



UNIVERSITÀ
DI PAVIA

Dipartimento di Fisica "Alessandro Volta"

Corso di Laurea Magistrale in Scienze Fisiche

Synthetic Altermagnetism in Photonic Crystal Slabs

Tesi per la Laurea Magistrale di:
Paolo Nacci

Relatore:
Prof. Dario Gerace

Correlatore:
Prof. Lucio C. Andreani

Anno Accademico 2025 – 2026

Abstract

Altermagnetism is a recently discovered magnetic phase that combines collinear anti-ferromagnetic ordering with spin-dependent band splitting, while preserving a vanishing net magnetization. Formally described through spin-group theory and crystalline symmetries, this unconventional magnetic state bridges the properties of ferromagnets and antiferromagnets, opening new perspectives for spintronics and the study of magnetic materials.

This work proposes and investigates a synthetic photonic realization of altermagnetism by exploiting the analogy between electronic and photonic band structures. First, a minimal theoretical model has been identified, consisting of a two-dimensional Rashba Hamiltonian supplemented by an altermagnetic term with rotation and inversion symmetries, on which the third component of Stokes parameters (i.e., S_3), formally associated with the degree of circular polarization of light, is proposed as the photonic counterpart of spin polarization in electronic systems. Analytical calculations of the Hamiltonian eigenvalues and eigenvectors showed that the resulting band dispersion reproduces the characteristic exchange of circular polarization along orthogonal spatial directions, which is one of the key signatures of electronic altermagnetism. These theoretical predictions are then validated through numerical simulations of a silicon photonic crystal slab consisting of a hexagonal lattice of triangular holes, using the Guided Mode Expansion (GME) method. By introducing controlled geometric symmetry breaking through the rotation of the triangular holes, a non-trivial distribution of the relevant Stokes parameter can be induced. A figure of merit based on the magnitude of S_3 has been employed to optimize the geometric parameters of the structure, thus allowing for an algorithmic identification of the optimal configurations maximizing the synthetic altermagnetic effect in the vicinity of the Γ point and along orthogonal directions in reciprocal space.

The results obtained in this thesis demonstrate that the characteristic band splitting of altermagnetic materials can be reproduced in a purely photonic system through geometric symmetry breaking alone, without the need for magnetic ordering or time-reversal symmetry breaking. Ultimately, these findings establish a versatile platform to investigate the physics of synthetic altermagnetism, paving the way for future applications in topological photonics, polarization engineering, and spin-inspired photonic devices.

Sommario

L'Altermagnetismo è una fase magnetica scoperta recentemente che combina un ordinamento antiferromagnetico collineare con una separazione delle bande dipendente dallo spin, mantenendo al contempo una magnetizzazione netta nulla. Descritto formalmente attraverso la teoria dei gruppi di spin e le simmetrie cristalline, questo stato magnetico non convenzionale mette in relazione le proprietà dei ferromagneti e degli antiferromagneti, aprendo nuove prospettive per la spintronica e lo studio dei materiali magnetici.

Questo lavoro propone e studia una realizzazione fotonica sintetica del Altermagnetismo, sfruttando l'analogia tra le strutture a bande elettroniche e fotoniche. In primo luogo, è stato identificato un modello teorico minimale, costituito da una Hamiltoniana di Rashba bidimensionale integrata con un termine altermagnetico dotato di simmetrie di rotazione e inversione. In tale modello, la terza componente dei parametri di Stokes (ovvero S_3), formalmente associata al grado di polarizzazione circolare della luce, viene proposta come controparte fotonica della polarizzazione di spin nei sistemi elettronici.

I calcoli analitici degli autovalori e degli autovettori dell'Hamiltoniana hanno mostrato che la dispersione delle bande risultante riproduce il caratteristico scambio della polarizzazione circolare lungo direzioni spaziali ortogonali, che costituisce una delle principali firme dell'Altermagnetismo elettronico. Queste previsioni teoriche sono state successivamente validate mediante simulazioni numeriche di un cristallo fotonico in guida planare di silicio, costituita da un reticolo esagonale di fori triangolari, utilizzando il metodo della Guided Mode Expansion (GME). Introducendo una rottura controllata della simmetria geometrica attraverso la rotazione dei fori triangolari, è possibile indurre una distribuzione non banale di S_3 . Per ottimizzare i parametri geometrici della struttura è stata utilizzata una figura di merito basata sul modulo di S_3 , consentendo così l'identificazione algoritmica delle configurazioni ottimali che massimizzano l'effetto altermagnetico sintetico in prossimità del punto Γ e lungo direzioni ortogonali nello spazio reciproco.

I risultati ottenuti in questa tesi dimostrano che la caratteristica separazione delle bande dei materiali altermagnetici può essere riprodotta in un sistema puramente fotonico attraverso la sola rottura della simmetria geometrica, senza la necessità di un ordinamento magnetico o della rottura della simmetria di inversione temporale. In ultima analisi, questi risultati stabiliscono una piattaforma ideale per studiare la fisica dell'altermagnetismo sintetico in modo controllato, aprendo la strada a future applicazioni nella fotonica topologica, nell'ingegneria della polarizzazione e nei dispositivi fotonici ispirati allo spin.

Contents

Introduction	2
1 Introduction to Altermagnetism	5
1.1 Spin groups of magnetic materials	5
1.2 Hamiltonian models of Altermagnetism	9
1.2.1 General altermagnetic Hamiltonian model	9
1.2.2 Effective two-dimensional Hamiltonian	11
1.3 Altermagnetic material candidates	14
2 Photonic Crystals and Photonic Crystal Slabs	16
2.1 The electromagnetic problem in linear media	19
2.1.1 Symmetry properties of two-dimensional photonic systems	25
2.2 Two-Dimensional Photonic Crystals	31
2.2.1 Master equation and TE/TM decoupling	32
2.3 Photonic Crystal Slabs	40
2.3.1 Guided-mode solution and frequency dispersion	41
2.4 Guided Mode Expansion method	46
3 Optical Analogue of a Minimal Altermagnetic Model	54
3.1 Jones Calculus	55
3.2 Stokes Parameters	57
3.3 Analogy between spin-1/2 and polarization	60
3.4 The altermagnetic analogue	61
4 Numerical Results	64
4.1 Photonic crystal of triangular holes in Si membrane	65
4.1.1 Geometric structure	65
4.1.2 Photonic band dispersion by GME simulation	68
4.1.3 Stokes parameters from GME	72
4.1.4 Numerical optimization by differential evolution	74
4.1.5 Tolerance to imperfections	78
4.2 Photonic crystal of triangular rods in glass slab	79
Conclusion	83
A Python codes developed for the thesis	86
A.1 Code modules and functions	86
A.2 Optimization Process	90
A.3 Creation of optimized structure	90
A.4 Calculation of Stokes parameters and plotting	92

<i>CONTENTS</i>	1
Bibliography	95

Introduction

In recent years, it has been observed that materials exhibiting time-reversal symmetry breaking and spin-polarization phenomena (e.g., the Rashba–Dresselhaus effect), typically associated with ferromagnetic materials, may simultaneously display antiparallel spin ordering and a vanishing net magnetization, features that are characteristic of antiferromagnets. A description based on spin groups and crystalline symmetries has resolved this apparent contradiction, establishing the existence of a new magnetic phase known as *altermagnetism*, named after the alternating spin arrangement along orthogonal directions.

Starting from this formal definition, it is possible to construct a general minimal model of the altermagnetic Hamiltonian. This Hamiltonian depends on the sublattice energy dispersion, the hopping coefficients between orthogonal planes, the spin–orbit coupling terms, and the Pauli matrices associated with the ions within the unit cell and the electrons, respectively.

Altermagnetism can emerge in crystal structures characterized by collinear antiparallel spin ordering, band degeneracies induced by spin–orbit coupling effects, and a vanishing net magnetization. Promising candidate materials include rutile compounds of the form XY_2 , where X is a magnetic element (e.g., CuO_2 or FeO_2). Since the experimental observation of this phenomenon has proven particularly challenging, an alternative approach has been proposed, namely the investigation of altermagnetism in a system analogous to electronic ones. In particular, this work exploits the close analogy between photonic bands, generated by light confined within photonic crystals, and electronic band structures. The first part of the project thus focused on identifying the most suitable form of the altermagnetic Hamiltonian. The general minimal Hamiltonian is rather complex and difficult to reproduce in a photonic system due to the presence of hopping terms between sublattices. The selected model is a simplified two-dimensional Rashba Hamiltonian supplemented by an altermagnetic term with symmetry $[C_2||C_{4z}]$, which is particularly suitable for its implementation in a two-dimensional slab geometry.

Having identified the most appropriate Hamiltonian, the work then concentrated on finding a photonic parameter capable of playing the same role as the spin plays in electronic systems. The chosen quantity is the Stokes parameter, S_3 , which characterizes

the helicity of light at each (k_x, k_y) point within the Brillouin zone. More specifically, the choice of S_3 as characteristic parameter is due to the explicit demonstration that, for an energy dispersion of the minimal model consisting of two bands crossing at Γ , its value is exchanged between the two bands along orthogonal directions, thus reproducing one of the key signatures of the altermagnetic electronic band structure.

The final part of the work was devoted to the numerical simulation of an actual photonic system capable of reproducing the synthetic altermagnetic behavior. The starting point was a highly symmetric structure, namely a silicon slab perforated with triangular holes arranged in a hexagonal lattice. Once the reference system had been defined, the relevant symmetries were intentionally broken in order to induce non-trivial effects on the chosen parameter, S_3 . To this end, the triangular holes were rotated by an angle ϑ with respect to the $\hat{y}$ axis. As described throughout this research work, the study is founded on two main pillars: the identification of the Stokes parameter (S_3) as the photonic analogue of electronic spin polarization, and the verification that the dispersion of the photonic crystal slab follows the behavior predicted by the minimal theoretical model reproducing altermagnetic features.

To address the first objective, it was necessary to evaluate the S_3 parameter for the chosen effective Hamiltonian. The analysis began with the calculation of the Hamiltonian eigenvalues and eigenvectors. The eigenvalues were used to derive the dispersion relation, while the eigenvectors enabled the reconstruction of the electric-field components by expressing them as linear combinations of the transverse-electric (TE) and transverse-magnetic (TM) modes. Once the field components were obtained, the Stokes parameter was calculated directly from them.

The second objective was pursued through the full numerical calculation of the photonic band structure using the Guided Mode Expansion (GME) method, a purely numerical approach providing the solution of stationary Maxwell equations in layered photonic structures embedded between two low-index claddings, and typically employed for the analysis of photonic crystal slabs. After computing the guided modes folded within the first Brillouin zone of the 2D lattice, and identifying the modes of interest, namely the lowest-energy bands exhibiting a crossing at the (Γ) point, the corresponding values of S_3 were numerically evaluated.

Since the determination of the behavior of the relevant Stokes parameter requires the calculation of the GME spectrum of the photonic system, and the subsequent extraction of the polarization properties of the radiative modes, it became necessary to introduce a figure of merit capable of identifying the optimal values of ϑ and of the hole area (A), such that the magnitude of S_3 would be maximized. This optimization procedure allowed the parameter space to be systematically explored, and the configurations exhibiting the strongest spin-like photonic response to be identified. The optimization procedure yielded the values that maximize the parameter in the vicinity

of the Γ point (i.e., the origin in reciprocal space) along orthogonal directions, namely along a path of the form $Y \rightarrow \Gamma \rightarrow P$. This choice enables a direct comparison between the results obtained for the photonic analogue and those known for the electronic altermagnetic case, in which the band structure is conventionally evaluated along two orthogonal directions $k_x \rightarrow \Gamma \rightarrow k_y$.

The presentation of the thesis is organized as follows. In Chapter 1 we will discuss altermagnetism and altermagnetic materials, introducing the concept of spin groups and presenting minimal models that capture the essential properties of altermagnetic systems. We will then introduce the corresponding effective Hamiltonian used during the work, and discuss possible material candidates exhibiting altermagnetic order.

In Chapter 2, we will introduce the theoretical framework of photonic crystals and photonic crystal slabs. We will first formulate the electromagnetic problem in periodic dielectric media, and discuss the symmetry properties of two-dimensional photonic systems with discrete translational invariance. We will then describe two-dimensional photonic crystals and photonic crystal slabs, concluding with a detailed description of the Guided Mode Expansion (GME) method, which will be employed for the numerical simulations presented in the following chapters.

Chapter 3 will then focus on demonstrating the optical analogue of the previously presented altermagnetic model. We will introduce the Jones calculus and the Stokes parameters as viable tools for describing the polarization properties of light, and establish the formal analogy that stands between optical algebra and spin algebra. Based on this framework, we will present the theoretical prediction underlying the proposed photonic realization of altermagnetism.

In Chapter 4, we will discuss the numerical methods and results obtained for photonic crystal slabs displaying synthetic altermagnetic properties. We will first consider a silicon slab patterned with triangular holes, discussing its geometrical structure, the results of GME simulations, and the calculation of the corresponding Stokes parameters. We will then investigate the optimization of the structure in order to enhance the predicted optical effects and the study of tolerance for an eventual experimental realization. Finally, we will consider a complementary geometry consisting of a slab of triangular rods embedded in a glass matrix, and presents its results obtained using the same simulation method used for the silicon slab with triangular holes.

Chapter 1

Introduction to Altermagnetism

Magnetism is one of the most fundamental and widely studied subjects in Condensed Matter Physics. Traditionally, two stable magnetic phases have been mainly distinguished in manybody magnetic materials: ferromagnetism and antiferromagnetism. Ferromagnets are characterized by a non-vanishing macroscopic magnetization and, due to their parallel magnetic order, exhibit a wide range of remarkable phenomena, ranging from time-symmetry breaking to spin-split electronic band structures.

Antiferromagnets, on the other hand, are characterized by a compensated antiparallel magnetic ordering, which results in a vanishing net magnetization and makes them, in several respects, similar to non-magnetic systems.

In recent years, a growing number of theoretical and experimental studies have highlighted the simultaneous presence of time-symmetry breaking effects and momentum-dependent spin splitting in systems with a collinear compensated magnetic order.[16] In these systems, the sign of the spin splitting alternates depending on both the crystal momentum and the spin orientation. Such materials cannot be classified within the conventional framework of either ferromagnetism or antiferromagnetism. Instead, they constitute a new distinct magnetic phase defined, due to their characteristic alternating spin-splitting behavior, as *altermagnetism*. A schematic comparison between these three main magnetic phases is shown in Fig. 1.1.

1.1 Spin groups of magnetic materials

To properly establish the fundamental characteristics of *altermagnetism*, it is first necessary to resolve the apparent ferromagnetic–antiferromagnetic dichotomy underlying the traditional classification of magnetic materials. To this end, we introduce the concept of the *magnetic group*, and, more specifically, the notion of the *spin group*.

Magnetic groups are symmetry groups allowing to classify the symmetries of magnetic systems in both real and spin space. However, the relativistic nature of the symmetry operations encoded in conventional magnetic groups makes them only partially suit-

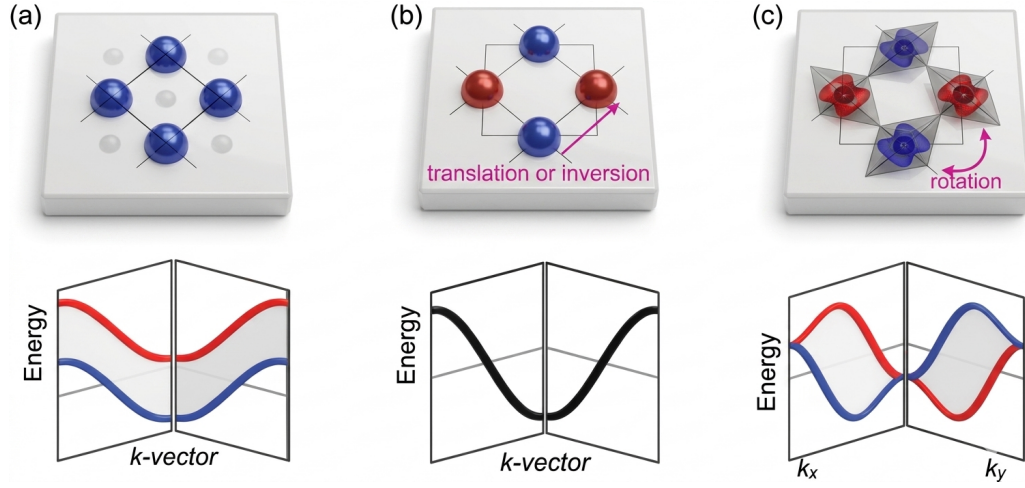


Figure 1.1: Schematic representation of the three different magnetic phases, shown both in real-space configuration and in the corresponding energy dispersion along two orthogonal directions in reciprocal space.[15] (a) Ferromagnetic phase, (b) antiferromagnetic phase, and (c) altermagnetic phase.

able for describing non-relativistic magnetic systems. For this reason, it is necessary to consider a symmetry group characterized by a richer structure, namely the *Non-relativistic Spin Group* (Fig. 1.2). This group can be regarded as a generalization of conventional magnetic groups and allows for independent rotation transformations acting in spin space and real space, while only the real-space operations remain crystallographically constrained.

In general, spin groups can be expressed as the direct product of a spin-only symmetry group r_s and a non-trivial spin group R_s :

$$r_s \times R_s, \quad (1.1)$$

for collinear spin arrangement on crystals the spin-only group can be written as a sum of two terms

$$r_s = C_\infty + \bar{C}_2 C_\infty, \quad (1.2)$$

in which C_∞ denotes the group of all spin-space rotations around the common spin axis, and $\bar{C}_2$ represents a 180° rotation around an axis perpendicular to the spin direction combined with spin-space inversion, originating from time-reversal symmetry.

Equation 1.2 has two important consequences. First, the presence of C_∞ implies that spin remains a good quantum number with a well-defined common quantization axis. Second, the presence of $\bar{C}_2$ introduces a symmetry operation related to time reversal, acting not only in spin space but also in real space.

As discussed above, time-reversal symmetry ($\mathcal{T}$) plays a crucial role in determining the spin symmetry of the electronic band structure. Indeed, applying the spin-group

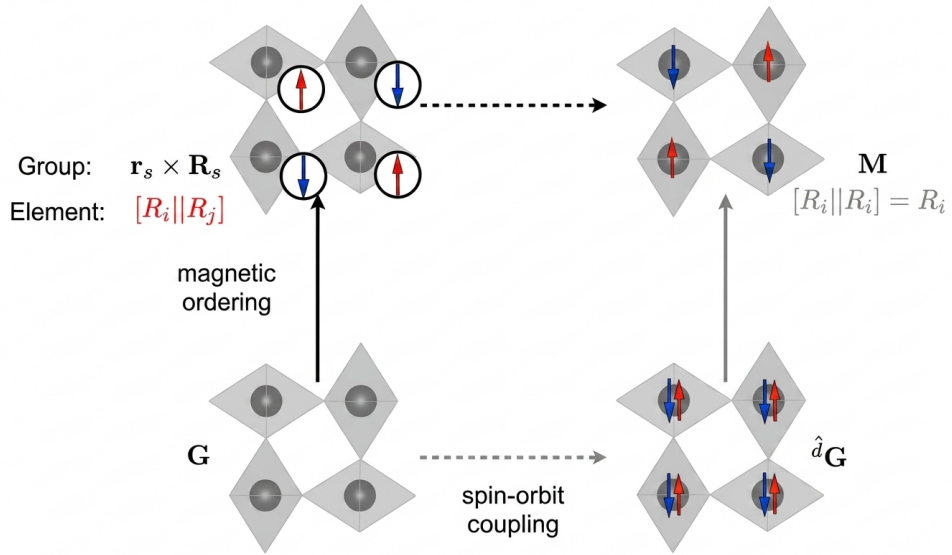


Figure 1.2: Bottom left: absence of magnetic order described by non-relativistic crystallographic groups (G). Bottom right: absence of magnetic order described by relativistic double crystallographic groups (dG), obtained by extending G with the spin-rotation operation R . Top right: magnetic order described by relativistic magnetic groups. Top left: magnetic order described by non-relativistic spin groups (see Supplementary material of Ref. [15]).

time-reversal transformation $[\bar{C}_2 || \mathcal{T}]$ to a spin- and momentum-dependent energy band yields:

$$[\bar{C}_2 || \mathcal{T}] \varepsilon(s, \mathbf{k}) = \varepsilon(s, -\mathbf{k}).$$

This transformation acts in the same way as spatial inversion. Since $[\bar{C}_2 || \mathcal{T}]$ is a symmetry operation of non-relativistic collinear spin arrangements in crystals, it can be shown that the electronic bands of all non-relativistic collinear magnets are invariant not only under spin inversion but also under inversion in real space.

The non-trivial spin groups R_s are constructed by combining spin-space and real-space transformations. For the spin-space part, we consider the group

$$S_2 = \{E, C_2\},$$

which contains the identity operation (E) and a 180° rotation about an axis perpendicular to the spin direction. For the real-space transformations, an analogous construction is employed, generally based on the *crystallographic Laue group*. While the spin-only group introduced above is common to all non-relativistic collinear magnets, the non-trivial spin group can assume different forms depending on the symmetry relating the opposite-spin sublattices. This leads to three distinct classes of *spin Laue groups*, corresponding to the three magnetic phases summarized in Table 1.1. The

	Type I	Type II	Type III
Opposite-spin sublattice transformation	None	$[C_2 \bar{E}]$ or $[C_2 t]$	$[C_2 A]$
Non-trivial spin group	$[E G]$	$[E G] + [C_2 G]$	$[E H] + [C_2 AH]$
$\mathcal{T}$ -symmetry breaking	Yes	No	Yes
Spin-split bands	Yes	No	Yes
Net magnetization	Yes	No	No

Table 1.1: Comparison of the three magnetic phases, outlined in Fig. 1.1 based on their symmetry properties and macroscopic characteristics.[15]

first class is obtained from the spin-space group

$$S_1 = \{E\},$$

whose only possible coset decomposition consists of the group itself. The corresponding isomorphic decomposition of the crystallographic Laue group G is therefore simply G . As a result, the associated non-trivial spin group is

$$R_s^I = [E||G].$$

Since there exist eleven *crystallographic Laue groups*, there are also eleven possible realizations of R_s^I . This class describes systems characterized by broken time-reversal symmetry, spin-split electronic bands, and a non-vanishing net magnetization. It therefore corresponds to collinear **ferromagnetism**, whose magnetic structure consists of a single spin sublattice.

The second class is obtained from the coset decomposition of

$$S_2 = \{E, C_2\},$$

the two coset are $\{E\}$ and $\{C_2\}$. By pairing with the crystallographic Laue group G , the non-trivial spin group yields

$$R_s^{II} = [E||G] + [C_2||G]$$

The presence of the additional symmetry operation enforces spin degeneracy of the electronic bands $\varepsilon(s, \mathbf{k})$ for all wave vectors in the Brillouin zone. Consequently, the eleven possible R_s^{II} groups describe non-relativistic systems with spin-degenerate, time-reversal-invariant band structures and vanishing net magnetization. This class corresponds to collinear **antiferromagnetism**.

At last, the third class is defined by a more complex spin-group and crystallographic

Laue group. In this case the non-trivial spin group is defined by:

$$R_s^{\text{III}} = [E || \mathbf{H}] + [C_2 || \mathbf{A}] [E || \mathbf{H}] = [E || \mathbf{H}] + [C_2 || (\mathbf{G} - \mathbf{H})]$$

where $\mathbf{H}$ is a halving subgroup of the *crystallographic Laue group* $\mathbf{G}$ containing only real-space operations, while $(\mathbf{G} - \mathbf{H}) = \mathbf{A}\mathbf{H}$ with $\mathbf{A}$ representing a proper or improper rotational operation that cannot be the inversion operator. The defining characteristic of this class is that the opposite-spin sublattices are related by a rotational symmetry operation rather than by translation or inversion. As a consequence, these systems simultaneously exhibit a compensated magnetic order, broken time-reversal symmetry, and spin-split electronic bands. This third class corresponds to **altermagnetism**.

1.2 Hamiltonian models of Altermagnetism

So far, we have established the existence of a novel magnetic phase, namely *altermagnetism*. This phase is characterized by a vanishing net magnetization and a collinear antiferromagnetic spin arrangement, while simultaneously exhibiting broken time-reversal symmetry and spin-split electronic bands. We now seek to identify a minimal theoretical model capable of describing these distinctive features. Such a model should capture the essential physics of altermagnetism while remaining sufficiently general to be applicable to a broad class of materials.

1.2.1 General altermagnetic Hamiltonian model

The construction of a minimal model is based on the interplay between two fundamental symmetries of the system: the on-site Wyckoff symmetry ($\mathbf{S}$) and the point-group symmetry ($\mathbf{P}$). In particular, the breaking of the latter is responsible for the emergence of spin splitting in the electronic band structure. Starting from these considerations, the construction of a minimal tight-binding Hamiltonian requires the identification of the irreducible representations of the point group Γ_N [12].

For simplicity, we restrict our discussion to structures with hexagonal, tetragonal, orthorhombic, or cubic unit cells. The electronic degrees of freedom are represented using Pauli matrices, where τ_i acts on the magnetic-sublattice degrees of freedom and σ_i acts on the spin degrees of freedom.

$$\tau_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}.$$

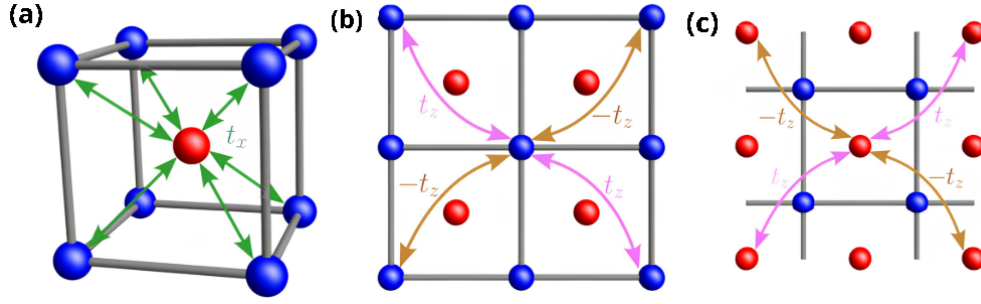


Figure 1.3: (a) Three-dimensional representation of the crystalline lattice illustrating the hopping parameter $t_{x,k}$ between magnetic sublattices within the unit cell. (b) and (c) Top views of the lattice showing the hopping parameter $t_{z,k}$ and the sign changes imposed by the crystal symmetry.[12]

The minimal Hamiltonian can then be written as

$$H = \varepsilon_{0,k} + t_{x,k} \tau_x + t_{z,k} \tau_z + \tau_y \vec{\lambda}_k \cdot \vec{\sigma} + \tau_z \vec{J} \cdot \vec{\sigma} , \quad (1.3)$$

where $\varepsilon_{0,k}$ denotes the sublattice-independent electronic dispersion, $t_{x,k}$ and $t_{z,k}$ represent symmetry-allowed hopping amplitudes, λ_k is the spin-orbit coupling (SOC) term, and J is the magnetic order parameter. In altermagnetic systems, this term is directly associated with the spin splitting of the electronic bands.

We now discuss some general properties of the Hamiltonian. First, we consider its invariance under lattice translations and spatial inversion. The matrices τ_i transform according to the irreducible representations of the point group Γ_N and therefore preserve the symmetry of the crystal. For a reciprocal-lattice translation by a vector $\mathbf{G}$, the lattice-dependent coefficients satisfy

$$\begin{aligned} \varepsilon_{0,k+\mathbf{G}} &= \varepsilon_{0,k} \quad , & t_{x,k+\mathbf{G}} &= e^{i\mathbf{G} \cdot \mathbf{t}_{12}} t_{x,k} \quad , \\ t_{z,k+\mathbf{G}} &= t_{z,k} \quad , & \lambda_{k+\mathbf{G}} &= e^{i\mathbf{G} \cdot \mathbf{t}_{12}} \lambda_k \quad . \end{aligned}$$

Where $\mathbf{t}_{12}$ denotes the translation vector connecting the two magnetic atoms within the unit cell. Consequently, all coefficients are periodic in reciprocal space, ensuring the translational invariance of the Hamiltonian. The parameter $\vec{J}$ describes an antiferromagnetic spin configuration in which the magnetic moments on the two sublattices are oppositely oriented. An important feature of this term is its independence from the crystal momentum $\mathbf{k}$. Since the operator τ_y is odd under time-reversal symmetry, the spin-orbit coupling term λ_k must also be odd under time-reversal transformation. The energy spectrum corresponding to Eq. 1.3 can be written in the general form of

the eigenvalues, as:

$$E_{\alpha=\pm, \beta=\pm} = \varepsilon_{0,\mathbf{k}} + \alpha \left[t_{x,\mathbf{k}}^2 + t_{z,\mathbf{k}}^2 + |\vec{\lambda}_{\mathbf{k}}|^2 + |\vec{J}|^2 + 2\beta \sqrt{t_{z,\mathbf{k}}^2 |\vec{J}|^2 + (\vec{\lambda}_{\mathbf{k}} \times \vec{J})^2} \right]^{1/2}. \quad (1.4)$$

In the limit $\lambda_{\mathbf{k}} = 0$, the four electronic bands are generally non-degenerate, except for two special cases. The first occurs when $t_{z,\mathbf{k}} = 0$, yielding two twofold degenerate bands. The second occurs when $t_{x,\mathbf{k}} = 0$ and $t_{z,\mathbf{k}} = \pm |\vec{J}|$, where the spectrum consists of one twofold degenerate band and two non-degenerate bands.

To illustrate the physical content of the model, let us consider a simple theoretical model consisting of a square lattice with four electronic bands. In this case, the Hamiltonian reduces to

$$H' = t (\cos k_x + \cos k_y) \pm \Delta (\cos k_x - \cos k_y) \pm J \sigma_z \tau_z. \quad (1.5)$$

The first term describes conventional nearest-neighbor hopping on a square lattice. The second term represents the orbital (or sublattice) degree of freedom, while the third term accounts for the magnetic order. The corresponding band energies are

$$E_{\pm}^{A/B} = t (\cos k_x + \cos k_y) \pm \Delta (\cos k_x - \cos k_y) \pm J. \quad (1.6)$$

In the non-magnetic phase, the electronic potential, Δ , is responsible for the splitting of the bands. When magnetic order is introduced, the bands become also spin polarized. A plot of these eigenvalues, and their corresponding spin texture, is reported in Fig. 1.4. As it is evidently shown in the Figure, the splitting near the Γ point is not determined solely by the magnetic contribution, J , but it is strongly influenced by the momentum-dependent electronic potential Δ .

1.2.2 Effective two-dimensional Hamiltonian

We now turn our attention to a more specific system, namely a thin slab of an altermagnetic material. This reduced-dimensional geometry gives rise to additional physical effects associated with quantum confinement, broken spatial symmetries, and surface-induced interactions.

At both the macroscopic and unit-cell levels, the system closely resembles a conventional antiferromagnet, exhibiting a vanishing net magnetization together with antiparallel local spin ordering (see, e.g., Fig. 1.1). The essential difference emerges in reciprocal space, where broken time-reversal symmetry, uncompensated by inversion symmetry, lifts the spin degeneracy of the electronic bands.

Let us consider the case in which the magnetic order parameter possesses d -wave symmetry. The resulting spin splitting between the two non-degenerate bands takes the

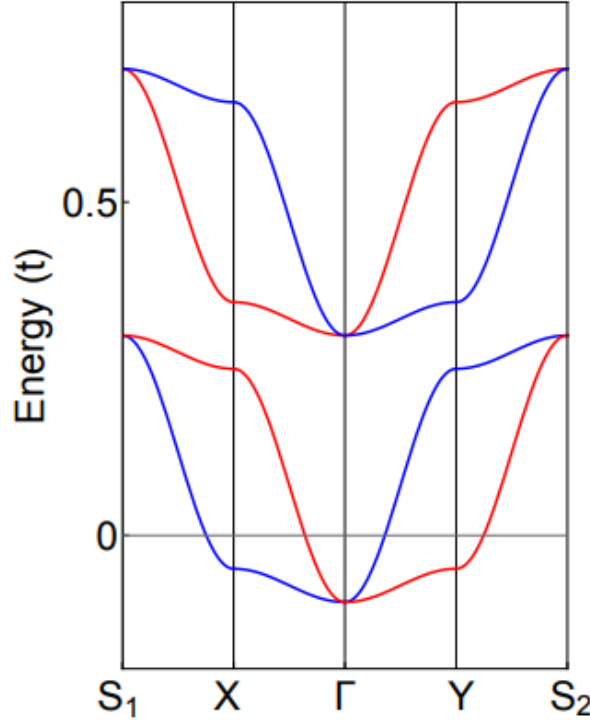


Figure 1.4: Electronic band structure of the four-band mode of Eq. 1.6l. The parameters are $J = 0.1$, $t = 0.2$, and $\Delta = 0.075$, corresponding to a total energy splitting of $\Delta E^{A/B} = 0.3$ eV. Here is rightfully depicted the alternating spin behavior: spin up is in red, while spin down is in blue [15].

form

$$E_+(\mathbf{k}) - E_-(\mathbf{k}) \propto k_x^2 - k_y^2, \quad (1.7)$$

which vanishes along the Brillouin-zone diagonals ($k_x = \pm k_y$) and reaches its maximal magnitude along the principal crystallographic directions. The sign of the splitting alternates between these directions, producing the characteristic altermagnetic band structure.

