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Ground State Preparation of the Schwinger Model on Digital Quantum Computers

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Abstract

Quantum simulation offers a new perspective for studying physical systems whose complexity makes them difficult to simulate using conventional computational methods. In particular, the use of digital quantum computers makes it possible to implement universal quantum simulators. A particularly interesting potential application is the study of gauge theories, which underlie the description of fundamental interactions and for which non-perturbative phenomena can be difficult to address using traditional analytical and numerical methods. This thesis investigates the digital quantum simulation of the Schwinger model, a gauge theory in one spatial and one temporal dimension, which represents an important theoretical toy model for the study of such phenomena.

The problem is formulated on a spatial lattice and mapped, through Gauss's law and the Jordan–Wigner transformation, onto an equivalent spin system. The ground state, corresponding to the vacuum state of the theory, is prepared using an adiabatic procedure, whose evolution is discretized using a Trotter formula. Numerical results show that the protocol allows the ground state to be prepared with high accuracy, reaching fidelities of up to 99% for different sizes of the discrete lattice. Additionally, the evaluation of spatial two-point correlation functions of the electric field demonstrates that the autocorrelation is accurately estimated within a 3% relative error, while cleanly isolating the deterministic Trotter discretization error. However, the analysis of the quantum circuits required for this digital quantum simulation reveals a rapid increase in circuit depth as the system size grows, highlighting one of the main challenges for implementation on real quantum devices.

Finally, the thesis investigates how the vacuum state prepared in this way can be used to extract the mass of the Schwinger boson, a quantity that can be calculated analytically in the continuum limit and can therefore provide a benchmark for assessing the accuracy of the simulation. Two strategies are considered: extracting the mass from correlation functions and determining it from the energy difference between the ground state and the first excited state. The main challenges associated with these procedures are discussed, together with possible strategies for improving their accuracy, including the use of higher-order Trotter formulas and symmetry-guided protocols.

Abstract

La simulazione quantistica offre una nuova prospettiva per studiare sistemi fisici la cui complessità rende difficile la simulazione con i metodi di calcolo convenzionali. In particolare, l'utilizzo di calcolatori quantistici digitali consente di realizzare dei simulatori quantistici universali. Un ambito di potenziale applicazione particolarmente interessante è quello delle teorie di gauge, alla base della descrizione delle interazioni fondamentali, per le quali i fenomeni non perturbativi possono essere difficili da trattare con metodi analitici e numerici tradizionali. Questa tesi studia la simulazione quantistica digitale del modello di Schwinger, una teoria di gauge in una dimensione spaziale e una temporale, considerata un importante modello teorico per lo studio di questi fenomeni.

Il problema viene formulato su un reticolo spaziale e ricondotto, attraverso la legge di Gauss e la trasformazione di Jordan–Wigner, a un sistema equivalente di spin. Lo stato fondamentale, corrispondente allo stato di vuoto della teoria, viene preparato mediante una procedura adiabatica, la cui evoluzione viene discretizzata attraverso l'uso della formula di Trotter. I risultati numerici mostrano che il protocollo permette di preparare lo stato fondamentale con elevata accuratezza, raggiungendo valori di fedeltà fino al 99% per diverse dimensioni del reticolo discreto. Inoltre, la valutazione delle funzioni di correlazione spaziale a due punti del campo elettrico dimostra che l'autocorrelazione viene stimata con accuratezza entro un errore relativo del 3%, permettendo di isolare nettamente l'errore deterministico di discretizzazione di Trotter. L'analisi dei circuiti quantistici necessari per questa simulazione quantistica digitale evidenzia tuttavia una rapida crescita della loro profondità all'aumentare delle dimensioni del sistema, mettendo in luce una delle principali difficoltà per l'implementazione su dispositivi quantistici reali.

