurate benchmark values against which the other methods can be compared. However, this approach relies on the availability of analytical density formulas and remains practical only in relatively low-dimensional settings. For more general contracts, such as American options or other path-dependent derivatives for which no suitable density representation is available, numerical integration may become difficult or impossible to implement. In such cases, lattice methods and Monte Carlo simulation remain the most natural numerical alternatives.

The numerical results obtained in this thesis show that, when the state space remains computationally manageable, the random-walk approximation represents a competitive alternative to Monte Carlo simulation, combining deterministic convergence with the possibility of constructing dynamic hedging strategies within a unified lattice framework.

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