The *effective* low-energy Hamiltonian can be obtained from the two-dimensional form of Eq. 1.3 (see Appendix G of Ref. [12]) supplemented by a Rashba spin-orbit interaction term [17]:

$$\tilde{\mathcal{H}} = \frac{\hbar^2 k^2}{2m} \mathbb{I} + \gamma (k_x^2 - k_y^2) \sigma_z + \alpha (k_x \sigma_y - k_y \sigma_x), \quad (1.8)$$

where γ quantifies the altermagnetic spin splitting, and α denotes the Rashba coupling strength.

The introduction of the Rashba term is physically motivated by the slab geometry. The presence of surfaces or interfaces explicitly breaks the out-of-plane mirror symmetry ($z \rightarrow -z$), thereby allowing a Rashba spin-orbit interaction. Consequently, Eq. 1.8 contains both the characteristic altermagnetic spin-splitting term and the Rashba con-

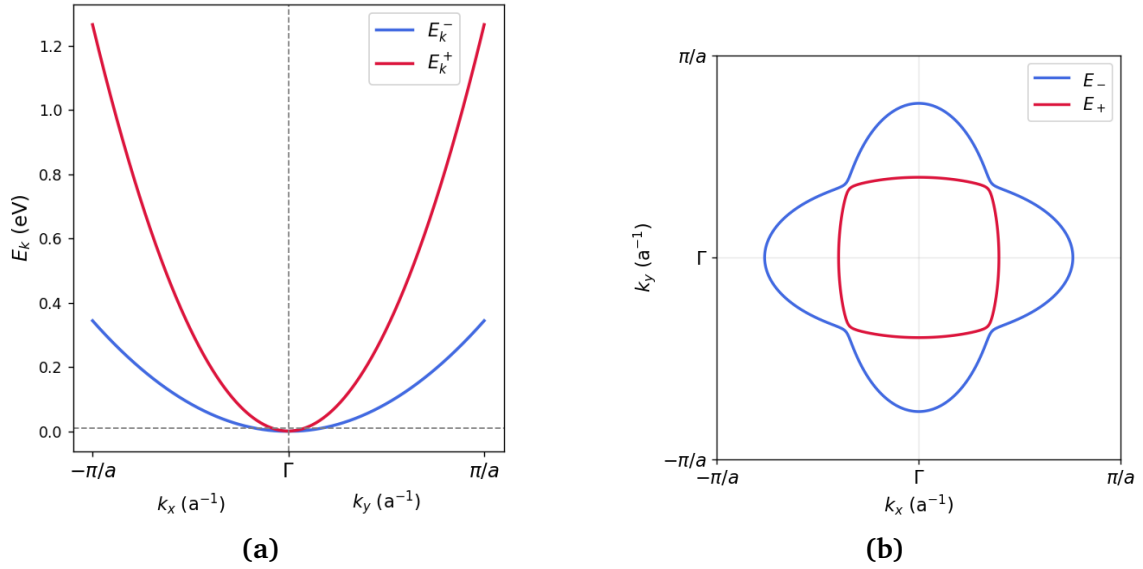


Figure 1.5: Effective Hamiltonian energy dispersion calculated from Eq. 1.10 (here, the color scheme indicates only the two eigenvalues, it is not related to the spin polarization), along (a) orthogonal directions $k_x \rightarrow \Gamma \rightarrow k_y$, and (b) 2D contour over all Brillouin Zone, at fixed energy $E_k = 0.2$ eV. The parameters for these plots are set as $m = 0.152 \text{ eV}^{-1}\text{\AA}^{-2}$, $\gamma = 1.881 \text{ eV}\text{\AA}^2$, $\alpha = 52 \text{ meV}\text{\AA}$, and $\hbar = 1$.

tribution. The latter induces a momentum-dependent modification of the band structure, as shown in Fig. 1.5. Notice that in this Figure, the color code does not correspond to the spin texture, but it just distinguishes between the $\pm$ eigenvalues. The latter can be straightforwardly determined by diagonalizing the effective Hamiltonian $\tilde{\mathcal{H}}$. In fact, by writing Eq. 1.8 in matrix form, one obtains

$$\tilde{\mathcal{H}} = \begin{pmatrix} \frac{\hbar^2 k^2}{2m} + \gamma(k_x^2 - k_y^2) & -\alpha(k_y + ik_x) \\ -\alpha(k_y - ik_x) & \frac{\hbar^2 k^2}{2m} + \gamma(k_x^2 - k_y^2) \end{pmatrix}. \quad (1.9)$$

The eigenvalues are obtained from the characteristic equation and by solving the resulting quadratic equation yields :

$$\varepsilon_{\mathbf{k}}^{\pm} = \frac{\hbar^2 k^2}{2m} \pm \sqrt{\alpha^2(k_x^2 + k_y^2) + \gamma^2(k_x^2 - k_y^2)^2}. \quad (1.10)$$

These eigenvalues are plotted in Fig. 1.5, and their spin texture is reported in Fig. 1.6, from which it is evident that the bands exhibit their largest spin splitting along the principal crystallographic directions (k_x and k_y), where the spin polarization is maximal. Along the Brillouin-zone diagonals, where $k_x = \pm k_y$, the altermagnetic contribution vanishes and the spin polarization is strongly reduced. Moving from one principal direction to the orthogonal one, the sign of the spin splitting reverses, resulting in

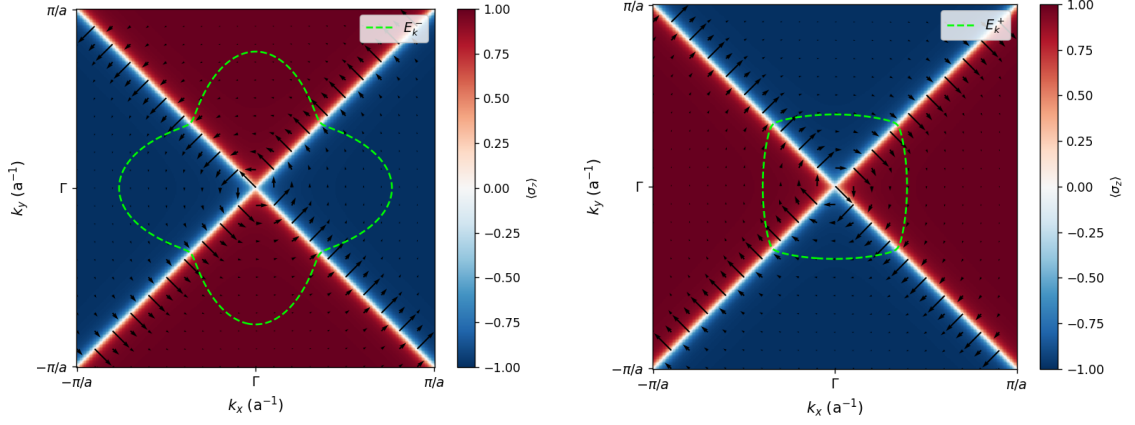


Figure 1.6: Illustration of the spin texture order over the whole Brillouin Zone: Left for $\epsilon_{\mathbf{k}}^+$ eigenvalues, right for $\epsilon_{\mathbf{k}}^-$ ones, showing the characteristic alternating behavior along orthogonal directions in reciprocal space.

an alternating spin texture in momentum space. As already mentioned, this alternating behavior constitutes one of the characteristic fingerprints of altermagnetism. In conclusion, this effective two-dimensional model accurately captures the low-energy electronic structure in the vicinity of the Γ point, and it can definitely be considered as a minimal Hamiltonian model capturing the essential physics of altermagnetism.

1.3 Altermagnetic material candidates

Having discussed the fundamental symmetry requirements and electronic properties associated with altermagnetism, we now briefly review some of the most promising material candidates proposed in the literature.

Among the most extensively studied systems are *rutile compounds*, crystal structures of the form XY_2 . In particular, RuO_2 has emerged as one of the flagship candidates for altermagnetism. Owing to its crystal symmetry and magnetic properties, this material exhibits several signatures that are consistent with altermagnetic order. Nevertheless, its classification as a genuine altermagnet remains the subject of ongoing theoretical and experimental debate.

Another important class of candidate materials includes compounds such as CrSb and MnTe . These systems crystallize in structures whose symmetry properties are compatible with altermagnetic ordering. Consequently, they provide an ideal platform for investigating the relationship between crystal symmetry, magnetic order, and momentum-dependent spin splitting.

Additional materials proposed as potential altermagnets include copper oxides and layered transition-metal compounds, among which La_2CuO_4 has attracted considerable attention. In these systems, the interplay between magnetic ordering and crystal

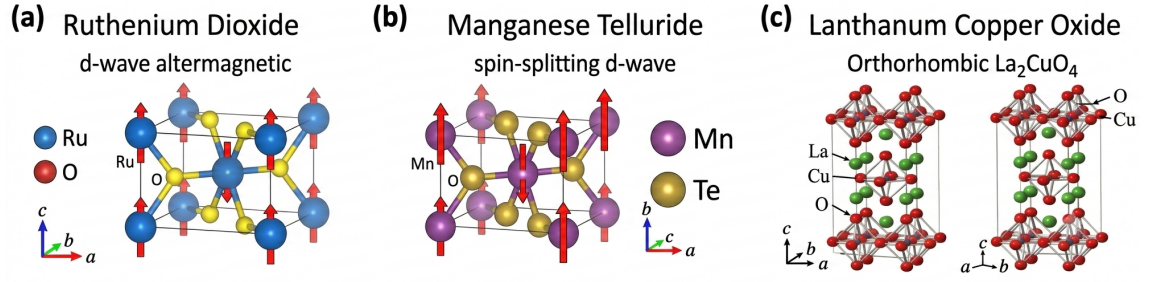


Figure 1.7: Crystalline structures of selected candidate altermagnetic materials: (a) CuO_2 , (b) MnTe , and (c) La_2CuO_4 . For these, the characteristic momentum-dependent spin splitting reflects the underlying altermagnetic order. See Ref. [15] for details.

symmetry can generate electronic structures that closely resemble the altermagnetic scenario predicted by symmetry analysis.

Despite the growing number of materials proposed as potential altermagnets ([15, 16]), and although several compounds exhibit experimental signatures compatible with altermagnetic behavior, an unambiguous demonstration capable of definitively distinguishing altermagnetism from competing magnetic scenarios remains an open challenge. The development of new experimental probes and the identification of robust material platforms therefore continue to represent major research directions in the field.

Chapter 2

Photonic Crystals and Photonic Crystal Slabs

In the last few decades, a new frontier has emerged in the field of Photonics: the ability to control and manipulate the optical properties of materials. An enormous range of technological applications would become possible if materials could be engineered to respond to light over selected frequency ranges. For example, by perfectly reflecting specific wavelengths, allowing light to propagate only in certain directions, or confining electromagnetic fields within a defined volume. Fiber-optic cables, which guide light over long distances, have already revolutionized telecommunications. Other fields, including laser engineering, high-speed computing, and spectroscopy, are also expected to benefit from continued advances in optical materials.

To understand which materials can provide such a high degree of control over light propagation, it is useful to draw an analogy with electronic materials. A crystal consists of a periodic arrangement of atoms or molecules. The spatial pattern in which these constituents are repeated is known as the *crystal lattice*. As an electron propagates through a crystal, it experiences a periodic potential, and both the nature of the constituent materials and the geometry of the lattice determine the electronic properties of the system.

Quantum mechanics in a periodic potential explains why, in a conducting crystal, electrons can behave as a diffusive gas of nearly free particles and propagate through the lattice without being scattered by its constituents. The key feature is that electrons propagate as waves, and waves satisfying the appropriate conditions can travel through a periodic potential without scattering, except in the presence of defects and impurities. At the same time, the periodic lattice can prohibit the propagation of waves within specific ranges of energy. These forbidden regions appear as **band gaps** in the electronic band structure, meaning that electrons cannot propagate with certain energies and, in some cases, in certain directions.

If the periodic potential is sufficiently strong, a band gap can extend over all possi-

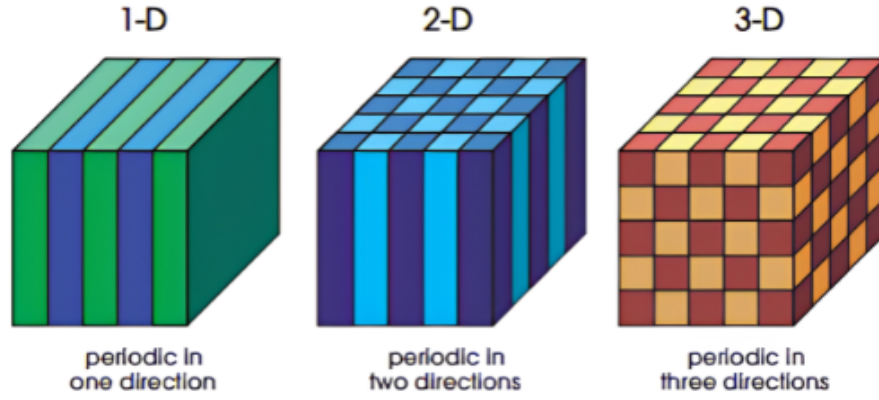


Figure 2.1: General overview of photonic crystal lattices for different types of electromagnetic periodicity along one, two, or three spatial directions, respectively.[6]

ble propagation directions, resulting in a **complete band gap**. A semiconductor, for example, exhibits a complete band gap between its valence and conduction bands, which is the reason for several characteristics phenomenologically observed in these materials.

The optical analogue of this concept brings to the definition of the so called **Photonic Crystals**. In this type of structures, contrary to their electronic counterparts, atoms and molecules are replaced by macroscopic materials with different dielectric constants. It is a theoretical model in which the periodic atomic potential is replaced by a periodic dielectric function or, equivalently, a periodic refractive index. If the dielectric contrast between the constituent materials is sufficiently large, and optical absorption is sufficiently low, the multiple reflections and refractions occurring at the interfaces can produce phenomena for photons (i.e., electromagnetic modes) analogous to those produced by the atomic potential for electrons.

Photonic crystals therefore provide a powerful approach to the control and manipulation of light in spatially inhomogeneous dielectric media. These low-loss periodic dielectric structures can be designed to exhibit, e.g., **photonic band gaps**, i.e., frequency ranges in which electromagnetic fields are prevented from propagating in certain spatial directions. In other words, a photonic crystal can be engineered to control specific ranges of wavelengths. Moreover, as it will be discussed throughout this thesis, photonic crystals can also enable light to propagate in anomalous and technologically useful ways.

A photonic crystal can be designed for different spectral regimes: structures with millimeter-scale dimensions can be used to control microwaves, whereas micron-scale structures can operate in the infrared and optical regimes. A simple and widely used example of a photonic crystal is a multilayer dielectric mirror, such as *Bragg's mirror*, consisting of alternating layers of materials with different dielectric constants. When light with the appropriate wavelength is incident on such a structure, it can be al-

most completely reflected. This occurs because the light is partially reflected at each interface between adjacent layers. When the layer spacing is periodic and properly designed, the multiple reflected waves interfere constructively in the backward direction and destructively in the forward direction, thereby suppressing transmission. Since these structures are periodic in only one spatial direction and consist of low-loss dielectric materials, they can be classified, according to the general definition, as *one-dimensional photonic crystals*.

If, over a given frequency range, a photonic crystal prohibits the propagation of electromagnetic waves of any polarization and in any direction, the crystal is said to possess a **complete photonic band gap**. A crystal with a complete band gap is necessarily an omnidirectional reflector; however, the converse is not necessarily true.

In particular, one-dimensional layered dielectric structures cannot exhibit a complete photonic band gap because their material interfaces are periodic along one spatial direction only. Nevertheless, they can be designed to achieve omnidirectional reflection for light incident from sufficiently distant sources.

In general, the realization of a complete photonic band gap requires the dielectric structure to be periodic along all three spatial directions, resulting in a *three-dimensional photonic crystal* (see, e.g., Fig. 2.2 for a schematic illustration). However, there exist important exceptions. A small degree of disorder introduced into an otherwise periodic medium does not necessarily destroy a band gap, while even highly disordered media can suppress light propagation in useful and controllable ways.[6, 13].

In this chapter, we will first introduce the macroscopic Maxwell equations as they apply to generic dielectric media. It will be shown that these equations can be arranged as a single Hermitian eigenvalue problem, useful to highlight several important properties, including the orthogonality of electromagnetic modes, the electromagnetic variational theorem, and the scaling laws governing dielectric systems.

We then introduce the basic concepts of solid-state physics and symmetry theory that are relevant to photonic crystals. Symmetry arguments are commonly used to understand the propagation of electrons in periodic crystal potentials, and we will show that the same concepts can be fully applied to electromagnetic waves propagating through photonic crystals. In particular, we will examine the consequences of translational, rotational, mirror-reflection, inversion, and time-reversal symmetries, while introducing the corresponding terminology from solid-state physics.

In the following, to develop the fundamental concepts underlying photonic crystals, we will focus our attention on photonic materials with two-dimensional periodicity, i.e., whose dielectric function is periodic in two spatial directions and homogeneous in the third, as illustrated in Fig. 2.1. These systems can exhibit photonic band gaps within the plane of periodicity. By analyzing the field distributions associated with electromagnetic modes in different crystal geometries, we can gain further insight

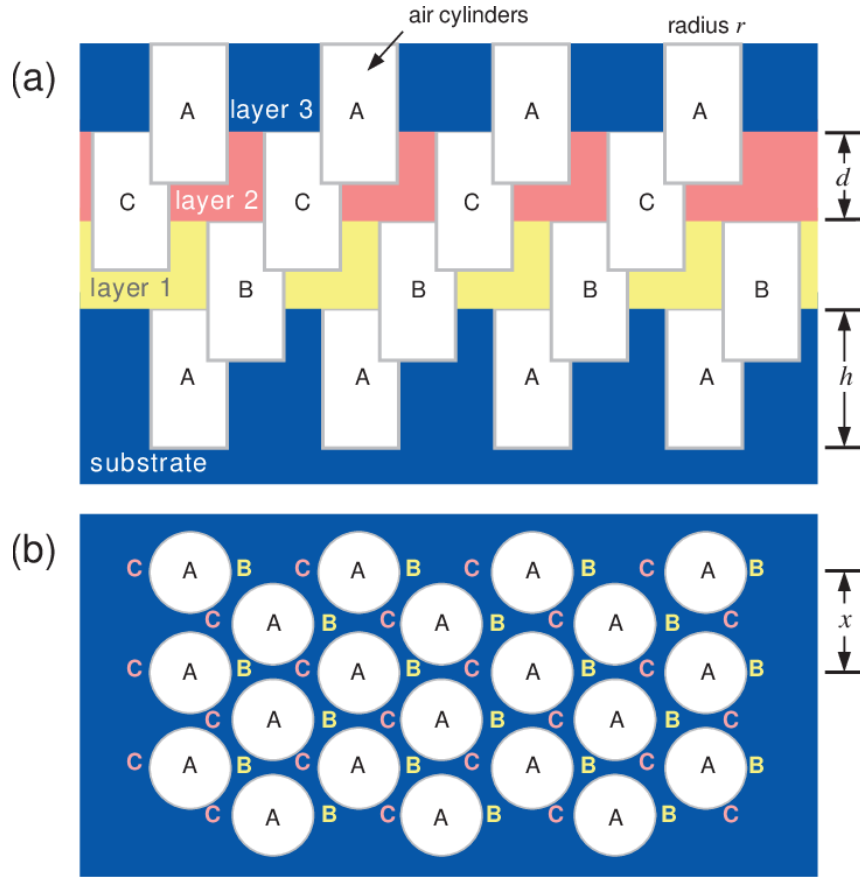


Figure 2.2: Schematic representation of a general three-dimensional structure of mixed dielectric materials. (a) The vertical side, showing the different material layers as they might be deposited during fabrication. (b) Planar view of a horizontal cross-section intersecting the different dielectric configurations of the different layers.[7]

into the physical origin and characteristics of band gaps in complex periodic media. Finally, we will turn our attention to the central structures and theoretical framework of this thesis: **photonic crystal slabs**, and the **Guided Mode Expansion (GME)** method to deal with them.

2.1 The electromagnetic problem in linear media

As it has been anticipated before, we now begin the theoretical analysis of general photonic systems starting from Maxwell equations in a linear medium. After specializing to the case of a mixed dielectric medium, Maxwell equations are recast as a linear Hermitian eigenvalue problem. This brings the electromagnetic problem, interpreted as a differential wave equation, into close analogy with the *Schrödinger* equation, and it allows us to take advantage of some well-established results from quantum mechanics, such as the orthogonality of modes, the variational theorem, and perturbation theory. In particular, this analogy can be further extended to deal with periodic systems in real space, in which a strict analogy to crystalline solids exists.

One key difference between the electromagnetic and the quantum-mechanical case, respectively is that photonic systems do not generally possess a fundamental scale, in either the spatial coordinate or in the potential strength, i.e., the magnitude of the dielectric constant. This makes photonic systems, and in particular photonic crystals, scalable in a way that atomic crystals are not.

Starting from the very basis, all of macroscopic electromagnetism, including the propagation of light in a photonic crystal, is governed by the four macroscopic Maxwell equations. In **SI** units, they are explicitly written in their differential form as:

$$\begin{aligned}\nabla \cdot \mathbf{H} &= 0, \\ \nabla \cdot \mathbf{D} &= 0, \\ \vec{\nabla} \times \vec{\mathbf{B}} - \frac{\partial \vec{\mathbf{D}}}{\partial t} &= \mathbf{J}, \\ \vec{\nabla} \times \vec{\mathbf{E}} + \frac{\partial \vec{\mathbf{H}}}{\partial t} &= 0.\end{aligned}\tag{2.1}$$

in which, respectively, $\mathbf{E}$ and $\mathbf{B}$ are the macroscopic electric and magnetic fields, $\mathbf{D}$ and $\mathbf{H}$ are the displacement and magnetic induction fields, and ρ and $\mathbf{J}$ (respectively, the free charge and current densities) are set to zero. The choice of dimensional units in which the Eq. 2.1 are written is mostly irrelevant (notice, for example, that in the *cgs units* form of Maxwell equations constants such as ϵ_0 and μ_0 are replaced by factors of 4π and c , without affecting the final form of our "Master" Equation).

Starting with the general case, we will restrict ourselves to light propagation within a **mixed dielectric medium**: a composite of regions made of homogeneous dielectric material as a function of the position vector r , in which the structure does not vary with time, and there are no free charges or currents, see Fig. 2.2. In general, $\epsilon(r)$ can have any prescribed spatial dependence, but our attention will focus on materials with patches of homogeneous dielectric. For now we assume this composite not to be necessarily periodic. With this type of medium in mind, in which light propagates but there are no sources of light, it is assumed $\rho = 0$ and $\mathbf{J} = 0$.

Next, we relate $\mathbf{D}$ to $\mathbf{E}$ and $\mathbf{B}$ to $\mathbf{H}$ with the so called constitutive relations. In general, the components D_i of the displacement field $\mathbf{D}$ are related to the components E_i of the electric field $\mathbf{E}$ via a power series, as:

$$\frac{D_i}{\epsilon_0} = \sum_j \epsilon_{ij} E_j + \sum_{j,k} \chi_{ijk}^{(2)} E_j E_k + O(E^3),\tag{2.2}$$

in which $\epsilon_0 \approx 8.85410^{-12}$ Farad/m is the vacuum permittivity. However, for many dielectric materials, it is reasonable to use the following approximations:

- First, we assume the field strengths are small enough such that we are in the

linear regime, so that the second-order susceptibility tensor, $\chi_{ijk}^{(2)}$, as well as all higher-order nonlinear terms, can be safely neglected.

- Second, we assume the material is *macroscopic* and *isotropic*, such that $\mathbf{E}(r, \omega)$ and $\mathbf{D}(r, \omega)$ are related by ϵ_0 multiplied by a scalar dielectric function $\epsilon(r, \omega)$, also called the *relative permittivity*.
- Third, we ignore any explicit frequency dependence of the dielectric constant. Instead, we simply choose the value of the dielectric constant appropriate to the frequency range of the physical system we are considering.
- Fourth, we focus primarily on *transparent materials*: which means we can treat $\epsilon(r)$ as purely real and positive. the assumption is necessary because: complex dielectric constants are used to account for absorption-loss and a negative dielectric constant is indeed a useful description of some materials, such as metals, where the limit $\epsilon \rightarrow -\infty$ corresponds to a perfect metal into which light cannot penetrate.

Assuming these four approximations to be valid, we have linear constitutive relations for the displacement field $\mathbf{D}(r)$ and similarly for $\mathbf{H}(r)$, where instead of ϵ_0 we have $\mu_0 = 4\pi 10^{-7}$ Henry/m which is the vacuum permeability.

$$\mathbf{D}(r) = \epsilon_0 \epsilon(r) \mathbf{E}(r) , \quad (2.3)$$

$$\mathbf{H}(r) = \mu_0 \mu(r) \mathbf{B}(r) . \quad (2.4)$$

For most dielectric materials of interest the *relative magnetic permeability*, $\mu(r)$, is very close to unity, and we may simply set $\mathbf{H}(r) = \mu_0 \mathbf{B}(r)$. In the end, with all of these assumptions in place, the Maxwell equations (2.1) become:

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 , \\ \nabla \cdot [\epsilon(r) \mathbf{E}(r)] &= 0 , \\ \vec{\nabla} \times \vec{\mathbf{H}} - \epsilon_0 \epsilon(r) \frac{\partial \vec{\mathbf{E}}(r)}{\partial t} &= 0 , \\ \vec{\nabla} \times \vec{\mathbf{E}}(r) + \mu_0 \frac{\partial \vec{\mathbf{H}}}{\partial t} &= 0 . \end{aligned} \quad (2.5)$$

In general, the fields $\mathbf{E}$ and $\mathbf{B}$ are complicated functions of both time and space. Since Maxwell equations are linear, however, we can separate the time dependence from the spatial dependence by expanding the fields into a set of harmonic modes. We employ the standard trick of using a complex-valued field and remembering to take the real part to obtain the physical fields. This allows us to write a harmonic mode as a time

dependent complex exponential:

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}) e^{-i\omega t} \quad , \quad \mathbf{B}(\mathbf{r}, t) = \mathbf{B}(\mathbf{r}) e^{-i\omega t} . \quad (2.6)$$

Instead of eliminating one of the two fields, as done in the derivation of the wave equation, we can retain both $\mathbf{E}$ and $\mathbf{H}$ and rewrite Eq.(2.5) as a first-order eigenvalue problem. To this end, the first equation can be rearranged as

$$\begin{aligned} \vec{\nabla} \times \vec{\mathbf{H}}(\mathbf{r}) + i \omega \varepsilon_0 \varepsilon(\mathbf{r}) \vec{\mathbf{E}}(\mathbf{r}) &= 0 , \\ \vec{\nabla} \times \vec{\mathbf{E}}(\mathbf{r}) - i \omega \mu_0 \vec{\mathbf{H}}(\mathbf{r}) &= 0 . \end{aligned} \quad (2.7)$$

We can divide the bottom equation by $\varepsilon(\mathbf{r})$, and then take the curl. Then by using the first equation to eliminate $\mathbf{E}(\mathbf{r})$ as variable; and combining the constants ε_0 and μ_0 to yield the vacuum speed of light (c). The result is an equation entirely in the magnetic field $\mathbf{H}(\mathbf{r})$:

$$\nabla \times \left(\frac{1}{\varepsilon(\mathbf{r})} \nabla \times \mathbf{H}(\mathbf{r}) \right) = \left(\frac{\omega}{c} \right)^2 \mathbf{H}(\mathbf{r}) . \quad (2.8)$$

This is what we will define the **Master Equation** of the electromagnetic problem. The choice to formulate the problem in $\mathbf{H}(\mathbf{r})$ is a matter of mathematical convenience: by solving for $\mathbf{H}(\mathbf{r})$ and then recovering $\mathbf{E}(\mathbf{r})$ through the original curl equation (Eq. 2.7), the transversality condition is automatically satisfied (since the divergence of a curl is always zero), avoiding as such the need to impose two separate transversality constraints.

This type of equation is what is typically defined as an *eigenvalue problem*. In fact, we can identify the left side of the master equation as a *differential operator* $\hat{\Theta}$ acting on $\mathbf{H}(\mathbf{r})$ to make it look more like a traditional eigenvalue problem:

$$\hat{\Theta} \mathbf{H}(\mathbf{r}) = \left(\frac{\omega}{c} \right)^2 \mathbf{H}(\mathbf{r}) \quad (2.9)$$

Here the eigenvectors $\mathbf{H}(\mathbf{r})$ are the spatial patterns of the harmonic modes, and the eigenvalues are proportional to the squared frequencies of those modes. The eigenvalues of Eq. 2.9 are the so called **photonic modes**.

Scaling properties

An important aspect to highlight is that the operator $\hat{\Theta}$ is **Hermitian**, meaning that any linear combination of solutions is itself a solution. By using this property, given a certain mode profile, we can construct another legitimate mode profile with the same frequency by simply multiplying the field strength by an overall constant. For this reason, we consider two field patterns that differ only by an overall multiplier to

represent the same mode.

Being an Hermitian operators guarantee for $\hat{\Theta}$ several important properties:

1. The eigenvalues $(\omega/c)^2$ are strictly real.
2. Eigenmodes corresponding to distinct frequencies are orthogonal, while degenerate modes can always be orthogonalized (e.g., via a Gram-Schmidt procedure).
3. The system can be analyzed through a variational principle.

Since $\varepsilon(\mathbf{r}) > 0$ everywhere, the integrand on the right-hand side is everywhere non-negative. The operator is therefore said to be **positive**. Consequently, all of the eigenvalues are nonnegative, and real.

In addition, let us consider two normalized modes, $H_1(\mathbf{r})$ and $H_2(\mathbf{r})$, with frequencies ω_1 and ω_2 . If $\omega_1 \neq \omega_2$, then $H_1(\mathbf{r})$ and $H_2(\mathbf{r})$ are orthogonal modes. If two harmonic modes have equal frequencies, instead, they are said to be degenerate and are not necessarily orthogonal. Degeneracy requires two different field patterns to have precisely the same frequency. Usually, there is a symmetry that is responsible for this apparent coincidence. For example, if the dielectric configuration is invariant under a 120° rotation, modes that differ only by the same rotation angle are expected to have the same frequency. Such modes are degenerate and are not necessarily orthogonal. However, since $\hat{\Theta}$ is linear, any linear combination of degenerate modes is itself a mode with the same frequency. As in quantum mechanics, we can therefore always choose linear combinations that are orthogonal. This allows us to state quite generally that different modes are orthogonal, or can be arranged to be orthogonal.

As already mentioned, one interesting aspect of electromagnetism in dielectric media is that there is no fundamental length scale, other than the assumption that the system is macroscopic. In atomic physics, the spatial scale of the potential function is generally set by a fundamental length scale, such as the *Bohr radius*. Consequently, configurations of materials that differ only in their overall spatial scale can nevertheless have very different physical properties. For photonic crystals, where there is no fundamental constant with the dimensions of length, the master equation is considered *scale invariant*. This leads to the imposition of simple relationships between electromagnetic problems that differ only by a contraction or expansion of all distances. As an example, suppose that we have an electromagnetic eigenmode $H(\mathbf{r})$ with frequency ω in a dielectric configuration $\varepsilon(\mathbf{r})$. We are interested in the harmonic modes of a dielectric configuration $\varepsilon'(\mathbf{r})$ that is simply a compressed or expanded version of the one before, such as:

$$\varepsilon'(\mathbf{r}) = \varepsilon\left(\frac{\mathbf{r}}{s}\right),$$

where s is a scale parameter. The spatial variables in the master equation will, also, be modified as:

$$\mathbf{r}' = s\mathbf{r} \quad \rightarrow \quad \nabla' = \frac{1}{s} \nabla .$$

After substituting these relations into the Eq. 2.8 and dividing out the factors s , we obtain the same equation again, but with the mode profile and frequency rescaled, respectively:

$$H'(\mathbf{r}') = H\left(\frac{\mathbf{r}'}{s}\right) , \quad \omega' = \frac{\omega}{s} .$$

This means that the new mode profile and its corresponding frequency can be obtained simply by rescaling the old mode profile and frequency. The solution of the problem at one length scale therefore determines the solutions at all other length scales.

This simple fact is of considerable practical importance. For example, the microfabrication of complex micron-scale photonic crystals can be quite difficult. However, models can be easily made and tested in the microwave regime, at the much larger length scale of centimeters, provided that materials with nearly the same dielectric constant can be found. The considerations in this section guarantee that the model will have the same electromagnetic properties, apart from the corresponding frequency scaling. Just as there is no fundamental length scale, there is also no fundamental value of the dielectric constant. Suppose that we know the harmonic modes of a system with dielectric configuration $\varepsilon(\mathbf{r})$, and we are interested in the modes of a system whose dielectric configuration differs by a constant factor everywhere:

$$\varepsilon'(\mathbf{r}) = \frac{\varepsilon(\mathbf{r})}{s^2}$$

By substituting in the Eq. 2.8 we get the same harmonic modes, while the frequencies are scaled by a factor s . Also, the relationship between $\mathbf{E}$ and $\mathbf{H}$ has changed by a factor of s . Thus, if the dielectric constant is multiplied everywhere by a factor of $1/4$, the mode patterns remain unchanged, while their frequencies double. Combining the two scaling relations, we see that if we scale the dielectric constant by s^2 while also rescaling the coordinates by s , the frequency remains unchanged.