Infine, viene studiato come utilizzare lo stato di vuoto così preparato per estrarre la massa del bosone di Schwinger, una quantità che può essere calcolata analiticamente nel limite continuo e quindi fornire un riferimento per valutare l'accuratezza della simulazione. Vengono considerate due strategie: l'estrazione della massa dalle funzioni di correlazione e la sua determinazione attraverso la differenza di energia tra lo stato fondamentale e il primo stato eccitato. Sono infine discusse le principali difficoltà associate a queste procedure e alcune possibili strategie per migliorarne l'accuratezza, tra cui l'impiego di formule di Trotter di ordine superiore e di protocolli che sfruttano le simmetrie del sistema.

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Introduction

Modern theoretical physics relies on Quantum Field Theory as the fundamental mathematical framework for describing elementary particles and their fundamental interactions, except for gravity. Since the foundational introduction of quantum perturbation theory by Paul Dirac in 1927 [1, 2, 3] and its subsequent systematic diagrammatic formulation for field theories by Richard Feynman in 1949 [4, 2, 3], perturbative methods have achieved remarkable success in weak-coupling regimes such as Quantum Electrodynamics. However, perturbative expansions systematically fail when applied to non-perturbative problems. In regimes where the interaction strength is large or where physical observables depend non-analytically on the coupling constant, physical quantities cannot be accessed through series expansions in the coupling parameter. To investigate these non-perturbative regimes without relying on perturbative approximations, Kenneth Wilson in 1974 introduced the formulation of Lattice Gauge Theories [5, 2, 3]. By discretizing continuous spacetime onto a grid with finite lattice spacing a , Lattice Gauge Theories provide the only mathematically rigorous and non-perturbative regularization scheme for describing strongly coupled regimes such as Quantum Chromodynamics (QCD). In Hamiltonian formulations, such as the Kogut-Susskind staggered lattice approach [6, 2, 3] developed in 1975, spatial coordinates are discretized while continuous time evolution is preserved, explicitly retaining local gauge symmetry on the lattice links.

For several decades, classical supercomputers employing Markov Chain Monte Carlo sampling techniques within Euclidean spacetime path integrals have served as the primary computational approach for Lattice Gauge Theories. Numerical calculations on Euclidean grids have successfully delivered high-precision static predictions. However, classical computational methods encounter severe intrinsic barriers when tasked with probing real-time dynamics and non-equilibrium phenomena. For fermionic fields, the fundamental bottleneck of classical Monte Carlo simulations stems from the sign problem, or complex action problem. When Wick-rotated to real time, the quantum evolution operator manifests as an oscillating complex phase in the path integral integrand, rather than a positive-definite Boltzmann probability weight. Consequently, probabilistic importance sampling degrades catastrophically because positive and negative statistical terms cancel each other out, causing the statistical signal-to-noise ratio to decay exponentially with spacetime volume and system size. Furthermore, the state space of quantum many-body systems scales exponentially with the number of constituents N . A quantum state of N spin-1/2 degrees of freedom or staggered fermionic sites inhabits a Hilbert space of dimension 2^N , requiring 2^N complex amplitudes for an exact classical state representation. Storing a state vector of even 50 qubits saturates the memory capacity of modern high-performance computing clusters. In spatial dimensions $d \geq 2$, where spatial entanglement scales extensively, classical approximation techniques