This simple scaling invariance is a special case of more general coordinate transformations. Remarkably, any coordinate transformation can be replaced simply by a corresponding change in ε and μ while keeping ω fixed. This can be a powerful conceptual tool because it allows us to warp and distort a structure in complicated ways while retaining a similar form for Maxwell's equations. In general, however, this transformation of ε and μ is not merely a multiplication by a constant.

Symmetry of photonic eigenmodes

Having established the general properties of the three-dimensional Master Equation, we now turn to systems with reduced spatial dimensionality, specifically two-dimensional media, and their consequence on the symmetry properties of their eigenmodes. Specifically, a two-dimensional photonic crystal is characterized by a dielectric permittivity $\varepsilon(\mathbf{r}_{\parallel})$ that is periodic in the transverse plane $\mathbf{r}_{\parallel} = (x, y)$ while remaining strictly uniform along the longitudinal axis (z), such that $\partial \varepsilon / \partial z = 0$.

When propagation is restricted to the plane of periodicity ($\mathbf{k} = (k_x, k_y, 0)$), Eqs. 2.8 decouple into two independent scalar sub-problems, respectively defined by their polarization symmetries with respect to the horizontal plane. Each of these sub-problems define specific eigenmodes, namely:

- **Transverse Electric (TE / E-polarized) Modes:** The electric field is polarized entirely along the uniform axis, $\mathbf{E}(\mathbf{r}) = E_z(\mathbf{r}_{\parallel})\hat{\mathbf{z}}$, while the magnetic field lies within the xy -plane ($\mathbf{H} = (H_x, H_y, 0)$). The Master Equation simplifies to the scalar wave equation:

$$-\frac{1}{\varepsilon(\mathbf{r}_{\parallel})} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) E_z(\mathbf{r}_{\parallel}) = \left(\frac{\omega}{c} \right)^2 E_z(\mathbf{r}_{\parallel}). \quad (2.10)$$

- **Transverse Magnetic (TM / H-polarized) Modes:** The magnetic field is aligned with the invariant axis, $\mathbf{H}(\mathbf{r}) = H_z(\mathbf{r}_{\parallel})\hat{\mathbf{z}}$, while the electric field lies in the plane ($\mathbf{E} = (E_x, E_y, 0)$). The corresponding differential equation takes the form:

$$-\left[\frac{\partial}{\partial x} \left(\frac{1}{\varepsilon(\mathbf{r}_{\parallel})} \frac{\partial}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{1}{\varepsilon(\mathbf{r}_{\parallel})} \frac{\partial}{\partial y} \right) \right] H_z(\mathbf{r}_{\parallel}) = \left(\frac{\omega}{c} \right)^2 H_z(\mathbf{r}_{\parallel}). \quad (2.11)$$

This TE/TM decoupling considerably reduces the computational complexity of solving for the eigenmodes and photonic band structures. More importantly, it provides a clear mathematical bridge: the continuous translational symmetry along z isolates the in-plane dynamics, allowing the remaining spatial symmetries of $\varepsilon(\mathbf{r}_{\parallel})$ (e.g., translations, rotations, and reflections) to be analyzed systematically via group theory.

2.1.1 Symmetry properties of two-dimensional photonic systems

As shown above, symmetry plays a fundamental role in determining, classifying, and governing the interactions of electromagnetic modes in photonic systems. Whenever a geometric transformation leaves the spatial dielectric distribution invariant, the corresponding symmetry operator $\hat{O}$ commutes with the Master operator, $[\hat{\Theta}, \hat{O}] = 0$. Consequently, the eigenmodes of $\hat{\Theta}$ can be simultaneously classified according to the irreducible representations of the crystal's symmetry group.

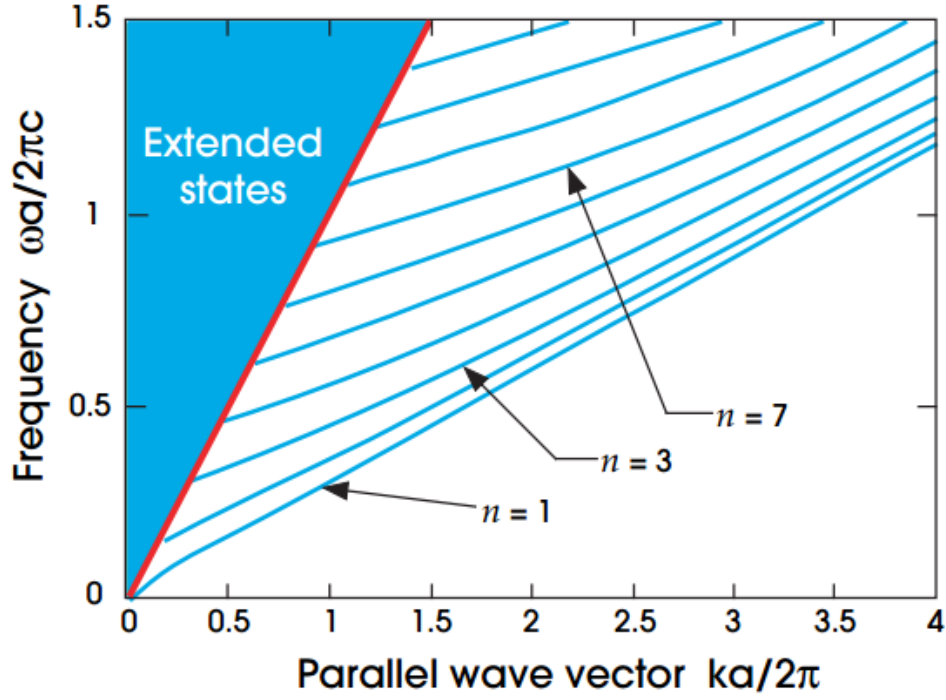


Figure 2.3: Example of frequency-wavevector dispersion of harmonic mode frequencies, $\omega_n(k)$, for a glass slab suspended in air. The blue lines correspond to modes that propagate through the glass. The shaded blue region is a continuum of states that extend into both the glass and the air around it. The red line is the so called **light line**, i.e., light dispersion in the uniform cladding medium (air in this case). [6]

For instance, consider a two-dimensional photonic system invariant under spatial inversion about its origin: $\varepsilon(-\mathbf{r}_{\parallel}) = \varepsilon(\mathbf{r}_{\parallel})$. If $\mathbf{H}(\mathbf{r}_{\parallel})$ is an eigenmode at frequency ω , then $\hat{O}_I \mathbf{H}(\mathbf{r}_{\parallel}) = \mathbf{H}(-\mathbf{r}_{\parallel})$ must also be a solution at the same frequency. For non-degenerate modes, the field profile can only change by an overall parity phase, categorizing modes strictly as **even** ($\mathbf{H}(-\mathbf{r}_{\parallel}) = \mathbf{H}(\mathbf{r}_{\parallel})$) or **odd** ($\mathbf{H}(-\mathbf{r}_{\parallel}) = -\mathbf{H}(\mathbf{r}_{\parallel})$). In degenerate cases, individual modes may not initially exhibit pure parity, but a linear combination can always be chosen to form a basis with well-defined symmetry properties.

In two-dimensional photonic crystal lattices, three fundamental spatial transformations preserve the dielectric lattice:

- **Discrete Translations** by a lattice vector $\mathbf{R} = m_1 \mathbf{a}_1 + m_2 \mathbf{a}_2$.
- **Point-Group Rotations** about a high-symmetry axis perpendicular to the plane.
- **Reflections** across mirror planes parallel to the axial direction.

Below, we review how these elementary spatial symmetries shape the dispersion relations and topological characteristics of two-dimensional photonic crystals.

Translational symmetry and index guiding

In direct analogy to solid-state electronic systems, continuous or discrete translational invariance dictates the conserved momentum parameters of the field modes.

If a structure exhibits continuous translational symmetry along an axis (e.g., z), the wave vector component k_z acts as a *conserved quantum number*, with fields scaling as $\mathbf{H}(\mathbf{r}) \propto e^{ik_z z}$. In a homogeneous medium, this continuous symmetry holds along all directions, yielding standard plane-wave eigenmodes

$$\mathbf{H}(\mathbf{r}) = \mathbf{H}_0 e^{i\mathbf{k}\cdot\mathbf{r}}, \quad (2.12)$$

with linear dispersion $\omega = c|\mathbf{k}|/\sqrt{\epsilon}$.

In planar dielectric waveguides, being a flat three-layer structures uniform in the xy -plane but bounded in z (see Fig.2.12), the in-plane translation conserves the wave vector $\mathbf{k}_{\parallel} = (k_x, k_y)$. Modes that fall below the **light line** of the cladding medium become exponentially evanescent outside the high-index core, realizing bound, spatially confined modes via **index guiding**, see Fig. 2.3.

When continuous translational symmetry is broken down to a discrete spatial periodicity as in a photonic crystal where :

$$\epsilon(\mathbf{r}_{\parallel} + \mathbf{R}) = \epsilon(\mathbf{r}_{\parallel}), \quad (2.13)$$

for any lattice vector $\mathbf{R}$, continuous momentum is no longer conserved. Instead, according to **Bloch's Theorem**, the eigenmodes assume the plane-wave form:

$$\mathbf{H}_{n,\mathbf{k}}(\mathbf{r}_{\parallel}) = e^{i\mathbf{k}\cdot\mathbf{r}_{\parallel}} u_{n,\mathbf{k}}(\mathbf{r}_{\parallel}), \quad \text{where} \quad u_{n,\mathbf{k}}(\mathbf{r}_{\parallel} + \mathbf{R}) = u_{n,\mathbf{k}}(\mathbf{r}_{\parallel}). \quad (2.14)$$

Bloch's theorem effectively reduces the eigenvalue problem defined on an infinite periodic domain to a bounded problem over a single **unit cell**. Because wave vectors differing by a reciprocal lattice vector $\mathbf{G}$ describe physically identical states, $\mathbf{k}$ can be restricted entirely to the **first Brillouin zone (BZ)** (see, e.g., Fig. 2.4 for examples in 2D).

For each Bloch wave vector $\mathbf{k}$ within the Brillouin zone, solving the Master Equation yields a discrete set of frequencies $\omega_n(\mathbf{k})$, indexed by the band number n . Plotting these dispersion relations yields the **photonic band structure**. The periodic dielectric modulation mixes plane-wave states separated by reciprocal lattice vectors $\mathbf{G}$, opening up frequency regions where no propagating Bloch modes can exist in any direction, giving rise to full or partial **photonic band gaps**.

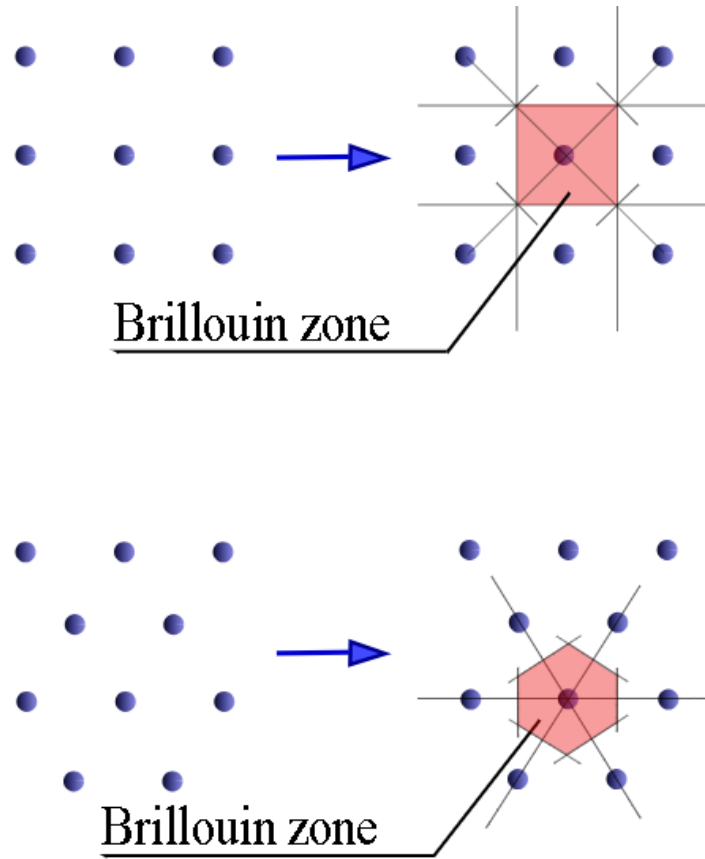


Figure 2.4: Schematic view of two-dimensional photonic crystal made from dielectric rods in air. Depicting from left to right: real space rod configuration, corresponding reciprocal space and the geometric definition of the first Brillouin Zone. Two cases are presented: Square lattice (upper panel); triangular, or hexagonal, lattice (lower panel).

Point-group symmetry: Rotations and mirror planes

Beyond translational invariance, the unit cell of a two-dimensional photonic crystal often displays *point-group symmetries*, such as rotations and mirror reflections, that fix at least one central point in the cell. The point groups most pertinent to two-dimensional lattices belong to the C_{nv} family.

The cyclic group C_n comprises discrete rotations by $2\pi/n$ about the central axis, see Fig. 2.5. Adjoining a mirror plane parallel to the symmetry axis automatically generates $n - 1$ additional mirror planes spaced by angles of π/n , assembling the point group C_{nv} . Representative examples include:

- Square lattices of circular rods/holes (C_{4v} point symmetry).
- Triangular/hexagonal lattices of circular elements (C_{6v} point symmetry).
- Reduced-symmetry configurations, such as square lattices with asymmetric bases (C_{1v}) or stretched triangular lattices (C_{2v}).

Point-group representations directly govern the physical properties of photonic bands. As a particular example, point-group selection rules determine the allowed topological

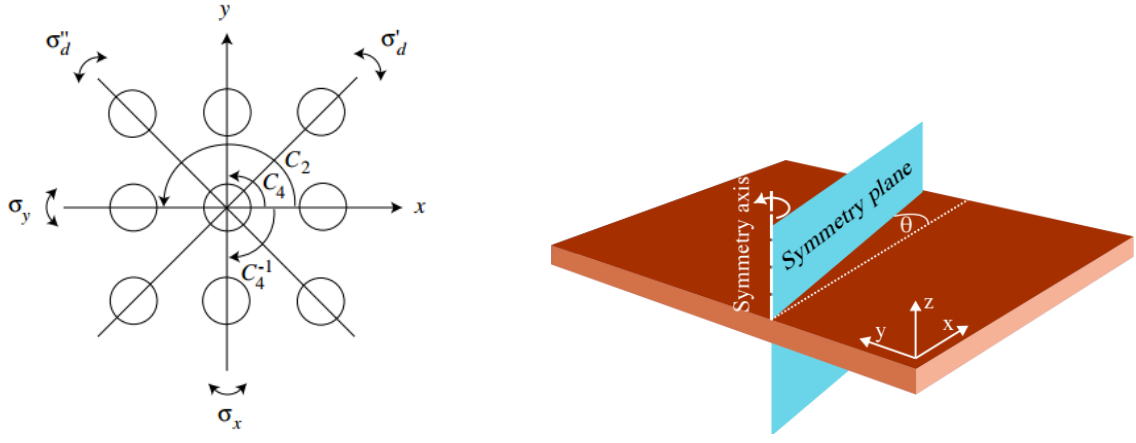


Figure 2.5: Different types of symmetry that characterize two-dimensional photonic crystals. (a) Visual representation of **point group symmetry** for the case of two-dimensional photonic crystals composed of a square array of dielectric rods in air, showing the main transformations that are proper of such system.[13] (b) Visual representation of **mirror symmetry** around the z -axis for the specific of dielectric slab suspended in air.[19]

charges, q , of the so called **Bound States in the Continuum (BICs)** at the Γ point, and dictate mode conversion under symmetry-lowering perturbations (e.g., transitioning from C_{6v} to C_{2v}) [19].

Furthermore, point-group operations impose conditions on $\omega(\mathbf{k})$ across the Brillouin zone. To this end, let us consider an operation of the point group $\hat{R}$, then

$$\omega(\hat{R}\mathbf{k}) = \omega(\mathbf{k}) . \quad (2.15)$$

Exploiting these geometrical redundancies allows numerical band calculations to be restricted to the smallest non-redundant domain, i.e., the **irreducible Brillouin zone**.

Mirror-symmetry operator and classification of Bloch eigenmodes

Mirror operations deserve targeted considerations, since they provide commuting quantum numbers for mode classification. Let us consider a mirror reflection plane passing through the origin along an angle θ relative to the x -axis, see Fig. 2.5. Its action on the transverse coordinates $(x, y)^\top$ is given by the transformation matrix $M(\theta)$:

$$M(\theta) = \frac{1}{1 + \tan^2 \theta} \begin{pmatrix} 1 - \tan^2 \theta & 2 \tan \theta \\ 2 \tan \theta & \tan^2 \theta - 1 \end{pmatrix}. \quad (2.16)$$

For instance, reflection across the yz -plane ($x \rightarrow -x$) corresponds to $M_x \equiv M(\pi/2)$. The corresponding unitary operator $\hat{O}_{M_x}$ acting on a vector field $\mathbf{f}(\mathbf{r})$ is defined as:

$$\hat{O}_{M_x} \mathbf{f}(\mathbf{r}) = M_x \mathbf{f}(M_x \mathbf{r}). \quad (2.17)$$

Since $[\hat{\Theta}, \hat{O}_{M_x}] = 0$, applying $\hat{O}_{M_x}$ to a Bloch state $\mathbf{H}_{\mathbf{k}}$ yields another valid eigenmode at the reflected wave vector $M_x \mathbf{k}$:

$$\hat{O}_{M_x} \mathbf{H}_{\mathbf{k}} = e^{i\phi} \mathbf{H}_{M_x \mathbf{k}}. \quad (2.18)$$

When the wave vector lies directly *on* the mirror line (such that $M_x \mathbf{k} = \mathbf{k}$), Eq. (2.18) simplifies to a localized eigenvalue equation:

$$\hat{O}_{M_x} \mathbf{H}_{\mathbf{k}}(\mathbf{r}) = \pm 1 \mathbf{H}_{\mathbf{k}}(\mathbf{r}), \quad (2.19)$$

allowing high-symmetry Bloch modes to be classified definitively as **even** (+1) or **odd** (-1) with respect to mirror reflection.

It is worth noting that parity transformations act differently on proper vectors (like the electric field $\mathbf{E}$) and pseudovectors (like the magnetic field $\mathbf{H}$). Under pure coordinate inversion $\mathbf{I} \mathbf{r} = -\mathbf{r}$, $\mathbf{E}$ flips sign ($\mathbf{E} \rightarrow -\mathbf{E}$) while $\mathbf{H}$ remains invariant ($\mathbf{H} \rightarrow \mathbf{H}$). Consequently, an even mode under inversion satisfies $\mathbf{H}(-\mathbf{r}) = \mathbf{H}(\mathbf{r})$ and $\mathbf{E}(-\mathbf{r}) = -\mathbf{E}(\mathbf{r})$.

Classifying modes by mirror eigenvalues drastically reduces computational burden by block-diagonalizing the eigenvalue problem into independent symmetry sub-sectors (a property heavily exploited in Guided-Mode Expansion methods). Experimentally, it enables selective mode excitation through controlled input light polarization.

Time-Reversal symmetry

Finally, we would now like to discuss a specific type of symmetry, which is indeed typical of all classical systems, such as photonic crystals. In lossless, linear, non-magnetic materials ($\mu = \mu_0, \varepsilon \in \mathbb{R}$), i.e., systems with the characteristics we are dealing with, Maxwell equations obey **time-reversal symmetry**. For time-harmonic fields, namely any physical field such as an electric or magnetic field that varies sinusoidally with time at a constant frequency, time reversal corresponds to complex conjugation. Applied to a Bloch mode, complex conjugation flips the momentum vector $\mathbf{k} \rightarrow -\mathbf{k}$, thus guaranteeing that:

$$\omega_n(\mathbf{k}) = \omega_n(-\mathbf{k}). \quad (2.20)$$

Thus, the photonic band structure is universally symmetric under inversion in reciprocal space, regardless of whether the spatial unit cell itself possesses inversion symmetry.

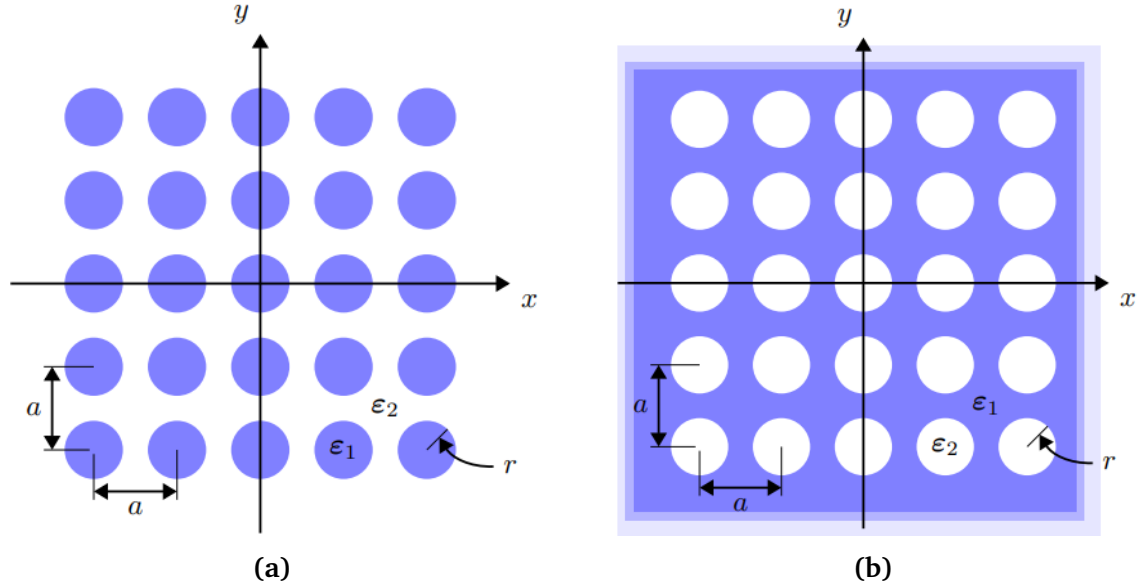


Figure 2.6: Examples of two complementary two-dimensional photonic crystals. (a) Dielectric rods (ϵ_1), with a radius of $r = 0.2a$, suspended in another material (ϵ_2); arranged in a square lattice configuration with lattice constant a . (d) Surface of dielectric constant ϵ_1 with a square lattice of dielectric holes (ϵ_2). Each hole with a radius of $r' = 0.3a$.

2.2 Two-Dimensional Photonic Crystals

Now we will focus our attention on a particular kind of photonic crystal. Two-dimensional photonic crystals are structures characterized by being periodic in two spatial directions and homogeneous along the third. In our case, the periodicity is limited to the xy -plane, while the structure is invariant along the z -direction (Fig. 2.6).

As already mentioned, this periodic behavior of the dielectric function can give rise to *photonic band gaps*, i.e., frequency intervals in which no extended electromagnetic modes are allowed to propagate through the crystal, in analogy to the electronic bandgap appearing in semiconducting or insulating crystals. In contrast to one-dimensional photonic structures, in which light can be confined along a single direction, defects introduced into a two-dimensional photonic crystal can be used to localize electromagnetic modes in two dimensions.

A typical realization of a two-dimensional photonic crystal consists of an array of dielectric columns extending infinitely along the z -direction, see Fig. 2.6. For appropriate choices of the lattice constant, dielectric contrast, and filling fraction, such a structure can exhibit a photonic band gap for in-plane propagation. Within this frequency range, extended states are forbidden and incident radiation is reflected. Thus, unlike a multilayer structure, a two-dimensional photonic crystal can inhibit propagation in arbitrary directions within the plane of periodicity. The electromagnetic modes of such a crystal can be classified by exploiting the symmetries of the structure. Since the system is homogeneous along the z -direction, there is no periodicity in this direc-

tion, and consequently the wave vector component k_z is not restricted by the lattice. Conversely, the dielectric function is periodic in the xy -plane, thus satisfying Eq. 2.13, where $\mathbf{R}$ is any lattice translation vector in the periodicity plane. For a square lattice, it can generally be written in terms of the lattice constant (a) as

$$\mathbf{R} = n_x a \hat{x} + n_y a \hat{y},$$

in which (n_x, n_y) are integers. By applying Bloch's theorem, the in-plane wave vector $k_{\parallel}$ can be restricted to the first Brillouin zone. The electromagnetic modes can therefore be indexed by the band number n , the unrestricted wave vector component k_z , and the in-plane wave vector $k_{\parallel}$. The magnetic field can be written in Bloch form as

$$H_{nk_z k_{\parallel}}(\mathbf{r}) = e^{i k_{\parallel} \cdot \rho} e^{i k_z z} u_{nk_z k_{\parallel}}(\rho)$$

where ρ denotes the projection of $\mathbf{r}$ onto the xy -plane. The function $u(\rho)$ has the periodicity of the lattice, as in Eq. (2.14). This expression highlights the main difference between a 2D photonic crystal and others multilayer structures. In the present case, $k_{\parallel}$ is restricted to the Brillouin zone, whereas k_z is unrestricted. Furthermore, the periodic part of the Bloch function depends on the in-plane coordinates rather than on z .

2.2.1 Master equation and TE/TM decoupling

It necessary to derive explicitly how, for in-plane propagation ($k_z = 0$), the vector problem collapses into two independent scalar eigenvalue problems. The starting point is the observation that for a two-dimensional crystal the dielectric function, $\epsilon(\mathbf{r}) = \epsilon(\rho)$, does not depend on z , so we restrict ourselves to in-plane propagation, $\mathbf{k} = k_{\parallel}$ (i.e. $\partial/\partial z = 0$). Under these conditions the Eq.(2.8) equations split into two mutually decoupled subsets, each involving only three field components:

$$\begin{aligned} \frac{\partial E_z}{\partial y} &= -\mu_0 \frac{\partial H_x}{\partial t}, \\ \frac{\partial E_z}{\partial x} &= \mu_0 \frac{\partial H_y}{\partial t}, \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= \epsilon_0 \epsilon(\rho) \frac{\partial E_z}{\partial t}. \end{aligned} \tag{2.21}$$

$$\begin{aligned}
\frac{\partial H_z}{\partial y} &= \varepsilon_0 \varepsilon(\rho) \frac{\partial E_x}{\partial t}, \\
\frac{\partial H_z}{\partial x} &= -\varepsilon_0 \varepsilon(\rho) \frac{\partial E_y}{\partial t}, \\
\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -\mu_0 \frac{\partial H_z}{\partial t}.
\end{aligned} \tag{2.22}$$

The first set involves only (E_z, H_x, H_y) : this is exactly the **TM** (or E -) polarization introduced above, since the electric field has only a z -component while $\mathbf{H}$ lies in the xy -plane. The second set involves only (H_z, E_x, E_y) : this is the **TE** (or H -) polarization, for which $\mathbf{H} = H_z \hat{z}$ and $\mathbf{E}$ lies in the plane. No component of one set enters the equations of the other, which is the formal reason why TE and TM modes never mix for strictly in-plane propagation. Now we have that the vector problem has been reduced to two independent scalar problems for $E_z(\rho)$ and $H_z(\rho)$.

By eliminating the in-plane components, each subset yields a single scalar wave equation, and taken the harmonic fields $E_z(\rho, t) = E_z(\rho)e^{-i\omega t}$, $H_z(\rho, t) = H_z(\rho)e^{-i\omega t}$ one obtains the two scalar master equations (already depicted in Eq.2.10, 2.11):

$$-\frac{1}{\varepsilon(\rho)} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) E_z(\rho) = \frac{\omega^2}{c^2} E_z(\rho), \quad (\text{TM}) \tag{2.23}$$

$$-\left[\frac{\partial}{\partial x} \frac{1}{\varepsilon(\rho)} \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \frac{1}{\varepsilon(\rho)} \frac{\partial}{\partial y} \right] H_z(\rho) = \frac{\omega^2}{c^2} H_z(\rho). \quad (\text{TE}) \tag{2.24}$$

The asymmetric placement of $\varepsilon(\rho)^{-1}$ in the two equations, in TM case outside the double derivative, but inside for the TE modes, is the mathematical root of the very different behavior between the two polarizations discussed qualitatively above: it makes the TM operator most sensitive to the value of the field at high-index regions, while the TE operator is most sensitive to the field gradient across the boundaries of connected high-index regions, see Fig. 2.8.

To turn the Eqs.2.23, 2.24 into an algebraic eigenvalue problem solvable numerically, one exploits the periodicity of $\varepsilon(\rho)$ and expands $1/\varepsilon(\rho)$ in a Fourier series over the reciprocal lattice vectors $\mathbf{G}_{\parallel}$,

$$\frac{1}{\varepsilon(\rho)} = \sum_{\mathbf{G}_{\parallel}} \kappa(\mathbf{G}_{\parallel}) e^{i\mathbf{G}_{\parallel} \cdot \rho},$$

and writes the Bloch eigenfunctions as a superposition of plane waves compatible with the lattice,

$$E_z(\rho) = \sum_{\mathbf{G}_{\parallel}} E_z(\mathbf{G}_{\parallel}) e^{i(k_{\parallel} + \mathbf{G}_{\parallel}) \cdot \rho}, \quad H_z(\rho) = \sum_{\mathbf{G}_{\parallel}} H_z(\mathbf{G}_{\parallel}) e^{i(k_{\parallel} + \mathbf{G}_{\parallel}) \cdot \rho}.$$

Substituting these expansions into the two master equations and projecting onto each plane-wave component gives rise to the plane-wave-expansion (PWE) eigenvalue problem actually solved numerically,

$$\begin{aligned} \sum_{\mathbf{G}'_{\parallel}} \kappa(\mathbf{G}_{\parallel} - \mathbf{G}'_{\parallel}) |k_{\parallel} + \mathbf{G}'_{\parallel}|^2 E_z(\mathbf{G}'_{\parallel}) &= \frac{\omega_n^2}{c^2} E_z(\mathbf{G}_{\parallel}) \\ \sum_{\mathbf{G}'_{\parallel}} \kappa(\mathbf{G}_{\parallel} - \mathbf{G}'_{\parallel}) (k_{\parallel} + \mathbf{G}_{\parallel}) \cdot (k_{\parallel} + \mathbf{G}'_{\parallel}) H_z(\mathbf{G}'_{\parallel}) &= \frac{\omega_n^2}{c^2} H_z(\mathbf{G}_{\parallel}) \end{aligned} \quad (2.25)$$

For each $k_{\parallel}$ in the first Brillouin zone, the sum over $\mathbf{G}_{\parallel}$ is truncated to a finite, but sufficiently large, plane-wave basis turning this into a standard Hermitian matrix eigenvalue problem, whose eigenvalues $\omega_n(k_{\parallel})$, sorted by band index n , are precisely the dispersion curves shown in Fig. 2.7. This is the formal justification, beyond the qualitative field-concentration argument given above, for why the TE and TM band structures of the same lattice can differ so markedly: they are eigenvalues of two structurally different operators, coupled to the geometry of $\varepsilon(\rho)$ in the same ways.

As already described, for propagation strictly within the xy -plane, corresponding to $k_z = 0$, the structure possesses mirror symmetry with respect to the xy -plane, see Fig. 2.5. This symmetry allows the electromagnetic modes to be separated into two independent polarization classes: **transverse-electric (TE)** and **transverse-magnetic (TM) modes**. For TE modes, the magnetic field is perpendicular to the plane of periodicity; while the electric field lies entirely in the xy -plane. The magnetic field can therefore be treated as a scalar quantity. For TM modes, the situation is actually reversed: the electric field is polarized along the z -direction, while the magnetic field lies in the xy -plane. An important consequence of this polarization separation is that the TE and TM modes can possess substantially different band structures, see Fig. 2.7.

In general, a given photonic crystal may exhibit a band gap for one polarization while remaining transparent for the other. As a representative example, consider a square lattice of dielectric columns with lattice constant a . The columns are assumed to be infinitely extended along the z -direction and embedded in air, see Fig. 2.6. For a square lattice, the first Brillouin zone has a square geometry. Due to the rotational and reflection symmetries of the lattice, the band structure can be conveniently represented along the boundary of the irreducible Brillouin zone, which is a triangular region. The

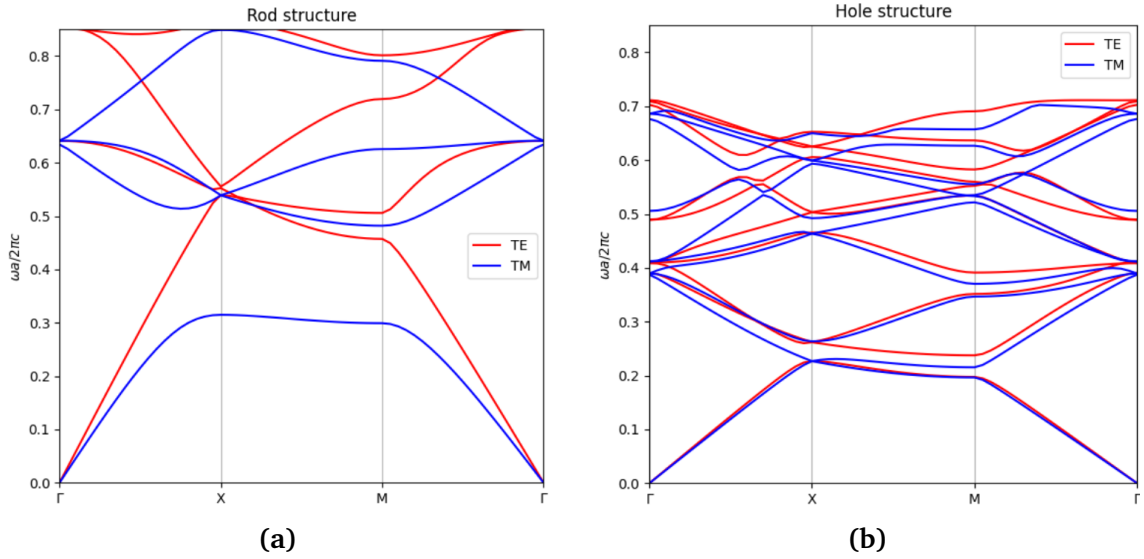


Figure 2.7: The photonic band structure calculated for a square array of dielectric columns with $r = 0.2a$ (a). The blue bands represent TM modes and the red bands represent TE modes. The irreducible Brillouin zone is a triangle of edges Γ, X, M , so the path in the reciprocal space is set as $\Gamma \rightarrow X \rightarrow M \rightarrow \Gamma$. In this specific case, alumina columns ($\epsilon_1 = 8.9$) are embedded in air ($\epsilon_2 = 1$). [6] (b) Same is done in for the complementary system, i.e., an alumina substrate patterned with a square array of air holes, with a radius of $r = 0.3a$, and keeping the same path in the reciprocal space for direct inspection and comparison.

three high-symmetry points of interest are denoted by Γ, X and M , whose coordinates are as follows:

$$(k_x, k_y) \left| \begin{array}{ccc} \Gamma & X & M \\ (0, 0) & (\frac{\pi}{a}, 0) & (\frac{\pi}{a}, \frac{\pi}{a}) \end{array} \right.$$

The band structure is therefore commonly plotted along the path $\Gamma \rightarrow X \rightarrow M \rightarrow \Gamma$. This representation is particularly useful because the limits of the photonic bands, which determine the boundaries of the band gaps, frequently occur at the edges or corners of the Brillouin zone. Although this is not a general mathematical requirement, it is a good approximation for many photonic crystal geometries of practical interest.

The spatial structure of the electromagnetic modes can be understood by examining the field distributions at the high-symmetry points of the Brillouin zone. At the Γ point, where $k_{\parallel} = 0$, the Bloch phase factor is unity. Consequently, the field distribution is identical from one unit cell to the next.

- At the X point, the phase changes by π between neighboring unit cells along the x -direction. The field therefore alternates in sign from one cell to the next, producing wave-fronts parallel to the y -direction.
- At the M point, the phase alternates along both x and y , resulting in a checker-

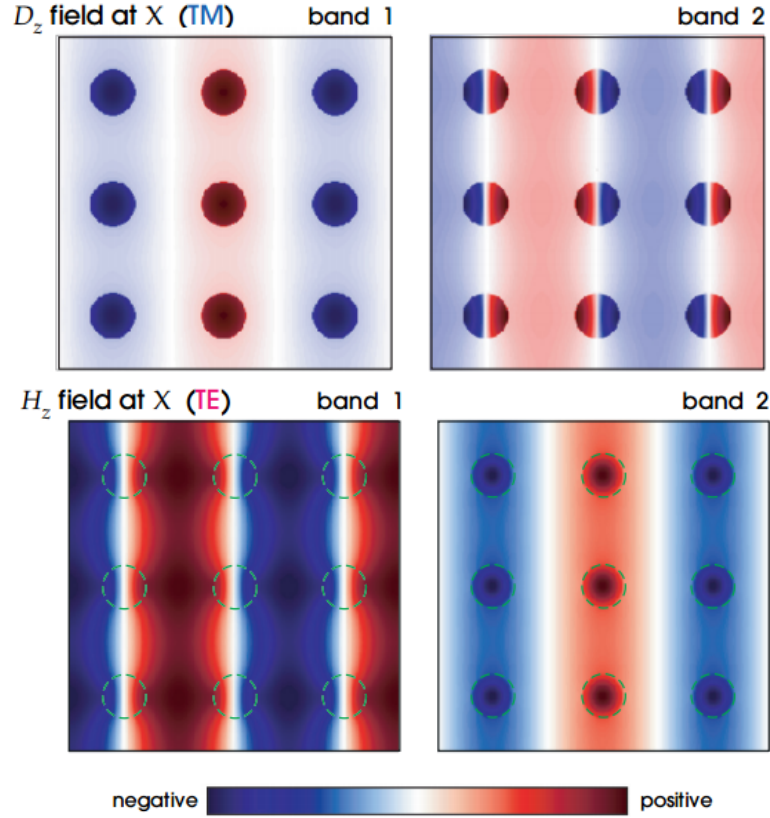


Figure 2.8: Displacement fields (upper figure) of TM states inside a square array of dielectric alumina ($\epsilon = 8.9$) columns in air, with radius $r = 0.2a$. The color indicates the amplitude of the displacement field, which points in the z direction. Magnetic fields (lower figure) of X-point for the same structure. TE states inside a square array of columns. The rod positions are indicated by dashed green outlines, and the color indicates the amplitude of the magnetic field, which points in the z direction. In both cases the dielectric band (band 1) is depicted on the left, while the air band (band 2) is on the right.[6]

board like field distribution.

Although the field patterns at X and M resemble propagating waves, these modes are in fact standing waves. Their group velocity vanishes at these high-symmetry points, meaning that they do not transport energy through the crystal with a non-zero group velocity.

Consider now a square lattice of alumina rods embedded in air, characterized by $\epsilon = 8.9$ with a normalized radius $r/a = 0.2$. The frequency is conveniently expressed in dimensionless form as

$$\frac{\omega a}{2\pi c}, \quad (2.26)$$

in which c is the speed of light in vacuum. The resulting TE and TM band structures exhibit markedly different behaviors. In particular, the TM modes display a complete band gap between the first and second bands, as already shown in Fig. 2.7 and Fig. 2.9. No analogous complete band gap is observed for the TE polarization in this system. The origin of this difference can be understood by examining the spatial distribution

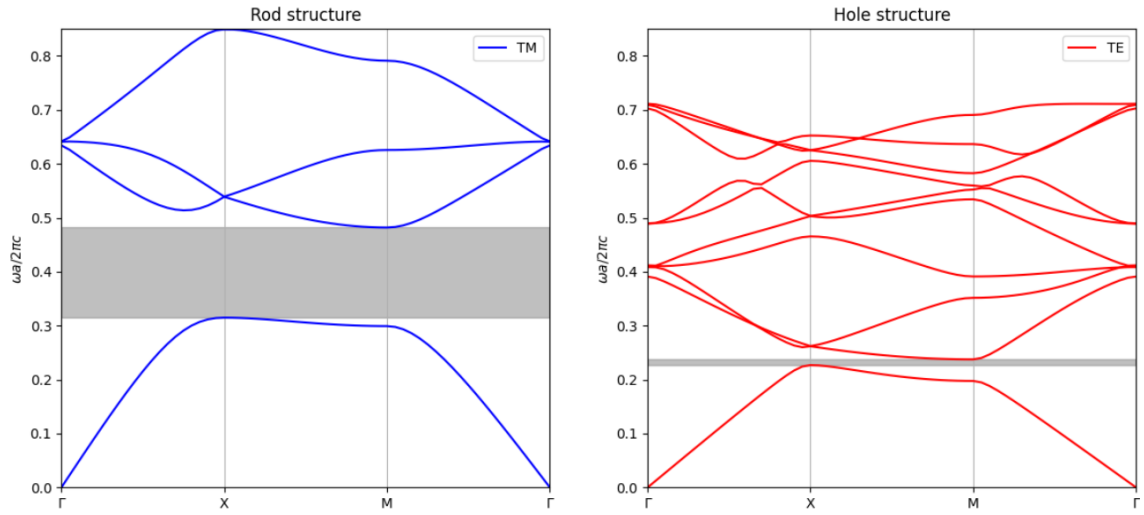


Figure 2.9: Photonic band structure for the TM modes of a square array of alumina columns ($\epsilon = 8.9$ and radius $r = 0.2a$) in air (left panel), directly compared to the photonic band structure for the TE modes of a square array of air holes ($r = 0.3a$) in an alumina substrate (right panel). Both cases show an examples of band gap (gray zone) specific only for one of the polarization modes.

of the electromagnetic fields associated with the first few bands. In fact, for the TM polarization, the mode belonging to the lowest-frequency band, often referred to as the dielectric band, is strongly concentrated within the high-permittivity dielectric rods. The second band, commonly referred to as the air band, exhibits a substantially different field distribution, see Fig. 2.8. In particular, a nodal region crosses the dielectric rods, significantly reducing the displacement-field amplitude within the high-permittivity material. The two modes therefore exhibit markedly different field concentration factors. The first mode efficiently localizes the electromagnetic field within the dielectric regions, whereas the second mode is forced to redistribute a larger fraction of the field into the surrounding air. This strong difference in field localization leads to a significant separation between the corresponding frequencies and, consequently, to the formation of a photonic band gap. The presence of isolated high-permittivity regions is therefore particularly favorable for the formation of TM band gaps. The situation is substantially different for the TE polarization. For such modes, it is convenient to examine the magnetic field component H_z , which is a scalar quantity in a two-dimensional structure. The electric displacement field D tends to attain large values near the nodal regions of the magnetic field.

In the square lattice of isolated dielectric rods, both of the lowest TE modes exhibit a significant displacement-field amplitude in the air regions. This behavior is a direct consequence of the topology of the dielectric material. In particular, the dielectric rods do not form a continuous path connecting neighboring lattice sites. Consequently, the electric displacement field cannot remain confined to the high-permittivity regions and is forced to cross the low-permittivity regions. As a result, the field concentration

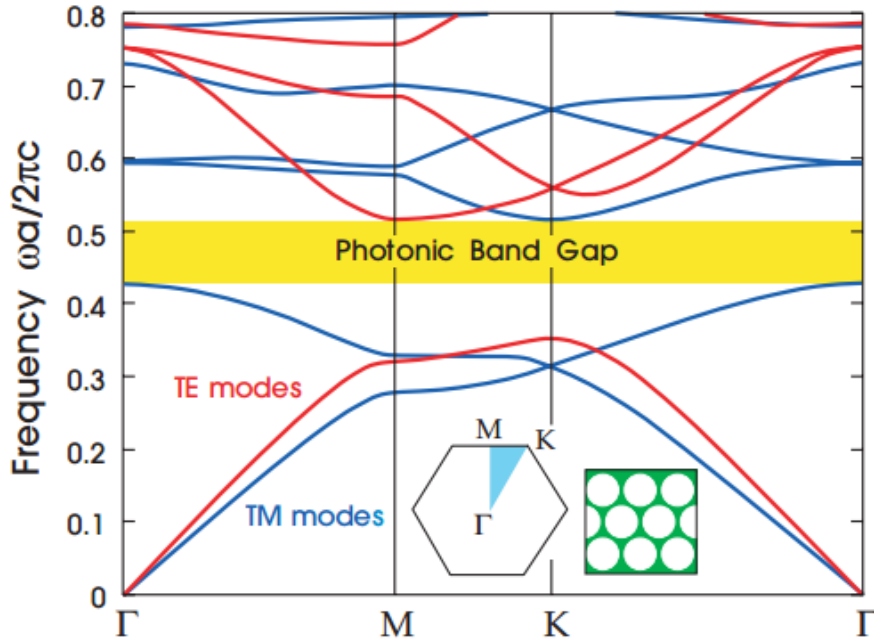


Figure 2.10: Example of photonic crystal structure that can show a **complete band gap**. The specific structure considered is a triangular lattice of air holes with $r/a = 0.48$ in a dielectric substrate ($\epsilon = 13$). [6]

factors of the first two TE modes are both relatively low and, more importantly, relatively similar. The resulting frequency separation is therefore insufficient to produce a complete TE photonic band gap. The absence of a TE band gap can thus be traced to the lack of connectivity of the high-permittivity material.

Such considerations lead to a useful qualitative rule for understanding the formation of photonic band gaps. For a lattice composed of isolated high-permittivity regions, such as the rod structure in Fig. 2.6, consecutive TM modes can exhibit substantially different field concentration factors, as shown in Fig. 2.8. The appearance of a nodal structure in the higher-frequency mode can significantly reduce its overlap with the dielectric material, favoring the opening of a TM band gap (see Fig. 2.9). Conversely, for TE modes, a connected high-permittivity network is advantageous. If the dielectric regions form continuous paths between neighboring lattice sites, the displacement field can remain predominantly within the high-permittivity material. The presence of an additional node in a higher-frequency mode can then produce a large change in field concentration and, consequently, a substantial frequency separation. The geometry and connectivity of the dielectric phase therefore play a fundamental role in determining which polarization exhibits a photonic band gap.

The previous discussion suggests that achieving a band gap for both polarizations requires a suitable compromise between dielectric isolation and connectivity. An effective strategy is to design a structure in which the high-permittivity regions are nearly isolated but remain connected through narrow dielectric channels. Such a geometry

combines the localization properties required to open TM gaps with the connectivity required to favor TE gaps. A representative example is a triangular lattice of air holes embedded in a high-permittivity dielectric medium. In this geometry, the low-permittivity air columns are arranged on a triangular lattice, see Fig. 2.10. When the radius of the air columns is sufficiently large, the remaining high-permittivity material forms localized regions between neighboring air columns. These regions resemble isolated dielectric sites while remaining connected through narrow dielectric necks. This combination makes it possible to open photonic band gaps for both polarizations within the same structure.

In particular for a triangular lattice of air columns with $r/a = 0.48$ and dielectric constant $\epsilon = 13$ the TE and TM band structures both exhibit photonic band gaps (Fig. 2.10). More importantly, the two gaps overlap in frequency. The resulting structure therefore possesses a **complete photonic band gap**, i.e., a frequency interval in which no propagating electromagnetic mode exists for either polarization. This result demonstrates that an appropriate combination of dielectric localization and connectivity can overcome the limitations of structures optimized for only one polarization. By carefully engineering the geometry of the dielectric phase, it is therefore possible to simultaneously satisfy the conditions required for the formation of TE and TM band gaps.

The properties of two-dimensional photonic crystals are strongly determined by the interplay between lattice symmetry, electromagnetic polarization, and the spatial distribution and connectivity of the dielectric material. As we have shown so far, for a square lattice of isolated dielectric columns, the TM polarization can exhibit a large photonic band gap because the first and second bands have substantially different field concentration factors, and it does not generally produce a TE band gap because the electric displacement field is forced to cross the low-permittivity regions, as depicted in Fig. 2.8. On the other hand, as it is explicitly shown in Fig. 2.9, its complementary structure, i.e., the square array of air holes in a high-dielectric constant substrate, shows a band gap for the TE modes, as a consequence of having a connected high-permittivity medium, and no TM band gap. So, by combining these two geometrical characteristics within the same structure, it is possible to obtain band gaps for both polarizations.

Thus, the design of a two-dimensional photonic crystal can be viewed as an optimization problem in which the geometry and connectivity of the dielectric phase are engineered to control the localization of the electromagnetic fields. By tailoring these properties, one can manipulate the relative positions of the TE and TM bands and, ultimately, design the desired photonic band structure and band-gap properties.

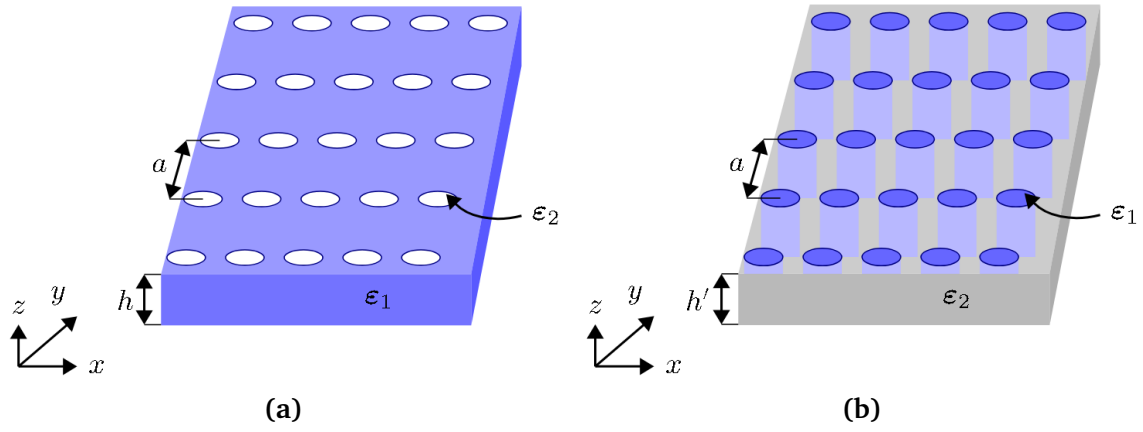


Figure 2.11: Examples of two complementary configurations for photonic crystal slabs. (a) Dielectric slab (ϵ_1) of thickness h perforated with a square array of air holes (radius r) and lattice constant a . (b) Photonic slab of thickness h' characterized by having a square array of dielectric cylindrical columns (ϵ_1) with radius r' suspended in air.

2.3 Photonic Crystal Slabs

We now take into consideration structures with two-dimensional periodicity but finite thickness along the orthogonal direction. These hybrid structures are commonly defined as **photonic-crystal slabs**, or **planar photonic crystals** (see schematic illustrations in Fig. 2.11).

Although they resemble two-dimensional photonic crystals, these systems are fundamentally different: the finite thickness in the vertical direction introduces qualitatively new behavior. As in three-dimensionally periodic photonic crystals, defects in photonic-crystal slabs can be used to form waveguides and cavities exploiting hybrid confinement mechanisms, due to total internal reflection along the vertical direction in addition to the interference mechanisms in the plane (see Fig. 2.12). These building blocks have enabled the experimental realization of many interesting optical devices using standard lithographic techniques based on two-dimensional patterns. This relative ease of fabrication, however, comes at the cost of requiring careful design to minimize losses associated with cavities and other disruptions of the periodicity. We will consider two basic geometries: first a square lattice of dielectric rods in air, referred to as the **rod slab**; second a square lattice of air holes in a dielectric slab, referred to as the **hole slab**, see Fig. 2.11. We initially consider suspended membrane structures surrounded entirely by air and later discuss the effect of introducing a substrate.

Because the structures possess discrete translational symmetry in two directions, the in-plane wave vector $k_{\parallel} = (k_x, k_y)$ is conserved, whereas the vertical wave vector k_z is not. As before, it is useful to plot the projected band structure, which represents ω as a function of $k_{\parallel}$ within the irreducible Brillouin zone of the two-dimensional lattice. The extended modes propagating in air form a **light cone**, defined by $\omega \geq c|k_{\parallel}|$, as properly shown before in Fig. 2.3. Below the light cone, the higher dielectric constant

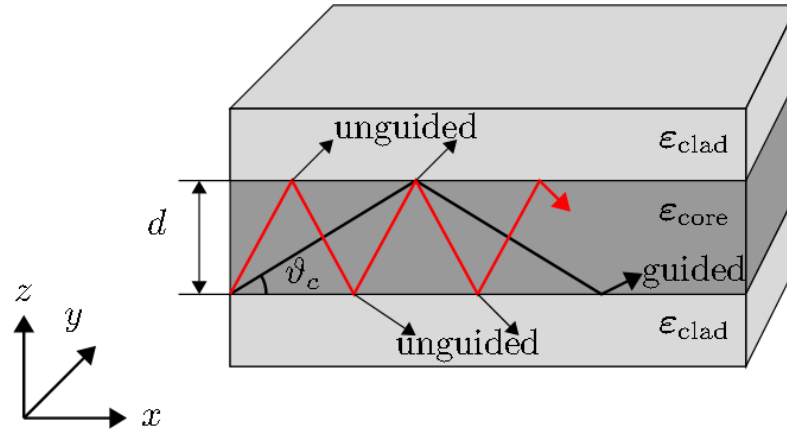


Figure 2.12: Cross-section of a uniform slab of thickness d and dielectric constant ϵ_{core} sandwiched between two dielectric materials with ϵ_{clad} above and below the core layer. The cladding dielectric constants can be, in general, different. The line inside the slab indicate the guided (**Black**), that fully propagate through the slab, and the unguided (**red**) light ray, which instead show radiation leaks in the cladding.

of the slab pulls down *discrete guided bands*. The eigenstates associated with these bands decay exponentially in the vertical direction.

2.3.1 Guided-mode solution and frequency dispersion

The origin of the light-cone condition $\omega \geq c|k_{\parallel}|$, and of the exponential vertical decay of guided modes, can be made explicit by solving the eigenvalue problem for the simpler case of a *uniform* slab of dielectric constant ϵ_b and thickness d , suspended in air and translationally invariant in x and y , see Fig. 2.12. Because of the mirror symmetry about the mid-plane $z = 0$, the guided modes separate into TE and TM types, each further split into modes symmetric or antisymmetric under $z \rightarrow -z$. Taking the propagation direction along x without loss of generality, the symmetric TE mode has an electric field of the form

$$\mathbf{E}_1(\mathbf{r}) = E_{1y} \hat{y} \cos(k_z z) e^{i(k_{\parallel} x - \omega t)}, \quad |z| < d/2,$$

where the in-plane and vertical wave numbers inside the slab are related to ω by the dispersion relation of a uniform medium of index $\sqrt{\epsilon_b}$,

$$\omega^2 = \frac{c^2}{\epsilon_b} (k_{\parallel}^2 + k_z^2).$$

Outside the slab ($|z| > d/2$) the field must remain finite as $z \rightarrow \pm\infty$ for the mode to be guided, so it is taken as an evanescent wave (Fig. 2.12),

$$\mathbf{E}_2(\mathbf{r}) = E_{2y} \hat{y} e^{ik_{\parallel} x - \kappa|z| - i\omega t}, \quad \kappa > 0,$$

with κ related to ω and $k_{\parallel}$ by the free-space wave equation,

$$\omega^2 = c^2 (k_{\parallel}^2 - \kappa^2) \iff \kappa^2 = k_{\parallel}^2 - \frac{\omega^2}{c^2}. \quad (2.27)$$

Note that this last relation only admits a real, positive κ when $\omega < c k_{\parallel}$. Imposing continuity of the $\mathbf{E}$ and $\mathbf{H}$ fields at $z = d/2$ yields a homogeneous 2×2 linear system for (E_{1y}, E_{2y}) , whose determinant must vanish for a non-trivial solution to exist. This condition gives the transcendental dispersion relations

$$\begin{aligned} \kappa \cos\left(\frac{k_z d}{2}\right) - k_z \sin\left(\frac{k_z d}{2}\right) &= 0 \quad (\text{TE, symmetric}) \\ \kappa \sin\left(\frac{k_z d}{2}\right) + k_z \cos\left(\frac{k_z d}{2}\right) &= 0 \quad (\text{TE, antisymmetric}) \end{aligned} \quad (2.28)$$

and, exchanging the roles of $\mathbf{E}$ and $\mathbf{H}$, the analogous pair of equations for TM modes (with an extra factor ε_b multiplying κ). For fixed $k_{\parallel}$, each branch $j = 0, 1, 2, \dots$ of these transcendental equations defines one so called guided band $\omega_j(k_{\parallel})$; this is precisely the set of discrete guided bands referred to above, and the four families (TE/TM, symmetric/antisymmetric) correspond to the four lowest branches plotted against $k_{\parallel}$. The boundary of the guided-mode region, i.e. the *light line*, is recovered as the limiting case in which the evanescent decay outside the slab vanishes, $\kappa \rightarrow 0$. Setting $\kappa = 0$ in $\omega^2 = c^2(k_{\parallel}^2 - \kappa^2)$ immediately gives

$$\omega = c k_{\parallel},$$

which, promoted to two in-plane dimensions $k_{\parallel} \rightarrow |\mathbf{k}_{\parallel}|$, is exactly the light-cone condition $\omega = c|\mathbf{k}_{\parallel}|$ quoted above. This also clarifies its physical meaning: for $\omega < c|\mathbf{k}_{\parallel}|$ (below the light line) κ is real and the mode decays exponentially away from the slab, a truly guided, lossless mode; for $\omega > c|\mathbf{k}_{\parallel}|$ (above the light line) κ becomes purely imaginary, the assumed evanescent solution turns into an outgoing radiating wave in air, and the "mode" is no longer a true bound eigenstate but a leaky resonance with complex frequency, as graphically depicted in Fig.2.12. Substituting $\kappa = 0$ back into the transcendental equations above also fixes the frequencies at which each guided branch touches the light line, e.g. $\omega = 2j\pi c / (d\sqrt{\varepsilon_b - 1})$ for the symmetric TE family, showing explicitly why only the lowest one or two branches remain guided down to $k_{\parallel} \rightarrow 0$ while higher branches acquire a finite cut-off frequency.

When the uniform slab is replaced by the actual periodic photonic-crystal slab, the same qualitative picture survives thanks to Bloch's theorem in the plane: $k_{\parallel}$ is still a good quantum number (restricted to the two-dimensional irreducible Brillouin zone), the light cone $\omega \geq c|\mathbf{k}_{\parallel}|$ is unchanged because it only depends on the properties of the surrounding air, and the periodic in-plane modulation of ε folds the uniform-slab

guided bands back into the Brillouin zone and opens gaps between them wherever bands of the same symmetry would otherwise cross, exactly as the plane-wave expansion opens gaps in the purely two-dimensional case. This is the mechanism by which the rod slab and the hole slab develop a TM-like and a TE-like guided-mode gap, respectively, below their own light cone.

For a suspended slab, the system is invariant under reflection through the $z = 0$ plane, see Fig. 2.5. Consequently, the modes can be classified as **TE-like (even)** or **TM-like (odd)** with respect to this *reflection symmetry*. In general, however, there are no corresponding x - or y -mirror symmetries because these are broken by the in-plane wave vector, $k_{\parallel}$. The rod slab favors a TM-like band gap, whereas the hole slab favors a TE-like band gap, see Fig. 2.13, as expected from the behavior of the corresponding two-dimensional photonic crystals. However these gaps are incomplete: they apply only to the guided modes and do not include the modes inside the light cone.

An important observation lies in the reason why the rod slab was chosen to be nearly four times thicker than the hole slab. This can be understood by considering the limiting cases of very thin and very thick slabs. If the slab is too thin, the bands are only weakly guided, if they are guided at all. States lying just below the light line decay very slowly into the surrounding air. Consequently, the frequency separation between the lowest guided band and the light line becomes too small to support a substantial gap. The modes also become highly disperse, making the underlying periodicity less important. In contrast, for a very thick slab, the band gap of the fundamental, lowest energy, guided mode approaches that of the corresponding infinite two-dimensional system. However, higher-order modes with additional vertical nodes are progressively pulled down into the gap. Within the framework of the variational theorem, the energetic cost associated with introducing additional vertical nodes becomes increasingly small. These considerations suggest that the ideal slab thickness should be approximately half a wavelength. Such a thickness is sufficient to confine the fundamental mode effectively while remaining thin enough to prevent higher-order modes from fitting inside the slab.

The relevant wavelength is not simply the wavelength in air or in the dielectric. Instead, it lies somewhere between the two because the effective dielectric constant is mode-dependent. It depends on a spatial average of the dielectric profile weighted by the electromagnetic field. The precise treatment of this weighting belongs to effective-medium theory and depends strongly on the polarization. For TE-like modes, the effective vertical wavelength is determined predominantly by the regions of high dielectric constant, whereas for TM-like modes the low-dielectric regions play a more important role. In practice, the limiting factor in slab structures is not usually the size of the band gap itself, so it is unnecessary to choose a strictly optimal thickness. Instead, the design is constrained by several other considerations, including the light cone, higher-

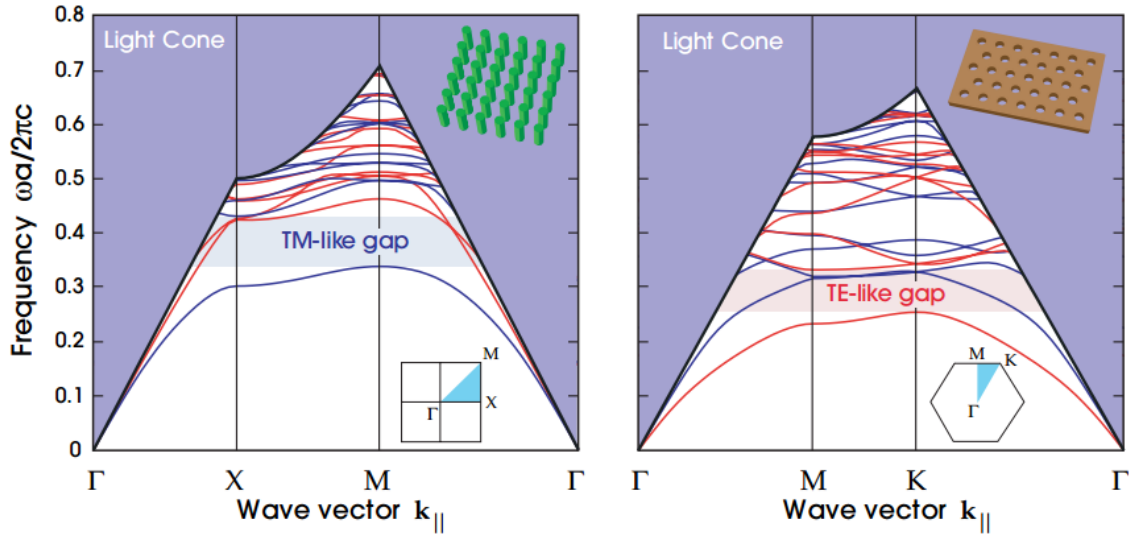


Figure 2.13: Band diagrams for photonic crystal slabs suspended in air. The rod slab (left), consisting of dielectric columns of height h suspended in air, and the hole slab (right). The blue shaded area is the light cone, all of the extended modes propagating in air. Below it are the guided bands localized to the slab: blue/red bands indicate TM/TE-like modes.[6]

order defect modes, and the presence of substrates. Therefore, a smaller thickness is often preferred because it is easier to fabricate while still providing a sufficiently large band gap. This explains the choice of the experimentally convenient parameters used here. Nevertheless, a substantial difference in thickness is required between the rod and hole slabs. This has contributed to making hole-slab structures particularly attractive for experimental applications.

A waveguide mode can be created by introducing a linear defect into the periodic structure. In the rod slab, for example, this can be achieved by reducing the radius of all the rods along a particular row. The *projected band diagram* can then be calculated for different values of the reduced rod radius, considering only TM-like modes. The extended modes of the crystal occupy the dark-blue region, while the band gap appears in the light-blue region. Within the band gap, a single guided band appears for each choice of rod radius, corresponding to a TM-like waveguide mode localized around the line defect. This mode cannot couple to the extended modes of the crystal because its frequency lies inside the photonic band gap. Nor can it couple to extended modes in air because it lies below the light line. Therefore, even though the band gap is incomplete, the waveguide mode remains guided because k_x is conserved.

A fundamental difference between a photonic-crystal slab and a truly two-dimensional photonic crystal is therefore the presence of the light cone. This imposes an important constraint on the design of waveguides. In particular, we cannot simply remove an entire row of rods and expect a guided mode to remain.

As in the previous section, the waveguide supports a series of guided modes that are confined horizontally by the *photonic band gap* and vertically by *index guiding*. How-

ever, there is a second category of guided modes. These arise because the waveguide has a higher average dielectric constant than the surrounding air. The corresponding guided bands lie below the extended modes of the crystal and are index-guided in all directions. Since all the guided modes considered here are TE-like and are fundamental, meaning that they have no nodes in the z direction, they can be visualized by plotting H_z in the $z = 0$ plane.

The system is also invariant under reflection about the y plane. Therefore, the modes can be classified as either even or odd under this reflection. It is important to remember, however, that the magnetic field $\mathbf{H}$ is a pseudovector. Consequently, an H_z field pattern that appears even under reflection in the y direction is actually odd under that symmetry operation.

The discussion so far has focused on *photonic-crystal slabs* suspended in air. Such structures are special for two reasons: they maximize the refractive-index contrast between the slab and its surroundings, and they possess reflection symmetry about the $z = 0$ plane. Placing the slab on a substrate breaks this reflection symmetry. As a result, TE-like and TM-like modes can couple to one another, eliminating any band gap that exists exclusively for one polarization. Consequently, when a photonic-crystal slab containing a linear waveguide is placed on a substrate, waveguide modes that were previously confined by the band gap can become leaky. A formerly guided TE-like mode can couple to extended TM-like modes, and vice versa, causing the mode to radiate away within the plane of periodicity.