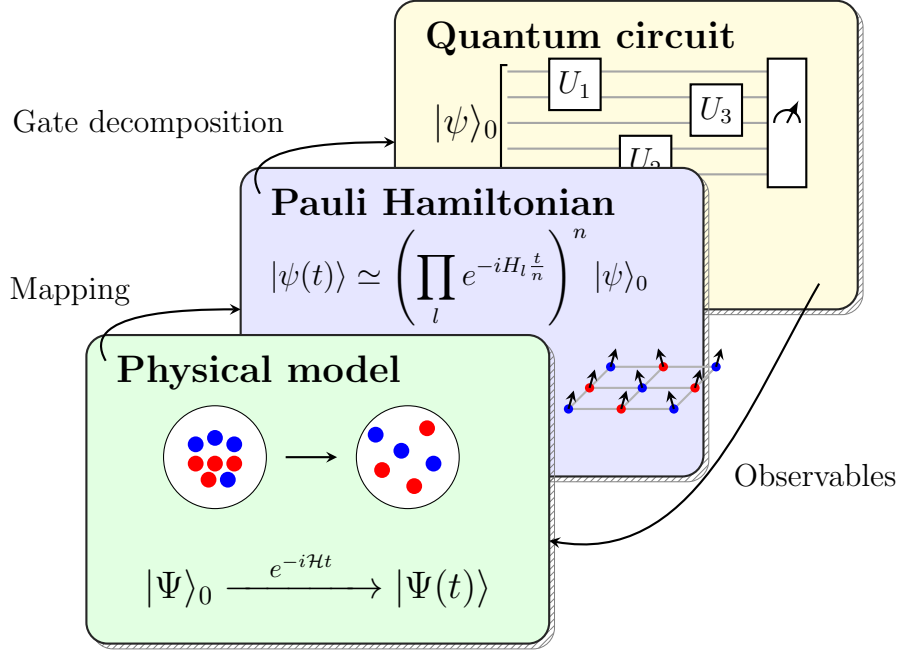


Figure 1: Conceptual illustration of a universal simulator implemented on a digital quantum computing device, inspired from Fig. 1 in Ref. [7]. The quantum state evolution of a physical model can be approximated to arbitrary precision by mapping the given model on a spin-type model (which can be easily encoded, e.g., onto a qubit-based register), and slicing the time evolution according to the Trotter formula (see Sec. 2.2.4); the sequence of unitary operations can then be programmed through a quantum circuit model to be directly run on a quantum computer, giving the approximated evolved state as an output.

such as tensor networks suffer from an exponential growth in bond dimension during real-time evolution, preventing efficient long-time classical simulation.

To circumvent these computational barriers, Richard Feynman in 1982 [8, 7] conjectured that physical quantum systems could be efficiently simulated using computational devices operating according to the laws of quantum mechanics, a concept later formalized by Seth Lloyd in 1996 [9, 7], who proved that universal quantum computers can simulate local quantum Hamiltonians with polynomial resource overhead. In Digital Quantum Simulation, target physical degrees of freedom are mapped onto a register of physical qubits. Continuous unitary time evolution is approximated by a discrete sequence of quantum logic gates using Suzuki-Trotter operator decompositions [10, 11, 7]. This workflow is summarized in Fig. 1. Because quantum hardware directly manipulates quantum superpositions and entanglement in real physical time, the dynamics and ground state properties of field theories can be simulated without encountering a sign problem. To ensure that quantum algorithms remain applicable across evolving hardware platforms, it is vital to design platform-agnostic, hardware-independent digital protocols. Rather than relying on the fixed physical interactions of an analog emulator, digital quantum simulation formulates algorithms within the general-purpose gate-based model satisfying DiVincenzo’s criteria [12, 7, 13]. In a platform-agnostic digital protocol, the target field theory is mapped onto an abstract Pauli spin algebra via canonical transformations such as the Jordan-Wigner transformation [14, 13]. The