One possible way to reduce this polarization mixing is to etch the periodic pattern into the substrate as well as into the slab, in a configuration similar to the monorail substrate. Alternatively, the z -symmetry can be partially restored by depositing a superstrate on top of the slab with a refractive index similar to that of the substrate.

Even in a symmetric structure, the guided modes within the band gap and below the light cone typically lie relatively close to the edge of the Brillouin zone. As the bands flatten near this edge, the **group velocity** approaches zero. At the same time, the group-velocity dispersion, which is related to

$$\frac{dv_g}{d\omega}$$

becomes very large near the Brillouin-zone edge. This quantity determines how rapidly optical pulses spread during propagation. It is conventional to define the dispersion parameter as

$$D = -\frac{2\pi c}{v_g^2 \lambda^2} \frac{dv_g}{d\omega}$$

This parameter describes the pulse spreading per unit bandwidth and per unit propagation distance. Therefore, the waveguide modes may exhibit both low group velocity

and strong dispersion over a significant portion of their bandwidth. In some applications, this behavior is desirable.

For example, a low group velocity, commonly referred to as slow light, can enhance the effects of *optical nonlinearities*. A waveguide mode in this structure is not fundamentally very different from the mode of an isolated dielectric strip. Nevertheless, it remains useful as a channel for accessing point defects and other optical devices embedded within the photonic crystal.

Although the waveguide modes of a perfect and symmetric slab are theoretically lossless, practical structures inevitably exhibit some degree of loss. In addition to the substrate losses discussed above, losses can arise from material absorption and from radiative scattering caused by fabrication imperfections, commonly referred to as fabrication disorder. Disorder breaks the translational symmetry of the slab and therefore allows waveguide modes to couple to states with different values of the wave vector k . It can also produce reflections into the counter-propagating waveguide mode at $-k$. Although a detailed analysis of disorder is beyond the scope of this discussion, a simple scaling relation can be derived for modes near the $v_g = 0$ band edge.

2.4 Guided Mode Expansion method

The **guided-mode expansion (GME)** is a numerical method for solving Maxwell equations in structures that are periodic in the in-plane directions but *finite*, or layered, along the vertical direction, such as photonic crystal (*PhC*) slabs, *PhC* waveguides, and *PhC* slab cavities [20]. It is a frequency-domain method based on the separation of the three-dimensional electromagnetic problem into two simpler ones: the in-plane dynamics, where the structure is periodic, and the confinement along the out-of-plane z direction.

First formulated for a three-layer patterned waveguide [1], and later generalized to an arbitrary multilayer stack [11], the GME approach has become a standard tool for the rapid and accurate calculation of photonic mode dispersion and radiative (perturbative) losses in *PhC* slabs. Its efficiency, compared to fully numerical methods such as the *finite-difference time-domain (FDTD)* or the *Fourier modal method*, is especially evident when the photonic modes under study have small radiative losses, i.e. a large quality factor. The implementation of GME used in this thesis is a modified version of the open-source Python package *legume* [11, 20].

After a general introduction to the method, we discuss the symmetry properties of the photonic modes in *PhCs*, exploiting the possibility offered by GME to separate modes according to their mirror symmetry with respect to a vertical plane.

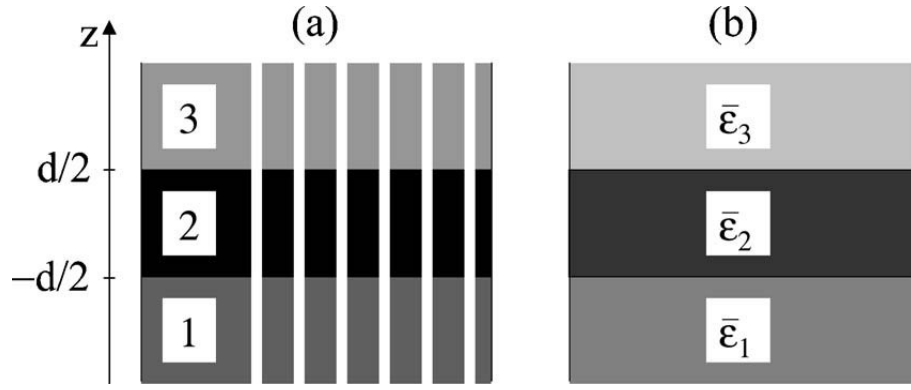


Figure 2.14: Schematic picture of the vertical waveguide structure for (a) the photonic crystal slab and (b) the effective waveguide. The lower and upper claddings, respectively layers 1 and 3 are taken to be semi-infinite. The change in gray scales in going from (a) to (b) suggests that the effective waveguide is characterized by an averaged dielectric constant.[1]

Formulation of the eigenvalue problem

GME relies on an approximation that restricts its applicability to a specific, though broad, type of structures. The dielectric function must be periodic in the xy plane, a condition that can sometimes be relaxed by means of the supercell technique, which effectively allows the study of *PhC* waveguides and *PhC* slab cavities [1, 20]. As a typical example, we consider a *PhC* slab, two instances of which are shown in the top panels of Fig. 2.11. The two cross-section views show how the slab has a finite thickness along z , while the in-plane pattern creates a periodic dielectric modulation for the photons.

Given this geometry, the dielectric function is decomposed into an in-plane, $\boldsymbol{\rho} = (x, y)$, and an out-of-plane, z , dependent part, $\varepsilon(\mathbf{r}) = \varepsilon(\boldsymbol{\rho}, z)$. Maxwell's equations can then be cast as an eigenvalue problem for the magnetic field $\mathbf{H}$, with harmonic time dependence $\mathbf{H}(\mathbf{r}, t) = \mathbf{H}(\mathbf{r})e^{-i\omega t}$, see Eq. (2.8). Where the eigenvalues $(\omega/c)^2$ of the Hermitian operator give the squared frequencies of the electromagnetic eigenmodes. In the GME method, the solutions of Eq. (2.8) are expanded as

$$\mathbf{H}_{\mathbf{k}}(\mathbf{r}) = \sum_{\mathbf{G}, \alpha} c_{\mathbf{g}, \alpha} \mathbf{H}_{\mathbf{g}, \alpha}^{\text{guid}}(\mathbf{r}) = \sum_{\mathbf{G}, \alpha} c_{\mathbf{g}, \alpha} \mathbf{h}_{\mathbf{g}, \alpha}^{\text{guid}}(z) e^{i(\mathbf{k} + \mathbf{G}) \cdot \boldsymbol{\rho}} \quad (2.29)$$

where $\mathbf{k} = (k_x, k_y)$ is the in-plane wavevector, restricted to the first Brillouin zone, the sum runs over the reciprocal lattice vectors $\{\mathbf{G}\}$ with $\mathbf{g} = \mathbf{k} + \mathbf{G}$, and $\{\alpha\}$ labels the guided modes used as basis functions. The functions $\mathbf{h}_{\mathbf{g}, \alpha}^{\text{guid}}(z)$ are the normalized profiles of the guided modes of an *effective* homogeneous waveguide, obtained by replacing each layer j of the real structure with an effective layer of a spatially averaged dielectric constant $\bar{\varepsilon}_j$; as shown in Fig. 2.14. It can be demonstrated that the results are essentially independent of this choice. Each guided mode is further labeled by a

polarization index $p = (\text{TE}, \text{TM})$ and a band index. Because the in-plane patterning mixes TE and TM polarizations, as a consequence, *PhC* slab modes cannot in general be labeled as purely TE or TM.

Now let's group all quantum numbers $(\mathbf{g}, \alpha)$ into a single index μ , and imposing the normalization $(\mathbf{H}_\mu, \mathbf{H}_\nu) = \delta_{\mu\nu}$ with respect to the scalar product over the simulated unit cell S , the eigenvalue problem of Eq. (2.8) takes the matrix form:

$$\sum_{\nu} \mathcal{H}_{\mu\nu} c_{\nu} = \lambda c_{\mu} \quad (2.30)$$

with $\lambda = \omega^2/c^2$ and matrix elements

$$\mathcal{H}_{\mu\nu} = (\mathbf{H}_\mu, \hat{\Theta} \mathbf{H}_\nu) \quad (2.31)$$

Once the eigenvalues and eigenvectors of $\mathcal{H}$ are found, the frequencies of the *PhC* are directly obtained, while the eigenvectors give the expansion coefficients c_{μ} needed to reconstruct the field profile through Eq. (2.29). Since $\mathcal{H}$ is Hermitian, all the frequencies $\omega_{\mathbf{k}}$ are real, and no information on radiative losses is available at this stage.

The coupling between *PhC* modes lying above the light line and the radiative modes propagating in the **semi-infinite claddings** is obtained by means of time-dependent perturbation theory, which is justified because the imaginary part of the frequencies is typically much smaller than the real part, $\Im(\omega)/\Re(\omega) \ll 1$. The imaginary part of the eigenvalue $\omega_{\mathbf{k}}^2/c^2$ is then defined as:

$$-\Im\left(\frac{\omega_{\mathbf{k}}^2}{c^2}\right) = \pi \sum_{\mathbf{G}', \lambda} \sum_{o=l, u} |\mathcal{H}_{\mathbf{k}, r}|^2 \rho_o(\mathbf{k} + \mathbf{G}', \omega_{\mathbf{k}}) \quad (2.32)$$

where the **Heaviside function** implicit in the density of states $\rho_o(\mathbf{g}, \omega)$ guarantees that no coupling to radiative waves occurs below the light lines of the claddings, where the field is confined by total internal reflection.

Reciprocal lattice truncation

In principle, the basis of Eq. (2.29) should contain an infinite number of reciprocal lattice vectors $\mathbf{G}_i$; in practice, the sum must be truncated at a finite number $N_{\mathbf{G}}$. Any reasonable truncation must converge to the same result as $N_{\mathbf{G}} \rightarrow \infty$, but for a finite basis the choice of truncation rule matters, especially when the symmetry properties of the modes are of interest. As such a truncation that does not respect the symmetry of the structure can lead to incorrect predictions about the modes.

Two truncation rules are commonly used [20]:

- **"abs"**: a circular cut in reciprocal space, keeping all vectors with $|\mathbf{G}_i| \leq G_{\max}$;

- **"tbt"**: a parallelogram-shaped cut, which does not in general preserve the point-group symmetry of the lattice, but produces matrices in Toeplitz form that can be exploited to speed up simulations with a large number of plane waves.

Since the symmetry separation discussed below requires that every $\mathbf{G}_i$ in the basis be mapped onto another vector of the same basis under the relevant mirror operation, the "abs" truncation is the natural choice whenever the symmetry properties of the modes need to be resolved, and is the one adopted throughout this thesis. The in dept description of the truncation methods is done in Chapter 4.

Symmetry separation of the GME matrix

The guided modes used as the basis of the expansion, being eigenmodes of an effective homogeneous waveguide, are invariant under reflection with respect to *any* vertical plane containing the guided-mode wavevector $\mathbf{g}$. In a patterned *PhC* slab, however, the in-plane modulation reduces the set of vertical mirror planes of the structure to a finite number, fixed by the point group of the lattice, e.g. C_{4v} for a square lattice of circular holes, C_{6v} for a triangular one.[18, 19] To this end, symmetry operations that leave at least one point fixed must be considered, i.e., proper and improper rotations, the latter including spatial inversion and mirror reflection.

Only when the in-plane wavevector, $\mathbf{k}$, points along one of these high-symmetry directions can the corresponding photonic eigenmodes be classified as even or odd with respect to the corresponding mirror symmetry. To separate even and odd modes with respect to a chosen vertical symmetry plane, it is sufficient to rewrite the matrix elements $\mathcal{H}_{\mu\nu}$ of Eq. (2.31) in a suitable basis. To this end, let us consider the reflection operator with respect to a plane parallel to the z -axis, when rotated by an angle θ from the x -axis, it has the explicit form:

$$\hat{\sigma}(\theta) = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) & 0 \\ \sin(2\theta) & -\cos(2\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \det \hat{\sigma}(\theta) = -1 \quad (2.33)$$

If the plane defined by θ is a genuine symmetry plane of the structure, every reciprocal lattice vector $\mathbf{G}_i$ in the truncated "abs" basis is mapped by $\hat{\sigma}(\theta)$ onto another vector $\mathbf{G}_j$ of the same basis (see Fig. 2.5): some vectors are mapped onto themselves, others exchange pairwise. This information is used to build a change-of-basis matrix C that block-diagonalizes $\mathcal{H}$ into an even and an odd sector. The procedure can be summarized as follows:

- initialize a matrix C , of the same dimension $(N_G \cdot N_\alpha) \times (N_G \cdot N_\alpha)$ as $\mathcal{H}_{\mu\nu}$, filled with zeros;

- compute the reflection matrix $\hat{\sigma}(\theta)$ of Eq. (2.33) for the chosen symmetry plane and verify that every $\mathbf{G}_i$ is mapped onto another vector $\mathbf{G}_j$ of the basis, for each guided mode α_k ;
- if $\mathbf{G}_i$ is mapped onto itself, set $C_{a,a} = 1$ with $a = (k-1)N_G + i$;
- if $\mathbf{G}_i$ is mapped onto $\mathbf{G}_j$ with $i \neq j$, set $C_{a,a} = -1/\sqrt{2}$, $C_{b,b} = C_{a,b} = C_{b,a} = 1/\sqrt{2}$, with $a = (k-1)N_G + i$ and $b = (k-1)N_G + j$.

The resulting linear combinations, given by

$$\mathbf{h}_{\mathbf{g}_i, \alpha}^{\text{guid}}(z) \pm \mathbf{h}_{\mathbf{g}_j, \alpha}^{\text{guid}}(z), \quad (2.34)$$

are either even (+) or odd (−) under the reflection. To obtain the correct parity of the full photonic eigenmode, however, one must also account for the polarization of the guided mode: a TE guided mode is odd under the reflection, while a TM guided mode is even, so that the final parity is given by the product of the parities in the $\{\mathbf{G}\}$ and $\{\alpha\}$ bases. Once C has been constructed, the Hamiltonian is rewritten in the symmetrized basis as $C^{-1} \mathcal{H} C$, which is block-diagonal, up to numerical noise, in the even and odd sectors.

As a numerical check, the off-diagonal blocks of $C^{-1} \mathcal{H} C$ should vanish identically; in practice they are small but finite due to floating-point arithmetic, and a threshold of 10^{-8} is typically adopted to confirm that the symmetry separation has been achieved correctly. Larger structures (more layers, larger G_{max}) can produce larger numerical residuals, in which case the threshold can be increased accordingly.

To illustrate the procedure outlined above, we computed all matrix elements $\mathcal{H}_{\mu\nu}$ for two representative structures, with the results shown in Fig. 2.15: a suspended membrane and a slab on a glass substrate, respectively, both patterned with a square lattice of circular air holes ($r/a = 0.2$) etched in a core of refractive index $n = 3.54$ and thickness $d/a = 0.5$. Before the change of basis, no separation between even and odd modes is visible in the plot.

For the suspended structure, vertically symmetric membrane an additional block diagonal structure is already present, due to the extra reflection symmetry with respect to the horizontal (xy) mid-plane of the core, which is absent in the asymmetric structure with the glass substrate. After the change of basis, both structures display two fully decoupled diagonal blocks, corresponding to the even and odd sectors with respect to the vertical mirror plane, which can then be diagonalized separately to obtain the corresponding photonic bands, along the high-symmetry path $M \rightarrow \Gamma \rightarrow X$. Note that the symmetry plane changes discontinuously at Γ ($\theta = 0^\circ$ along $\Gamma \rightarrow X$, $\theta = 45^\circ$ along

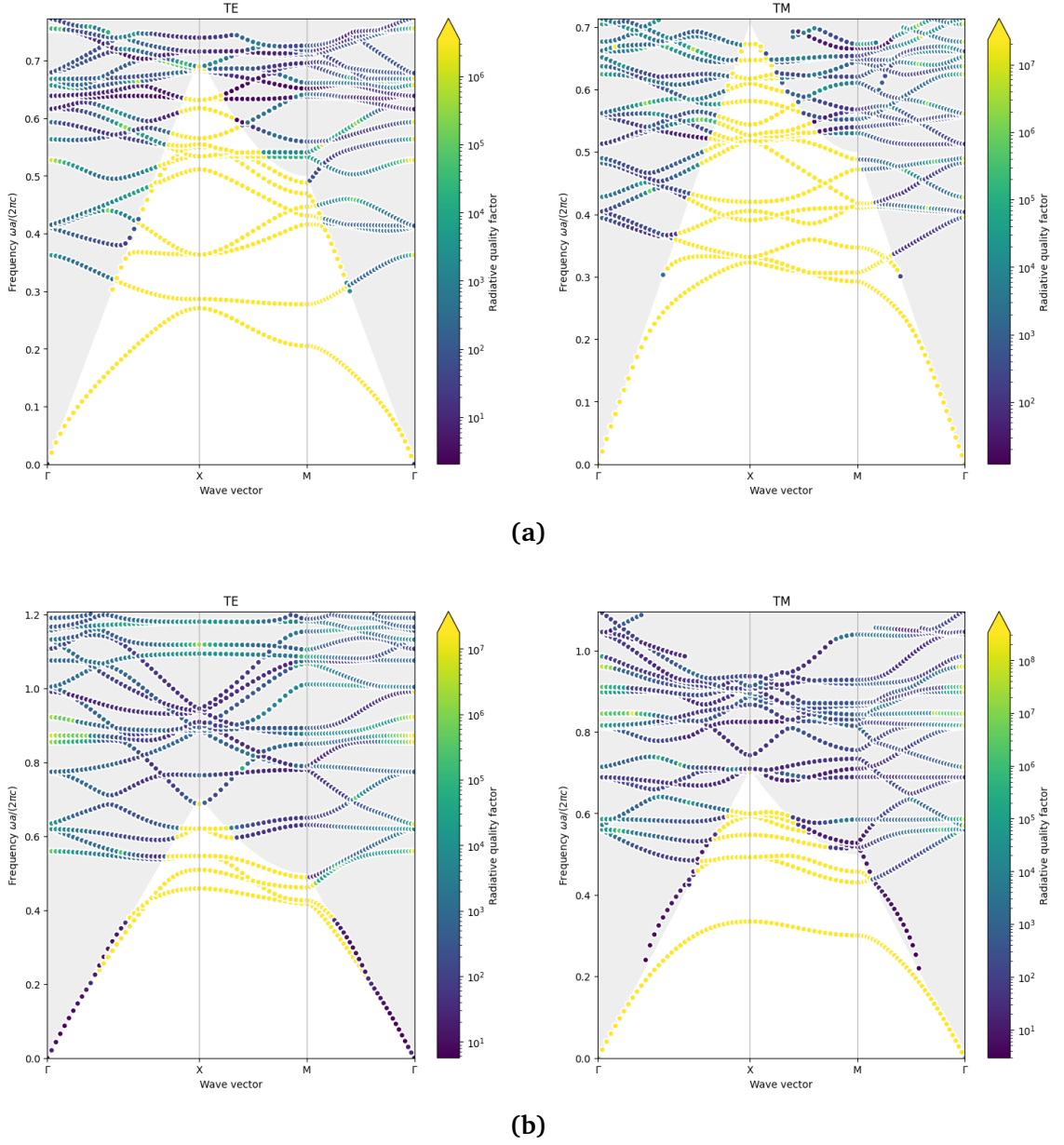


Figure 2.15: Band structure calculated for (a) silicon slab with a square lattice of air holes, and (b) silicon columns suspended in air. As it is mentioned in the text, the two slabs have different thickness and radius: for the hole case $h = 0.6a$ and $r = 0.3a$; while for the rod case $h' = 2a$ and $r' = 0.2a$. In the figure are reported the GME results (with $g_{\max} = 6$ and a base of 113 plane waves) for both even and odd modes, respectively. The color scale of each point corresponds to the quality factor of the eigenmode (in particular, yellow color identifies lossless photonic eigenmodes).

$\Gamma \rightarrow M$); as a consequence, a band that is even along one high-symmetry direction can be odd along another.

Numerical calculation of Stokes parameters within GME

In the original GME release, no calculation of Stokes parameters was available. It is an original contribution of this thesis to allow for the determination of the Stokes parameters within GME. In fact, GME allows for the calculation of photonic eigenmodes,

which in turn determine the definition of the following normalized Stokes vector components as functions of the Cartesian components of the displacement field (d_x, d_y):

$$\begin{aligned}\frac{S_1}{S_0} &= \frac{|d_x|^2 - |d_y|^2}{|d_x|^2 + |d_y|^2} \\ \frac{S_2}{S_0} &= 2 \frac{\text{Re}[(d_x)^* d_y]}{|d_x|^2 + |d_y|^2} \\ \frac{S_3}{S_0} &= \frac{2 \text{Im}[(d_x)^* d_y]}{|d_x|^2 + |d_y|^2},\end{aligned}\tag{2.35}$$

in which $S_3/S_0 = 0$ corresponds to purely linear polarization, while $S_3/S_0 = \pm 1$ corresponds to purely circular polarization, see Chapter 4.1.3 for details. In the literature, two independent numerical methods have been employed to extract these quantities: *finite-difference time-domain (FDTD)* simulations on one side, in which the near-field profile of a resonance excited by dipole emitters at each $\mathbf{k}$ is projected onto the far field, and **rigorous coupled-wave analysis (RCWA)** on the other side, from which the polarization field is obtained through *temporal coupled-mode theory (TCMT)*. [4, 20, 19] Within TCMT, the reflected/transmitted field is written as

$$\frac{dA}{dt} = (i\omega_0 - \gamma)A + \mathbf{K}^T \mathbf{a}, \tag{2.36}$$

$$\mathbf{b} = \mathbf{C}\mathbf{a} + \mathbf{D}\mathbf{A}, \tag{2.37}$$

where $\mathbf{a}$ ($\mathbf{b}$) is the incoming (outgoing) field, A is the resonant-mode amplitude, $\mathbf{K}$ and $\mathbf{D}$ are the in-coupling and out-coupling vectors, and $\mathbf{C}$ is a direct (non-resonant) scattering term. For a mirror-symmetric slab supporting a TE-like mode even under σ_{xy} , the polarization-conserving (TE–TE) reflectivity displays *Fano resonances*, since it contains both the resonant and the direct scattering contributions, While the polarization-mixing (TE–TM) reflectivity is purely resonant and shows nearly *Lorentzian peaks*. Using such consideration yields the complex products $d_s k_s$, $d_p k_s$ from which the Cartesian components d_x, d_y , and hence the Stokes parameters, are reconstructed. [18, 4]

The counterpart for the Stokes parameter S_3 , used for the validation of GME results against the two independent methods above, is **circular dichroism**. The circular dichroism observed in reflectivity, is determined by the coupling of the incident far field to the resonance *and* by the direct (non-resonant) scattering term of Eq. (2.37), as:

$$\text{CD}_R = \frac{R_{\text{lcp}} - R_{\text{rcp}}}{R_{\text{lcp}} + R_{\text{rcp}}} \tag{2.38}$$

interference between these two channels means that CD_R does not coincide exactly with S_3 , although the two quantities show qualitatively similar behavior, including a

change of sign as a function of incidence angle, with the sign reversal occurring at slightly different wavevectors for the two quantities. The correspondence becomes exact only for the CD measured in absorption or emission, which probes the resonant mode alone [3]. The details on the Stokes parameters formalism will be given in the next chapter, while the GME results will be reported in the last chapter of the thesis. The RCWA implementation used in this work to extract the CD, and benchmark the GME results, is the one reported in Ref. [9].

Chapter 3

Optical Analogue of a Minimal Altermagnetic Model

The purpose of this chapter is to formally establish the close analogies existing between photonic and electronic systems in terms of light polarization and the electron spin. Crucially, this analogy sets the basis to define a synthetic analog of altermagnetic systems.

As discussed in the previous chapters, photonic crystals, similarly to electronic crystals, exhibit a periodic structure characterized by the alternation of two refractive indices (n_1 and n_2). This periodicity gives rise to a band structure for the electromagnetic modes (photonic modes), analogous to the electronic band structure found in crystals. One could argue that several other physical systems, such as polaritons or excitons, could display analogous band structures and could therefore be employed as electronic analogues. However, the photonic platform is particularly appealing because it is scalable, experimentally reproducible, artificially engineered, and, unlike electronic systems, it does not rely on quantum physics. Having already discussed the structural and energetic similarities between photonic and electronic systems, we now seek a photonic degree of freedom capable of playing the role of the electronic spin. First, let us consider an electromagnetic wave. From the quantum-mechanical perspective, the photon is a spin-1 particle possessing two degrees of freedom when quantized in the Coulomb gauge, corresponding to the two transverse polarization states. Starting from this observation, the analogue of the electronic spin can be identified with the polarization state of light, or more precisely with the parameters describing its polarization. To make this correspondence explicit, we will hereby present two equivalent descriptions of the polarization state: the first based on real electromagnetic fields and trigonometric functions, and the second formulated in terms of complex fields. We then demonstrate that the algebra governing the transverse polarization is formally identical to the *spinor* algebra describing spin-1/2 particles.

3.1 Jones Calculus

We begin with the geometrical description of polarization states. Since the present work is restricted to two-dimensional systems, we limit the discussion to the plane waves.

Lets Consider a monochromatic plane wave propagating in vacuum or in a dielectric medium of permittivity $\varepsilon = \varepsilon_0 \varepsilon_r$ along the z direction, with angular frequency ω , wavevector $k = n\omega/c$, where $n = \sqrt{\varepsilon_r}$ is the refractive index. From classical electromagnetism, the polarized electric field can be written as

$$\mathbf{E}(z, t) = E_{0,x} \cos(kz - \omega t + \delta_x) \hat{x} + E_{0,y} \cos(kz - \omega t + \delta_y) \hat{y}, \quad (3.1)$$

where $(E_{0,x}, E_{0,y})$ denote the real transverse components of the electric field and (δ_x, δ_y) represent their respective phase factors.[10][5] By rewriting Eq. 3.1 in terms of the relative phase difference between the two components($\delta = \delta_y - \delta_x$) immediately yields the three fundamental polarization states:

- $\delta = 0$ and $\delta = \pi$ it yields linear polarization;
- $\delta = \pm \frac{\pi}{2}$ and $E_{0,x} = E_{0,y}$ it yields circular polarization;
- $\delta = \pm \frac{\pi}{2}$ and $E_{0,x} \neq E_{0,y}$, or $\delta \neq 0, \pi, \pm \frac{\pi}{2}$ it yields elliptical polarization.

Extending Eq. 3.1 to the complex representation, while adopting the convention that the physical electric field corresponds to the real part of the complex field, leads to the more general expression

$$\mathbf{E}(z, t) = [E_{0,x} \hat{x} + E_{0,y} \hat{y}] e^{i(kz - \omega t)} = E_0 \hat{\varepsilon} e^{i(kz - \omega t)} \quad (3.2)$$

In this formulation, the field components can be conveniently expressed in vector form, where $\hat{\varepsilon}$ denotes an arbitrary complex number. Within the complex formalism, the polarization states may therefore be reinterpreted as (see Fig. 3.1):

- $\hat{\varepsilon} = \cos \theta \hat{x} + \sin \theta \hat{y}$ linear polarization, characterized by an arbitrary polarization angle θ .
- circular polarization, where $\hat{\varepsilon}_+$ describes *positive helicity* states, corresponding to a clockwise rotation of the electric field and carrying a photon angular momentum of $+\hbar$; conversely, $\hat{\varepsilon}_-$ represents *negative helicity* states.
- elliptical polarization, whose principal axis lies along the x direction and is parameterized by $\hat{\varepsilon} = \cos \chi \hat{x} + i \sin \chi \hat{y}$. In this case, the electromagnetic wave exhibits *positive* or *negative helicity* depending on the sign of χ .

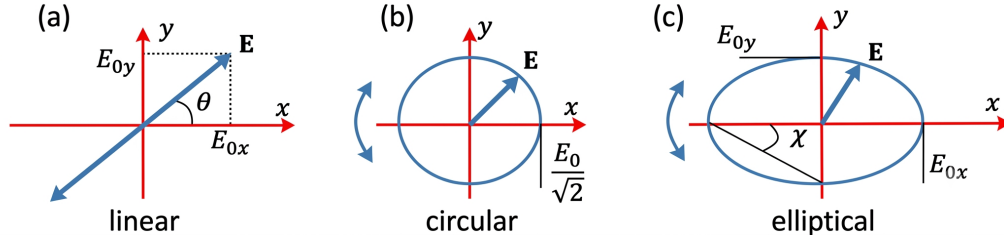


Figure 3.1: Schematic view of the principal polarization states of a light beam. (a) Linear polarization, (b) circular polarization, (c) elliptical polarization.

A brief remark on terminology is appropriate. The term *helicity* is sometimes used ambiguously, as it refers to the angular momentum projection of the photon. For this reason, many optics textbooks instead adopt the terms right-hand circular polarization (RCP) and left-hand circular polarization (LCP).

So far, polarization has been described in terms of real electromagnetic fields and trigonometric functions. We have also introduced the complex-field formalism, which provides a more general and compact representation. Building upon this description, we now introduce an alternative notation known as the **Jones calculus**.

This formalism employs a basis of polarization vectors commonly used in optical experiments and is particularly convenient because optical components can be represented as matrices, known as *Jones matrices*.

Starting from Eq. 3.2 the electric field can be expressed in terms of the complex Jones vector as

$$\mathbf{J} = \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} E_{0,x} e^{i\delta_x} \\ E_{0,y} e^{i\delta_y} \end{bmatrix} = \begin{bmatrix} E_{0,x} \\ E_{0,y} e^{i\delta} \end{bmatrix}, \quad \text{with } E_x, E_y \in \mathbb{C}, \quad (3.3)$$

which may be written either in terms of the individual phases or solely as a function of their phase difference. Apart from an overall multiplicative factor ($\varepsilon/2$), the field intensity is given by $I = \mathbf{J}^* \mathbf{J}$.

In practice, Jones vectors are usually normalized so that $\mathbf{J}^* \mathbf{J} = 1$ thereby representing a plane wave of unit intensity. In Tab. 3.1 we summarized the Jones vectors corresponding to the fundamental polarization states. It can be shown that, by considering

Linear	Circular	Elliptical
$\mathbf{J}_H = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\mathbf{J}_D = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\mathbf{J}_L = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}$
$\mathbf{J}_V = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\mathbf{J}_A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$	$\mathbf{J}_R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}$

Table 3.1: Jones vectors for common polarization states.

the generic two-dimensional rotation matrix

$$U_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \quad (3.4)$$

and taking J_H and J_V as the basis vectors, all other polarization states can be generated through an appropriate rotation, namely $J_\theta = U_\theta J_H$, here J_θ indicates the general polarization state.

$$J_\theta = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}. \quad (3.5)$$

As mentioned above, the Jones formalism is particularly useful because it allows optical elements to be represented algebraically as linear operators acting on Jones vectors. As an illustrative example, consider the action of a linear polarizer on a generic polarization state. The Jones matrix describing a linear polarizer oriented at an arbitrary angle α is given by

$$LP_\alpha = \begin{bmatrix} \cos^2 \alpha & \cos \alpha \sin \alpha \\ \cos \alpha \sin \alpha & \sin^2 \alpha \end{bmatrix} = |J_\alpha\rangle\langle J_\alpha|.$$

Its action on a generic Jones vector is therefore $LP_\alpha J_\theta = |J_\alpha\rangle\langle J_\alpha | J_\theta\rangle = \cos(\alpha - \theta) J_\alpha$; where J_α remains a normalized Jones vector. By computing the transmitted intensity, one readily recovers *Malus law*.

3.2 Stokes Parameters

So far, we have described two different representations of polarization states: one based on real electric fields and trigonometric relations, and another based on the Jones vector formalism.