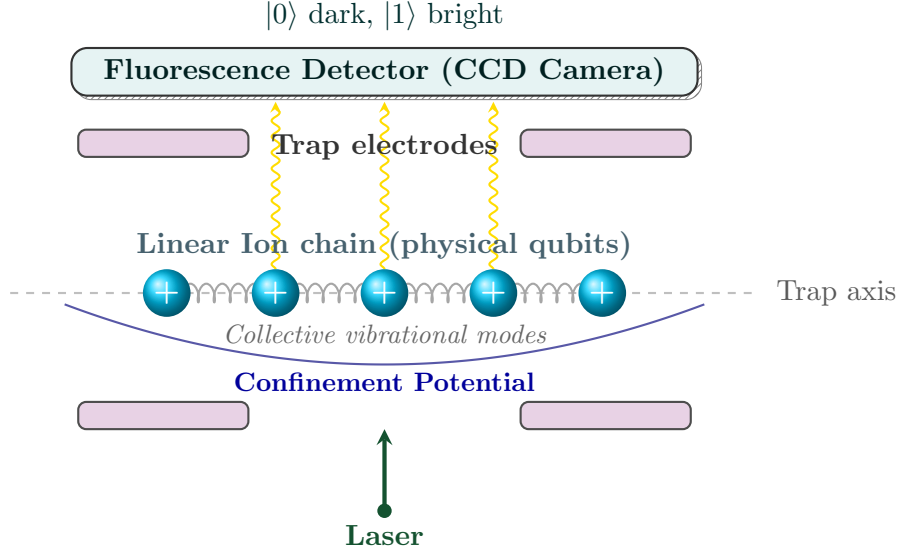


Figure 2: Schematic diagram of a linear trapped-ion quantum computer architecture. Physical qubits are encoded in the internal energy states of ions arranged in a 1D chain along the trap axis, confined by an electrode potential. Multi-qubit operations are mediated by collective vibrational modes, enabling long-range or all-to-all connectivity. Coherent operations and gate manipulations are addressed using targeted control laser beams, while state readout is achieved via state-dependent fluorescence detection captured by a CCD camera.

algorithm is constructed using a standard universal gate set composed of single-qubit rotations and entangling CNOT or CZ gates. Once codified as an abstract quantum circuit, the protocol can be executed on any gate-based quantum architecture via transpilation layers, irrespective of the underlying physical platform.

In the current Noisy Intermediate-Scale Quantum era, two physical architectures [7] lead the pursuit of practical digital quantum simulation: Trapped Atomic Ions and Superconducting Circuits.

Trapped-ion platforms [7, 15] confine charged atomic species in ultra-high vacuum using oscillating radio-frequency electric fields in linear Paul traps. Qubits are encoded into stable internal electronic energy states, specifically hyperfine ground-state splittings or optical quadrupole transitions. Single-qubit operations are implemented by driving resonant optical laser transitions or microwave Raman pulses with high precision; multi-qubit entangling gates, instead, are mediated by the collective vibrational modes, or phonons, of the ion string within the harmonic trapping potential. Trapped ions feature exceptionally long coherence times ranging from seconds to minutes and state-of-the-art gate fidelities exceeding 99.9% for single-qubit operations and 99.5% for two-qubit gates. Critically, phonon-mediated interactions enable flexible, reconfigurable all-to-all connectivity across the entire linear string, making trapped ions exceptionally suited for executing digital algorithms involving non-local or long-range spin interactions, such as those arising from gauge-field elimination.

Superconducting quantum computers [7, 16] utilize solid-state microelectronic circuits micro-fabricated on silicon or sapphire substrates, operated inside dilution refrigerators at cryogenic temperatures around 10 to 15 mK. Qubits are encoded in the non-linear

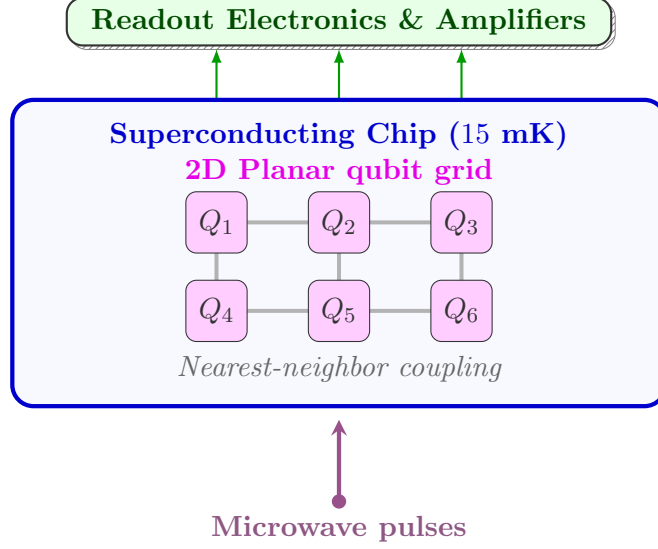


Figure 3: Schematic representation of a superconducting quantum processor architecture. A 2D planar grid of transmon qubits ($Q_1 \dots Q_6$) with nearest-neighbor couplings is integrated on a superconducting chip maintained at cryogenic temperatures (≈ 15 mK). Single-qubit rotations are driven via microwave pulses, while state readout is performed via room-temperature readout electronics and cryogenic amplifiers.