For the sake of completeness, it is worth introducing a third representation, fully consistent with the Jones formalism, namely the one that makes use of the so called *Stokes parameters*. These real-valued parameters are directly derived from the complex electric-field components introduced in Eq. 3.3, according to

$$\begin{aligned} S_0 &= |E_x|^2 + |E_y|^2, & S_2 &= 2 \Re \{E_x^* E_y\}, \\ S_1 &= |E_x|^2 - |E_y|^2, & S_3 &= 2 \Im \{E_x^* E_y\}. \end{aligned} \quad (3.6)$$

Unlike the Jones formalism, the Stokes representation is applicable not only to completely polarized light but also to partially polarized and unpolarized light. For the

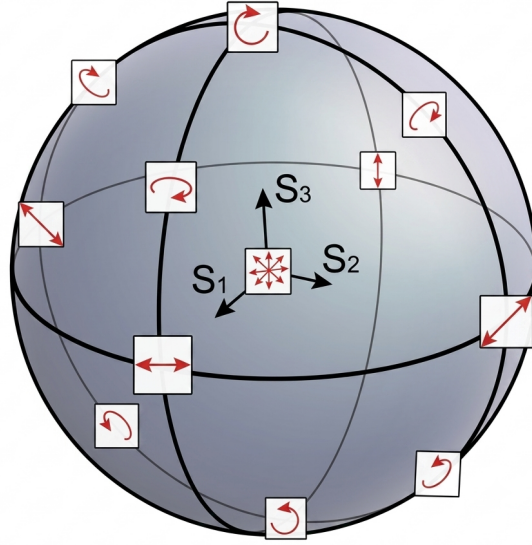


Figure 3.2: Schematic view of Poincaré Sphere, highlighting the polarization axis and the three components of the Stokes vector, i.e., the Stokes parameters.

particular case of fully polarized light, the Stokes parameters satisfy the relation

$$|S_1|^2 + |S_2|^2 + |S_3|^2 = |S_0|^2 .$$

If one considers a sphere of radius $|S_0|$, the Stokes parameters can be interpreted as basis for vectors lying on its surface. This geometrical representation is known as the *Poincaré sphere* (Fig.3.2). An interesting analogy with the electron spin immediately emerges when the normalized Stokes parameters are considered, the resulting geometrical representation is formally identical to the Bloch sphere. The *Poincaré sphere* provides an intuitive visualization of polarization states. Linear polarization states lie on the equator, whereas right- and left-handed circular polarization correspond to the north and south poles, respectively.

This geometrical interpretation also clarifies the physical meaning of the Stokes parameters: S_0 is proportional to the total intensity of the electromagnetic field; S_1 quantifies the degree of linear polarization along the $\hat{x}$ and $\hat{y}$ directions, respectively; S_2 measures the degree of linear polarization at 45° ; finally, S_3 characterizes the degree of circular polarization (see Table 3.2 for a summary). In general, an arbitrary point on the *Poincaré sphere* represents an elliptically polarized state. To demonstrate this result, one may start from the most general Jones vector describing elliptical polarization, with its principal axis aligned along the x direction (Eq. 3.5). The most general polarization ellipse is then obtained by rotating the Jones vector by an angle ψ (Fig. 3.3). From the rotated vector, the Stokes parameters can be expressed as functions of the total intensity and the two angles χ and ψ .

	S_0	S_1	S_2	S_3
J_H	1	1	0	0
J_V	1	1	0	0
J_D	1	0	1	0
J_A	1	0	-1	0
J_L	1	0	0	-1
J_R	1	0	0	1

Table 3.2: Values of the Stokes parameters corresponding to Jones vectors of the principal polarization states.

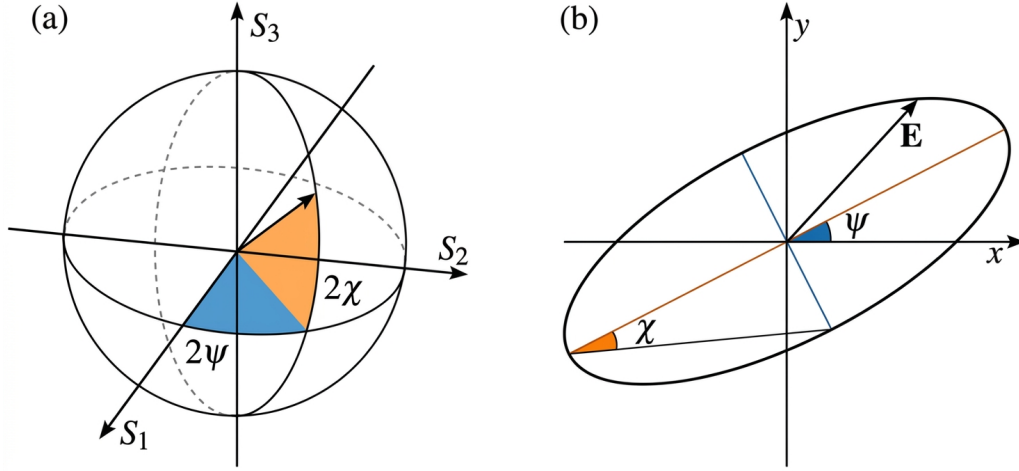


Figure 3.3: (a) The Poincaré sphere, with the three axes defined by the Stokes parameters; (b) general elliptical polarization as a function of the angles ψ and χ , respectively.

$$\begin{aligned}
 S_1 &= I \cos 2\psi \cos 2\chi , \\
 S_2 &= I \sin 2\psi \cos 2\chi , \\
 S_3 &= I \sin 2\chi .
 \end{aligned} \tag{3.7}$$

For simplicity, the present discussion is restricted to fully polarized states. A more general treatment introduces the *degree of polarization*, denoted by

$$p = \frac{\sqrt{S_1^2 + S_2^2 + S_3^2}}{S_0} \leq 1 ,$$

which is equal to unity for completely polarized light and satisfies $0 \leq p < 1$ for partially polarized states.

The main advantage in using *Stokes parameters* lies on the possibility of being experimentally measured. It can be demonstrated that by measuring the output intensity from a wave retarder, like a waveplate, and a linear polarizer, it yields the dependence from *Stokes parameters* (see, e.g., Fig. 3.4)

$$I(\vartheta, \Phi) = \frac{1}{2} [S_0 + S_1 \cos 2\vartheta + S_2 \sin 2\vartheta \cos \Phi + S_3 \sin 2\vartheta \sin \Phi] . \tag{3.8}$$

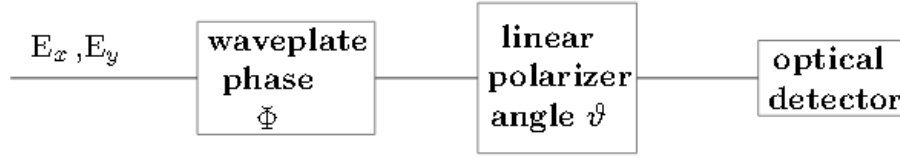


Figure 3.4: Schematic experimental setup used to measure Stokes parameters.[14]

Starting from Eq. 3.8, we need to operate four set of independent measurements. The first three without the wave retarder and the polarizer set at 0° , 45° , 90° . while in the last one is inserted the waveplate $\frac{\lambda}{4}$ ($\Phi = \frac{\pi}{2}$), oriented in such a way that its optic axis lies at 0° , then the linear polarizer is aligned again at 45° , like in the second measurement, but now the polarizer is preceded by the retarder. The *Stokes parameters* are then obtained by rearranging Eq. 3.8 as:

$$\begin{aligned} S_0 &= I(0,0) + I(90,0), & S_2 &= 2I(45,0) - I(0,0) - I(90,0), \\ S_1 &= I(0,0) - I(90,0), & S_3 &= I(45,90) - I(0,0) - I(90,0). \end{aligned}$$

The first three parameters S_0 , S_1 , S_2 are determined from measurements with the polarizer only, while S_3 requires a measurement with the retarder followed by the polarizer.

3.3 Analogy between spin-1/2 and polarization

Having introduced all the necessary background, we can now start arguing the main goal of this chapter, i.e., showing the analogy between Jones and spin algebras, respectively. It is worth pointing out that such an analogy emerges from an algebraic analogy for the transformation properties of the coefficients, and not a physical analogy of the wavefunctions.

To this end, let us consider how *Stokes parameters* change when a *Jones vector* (J_θ) is rotated by θ along z -axis.

$$\begin{bmatrix} S'_1 \\ S'_2 \\ S'_3 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & -\sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix}, \quad (3.9)$$

which describe a rotation of 2θ along S_3 -axis. This implies that *Stokes parameters* take a full turn of 2π every time that the electric field (*Jones vector*) rotates by π . It should not sound weird since opposite electric fields have the same polarization state. The doubling of rotation angle between J_θ and *Stokes parameters* is the same situation that arise between spin-wavefunctions and spin-vectors for spin-1/2 particles in quantum mechanics.

In general, it can be said that under transformation of the electric field (E_x, E_y) with 2×2 matrices U belonging to the $SU(2)$ group (special unitary group, made of complex unitary matrices with unit determinant), the Pauli matrices transform like a vector in $\mathbb{R}^3$ space. This implies that $\vec{S}$ also transforms like a vector, and it rotates with matrices R that belong to $SO(3)$ (special orthogonal group, made of real orthogonal matrices with unit determinant).

By taking a general input state $|E\rangle$, which indicates a general electric field, the *Stokes parameters* can be written as:

$$S_0 = \langle E | \mathbb{I} | E \rangle, \quad \vec{S} = \langle E | \vec{\sigma} | E \rangle. \quad (3.10)$$

Here $\mathbb{I}$ is the identity matrix, $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli's matrices and the polarization vector $\vec{S} = (S_2, S_3, S_1)$. To show that this formalism is correct, take for example S_3 :

$$S_3 = \langle E | \sigma_y | E \rangle = \begin{pmatrix} E_x^* & E_y^* \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = 2 \Im(E_x^* E_y),$$

which correspond to S_3 of Eq. 3.6, and by repetition one can also get the analogous results for S_1 and S_2 . The relation between *Stokes parameters* and Pauli matrices can thus be summarized as follows:

$$\begin{aligned} \sigma_z |H\rangle &= |H\rangle, & \sigma_x |D\rangle &= |D\rangle, & \sigma_y |L\rangle &= |L\rangle, \\ \sigma_z |V\rangle &= -|V\rangle, & \sigma_x |A\rangle &= -|A\rangle, & \sigma_y |R\rangle &= -|R\rangle. \end{aligned}$$

Everything that has been stated so far concurs to prove that the algebra of transverse polarizations is formally identical to the algebra of wavefunctions of a spin-1/2 particle, represented by *Spinors*. Since Spinors are sets of two complex numbers, very much like (E_x, E_y) , which have definite transformation properties under $SU(2)$, they are the coefficients of a wavefunction. The latter is interpreted a classical state for the electromagnetic polarization, while it represents a purely quantum state for the electron spin. To this end, the algebraic analogy stands because coefficients are set of numbers, neither classic nor quantum.

3.4 The altermagnetic analogue

So far, we have established the existence of an analogy between the *spinor* algebra and the algebra of transverse polarization, at least in two dimensions. It remains to determine whether this correspondence is preserved when a specific physical system is considered. In Chapter 1, we introduced a *Rashba-altermagnetic* model describing a

slab geometry. However, in order to compare the spin texture of the system (Fig. 1.6) with the *Stokes parameters*, it is first necessary to determine the latter for the photonic system under investigation.

To this end, we consider the optical TE and TM modes, which exhibit distinct polarization properties in the far-field emission. In particular, these modes form two orthogonal states with perfectly linear polarization. Within the Jones-vector formalism introduced previously, we choose the reference frame such that, when the in-plane wavevector is oriented along the x direction, the TE and TM modes radiate y - and x -polarized light, respectively. Under these conventions, the TE and TM modes are represented by the Jones vectors $|V\rangle$ and $|H\rangle$, respectively. Owing to the symmetry of the underlying photonic crystal lattice, these two modes remain completely decoupled and may therefore cross in the wavevector–energy dispersion without hybridization. In the following, we focus on this crossing point and describe the linear photonic dispersion in its vicinity by considering the Hamiltonian expressed in the guided-mode basis $\{|TE\rangle, |TM\rangle\}$. For simplicity, we retain only the altermagnetic contribution of Eq. 1.8:

$$\mathcal{H} = \frac{\vec{K}^2 \hbar^2}{2m} \mathbb{I} + \gamma (k_x^2 - k_y^2) \sigma_z, \quad (3.11)$$

where $\vec{K} = (k_x, k_y)$, with $[\mathbf{K}] = \text{\AA}^{-1}$, and γ denotes the altermagnetic coefficient. For plotting purposes, we have set $m = 0.152 \text{ eV}^{-1} \text{\AA}^{-2}$, $\gamma = 1.881 \text{ eV} \text{\AA}^2$, and we considered $\hbar = 1$. The eigenvalues of $\mathcal{H}$ determine the energy dispersion, whereas the corresponding eigenvectors $|\Psi\rangle$ encode the polarization state. The latter can be obtained by projecting the eigenvectors onto the guided-mode basis

$$|\Psi\rangle = \mu |TE\rangle + \eta \zeta |TM\rangle, \quad (3.12)$$

where μ and ζ are the projection coefficients onto the TE and TM modes, respectively. In realistic systems, however, the two guided modes generally couple with different efficiencies to the far-field radiation used to probe the eigenmodes. To account for this effect, we introduce the anisotropy parameter η (in our case fixed at $\eta = 0.5$), which phenomenologically describes the unequal coupling of the two modes to the far field. The transverse components of the radiated electric field, (E_x, E_y) , can then be evaluated by noting that, when the in-plane wavevector is rotated by an angle ϕ , the polarization directions of the TE and TM modes rotate accordingly while remaining

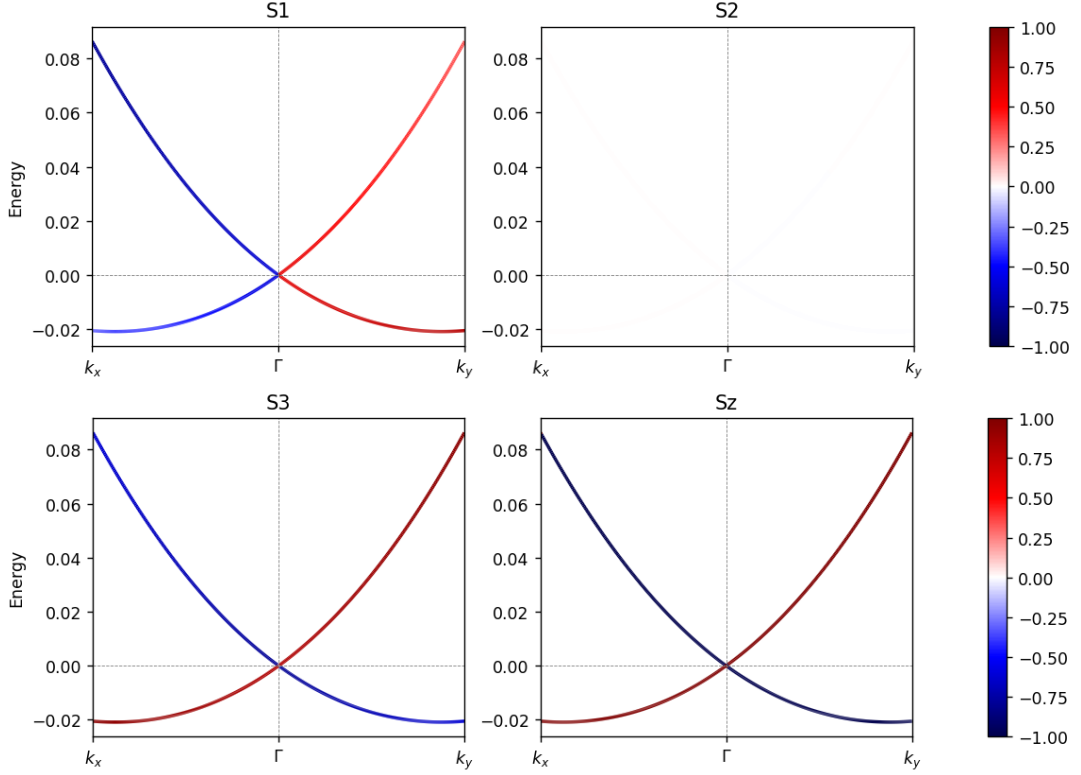


Figure 3.5: Comparison between the Stokes parameters and the spin polarization (i.e., σ_z) along the high-symmetry path $k_x \rightarrow \Gamma \rightarrow k_y$. In the lower panel, the component S_3 is directly compared to σ_z , for an immediate visualization of the analogy.

mutually orthogonal. This leads to

$$\begin{aligned}
 E_x &= \langle \text{TE} | \Psi \rangle, \sin \phi + \langle \text{TM} | \Psi \rangle, \cos \phi , \\
 E_y &= -\langle \text{TE} | \Psi \rangle, \cos \phi + \langle \text{TM} | \Psi \rangle, \sin \phi , \\
 \phi &= \arctan \left(\frac{k_y}{k_x} \right) .
 \end{aligned} \tag{3.13}$$

Since the altermagnetic phase exhibits its maximum spin splitting along two orthogonal directions, we evaluate the Stokes parameters along the same high-symmetry paths.

Specifically, we consider the Hamiltonian along the k_x direction, $\mathcal{H}(k_x, 0)$, corresponding to $\phi = 0$, and along the k_y direction, $\mathcal{H}(0, k_y)$, for which $\phi = \pi/2$. By substituting the corresponding electric-field components into the definition of the Stokes parameters given in Eq. 3.6, we obtain their values along both directions. From Fig. 3.5 it can be seen that the best analog for the spin, taken from the *Stokes parameters* is S_3 , which mimics not only the alternating magnetic order, but also the intensity.

We consider these results as the fingerprint of our synthetic analogy: a photonic system exhibiting an energy dispersion whose branches (crossing at the Γ point) are associated with an inversion of the S_3 component along orthogonal directions.

Chapter 4

Numerical Results

In this chapter, we report the numerical methods and techniques employed for the construction of the synthetic altermagnetic medium. In particular, we will show numerical evidencing the actual analogy between the photonic S_3 parameter calculated for the corresponding band structure and the spin polarization of electronic bands in altermagnetic materials. Before discussing in detail the methods used to construct both the geometrical structure of the photonic crystal slab under investigation and the procedure by which the radiative coupling coefficients are calculated, from which the value of S_3 is obtained at each k -point along the selected path, it is useful to first retrace the actual starting point of the thesis, as well as the rationale behind its development.

As it has been already discussed in Chapter 1, Altermagnetism is a magnetic phase characterized by spin-split bands along orthogonal paths in k -space, with a doubly degenerate Γ point, and exhibiting the distinctive property of reversing the spin orientation between the two orthogonal directions. Therefore, the starting point for constructing the synthetic analogue in photonic systems is the realization of a structure exhibiting a crossing at Γ along a path in k -space extending in two orthogonal directions. At this crossing, the photonic bands must be degenerate. Furthermore, it must be verified that S_3 follows the characteristic profile of altermagnetic spin along the two orthogonal directions.

To this end, the computational method employed is based on a dedicated Python package developed by this thesis Supervisors in collaboration with Dr. Simone Zanotti and Dr. Momchil Minkov. The package, called *Legume* (<https://github.com/fancompute/legume>), is a free and open-source electromagnetic solver that implements the already introduced guided-mode expansion (GME) method (see Chapter 2.4) for patterned multilayer waveguides, or photonic crystal slabs. This method is particularly well suited to the calculation of polarization states without explicitly relying on far-field radiative modes.

In fact, *Legume* does not solve Maxwell equations exactly. Instead, it expands the

electromagnetic field on a basis of guided modes of an effective homogeneous waveguide. This effective waveguide retains the same multilayer structure as the original system, while the permittivity of each layer is averaged over the in-plane directions, thereby removing the periodic pattern. The coupling to free-space radiation is then treated perturbatively. Nevertheless, the modes symmetries are preserved, which allows calculating the polarization properties (and, thus, the Stokes parameters) for each eigenmode.

4.1 Photonic crystal of triangular holes in Si membrane

To realize this photonic analogue, we have established to start from a structure that is as physically feasible as possible: a perforated silicon slab (see Fig. 4.1) suspended in air, namely the simplest dielectric multilayer structure. The choice for silicon as the paradigmatic material is mostly dictated by its sufficiently large refractive index at near-infrared wavelengths. In addition, the choice also fulfils the requirement to be one of the most conventional materials possible in current nanophotonics, thus avoiding, e.g., the use of gyromagnetic materials and relying exclusively on geometric symmetry breaking to achieve the desired analogue properties. In fact, to obtain $S_3 \neq 0$ for the band structure of a photonic crystal, it is necessary to break certain symmetries of the system, typically inversion symmetry and rotational symmetry. As already discussed in Chapter 1, the altermagnetic phase also requires the breaking of $\mathcal{T}$ -reversal symmetry. However, our choice of materials inherently prevents time-reversal symmetry breaking, since no gyromagnetic materials are employed.

Based on these considerations, we started with a highly symmetric photonic lattice: a hexagonal lattice of triangular air holes (see Fig. 4.1). Starting from this structure, we sought a suitable mechanism to break its symmetries. The most straightforward solution we identified was to rotate the triangular hole within the unit cell. [2] For this reason, the hole geometry was constructed parametrically as a function of the triangle area and its rotation angle, thereby obtaining a geometry that could be systematically optimized.

4.1.1 Geometric structure

The numerical simulation begins with the computational construction of the geometric structure underlying the photonic crystal, using the functionalities provided by the *Legume* library. The first step consists in defining the two-dimensional Bravais lattice, from which both the primitive vectors in real space and the corresponding reciprocal-lattice vectors are derived. The latter are subsequently used to construct the Brillouin zone and to define the reciprocal-space basis employed in the electromagnetic calcu-

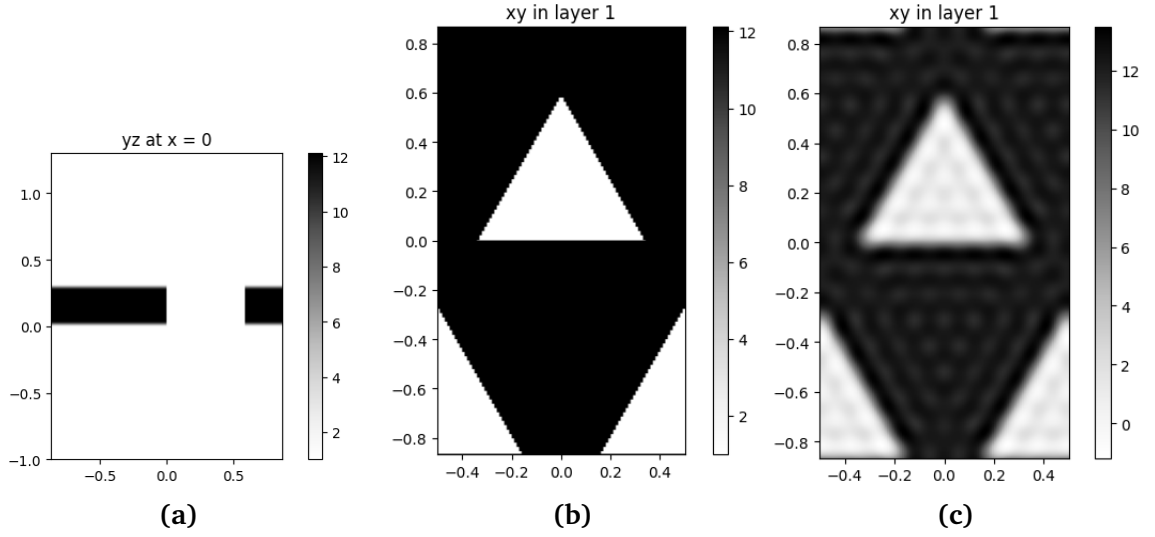


Figure 4.1: Dielectric profile of the patterned slab, represented by the relative dielectric constants in real space and within the unit cell. (a) Visualization of the vertical structure made of three dielectric layers: upper cladding (air), silicon slab (with constant value $\epsilon_{\text{Si}} = 12.1$), and lower cladding (air again). (b) Visualization of the planar unit cell ideally made of triangular air holes in the silicon slab. (c) Fourier transform of the dielectric profile obtained from the plane-wave expansion in the perforated slab layer structure, showing good approximation to the ideal structure.

lation. On this lattice, the perforated photonic slab is then constructed as a multilayer structure.

Once the geometry has been defined, the photonic band structure is calculated using the GME method, Fig.4.3. In this approach, the electromagnetic field is expanded in a truncated basis of guided modes of an *effective* homogeneous waveguide, with the extent of the reciprocal-space basis controlled by the chosen truncation parameters. The first step is the definition of the two-dimensional lattice. *Legume* provides three predefined options: ‘Hexagonal’, corresponding to a hexagonal lattice; ‘Square’, corresponding to a square lattice; or a fully custom lattice specified by manually defining the primitive vectors ($\mathbf{a}_1, \mathbf{a}_2$), for clarity see Appendix A.3. Once the lattice type has been selected, the ‘Lattice’ object calculates and stores the real-space primitive vectors ($\mathbf{a}_1, \mathbf{a}_2$) and the corresponding reciprocal-lattice vectors ($\mathbf{b}_1, \mathbf{b}_2$), which satisfy the duality relation

$$\mathbf{a}_i \cdot \mathbf{b}_j = 2\pi \delta_{ij}.$$

The perforated photonic slab is then constructed on this lattice. It consists of the multilayer structure forming the core of the system under investigation. In the examined case, the slab is a symmetrically suspended membrane: the central layer is surrounded above and below by two homogeneous semi-infinite media, referred to as the **claddings**, characterized by the same permittivity (air in this case). This symmetric configuration is important because it differs from the case of a slab supported by an asymmetric substrate, in which the upper and lower media would have different

permittivities.

The structure is defined through the *PhotCryst* object, which acts as the container for the multilayer geometry. The central layer, with thickness H , is introduced using the *'add_layer'* function, and it represents the dielectric slab itself, see Appendix A.3. It is initially composed entirely of the high-refractive-index material, characterized by a permittivity different from that of the upper/lower cladding. The holes are subsequently introduced within this material, locally modifying the permittivity of the layer. In the present implementation, the holes are treated as polygonal regions in which the permittivity is replaced by that of the surrounding cladding, as shown in Fig. 4.1. From a physical standpoint, this choice corresponds to a *through-hole*: the slab is completely etched through, exposing the cladding material.

In the considered geometry, the hole is an equilateral triangle and is parameterized not directly by its side length, but by its *area*. This choice is particularly convenient for geometric optimization, as it will be shown in the following, since the area can be used directly as a design variable. Starting from the geometrical relation between the side length and the area of an equilateral triangle,

$$A = \frac{\sqrt{3}}{4} L^2 ,$$

the side length is obtained by inversion. Therefore, the *Legume* variable *'edge'* represents the side length of the triangle, calculated from the prescribed area. The polygonal hole is then defined through the coordinates of its vertices. In the present case, the vertices are located at positions:

$$\mathbf{a} = \left(-\frac{\text{edge}}{2}, 0 \right), \quad \mathbf{b} = \left(\frac{\text{edge}}{2}, 0 \right), \quad \mathbf{c} = \left(0, \frac{\sqrt{3}}{2} \text{edge} \right).$$

This construction produces an equilateral triangle with its basis aligned with the x -axis, as shown in Fig. 4.1. An important aspect concerns the ordering of the vertices: in *Legume*, polygon vertices must be provided in counterclockwise order. This convention is related to the way in which the polygonal geometry is represented in reciprocal space, and it is required for the correct evaluation of its Fourier transform.

A key feature of the GME implementation provided by *Legume* concerns the calculation of the Fourier transform: the software does not calculate the Fourier transform of the polygon by discretizing the geometry on a numerical grid followed by a conventional fast Fourier transform (FFT) algorithm. Instead, geometric objects such as *'Poly'*, *'Circle'*, and *'FourierShape'* are associated with an analytic Fourier representation, obtained in closed form from the geometry of their boundaries and, in the case of polygons, from the coordinates of their vertices.

This approach provides an accurate representation of the spectral coefficients of the

periodic permittivity distribution, $\varepsilon(\mathbf{r})$, while avoiding the numerical errors associated with geometric discretization. At the same time, the analytical representation preserves the differentiability with respect to the geometrical parameters, such as the hole area and its rotation angle. This property is particularly important when the electromagnetic simulation is embedded within an automated optimization procedure. The `'add_shape'` command then inserts the hole geometry into the most recently defined layer. Mathematically, the contribution of the shape is incorporated into the Fourier representation of the permittivity through the dielectric contrast

$$\Delta\varepsilon = \varepsilon_{\text{shape}} - \varepsilon_{\text{background}},$$

which accounts for the local replacement of the core material by the hole material. The geometry is therefore directly translated into the Fourier coefficients, $\Delta\varepsilon(\mathbf{G})$, with $\mathbf{G}$ a reciprocal lattice vector, which are subsequently used in the calculation of the photonic dispersion.

4.1.2 Photonic band dispersion by GME simulation

The GME simulation is controlled by three main aspects: the set of k -points along which the band structure is evaluated, the truncation of the reciprocal-space basis, and the number of guided modes retained for each reciprocal-lattice vector. These parameters determine both the accuracy and the computational cost of the numerical calculation.

As discussed in the previous paragraph, the geometric structure of the photonic crystal slab is first defined in real space. The calculation is then executed in reciprocal space, where the reciprocal-lattice vectors are used to construct the high-symmetry points of the first Brillouin zone, as well as to define the set of reciprocal lattice vectors, $\mathbf{G}_i$, entering the plane-wave expansion. Here, we first consider the high-symmetry points, which are explicitly constructed from the reciprocal-lattice primitive vectors, $(\mathbf{b}_1, \mathbf{b}_2)$. This approach allows a customized convention to be adopted for the k -space path. In fact, the `'bz_path'` function subsequently constructs the path in reciprocal space by linearly interpolating between the selected high-symmetry points, namely $\text{Y} \rightarrow \Gamma$ and $\Gamma \rightarrow \text{K}$ (see Fig. 4.2), of coordinates:

Γ	Y or M	K or P
$(0, 0)$	$(0, \frac{\pi}{a})$	$(\frac{\pi}{a}, 0)$

Notice that the point $(\frac{\pi}{a}, 0)$ can be indicated in two ways, namely K and P, depending on the case considered. The same applies to the y -direction. In our case, for the GME simulations along orthogonal directions we have chosen to use the P and Y symbols,

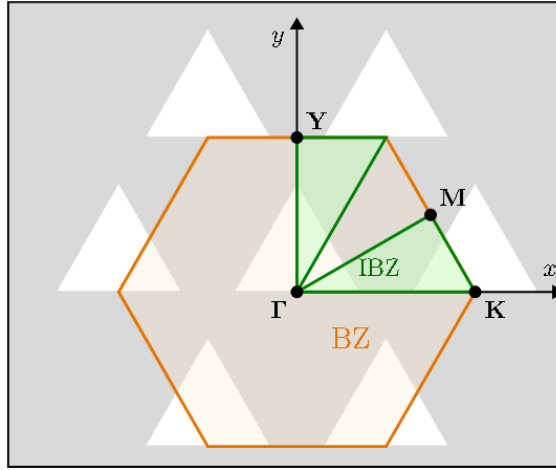


Figure 4.2: Visual representation of the Brillouin Zone (BZ), encircled within the orange border, and the Irreducible Brillouin Zone (IBZ), delimited by the green one, for the hexagonal array of triangular holes. The IBZ can be moved along the whole BZ.

while for the whole BZ we will typically define the K and M symbols. Apart from the high symmetry points, the number of interpolation points can be independently specified for each segment.

The GME method is then implemented through the ‘*GuidedModeExp*’ constructor. The method constructs a basis of reciprocal-lattice vectors ($\mathbf{G}_i$), which represent the in-plane plane waves entering the expansion. This basis is truncated according to two independent criteria. The first determines the number of reciprocal-lattice vectors retained in the expansion, while the second determines the number of local guided modes associated with each reciprocal vector. The parameter $g_{\max}$ that is passed to the constructor determines the extent of the reciprocal-space basis and, together with the selected truncation scheme, specifies the region of reciprocal space from which the vectors ($\mathbf{G}_i$) are selected. *Legume* provides two principal truncation schemes, denoted by ‘*abs*’ and ‘*tbt*’. The default ‘*abs*’ scheme selects reciprocal-lattice vectors according to their magnitude, effectively defining a circular truncation region in reciprocal space:

$$\mathbf{G}_{\text{selected}} = \{ \vec{\mathbf{G}}_i \quad \text{s.t.} \quad |\vec{\mathbf{G}}_i| < 2\pi g_{\max} \} .$$

This choice is particularly suitable for the present system because it preserves the rotational symmetry of the reciprocal-space basis. Since the photonic crystal possesses C_3/C_6 symmetry, avoiding the introduction of preferred reciprocal-space directions is important when investigating symmetry-protected modes and related phenomena.

The second option is the ‘*tbt*’ (i.e., *Toeplitz-block-Toeplitz*) truncation scheme. In this case, the reciprocal-space basis is constructed within a parallelogram-shaped region. This choice is particularly advantageous from a computational perspective because the Fourier components of the in-plane dielectric profile can be organized into a block-

Toeplitz structure, allowing their evaluation and storage to be significantly optimized. Consequently, when a large number of plane waves is required, the ‘*tbt*’ truncation can provide substantial computational advantages in terms of both execution time and memory consumption. However, this computational advantage comes at the cost of reduced rotational symmetry. For a hexagonal lattice, the parallelogram-shaped truncation introduces preferred reciprocal-space directions and therefore breaks the rotational symmetry of the basis. This can lead to artificial symmetry breaking and unexpected results when the symmetry properties of the modes are central to the physical analysis. For this reason, the ‘*abs*’ truncation is preferred in the present study.

The second truncation is conceptually different. For each selected reciprocal-lattice vector $\mathbf{G}_i$, the effective homogeneous waveguide possesses an infinite set of local guided modes, alternating between TE- and TM-like modes. *Legume* retains only a finite subset of these modes. Typically, modes indexed by $[0, 2, 4, \dots]$ are associated with TE-like modes, whereas those indexed by $[1, 3, 5, \dots]$ are associated with TM-like modes. This classification, however, is not necessarily preserved in the final photonic-crystal modes, since the periodic dielectric modulation can couple and hybridize different polarization components.

Unlike the truncation with respect to $\mathbf{G}_{\text{selected}}$, which can in principle be systematically converged by increasing g_{max} , the truncation of the local guided-mode basis introduces a strong approximation. The expansion is projected onto an incomplete basis because the radiative modes of the effective homogeneous waveguide are not included in the initial expansion. Their contribution is instead incorporated perturbatively at a later stage in order to evaluate the radiative losses of the photonic-crystal modes. Within this framework, the number of guided modes retained for each reciprocal vector therefore represents an additional convergence parameter of the GME calculation. Increasing this number enlarges the basis and can improve the accuracy of the calculated frequencies, at the expense of a higher computational cost.

Once the reciprocal-space basis and the guided-mode basis have been defined, the GME calculation is performed for each k -point along the selected path. The first step consists of constructing the electromagnetic operator matrix in the basis of the guided modes of the reference homogeneous slab. The matrix elements contain the Fourier coefficients $\Delta \epsilon(\mathbf{G})$ describing the periodic dielectric modulation, which, as discussed above, are obtained analytically from the geometry of the holes. The resulting matrix is then numerically diagonalized, yielding the frequencies and corresponding eigenvectors. The lowest-energy frequencies are retained to construct the real part of the photonic band structure under investigation.

The calculation is not limited to the real part of the frequencies. A major advantage of the GME approach implemented in *Legume* is that it also provides the imaginary part of the frequency for modes that lie above the cladding light line and can there-

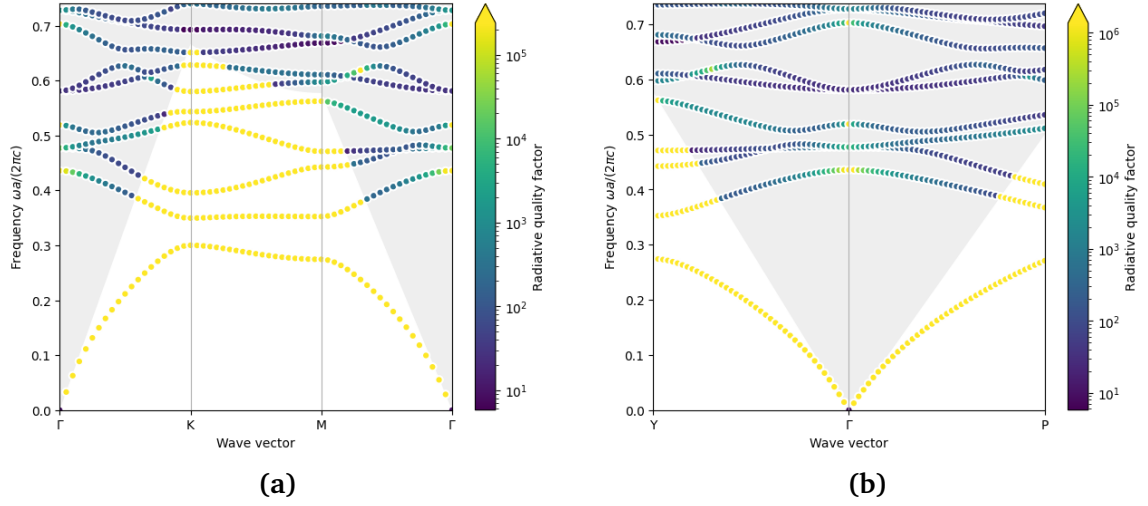


Figure 4.3: GME results for the complex (real and imaginary parts) photonic band dispersion obtained for the structure shown in Fig. 4.1, with a truncation parameter of $g_{\max} = 6$ (corresponding to a base of 97 plane waves) and 10 guided modes, plotted along different directions in reciprocal space: (a) $\Gamma \rightarrow K \rightarrow M \rightarrow \Gamma$, and (b) along the orthogonal path $Y \rightarrow \Gamma \rightarrow P$ (which coincides with $M \rightarrow \Gamma \rightarrow K$, in fact).

fore couple to radiative modes. The radiative loss is calculated perturbatively using a procedure based on Fermi's golden rule. This additional calculation makes it possible to determine the *radiative quality factor* of the modes without requiring a separate full-wave simulation.

The real part of photonic eigenfrequencies is subsequently plotted as a function of the position along the selected k -space path (see, e.g., Fig. 4.3). The corresponding *quality factor* for each point can be evaluated as:

$$Q_k = \frac{\text{Re}(\omega_k)}{2 \text{Im}(\omega_k)} .$$

This quantity provides a direct measure of the radiative confinement of the modes. Modes located below the cladding light line are genuinely guided and, within the lossless approximation, have $\text{Im}(\omega_k) = 0$, corresponding to $Q \rightarrow \infty$. The quality factor is reported as a color code for each plotted point in Fig. 4.3. Conversely, modes above the light line can couple to radiative states and therefore acquire a finite Q_k . The value of Q_k generally decreases as the coupling to the radiation continuum increases.

The analysis above is particularly relevant for the identification of so called *Bound States in the Continuum (BICs)*. Such modes can remain spatially confined even though their frequencies lie within the continuum of radiative states. In the band-structure representation, BICs therefore appear as modes with vanishing or strongly suppressed radiative losses, and consequently very large, ideally divergent, quality factors, even when being at Γ . Their existence is generally related to symmetry protection or to

destructive interference between different radiation channels.

Overall, the numerical workflow establishes a consistent chain connecting the parametric geometry of the photonic slab to its electromagnetic properties. The procedure begins with the construction of the real-space lattice and the suspended perforated membrane, proceeds through an analytical reciprocal-space representation of the dielectric modulation, and then constructs the GME basis from reciprocal-lattice vectors and local guided modes. The resulting eigenvalue problem provides the photonic band structure, while the perturbative treatment of radiative coupling yields the corresponding losses and quality factors.

A key advantage of this approach is that the geometrical parameters of the hole remain analytically connected to the Fourier coefficients of the dielectric structure. As a consequence, quantities such as the hole area and rotation angle can be treated as continuous design variables while preserving the differentiability of the computational pipeline. The GME simulation is therefore not only suitable for calculating the photonic band structure and the related radiative properties of the system, but also provides a natural framework for automated optimization of the photonic-crystal geometry.

4.1.3 Stokes parameters from GME

In the following, we describe the procedure used to obtain the value of S_3 by exploiting the eigenmode calculation by *Legume*, see Appendix A.4. This part is an original addition to the freely available software, and as such it is not distributed in the public version, yet. First, it is necessary to understand how the electric-field components, E_x and E_y , are calculated. As described in Chapter 3, both components are complex quantities, and they therefore consist of a real and an imaginary part, respectively. In general, the radiative loss rate of a mode with wavevector $\mathbf{k}$ can be obtained using time-dependent perturbation theory, in close analogy with Fermi's golden rule in quantum mechanics.

As such, the real part is obtained by diagonalizing the so-called *guided* Hamiltonian, which is constructed from the Fourier components of the periodic dielectric perturbation, $\Delta\epsilon(\mathbf{G})$, in the basis of the guided modes, see Eq. 2.29 in Section 2.4.1. This procedure yields the real eigenfrequencies and the corresponding Bloch eigenvectors of the guided modes.

The imaginary part, which is associated with the coupling of guided modes to the radiative continuum within the light cone, is instead calculated for each wavevector $\mathbf{k}$ by considering, among all reciprocal lattice vectors $\mathbf{G}$ included in the basis, those for

which $\mathbf{k} + \mathbf{G}$ lies within the light cone of the cladding, namely

$$|\mathbf{k} + \mathbf{G}| < |\mathbf{K}_{n_c}|,$$

where the quantity on the right-hand side represents the magnitude of the wavevector in the homogeneous external medium at the corresponding frequency. Wavevectors satisfying this condition are no longer evanescent but can propagate, giving rise to what are commonly referred to as *open diffraction orders*.

For each open diffraction order, the matrix element is then evaluated to first order in perturbation theory between the corresponding component of the guided-mode eigenvector at reciprocal vector $\mathbf{G}$ and a free-space plane wave, polarized either in the TE or TM basis, propagating with in-plane wavevector $\mathbf{k} + \mathbf{G}$ in either the upper or lower cladding.

The resulting quantities are the radiative coupling coefficients, or *leaky-mode coefficients*, calculated by Legume for each $\mathbf{k}$ -point (with the exception of the Γ point, which is treated separately) and for each mode. These are complex quantities representing the amplitude with which a guided mode couples to the radiative continuum, decomposed in the TE and TM far-field polarization basis.

To calculate the electric-field components for a specific mode, it is necessary to consider the free-space wavevectors associated with that mode. These can be written as $k_{fs} = (k_p, k_z)$ where the out-of-plane component k_z is obtained from the in-plane wavevector $\mathbf{k}_p$ and the normalized frequency according to

$$k_z = \sqrt{4\pi^2 \varepsilon_{\text{cladding}} E_{\text{mod}}^2 - k_p^2},$$

where E_{mod} denotes the normalized frequency and $\varepsilon_{\text{cladding}}$ is the dielectric permittivity of the cladding. Importantly, k_z must be treated as a complex quantity, since for wavevectors lying below the light cone it can become purely imaginary, corresponding to an evanescent field. A global reference phase, φ , is then chosen such that the TE contribution becomes real and positive. The same phase factor is applied to the TM contribution in order to preserve the relative phase between TE and TM modes, which is precisely the quantity that determines the circular polarization.

It is worth emphasizing that, apart from an arbitrary global phase, all relative phases must be well defined, since they correspond to physically observable quantities. In particular, when the radiative coupling is expressed in the s and p polarization basis, the relative phase between the two components has a physical meaning. It is precisely this relative phase that gives rise to a non-zero value of S_3 . A global phase, on the other hand, does not affect any of the Stokes parameters, and therefore the above phase convention is physically legitimate.

The Cartesian components of the electric field can then be obtained through the standard geometrical transformation for the far field of a plane wave with in-plane wavevector (k_x, k_y) . The TE (s) and TM (p) unit polarization vectors are given by

$$\begin{aligned}\hat{\text{TE}}(\hat{s}) &\perp \mathbf{k}_{//}, \quad \hat{s} = \left(-\frac{k_y}{k_p}, \frac{k_x}{k_p}\right), \\ \hat{\text{TM}}(\hat{p}) &\parallel \mathbf{k}_{//}, \quad \hat{p}_{xy} = \frac{(k_x, k_y) \cdot \mathbf{k}_z}{k_p \cdot |\mathbf{k}|}.\end{aligned}$$

Here, $|\mathbf{k}| = \sqrt{(k_p^2 + k_z^2)}$.

Finally, the two contributions are weighted by their respective complex amplitudes and summed up, thus yielding the Cartesian components (E_x, E_y) of the radiated electric field in the plane. These components contain both the amplitude information and the relative TE – TM phase. The value of S_3 can subsequently be calculated from Eq. 3.8 for each individual $\mathbf{k}$ -point along the selected path.

When the two Cartesian field components are in phase, corresponding to linear polarization, one obtains $S_3 = 0$. Conversely, when the two components have equal magnitude and a relative phase of $\pm 90^\circ$, the resulting polarization is purely circular and $S_3 = \pm 1$. At $k_p = 0$, corresponding to the Γ point, the azimuthal direction is not well defined, since the quantities k_x/k_p and k_y/k_p become singular. Similarly, at points where the radiative components vanish, S_3 is set to zero for numerical convergence and stability of the simulation.

4.1.4 Numerical optimization by differential evolution

The final step of the numerical simulation procedure consisted in identifying the optimal geometrical configuration that maximizes the value of S_3 around the band crossing at Γ . To this end, an optimization function from the *SciPy* registry, called ‘*differential_evolution*’ (*DE*), was employed, see Appendix A.2. This is a global, derivative-free optimization algorithm belonging to the family of evolutionary/metaheuristic algorithms.

The choice for this optimizer was motivated by the nature of the objective function, which is not, in principle, an analytic function: each evaluation requires a complete GME simulation of the band structure and the associated eigenmodes, including the identification of quasi-degenerate bands at Γ , the extraction of the radiative coupling coefficients, and the calculation of S_3 in a neighborhood of Γ defined as:

$$N = \{(k, k') \in \mathbf{k}_{\text{path}} \times \mathbf{k}_{\text{path}} \mid |\Gamma - k| = |\Gamma - k'| \leq n\}. \quad (4.1)$$

As a result, the objective function may exhibit numerical noise, discontinuities, or step-like features, for instance due to changes in the identification of the pair of degenerate bands, or singularities occurring when $S_0 = 0$ (Eq. 3.8). A gradient-based

method, such as Newton or BFGS, would therefore be unreliable or even inapplicable in this context, since the gradient cannot be computed analytically and numerical differentiation would be unstable for such a noisy function.

Furthermore, the objective landscape in the parameter space defined by the triangle area and rotation angle is plausibly with multiple local minima associated with the angular periodicity of the hexagonal structure. Since *DE* is a population-based global search algorithm, it is less likely to become trapped in a local minimum compared with local optimization methods. *DE* also explores the entire admissible parameter space from the outset through its initial random population, which is particularly useful in the present case, where there is no strong prior intuition regarding the location of the optimal configuration. Finally, given the small number of parameters and the high computational cost associated with each individual function evaluation, *DE* represents a good compromise between global exploration and an acceptable computational cost. The difficulty found with the optimizer was associated with the objective function output, which is a 2×20 vector (see the function 'compute_opt_s3' in Appendix A.1). In order to exploit the optimizer effectively, it is therefore convenient to define a figure of merit (F) as a dimensionless parameter capable of quantifying the deviation of S_3 from zero, for a given geometrical configuration in the vicinity of Γ . This figure of merit is then defined as follows:

$$F = \sum_{\substack{(k,k') \in N \\ k \neq k'}} [S_3^{\text{upper}}(k) - S_3^{\text{lower}}(k)][S_3^{\text{upper}}(k') - S_3^{\text{lower}}(k')]$$

This sum is defined by the radius of the circular neighborhood around Γ taken. In our case the sum was taken over 10 terms, corresponding to half the number of points considered in the neighborhood sampled by the objective function in Eq. (4.1), i.e., the product of the deviations evaluated at fixed k and k' points between the two bands selected by the objective function. To ensure a constant sign of the figure of merit, the k and k' points are chosen symmetrically with respect to Γ .

$$|\Gamma - k| = |\Gamma - k'|$$

Based on the expected profile of the Stokes parameter required to reproduce the alternating magnetic spin analogue (see, e.g., Fig. 3.5), and considering that $S_3 \in [-1, 1]$, in the limiting case of maximum circular polarization, i.e., $S_3 = \pm 1$, one obtains:

$$\begin{aligned} f_{(k,k')} &= [S_3^{\text{upper}}(k) - S_3^{\text{lower}}(k)][S_3^{\text{upper}}(k') - S_3^{\text{lower}}(k')] \\ &= [(-1) - (1)][(1) - (-1)] \\ &= (-2)(2) = -4. \end{aligned}$$

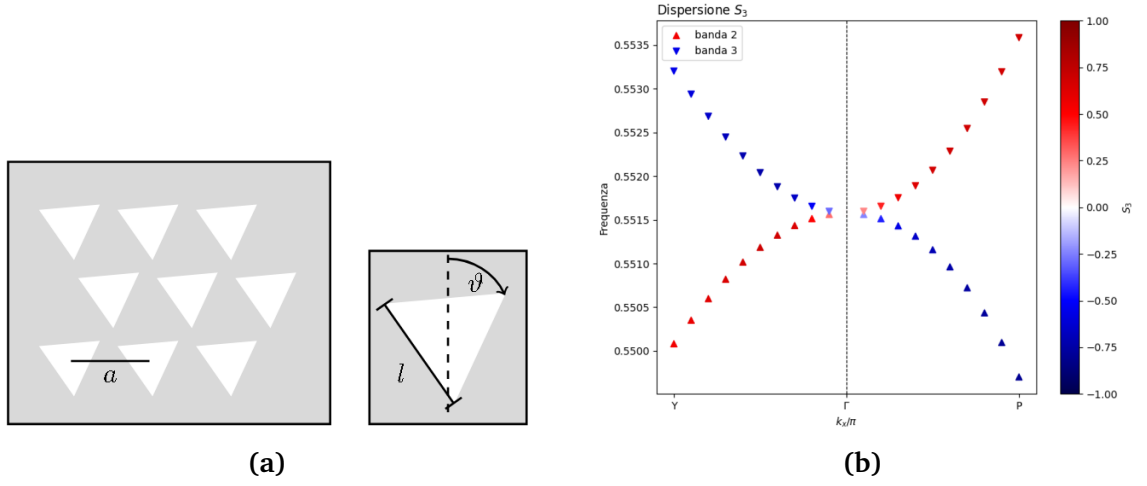


Figure 4.4: (a) Schematic representation of the geometric structure of the perforated slab along the orthogonal directions, with $a = 1$, $l = 0.9095 a$, $\vartheta = 65.2763^\circ$, and slab height $H = 0.3 a$. (b) Visualization of the two lowest-energy bands of the band structure, exhibiting a crossing at Γ . The color scale indicates the value of S_3 at each k -point in the selected neighborhood around Γ .

The optimization was therefore formulated by requiring the identification of the configuration in the parameter space defined by the triangle area and rotation angle that yields the most negative possible value of the figure of merit. The area was constrained to remain below $0.6 a^2$, corresponding to a triangle side length larger than the lattice constant a , while the triangle rotation angle was restricted to the interval $[0^\circ, 120^\circ]$, thereby exploiting the symmetry properties of the structure and reducing the computational cost.

Finally, by running the optimization procedure repeatedly, the optimal geometrical parameters of the structure were determined, yielding the following results (see Fig. 4.4(a) for the optimized structure):

Area	$0.3582 a^2$
Rotation Angle	$65,2763^\circ$
Figure of merit	$-15,107458$

The results for the band dispersion for this optimized structure along the two orthogonal directions, as well as their polarization texture quantified by the calculated magnitude of the S_3 parameter at each point in reciprocal space and for each of the two relevant bands are reported in Figs. 4.4(b) and 4.5, respectively. As it appears evidently, the analogy with the altermagnetic behavior in electronic systems, as described by the effective model in both Chapters 1 and 3, is complete and quite striking.

To further validate the electromagnetic results obtained by our numerical procedure, a second and independent computational method was employed: Rigorous Coupled-Wave Analysis (RCWA), implemented in the formulation given in Ref. [9]. This method, already mentioned in the previous chapter, is an electromagnetic simulation method

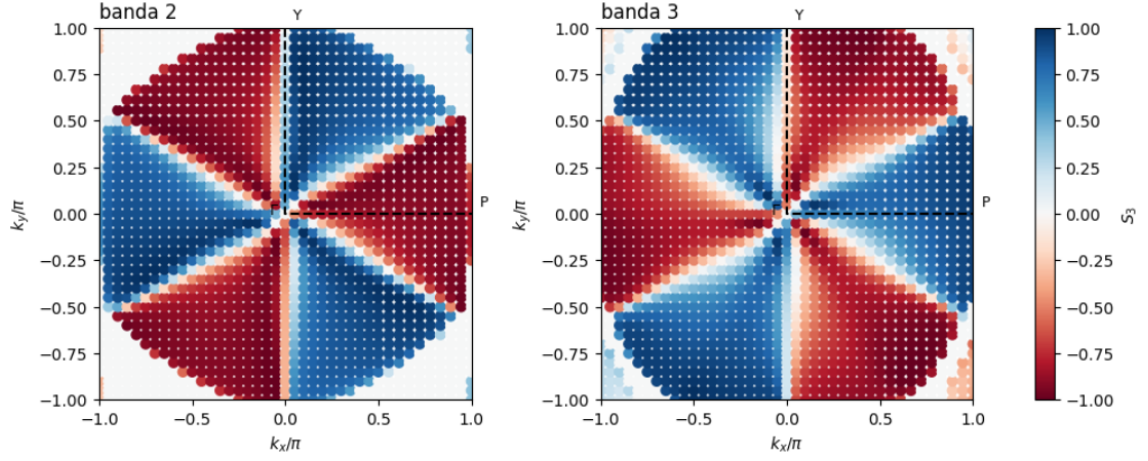


Figure 4.5: Color scale plot of the calculated of S_3 parameter within the whole Brillouin zone, respectively for bands 2 (left panel) and 3 (right panel). Black dashed lines highlight the path along which the dispersion of Fig. 4.4 is plotted. Here, the characteristic alternating behavior can also be appreciated, in analogy to Fig. 1.6.

conceptually different from GME, but still based on the Fourier decomposition of the periodic structure and the rigorous propagation of the electromagnetic fields through the different layers of the system. It allows the coupling between the structure and the far field to be calculated rigorously, without the approximations inherent to the guided-mode expansion on which GME is based. In particular, it allows calculating the optical response of the multilayer structure, such as reflection and/or transmission coefficient at varying angles, which give dispersive features corresponding to the external excitation of guided resonances supported by the structure.

It is interesting to point out that the RCWA method does not directly provide a calculation of S_3 , unlike the GME method described above, through the radiative coupling coefficients of the guided modes. Instead, RCWA provides a quantity that is directly related to the Stokes parameter S_3 , namely the circular dichroism, which in this case can be defined as the difference in absorption between the two incident circular polarizations (see, e.g., the description at the end of Chapter 2). Since circular dichroism is, by construction, closely related to the degree of circular polarization of the radiation emitted by the structure, and it therefore gives the same information as S_3 , a comparison between the two results remains physically meaningful, despite the fact that they correspond to distinct but physically correlated observables.

By using the optimized geometrical parameters obtained through the differential evolution algorithm, the hole area and rotation angle, the same structure was subsequently simulated using the RCWA, yielding a plot of the circular dichroism as a function of energy in the vicinity of the Γ point. The calculations were performed by Prof. L. C. Andreani, whose results are shown in Fig. 4.6. The qualitative comparison between the S_3 profile calculated using GME and the circular dichroism calculated using RCWA, for the same optimized geometrical parameters, shows a consistent agreement

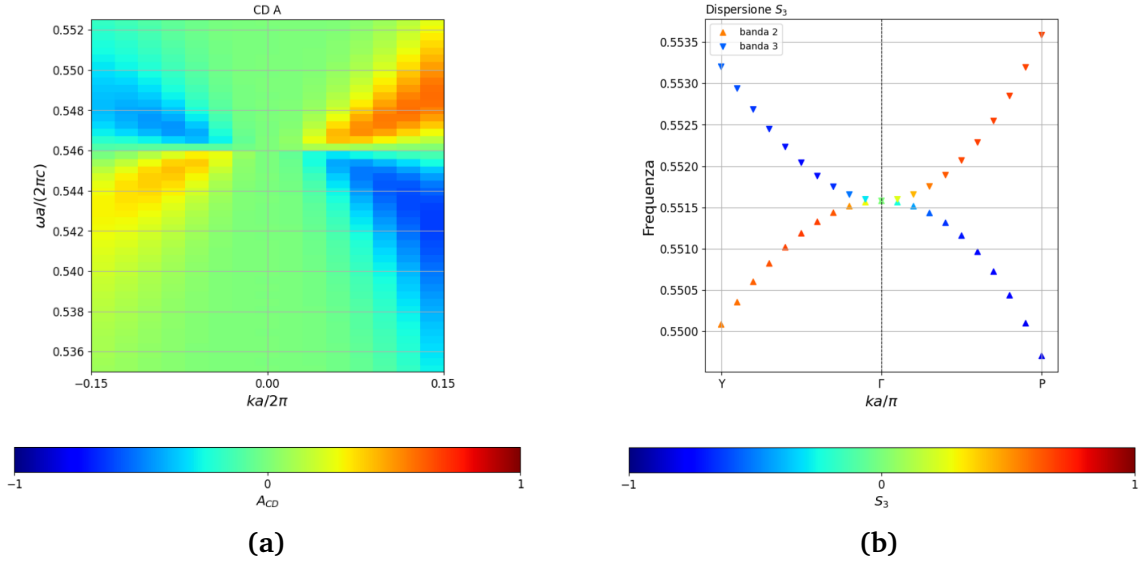


Figure 4.6: Comparison between the two numerical methods employed in this work: (a) RCWA simulation of circular dichroism (courtesy of L. C. Andreani), and (b) GME results for the two branches crossing in Γ , plotted in terms of the calculated S_3 parameter. The horizontal span is the same for both figures: RCWA in units of $ka/2\pi$; while GME in units of ka/π .

between the two approaches, both in terms of sign and of the behavior around the Γ point. The two methods exhibit a slight shift in energy, which is nevertheless consistent with the error associated with the different types of simulations employed by the two approaches.[20] Another noticeable feature is the different energy span between the extreme points in the two simulations. This discrepancy can likewise be attributed to the different simulation approaches, as well as to the fact that RCWA does not directly calculate S_3 , but rather the circular dichroism. Furthermore, the plotting conventions used for the two figures are different, which may also contribute to the observed discrepancies. Specifically, the RCWA results are represented using a mesh grid spanning both directions, with the color scale indicating the circular dichroism. In contrast, the GME results are obtained by plotting the two bands along a composite path in reciprocal space, defined as $(0, k_y) \rightarrow (0, 0) \rightarrow (k_x, 0)$. Despite these differences, the comparison between the two methods still shows a very good agreement, confirming the robustness and reproducibility of the optimal result obtained by differential evolution, independently of the electromagnetic simulation method employed.

4.1.5 Tolerance to imperfections

As anticipated at the beginning of the section, one of the guiding principles followed in the design of the structure was its potential experimental realization. A device optimized only theoretically but extremely sensitive to minor geometrical imperfections would have limited practical relevance. In the fabrication of this kind of structures

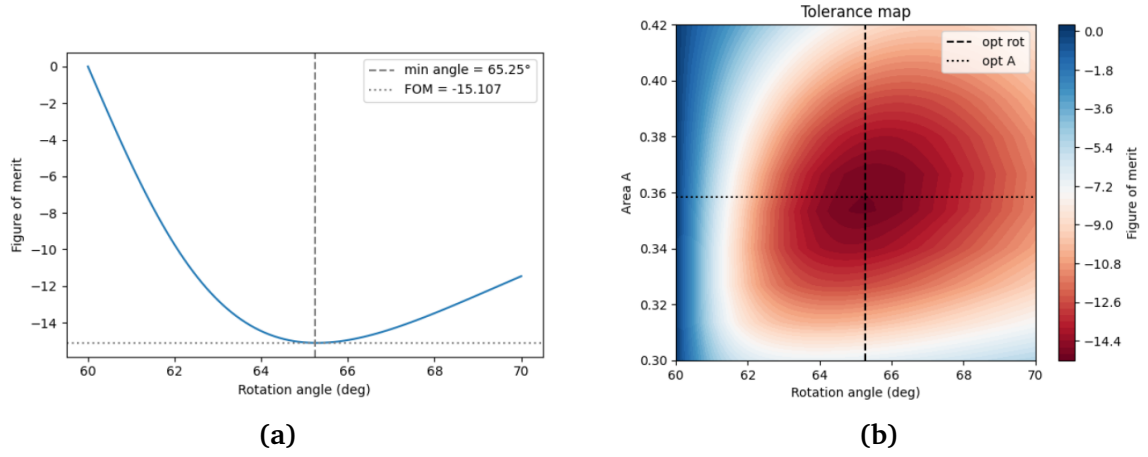


Figure 4.7: (a) Figure of merit as a function of the rotation angle, showing a minimum around 65° . (b) Figure of merit as a function of the rotation angle and the area of the triangular hole, showing a region of stability around the optimized parameters.

through processes such as lithography, etching, and deposition, small deviations from the nominal design parameters are in fact unavoidable.

For this reason, in addition to identifying the optimum through the differential evolution algorithm, a tolerance analysis was carried out to assess the robustness of the system with respect to variations in its fundamental geometrical parameters, namely the area of the triangular hole and its rotation angle. The analysis was performed at two levels. First, a one-dimensional scan was carried out around the optimal values, varying only the rotation angle and monitoring the evolution of the figure of merit. Subsequently, a two-dimensional scan was performed over the entire parameter space, allowing the region surrounding the optimum to be visualized and the extent of the parameter range over which the figure of merit remains close to its optimal value to be assessed (Fig. 4.7). Therefore, the presence of a sufficiently broad tolerance range provides further evidence of the experimental feasibility of the proposed design, rather than merely confirming its theoretical optimality.

4.2 Photonic crystal of triangular rods in glass slab

In the discussion presented so far, we have shown that the perforated silicon slab constitutes a suitable analogue of the targeted electromagnetic behavior. This result naturally raises the question of whether the complementary system, i.e., a photonic crystal slab made of triangular dielectric rods surrounded by a glass medium, could provide an equally valid candidate for the realization of the photonic analogue. The main motivation for this last study lies in the possibility to realize this system with currently available technology, by patterning triangular rods deposited on glass, and then filling the space between the rods with an index-matching dielectric, and then

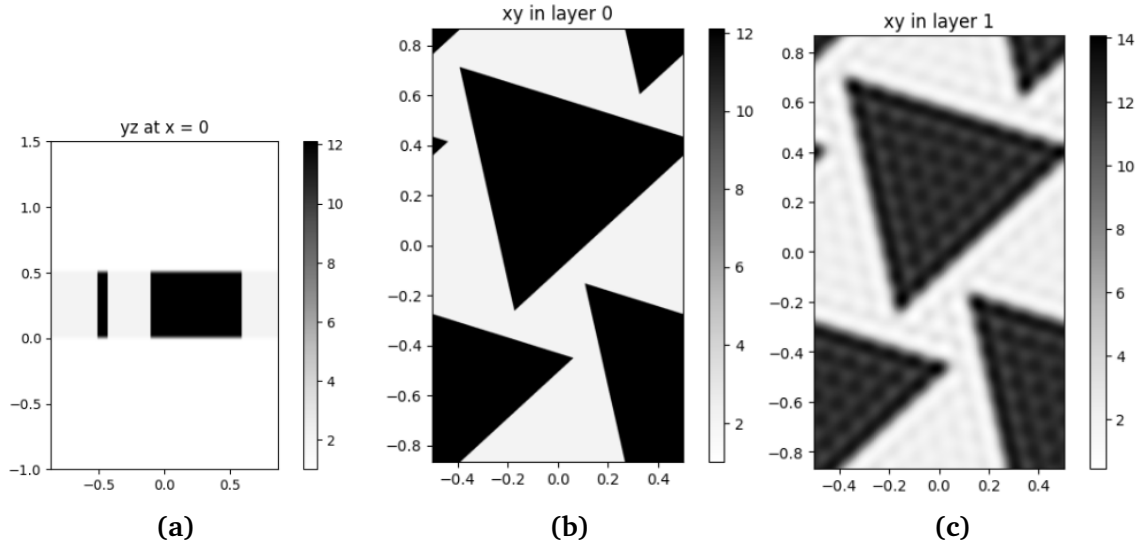


Figure 4.8: Representation of the slab permittivity. (a) Schematic representation of the three-layer structure, consisting of the upper/lower claddings (air) and the glass slab with silicon rods (core layer). (b) Visualization of the ideal dielectric profile corresponding to triangular silicon rods embedded in the glass slab. (c) Fourier transform of the dielectric profile obtained from the plane-wave expansion of the rod-in-glass structure, showing very good agreement with the ideal dielectric profile in (b).

removing the substrate to have a suspended layer of high-index rods (e.g., made of silicon) in glass. This is a realistic structure, since an air background would not be able to sustain an array of dielectric rods in air.

To address this question, we have repeated the same procedure already followed for the perforated slab, introducing the appropriate geometrical and physical modifications to the model. Specifically, the rod-based structure consisting of triangular silicon rods embedded in a glass matrix is defined in Fig. 4.8.

In this case, the modes considered in the GME calculation were the TM-like modes, consistently with the nature of the rod-based structure, in which field confinement is dominated by this polarization. The remaining optimization procedure, including the definition of the figure of merit, the use of the differential evolution algorithm, and the identification of the pair of quasi-degenerate bands at Γ , was kept unchanged with respect to the previous case. The only additional constraint was the need to impose a more stringent upper bound on the triangle area, in order to avoid entering a configuration analogous to that of the previous case.