anharmonic energy levels of superconducting LC-resonators, where non-dissipative Josephson junctions act as non-linear inductors. In this context, qubit states are manipulated via microwave pulses delivered through high-frequency transmission lines. Superconducting circuits offer ultra-fast gate operation times on the order of 10 to 100 ns, allowing large numbers of gate operations within short execution windows. Leveraging established semiconductor fabrication technologies, superconducting processors scale efficiently in two-dimensional planar grids, though coherence times are shorter, typically ranging between 50 and 150 μ s; moreover, connectivity is generally limited to nearest-neighbor geometry and spectral frequency crowding presents ongoing engineering challenges.

This thesis addresses the digital quantum simulation of the ground state preparation for an important theoretical toy model: the Schwinger model, representing Quantum Electrodynamics in 1+1 dimensions, formulated on a Kogut-Susskind staggered lattice with periodic boundary conditions. By analytically integrating out the gauge field degrees of freedom using Gauss’s Law, the gauge field is eliminated by the evolving dynamics, transforming the theory into an effective spin-1/2 Hamiltonian with long-range Coulomb-like interactions mapped via the Jordan-Wigner transformation. We develop a successful platform-agnostic digital pipeline employing Adiabatic Quantum State Preparation (AQSP), discretized through first-order Trotter product formulas, allowing the faithful preparation of the ground state of the theory.

The structure of this work is organized across three main chapters. The first Chapter provides the formal field-theoretic formulation of Lattice Gauge Theories, detailing the Kogut-Susskind staggered fermion discretization, and derives the long-range spin representation of the (1+1)D Schwinger model upon gauge field elimination. The second Chapter presents the framework of Digital Quantum Simulation, introducing

universal quantum circuits, gate decompositions, readout methods and the principles of Adiabatic Quantum State Preparation and Trotterization. The third chapter details the implementation of the AQSP protocol for the Schwinger vacuum state, presenting numerical results validating state preparation against exact classical diagonalization, achieving Uhlmann fidelities up to 99% across various system sizes N and lattice spacings a . Furthermore, we evaluate the spatial two-point correlation functions of the electric field, accurately estimating the autocorrelation at $r = 1$ within a 3% relative error and cleanly isolating the deterministic Trotter discretization error. Finally, we propose also a primary application of the simulated Schwinger vacuum, namely extracting the mass of the Schwinger vector boson m_S , which dynamically emerges via (1+1)D bosonization. We evaluate two extraction pathways, comparing spatial correlation decay against direct spectral gap measurements, and analyze operational bottlenecks and mitigation strategies for ideal digital quantum hardware.

Chapter 1

Lattice Gauge Theories and the Schwinger Model

This chapter establishes the theoretical foundation required for the non-perturbative formulation and digital quantum simulation of lattice gauge theories, focusing on Quantum Electrodynamics in $1 + 1$ dimensions, which is encoded in the Schwinger model. Beginning with the Path Integral representation of Green functions for scalar and Dirac fields, we introduce the discretization of continuous spacetime onto a discrete lattice as a mathematically rigorous regularization scheme. We detail the gauge-invariant formulation of Abelian gauge fields via Schwinger line-integrals and plaquette variables, followed by the Kogut–Susskind staggered fermion approach to overcome the fermion doubling problem. Finally, we analytically integrate out the gauge field degrees of freedom using Gauss’s Law in $1 + 1$ dimensions.