In addition, and in order to obtain a lower value of the figure of merit, an additional parameter was included in the optimization procedure: the slab thickness. The resulting optimized parameters are reported below:

Thickness	$0.5003 a$
Area	$0.4332 a^2$
Rotation Angle	$42,6528^\circ$
Figure of merit	-0.720724

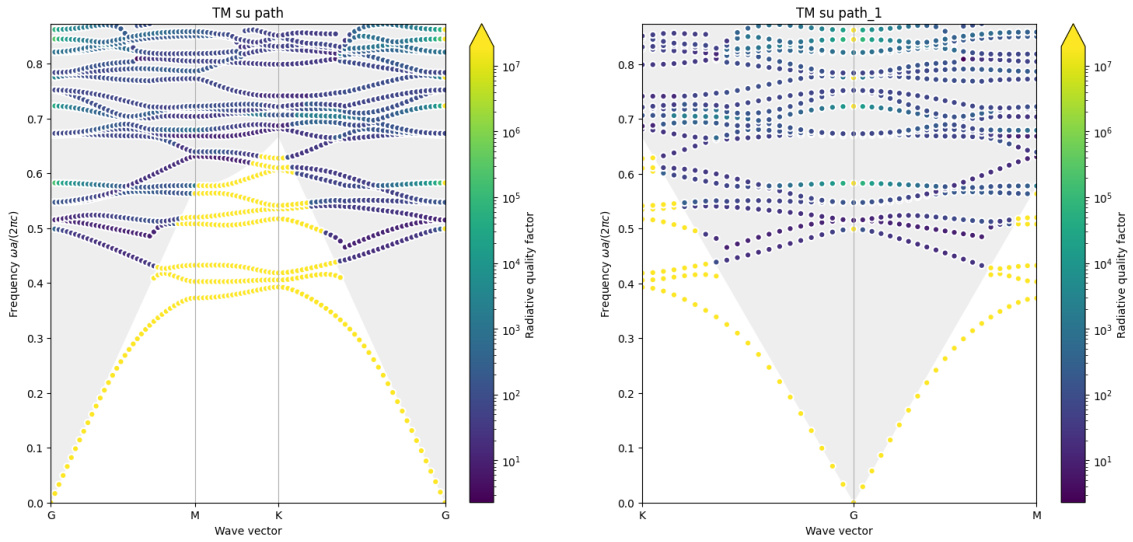


Figure 4.9: GME simulation ($g_{\max} = 6$ and a base of 97 plane waves) of the TM modes of the rod-based photonic crystal slab plotted along two different directions. The dispersion is shown over the whole Brillouin zone (left panel), as well as over two orthogonal spatial directions (right panel). In both cases, system parameters were fixed as: rod thickness $h = 0.5003a$; triangular rods area at $A = 0.4332a^2$; rotation angle for the rods at 42.6528° .

The optimization results obtained for this configuration were, however, significantly different from those of the perforated slab, since the two structures are fundamentally different from an electromagnetic point of view.

In the rod configuration, silicon is embedded in a glass matrix rather than in air, resulting in a slightly lower refractive-index contrast as compared to the photonic crystal slab of triangular air holes, in which the contrast is between high-index material (silicon) and air. In fact, the reduced dielectric contrast leads to weaker coupling with the electromagnetic field confined in different sites and, consequently, to an intrinsically weaker circular-polarization effect. However, and contrary to the initial expectation, the optimized system does not yield a sufficiently low value of the figure of merit. Since the figure of merit is negative, a lower value corresponds to a better-performing structure. The optimized configuration reaches a maximum S_3 of only approximately 10% (see Fig. 4.10). Therefore, this result suggests that, although the rod-based system shares with the perforated slab the same initial geometric symmetry as well as the same symmetry-breaking strategy based on basis rotation with respect to the primitive lattice vectors, its lower refractive-index contrast significantly limits its effectiveness as a photonic analogue of altermagnetism. As a consequence, we may conclude that the perforated slab represents the clearly preferable configuration for realizing the desired photonic analogue.

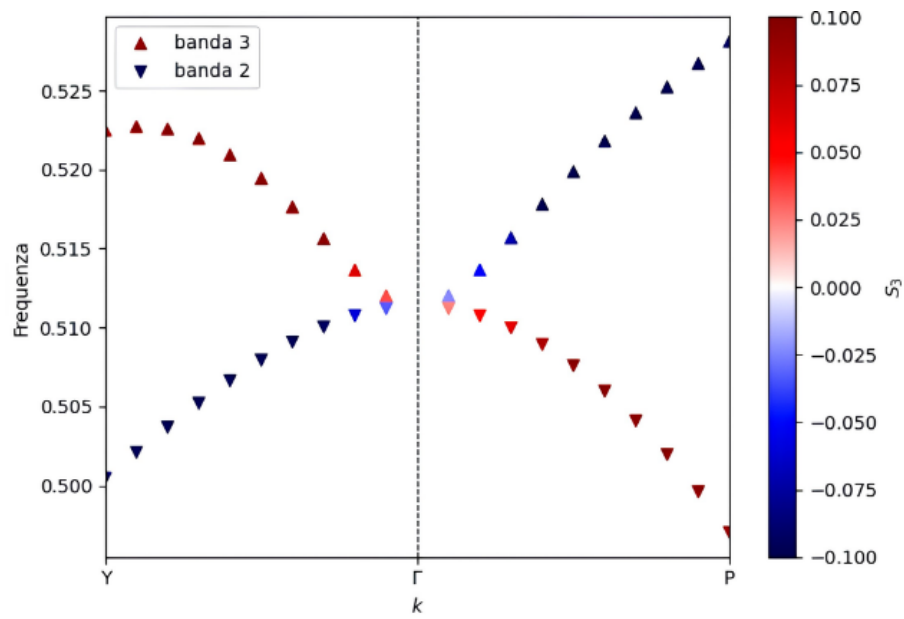


Figure 4.10: Visualization of the two lowest-energy bands from the full photonic band dispersion of the rod-in-glass structure, exhibiting a crossing at Γ . The color scale indicates the value of S_3 at each k -point in the selected neighborhood around Γ , corresponding to about 40% of the whole Brillouin Zone.

Conclusion

The results reported in the present thesis concern the successful reproduction, within a photonic system with two-dimensional periodic dielectric profile, of the spin-dependent band splittings that are characteristics of altermagnetic materials. To describe altermagnetism from a minimal theoretical model, we have analyzed a two-dimensional Rashba Hamiltonian, supplemented with a two-dimensional altermagnetic term. This effective Hamiltonian is characterized by having the required band splitting character, given by the Rashba term, as well as the typical altermagnetic spin exchange along orthogonal spatial directions ($k_x \rightarrow \Gamma \rightarrow k_y$).

For the analogue system realized in a photonic setting, the role played by spin polarization in the electronic system is found to be represented by the Stokes parameter S_3 , which is commonly associated with the degree of circular polarization of light. We have also demonstrated how such an analogy between spin and polarization degree directly originates from the similarities existing between their algebras.

In the final part of the thesis, we have simulated a realistic photonic structure, namely a photonic crystal slab, in which the behavior of the Stokes parameter S_3 across the Brillouin zone is found to reproduce all the characteristic spin textures of an altermagnetic band structure. Specifically, the exchange of polarization along orthogonal directions between two photonic bands, which in our case were chosen as the two lowest energy bands that cross at Γ point, is found to closely mimic the behavior of the spin polarization of electrons in altermagnetic models. The agreement between the theoretical assumptions, in particular considering the eigenvectors of the effective Hamiltonian as projected onto the polarization basis ($\{|TE\rangle, |TM\rangle\}$), is obtained by numerically calculating the band structure of the photonic crystal slab through the Guided Mode Expansion (GME) method. Such method further demonstrates that the essential phenomenology of the selected altermagnetic model can be encoded into specific types of crystal structures. Finally, we have directed our attention to the practical problem of realizing a photonic crystal slab structure with conventional materials, by studying the system tolerance to little variations of geometric parameters that define the structure.

For the numerical simulations shown in the work, my personal contribution has been mostly focused on building the S_3 calculation code and solving the optimization prob-

lem. In particular, the problem of S_3 calculations has been solved by taking the coupling coefficients from the GME simulation, and then using them to get the *effective* field components. A sample of the original code developed during the thesis to solve for this problem is reported in Appendix A.4, for completeness. For the optimization, the focus has been on developing the figure of merit presented in Chapter 4.1.4, which has been shown to successfully give a qualitative understanding about the goodness of the analogue realization for a given structure. The corresponding numerical code is reported in Appendix as well.

Despite constituting, to our knowledge, the first proposal for a synthetic realization of an altermagnetic system in a fully dielectric photonic environment, it is important to emphasize, however, that the effect observed in the present work does not originate from genuine time-reversal symmetry breaking, as it is the case for electronic systems. Rather, it is entirely induced by the deliberate breaking of geometric symmetries of the photonic crystal lattice. As discussed in Chapter 4.1, the design of the crystal structure was created by trying to keep the systems as classical and experimentally feasible as possible. Since in a linear and *non-gyromagnetic* optical system, the fundamental symmetries of the electromagnetic response, including time-reversal, are fully preserved, a genuine implementation of time-reversal symmetry breaking cannot be achieved through geometric symmetry breaking alone. In fact, it would require, for example, the introduction of gyromagnetic materials or other non-reciprocal elements. The latter configuration has been actually discussed very recently in the published work from Andrew S. Kim et al. [8], which has appeared only after the beginning of the present Master thesis project.

It is relevant to stress, once more, that the systems investigated in the present thesis do not constitute a faithful realization of altermagnetism, but rather a *synthetic analogue* in which the characteristic band-structure signatures of altermagnetism are reproduced through purely geometrical mechanisms. The realization of the photonic analog is not bound only by the non realization of time-reversal symmetry breaking but also by the fact that we only focused on recreating the characteristic alternating behavior for the polarization parameter, and not the exact band structure dispersion depicted by the effective Hamiltonian. Even then this result shows that geometric symmetry breaking can provide a powerful route for emulating the selected signatures of altermagnetic band structures in photonic systems, without requiring magnetic materials or intrinsic time-reversal symmetry breaking. It is precisely in this sense that the system can be regarded as a *synthetic altermagnetic analogue*: it reproduces the relevant symmetry-driven band and polarization features of altermagnetism while relying on a fundamentally different physical mechanisms. Which, in some sense, makes it even more interesting and worth further exploration.

The results presented in this Master's thesis were obtained for a slab geometry and

focused exclusively on reproducing the effects associated with the exchange characteristic. A natural continuation of this work would therefore be to further develop this photonic analogue by engineering a photonic crystal capable of more closely reproducing the dispersion relation of the effective Hamiltonian model, as shown in Fig. 3.5. In particular, the curvature of the lower energy branches should mimic two positive mass excitations, which is not the case in the results reported in Chapter 4 (see, e.g., Fig. 4.6). Such an approach would allow to analyze a more direct correspondence between the electronic model and its classical photonic counterpart. Furthermore, it may be of interest to extend the present framework to more complex systems, such as three-dimensional photonic crystal structures. However, such an extension would require the development of a new framework for simulating guided modes, since the plane-wave expansion approach employed in the present work cannot be directly applied to fully three-dimensional structures as the ones represented, e.g., in Fig. 2.2. In conclusion, this thesis has demonstrated the feasibility of realizing a purely classical photonic analogue capable of reproducing a band structure characteristic of a specific electronic system, namely altermagnetism. The analogue structure introduced in this work could serve as a useful platform for investigating existing altermagnetic systems, as well as structures that have yet to be realized.

Appendix A

Python codes developed for the thesis

Here it is reported the original Python code written specifically for this thesis work, and used to obtain one of the numerical calculations shown in the previous chapters. In particular, the present version of the code shows both the optimization process, performed by exploiting the function defined at the beginning, as well as the calculation of the Stokes parameter S_3 for each photonic eigenmode of interest. In particular, here the calculation is related to the optimized hole structure, along the two orthogonal spatial path, see Fig. 4.6. For the full description of how this code has been conceptually built, and its characteristics, we refer to the description in Chapter 4.

A.1 Code modules and functions

```
1 #-----
2 #   IMPORTING REQUIRED MODULES FOR THE SIMULATION
3 #-----
4 import sys
5 sys.path.append("/content/legume")
6 import legume
7 print(f"Version of the imported legume : {legume.__version__}")
8 from legume import PlaneWaveExp, Circle, ShapesLayer, Lattice, viz,
    Poly
9 import copy
10 import numpy as np
11 from scipy.optimize import differential_evolution
12 import pickle, os
13 import warnings
14 import matplotlib.pyplot as plt
15 import matplotlib
16 import matplotlib.gridspec as gridspec
```

```

17 import matplotlib.colors as mcolors
18 #-----
19 #   DEFINITION OF THE REQUIRED FUNCTIONS FOR THE SIMULATION
20 #-----
21 def compute_opt_s3(
22     ep_low, ep_up, high,
23     ep_bas, ep_lay, A, rot, gmode):
24     '''
25     Given the configuration parameters for the photonic crystal,
26     compute the S3 parameter for the two bands that are (quasi-)
27     degenerate at Gamma.
28     It returns an array of shape (10, N);
29     as the value of opt_S3 at points near gamma.
30
31     A, rot      : float type.
32     gmode       : list of integers for the mode indices (e.g., [0,3]
33     for TE like or [1,2,5] for TM likes).
34     '''
35     edge = (2/np.sqrt(np.sqrt(3))) * np.sqrt(A)
36     Y = np.array([0.0, np.pi*(2/np.sqrt(3))])
37     G = np.array([0.0, 0.0])
38     P = np.array([np.pi, 0.0])
39     k_til = 10
40
41     cry = legume.Lattice('hexagonal')
42     pcr = legume.PhotCryst(cry, eps_l=ep_low, eps_u=ep_up)
43     pcr.add_layer(d=high, eps_b=ep_bas)
44     khol = legume.Poly(
45         eps=ep_lay,
46         x_edges=[-edge/2, edge/2, 0],
47         y_edges=[0, 0, edge*(np.sqrt(3)/2)]
48     ).rotate(np.radians(rot))
49     pcr.add_shape([khol])
50
51     bz = cry.bz_path([Y, G, P], [40, 40])
52     Nk = int(bz['kpoints'].shape[1])
53     kspace = bz['kpoints']
54     arg_gamma = int(bz['indexes'][1]) # forces an int
55     kp = np.sqrt(kspace[0]**2 + kspace[1]**2)
56
57     if arg_gamma - k_til < 0 or arg_gamma + k_til + 1 > Nk:

```

```

56         raise ValueError(f"arg_gamma={arg_gamma} is out of range: Nk={
Nk}, k_til={k_til}")
57
58     gme = legume.GuidedModeExp(pcr, gmax=6 + 1e-10)
59     gme.run(kpoints=bz['kpoints'], gmode_inds=gmode, numeig=6, verbose
=False)
60     Nk_actual = gme.freqs.shape[0]
61     lower_en, higher_en = s3_bands(gme, n=8, banc=1e-4, gamma_idx=
arg_gamma)
62     if Nk_actual != Nk:
63         raise ValueError(f"legume has changed Nk: aspected {Nk} in {
Nk_actual}")
64     S3_bands = {}
65     for indmode in [lower_en, higher_en]:
66         S3 = np.zeros(Nk)
67         S0 = np.zeros(Nk)
68         d_x = np.zeros(Nk, dtype=complex)
69         d_y = np.zeros(Nk, dtype=complex)
70
71         kz = np.sqrt(4*np.pi**2 * epsh * gme.freqs[:, indmode]**2 - kp
**2 + 0j)
72         kz[arg_gamma] = np.sqrt(4*np.pi**2 * epsh * gme.freqs[
arg_gamma, 0]**2
73                                     - kp[arg_gamma]**2 + 0j)
74
75         du_s = np.array([safe_get(gme.rad_coup['u_te'], ik, indmode)
for ik in range(Nk)])
76         du_s[arg_gamma] = safe_get(gme.rad_coup['u_te'], arg_gamma,
indmode)
77
78         du_p = np.array([safe_get(gme.rad_coup['u_tm'], ik, indmode)
for ik in range(Nk)])
79         du_p[arg_gamma] = safe_get(gme.rad_coup['u_tm'], arg_gamma,
indmode)
80
81         for i in range(Nk):
82             if i == arg_gamma:
83                 continue
84             if du_p[i] != 0 or du_s[i] != 0:
85                 d_te_i = abs(du_s[i])

```

```

86         phase = (abs(du_s[i]) / du_s[i]) if du_s[i] != 0 else
1
87         d_tm_i = du_p[i] * phase
88
89         mod_k = np.sqrt(kp[i]**2 + kz[i]**2)
90         d_x[i] = (-d_te_i * kspace[1][i] / kp[i]
91                 + kspace[0][i] * kz[i] * d_tm_i / kp[i] /
mod_k)
92         d_y[i] = ( d_te_i * kspace[0][i] / kp[i]
93                 + kspace[1][i] * kz[i] * d_tm_i / kp[i] /
mod_k)
94
95         S0[i] = abs(d_x[i])**2 + abs(d_y[i])**2
96         if S0[i] != 0:
97             S3[i] = -2 * np.imag(d_x[i] * np.conj(d_y[i])) /
S0[i]
98
99         S3_bands[indmode] = S3
100
101         i_y = np.arange(arg_gamma - k_til, arg_gamma)
102         i_p = np.arange(arg_gamma + 1, arg_gamma + k_til + 1)
103
104         diff_Y = S3_bands[lower_en][i_y] - S3_bands[higher_en][i_y]
105         diff_P = S3_bands[lower_en][i_p] - S3_bands[higher_en][i_p]
106
107         opt_S3 = diff_Y[:-1] * diff_P
108         return opt_S3
109 def objective_hol(params):
110     A, rot = params
111     try:
112         opt_S3 = compute_opt_s3(epsh, epsh, H, epss, epsh, A, rot,
[0,3])
113         val = float(np.sum(opt_S3)) # need to sum it up cause the
optimazer need a scalar
114         return val
115     except Exception as e:
116         print(f"Raise error {params}: {e}")
117         return 1.0 # positive penality so the optimaizer ignores it
118 #-----
119 # DEFINITION OF THE UNIVERSAL CONSTANTS FOR THE SIMULATION
120 #-----

```

```

121 a =1      # lattice constant
122 epsh = 1**2      # hole permittivity
123 epss = 3.48**2    # SiN slab permittivity
124 H = 0.3      # slab thickness

```

A.2 Optimization Process

```

1  #-----
2  #   RUN THE OPTIMIZATION PROCESS USING DIFFERENTIAL EVOLUTION
3  #-----
4  bounds = [
5      (0.1, 0.6),    # A
6      (0, 120),      # rot (gradi)
7  ]
8  result = differential_evolution(
9      objective_hol,
10     bounds,
11     strategy      = 'best1bin',
12     maxiter       = 30,
13     popsize       = 6,
14     tol           = 1e-3,
15     mutation      = (0.5, 1.0),
16     recombination = 0.7,
17     seed          = 42,
18     workers       = 1,
19     disp          = True,
20     polish        = False,
21 )
22 A_opt, rot_opt = result.x
23 print(f"A           = {A_opt:.4f}")
24 print(f"rot         = {rot_opt:.4f} deg")
25 print(f"sum(opt_S3) = {result.fun:.6f}")

```

A.3 Creation of optimized structure

```

1  #
   -----
2  #   CREATION OF THE PHOTONIC CRYSTAL STRUCTURE

```

```

3  #   AND RUN THE SIMULATION
4  #
   -----

5  angle = rot_opt    # rotation angle for the triangle (in radians)
6  area = A_opt       # area of triangular holes
7  crlat = legume.Lattice('hexagonal')
8  phtcn = legume.PhotCryst(crlat,eps_l = epsh, eps_u = epsh)
9  phtcn.add_layer(d=H, eps_b=epss)
10 edge = (2/np.sqrt(np.sqrt(3)))*np.sqrt(area)
11 hole = legume.Poly(
12     eps=epsh,x_edges=[- edge/2, edge/2, 0],
13     y_edges=[0, 0, edge * (np.sqrt(3)/2)]).rotate(np.radians(angle))
14 phtcn.add_shape([hole])
15 b1 = np.array(crlat.b1)
16 b2 = np.array(crlat.b2)
17 # Coordinates of highest symmetry points
18 G = np.array([0.0, 0.0])
19 X = b1 / 2
20 Y = b2 / 2
21 P = np.array([np.pi, 0.0])          # (, 0)   proiection of M (or X)
   point on x-axis
22 M = (b1 + b2) / 2
23 camm = crlat.bz_path([Y, G, P], [40, 40])
24 camm['labels'] = ['Y', '', 'P']
25 # Grid 2D centered on Gamma
26 idx_gamma = camm['indexes'][1]
27 k_gamma = camm['kpoints'][:, idx_gamma] # take Gamma coordinates from
   path
28 npoi = 10
29 dk_step = np.linalg.norm(camm['kpoints'][:, idx_gamma+1] - camm['
   kpoints'][:, idx_gamma])    # same spacing of 1D path
30 delta_k = npoi * dk_step      # fraction of BZ
31 dk = np.linspace(-delta_k, delta_k, 2 * npoi + 1)
32 kx_grid = k_gamma[0] + dk
33 ky_grid = k_gamma[1] + dk
34 kspace_intorno = np.array(np.meshgrid(kx_grid, ky_grid)).reshape(2,
   len(dk)**2)
35 print(f"Shape of kspace_intorno: {kspace_intorno.shape}")
36 # Set some of the running options
37 options = {'gmode_inds': [0, 3], #(TE modes) while [1, 2, 5] (TM modes

```

```

    )
38         'numeig': 4,      # number of Bloch bands to store
39         'verbose': False #Print information at intermediate steps
40     }
41 # Run the simulation
42 guid = legume.GuidedModeExp(phtcn, gmax= 4 + 1e-10)
43 guid.run(kpoints=kspace_intorno, **options)
44 d_gamma = 0      # Gamma point (kx= 0, ky =0)
45 klen = len(dk)
46 for i in np.arange(klen**2):
47     if kspace_intorno[0][i]==0:
48         if kspace_intorno[1][i]==0:
49             d_gamma=i

```

A.4 Calculation of Stokes parameters and plotting

```

1  #-----
2  #   CALCULATION OF THE S3 PARAMETER FOR THE TWO BANDS
3  #-----
4  S0 = np.zeros(klen**2)
5  S3 = np.zeros((2,klen**2))
6  d_te = np.zeros(klen**2,dtype=complex)
7  d_tm = np.zeros(klen**2,dtype=complex)
8  d_x = np.zeros(klen**2,dtype=complex)
9  d_y = np.zeros(klen**2,dtype=complex)
10 for indm in [2,3]:
11     kdiag = np.sqrt(kspace_intorno[0]**2 + kspace_intorno[1]**2)      #
        elements on diagonal
12     kort = np.sqrt(4*np.pi**2*epsh*guid.freqs[:, indm]**2-kdiag**2+0j)
        # definition of kz to evaluate the loss
13     kort[d_gamma] = np.sqrt(4*np.pi**2*epsh*guid.freqs[d_gamma, 0]**2-
        kdiag[d_gamma]**2+0j) # for Gamma point
14     # Coupling constant to lower-cladding s modes
15     dl_s = np.array([guid.rad_coup['l_te'][ik][indm][0] if len(guid.
        rad_coup['l_te'][ik][indm]) > 0 else 0 for ik in range(klen**2)])
16     dl_s[d_gamma] = guid.rad_coup['l_te'][d_gamma][0][0] if len(guid.
        rad_coup['l_te'][d_gamma][0]) > 0 else 0
17     # Coupling constant to lower-cladding p modes
18     dl_p = np.array([guid.rad_coup['l_tm'][ik][indm][0] if len(guid.
        rad_coup['l_tm'][ik][indm]) > 0 else 0 for ik in range(klen**2)])

```

```

19     dl_p[d_gamma] = guid.rad_coup['l_tm'][d_gamma][0][0] if len(guid.
rad_coup['l_tm'][d_gamma][0]) > 0 else 0
20     for i in range(klen**2):
21         # Skip Gamma point to avoid division by zero (kp=0)
22         if i == d_gamma:
23             continue
24         # Check if the coupling is non-zero
25         if dl_p[i] != 0 or dl_s[i] != 0:
26             d_te[i] = abs(dl_s[i])
27             # Avoid division by zero if du_s is 0
28             phase_factor = (abs(dl_s[i])/dl_s[i]) if dl_s[i] != 0 else
1
29             d_tm[i] = dl_p[i] * phase_factor
30             mod_k = np.sqrt(kdiag[i]**2 + kort[i]**2)
31             d_x[i] = -d_te[i]*kspace_intorno[1][i]/kdiag[i] +
kspace_intorno[0][i]*kort[i]*d_tm[i]/kdiag[i]/mod_k
32             d_y[i] = d_te[i]*kspace_intorno[0][i]/kdiag[i] +
kspace_intorno[1][i]*kort[i]*d_tm[i]/kdiag[i]/mod_k
33             #phi[i] = np.angle(d_tm[i]) - np.angle(d_te[i])
34             S0[i] = (abs(d_x[i])**2 + abs(d_y[i])**2)
35             if S0[i] != 0:
36                 S3[indm - 2, i] = -2*np.imag(d_x[i]*np.conjugate(d_y[i
]))/S0[i]
37             if S0[i] == 0:
38                 S3[indm - 2, i] = 0
39     #-----
40     #   DEFINITION OF THE PATH TO PLOT S3 OVER BAND 2 AND 3
41     #-----
42     # Recreate the Y -> G -> P path but it can be any type
43     kx_flat = kspace_intorno[0]
44     ky_flat = kspace_intorno[1]
45     tol = (kx_flat[1] - kx_flat[0]) / 2
46     # Rebuild the path Y -> G -> P
47     # Y -> G
48     idx_YG = np.where(np.isclose(kx_flat, 0, atol=tol) & (ky_flat >= 0))
[0]
49     idx_YG = idx_YG[np.argsort(ky_flat[idx_YG])[:-1]]
50     # G -> P (except kx=0 to not replicate Gamma)
51     idx_GM = np.where(np.isclose(ky_flat, 0, atol=tol) & (kx_flat >= 0))
[0]
52     idx_GM = idx_GM[np.argsort(kx_flat[idx_GM])]

```

```

53 idx_path = np.concatenate([idx_YG, idx_GM[1:]])
54 dk = np.linalg.norm(np.diff(kspace_intorno[:, idx_path], axis=1), axis
    =0)
55 k_pa = np.concatenate([[0], np.cumsum(dk)])
56 k_gamma = k_pa[len(idx_YG) - 1]
57 n = int(np.sqrt(kspace_intorno.shape[1]))
58 lower_lim = -1
59 upper_lim = 1
60 #-----
61 #   PLOTTING THE PHOTONIC SLAB BAND STRUCTURE
62 #-----
63 fig, ax2d = plt.subplots(figsize=(14, 6))
64 sc2 = ax2d.scatter(k_pa, guid.freqs[idx_path, 2], c=S3[0, idx_path],
    cmap='jet', vmin=lower_lim, vmax=upper_lim, label='banda 2', marker
    = '^')
65 sc3 = ax2d.scatter(k_pa, guid.freqs[idx_path, 3], c=S3[1, idx_path],
    cmap='jet', vmin=lower_lim, vmax=upper_lim, label='banda 3', marker
    = 'v')
66 ax2d.axvline(k_gamma, color='k', linestyle='--', linewidth=0.8)
67 ax2d.set_xlabel(r'$k_x / \pi$')
68 ax2d.set_ylabel('Frequenza')
69 ax2d.set_title(r'Dispersione $S_3$', loc='left')
70 ax2d.set_xticks([k_pa[0], k_gamma, k_pa[-1]])
71 ax2d.set_xticklabels(['Y', r'$\Gamma$', 'P'])
72 fig.colorbar(sc2, ax=ax2d, label=r'$S_3$')
73 ax2d.legend()
74 plt.tight_layout()
75 plt.show()

```

Bibliography

- [1] L. C. Andreani and D. Gerace. “Photonic-crystal slabs with a triangular lattice of triangular holes investigated using a guided-mode expansion method”. In: *Phys. Rev. B* 73 (23 2006), p. 235114. DOI: 10.1103/PhysRevB.73.235114. URL: <https://link.aps.org/doi/10.1103/PhysRevB.73.235114>.
- [2] H.-C. Chan and G.-Y. Guo. “Tuning topological phase transitions in hexagonal photonic lattices made of triangular rods”. In: *Physical Review B* 97.4 (2018). ISSN: 2469-9969. DOI: 10.1103/physrevb.97.045422. URL: <http://dx.doi.org/10.1103/PhysRevB.97.045422>.
- [3] W. Chen, Q. Yang, Y. Chen, and W. Liu. “Extremize Optical Chiralities through Polarization Singularities”. In: *Phys. Rev. Lett.* 126 (25 2021), p. 253901. DOI: 10.1103/PhysRevLett.126.253901. URL: <https://link.aps.org/doi/10.1103/PhysRevLett.126.253901>.
- [4] S. Fan, W. Suh, and J. D. Joannopoulos. “Temporal coupled-mode theory for the Fano resonance in optical resonators”. In: *J. Opt. Soc. Am. A* 20.3 (2003), pp. 569–572. DOI: 10.1364/JOSAA.20.000569.
- [5] J. D. Jackson. *Classical electrodynamics, 3rd ed.* American Association of Physics Teachers, 1999.
- [6] J. D. Joannopoulos, R. D. Meade, J. N. Winn, and P. S. J. Russell. *Photonic crystals: Molding the flow of light*. 6580. London: Macmillan Journals, 1996, p. 290.
- [7] S. G. Johnson. *Photonic crystals : from theory to practice*. Massachusetts Institute of Technology, 2001. URL: <https://api.semanticscholar.org/CorpusID:61941261>.
- [8] A. S. Kim, Y. Huang, Z. Sun, Q.-H. Park, and H. Choi. “Photonic Altermagnets: Magnetic Symmetries in Photonic Structures”. In: *Nano Letters* 26.11 (2026), pp. 3681–3687. ISSN: 1530-6984. DOI: 10.1021/acs.nanolett.5c05516.
- [9] M. Liscidini, D. Gerace, L. C. Andreani, and J. E. Sipe. “Scattering-matrix analysis of periodically patterned multilayers with asymmetric unit cells and birefringent media”. In: *Phys. Rev. B* 77 (3 Jan. 2008), p. 035324. DOI: 10.1103/

- PhysRevB.77.035324. URL: <https://link.aps.org/doi/10.1103/PhysRevB.77.035324>.
- [10] P. Mazzoldi, M. Nigro, and C. Voci. *Fisica: Elettromagnetismo e Onde*, 3rd ed. EdiSES, 2021.
 - [11] M. Minkov, I. A. D. Williamson, L. C. Andreani, D. Gerace, B. Lou, A. Y. Song, T. W. Hughes, and S. Fan. “Inverse Design of Photonic Crystals through Automatic Differentiation”. In: *ACS Photonics* 7.7 (2020), pp. 1729–1741. ISSN: 2330-4022. DOI: 10.1021/acsp Photonics.0c00327. eprint: <https://pubs.acs.org/apchd5/article-pdf/7/7/1729/14415851/ph0c00327.pdf>. URL: <https://doi.org/10.1021/acsp Photonics.0c00327>.
 - [12] M. Roig, A. Kreisel, Y. Yu, B. M. Andersen, and D. F. Agterberg. “Minimal models for altermagnetism”. In: *Physical Review B* 110.14 (2024). ISSN: 2469-9969. DOI: 10.1103/physrevb.110.144412. URL: <http://dx.doi.org/10.1103/PhysRevB.110.144412>.
 - [13] K. Sakoda. *Optical Properties of Photonic Crystals*. Springer-Verlag Berlin and Heidelberg GmbH Co. KG, 2005, p. 492. DOI: 10.1016/j.ijleo.2005.07.002.
 - [14] B. Schaefer, E. Collett, R. Smyth, D. Barrett, and B. Fraher. “Measuring the Stokes polarization parameters”. In: *American Journal of Physics* 75.2 (2007), pp. 163–168.
 - [15] L. Šmejkal, J. Sinova, and T. Jungwirth. “Beyond Conventional Ferromagnetism and Antiferromagnetism: A Phase with Nonrelativistic Spin and Crystal Rotation Symmetry”. In: *Physical Review X* 12 (2022). DOI: 10.1103/PhysRevX.12.031042.
 - [16] L. Šmejkal, J. Sinova, and T. Jungwirth. “Emerging Research Landscape of Altermagnetism”. In: *Phys. Rev. X* 12 (4 2022), p. 040501. DOI: 10.1103/PhysRevX.12.040501. URL: <https://link.aps.org/doi/10.1103/PhysRevX.12.040501>.
 - [17] M. Trama, I. Gaiardoni, C. Guarcello, J. I. Facio, A. Maiellaro, F. Romeo, R. Citro, and J. van den Brink. “Nonlinear anomalous Edelstein response at altermagnetic interfaces”. In: *Phys. Rev. B* 112 (18 2025), p. 184404. DOI: 10.1103/sxyb-5rcy. URL: <https://link.aps.org/doi/10.1103/sxyb-5rcy>.
 - [18] L. Zagaglia, S. Zanotti, M. Minkov, M. Liscidini, D. Gerace, and L. C. Andreani. “Polarization states and far-field optical properties in dielectric photonic crystal slabs”. In: *Opt. Lett.* 48.19 (2023), pp. 5017–5020. DOI: 10.1364/OL.501595. URL: <https://opg.optica.org/ol/abstract.cfm?URI=ol-48-19-5017>.

- [19] S. Zanotti. “Light-matter interaction in photonic metasurfaces”. PhD thesis. University of Pavia, 2023.
- [20] S. Zanotti, M. Minkov, D. Nigro, D. Gerace, S. Fan, and L. C. Andreani. “Legume: A free implementation of the guided-mode expansion method for photonic crystal slabs”. In: *Computer Physics Communications* 304 (2024), p. 109286. ISSN: 0010-4655. DOI: <https://doi.org/10.1016/j.cpc.2024.109286>. URL: <https://www.sciencedirect.com/science/article/pii/S0010465524002091>.