1.1 Path Integral representation of Green functions

In classical mechanics, the evolution of a system may be described through either the Lagrangian formalism, relying on the Euler–Lagrange equations, or the Hamiltonian formalism. Both approaches arise naturally from the Principle of Least action, which postulates that the system follows the trajectory that minimizes the action functional $S(\vec{q})$, that is the path that guarantees that $\delta S(\vec{q}) = 0$. In non-relativistic Quantum Mechanics, within the Hamiltonian formalism, the dynamics of a system is governed by the Schrödinger equation, which determines the time evolution of the system’s state and the corresponding probability amplitudes. Consequently, in order to reestablish the connection with the classical action principle, it is necessary to reformulate the principle of Least Action in a manner that reflects the fact that the system exists simultaneously in a superposition of different possible states. This is achieved through the introduction of the Path Integral formalism [2, 13, 3], which describes the time evolution of a quantum system as a sum over all possible instantaneous trajectories, each weighted by a different probability amplitude. In order to reconstruct the evolution of a quantum system, we henceforth exploit the formalism of Green functions and analyze their meaningful representation under the Path Integral approach [2, 3, 13].

Given a quantum system in a Hilbert space $\mathcal{H}$, we can study it by choosing a set of physical observables $\{\hat{Q}_\alpha\}$ (where the index α accounts for the n degrees of freedom of the system) that share a common spectral resolution and act on their simultaneous

eigenstates $|q\rangle$ as

$$\hat{Q}_\alpha |q\rangle = q_\alpha |q\rangle \quad \forall \alpha = 1, \dots, n$$

In order to derive an expression for the Green functions, we can start by assuming that the Hamiltonian that describes the evolution of our system is time-independent so that we can determine its state at a time t' and the corresponding probability amplitudes in the following way. Taking the Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle$$

we can notice that it has the same mathematical form as a linear ordinary differential equation; therefore, we can solve it by separating the variables and then integrate it in order to find the logarithmic solution. In fact, by separating the variables t and $|\psi(t)\rangle$, we have that

$$\frac{d|\psi(t)\rangle}{|\psi(t)\rangle} = -\frac{i}{\hbar} H dt$$

and we can integrate this expression considering a generic time-interval (t, t') as

$$\int_{|\psi(t)\rangle}^{|\psi(t')\rangle} \frac{d|\psi(s)\rangle}{|\psi(s)\rangle} = - \int_t^{t'} \frac{i}{\hbar} H ds$$

obtaining that

$$\ln \left(\frac{|\psi(t')\rangle}{|\psi(t)\rangle} \right) = -iH(t' - t)$$

from which it follows that

$$|\psi(t')\rangle = U(t', t) |\psi(t)\rangle = e^{-\frac{i}{\hbar} H(t'-t)} |\psi(t)\rangle \quad (1.1)$$

Multiplying both sides of Eq. (1.1) equation by $\langle q'|$, that is the simultaneous eigenstate of the set $\{\hat{Q}_\alpha\}$ at a time t' , we obtain

$$\psi(q', t') \equiv \langle q' | \psi(t') \rangle = \langle q' | e^{-\frac{i}{\hbar} H(t'-t)} | \psi(t) \rangle$$

Then, exploiting the fact that we can express the Identity operator as $\mathbb{I} = \int dq |q\rangle \langle q|$ where the integration measure is given by $dq = \prod_{\alpha=1}^n dq_\alpha$, we can write the previous equation as

$$\begin{aligned} \psi(q', t') &= \int dq \langle q' | e^{-\frac{i}{\hbar} H(t'-t)} | q \rangle \langle q | \psi(t) \rangle = \int dq \langle q' | e^{-\frac{i}{\hbar} H(t'-t)} | q \rangle \psi(q, t) = \\ &=: \int dq G(q', t'; q, t) \psi(q, t) \end{aligned} \quad (1.2)$$

where we introduced the Green function $G(q', t'; q, t)$, which describes the propagation of the state $|\psi(t)\rangle$ with respect to the set of observables $\{\hat{Q}_\alpha\}$. More precisely, the integrating function of Eq. (1.2) represents [2, 13, 3] the conditional probability to find

the system in a state $|\psi(t')\rangle$ at a time t' having measured the system in the state $|\psi(t)\rangle$ at a time t . The Green functions [2, 13, 3] satisfy also the following composition law

$$G(q', t'; q, t) = \int dq'' G(q', t'; q'', t'') G(q'', t''; q, t)$$

where we have $t' > t'' > t$. This expression encodes the fact that the evolution of a quantum system can be splitted into intermediate steps and follows from the properties of the operator $U(t', t)$. In fact, since $U(t', t) = U(t', t'')U(t'', t)$ and exploiting the fact that we can write the Identity operator as $\mathbb{I} = \int dq'' |q''\rangle\langle q''|$, we have that

$$\begin{aligned} G(q', t'; q, t) &\equiv \langle q' | e^{-\frac{i}{\hbar} H(t'-t)} | q \rangle = \langle q' | e^{-\frac{i}{\hbar} H(t'-t'')} e^{-\frac{i}{\hbar} H(t''-t)} | q \rangle = \\ &= \int dq'' \langle q' | e^{-\frac{i}{\hbar} H(t'-t'')} | q'' \rangle \langle q'' | e^{-\frac{i}{\hbar} H(t''-t)} | q \rangle = \\ &= \int dq'' G(q', t'; q'', t'') G(q'', t''; q, t) \end{aligned}$$

At this point, we can derive a Path Integral representation for the Green functions [2, 13, 3]. First, we start by considering the matrix element given by

$$\langle q', t' | q, t \rangle = \langle q' | e^{-\frac{i}{\hbar} H(t'-t)} | q \rangle \quad \text{where} \quad |q, t\rangle := e^{\frac{i}{\hbar} Ht} |q\rangle \quad (1.3)$$

that is an eigenstate of the Heisenberg operators given by

$$\hat{Q}_\alpha(t) = e^{\frac{i}{\hbar} Ht} \hat{Q}_\alpha e^{-\frac{i}{\hbar} Ht} \quad \text{such that} \quad \hat{Q}_\alpha(t) |q, t\rangle = q_\alpha |q, t\rangle$$

To proceed towards a more suitable representation, we perform an analytic continuation of the time variable to the imaginary axis, a procedure formally known as Wick rotation. By substituting the real time t with the imaginary (or Euclidean) time τ via the mapping $t \mapsto -i\tau$ and $t' \mapsto -i\tau'$, the metric of the theory shifts from Minkowski to Euclidean space-time [2, 13]. This transformation has a profound impact on the dynamics. In fact, the oscillatory phase factor $e^{-\frac{i}{\hbar} Ht}$ is replaced by a decaying exponential $e^{-\frac{\tau}{\hbar} H}$ which represents the Euclidean statistical operator [2, 3, 13]. Therefore, the shift to imaginary time results in

$$\langle q', t' | q, t \rangle \xrightarrow[t' \mapsto -i\tau']{t \mapsto -i\tau} = \sum_n \psi_n(q') e^{-\frac{\tau}{\hbar} E_n(t'-t)} \psi_n^*(q) = \langle q' | e^{-\frac{\tau}{\hbar} H(\tau'-\tau)} | q \rangle$$

Then, we need to split the time interval $[\tau, \tau']$ into N infinitesimal segments of length $\epsilon = (\tau'-\tau)/N$, and explicitly develop the matrix element accordingly. The intermediate times are labeled as $\tau_1, \tau_2, \dots, \tau_{N-1}$ with $\tau < \tau_1 < \tau_2 < \dots < \tau_{N-1} < \tau'$. Moreover, recalling also that we can write the Identity operator as $\mathbb{I} = \int dq^{(l)} |q^{(l)}\rangle\langle q^{(l)}|$, where the index l refers to the l -th time step, we obtain

$$\begin{aligned} \langle q' | e^{-\frac{1}{\hbar} H(\tau'-\tau)} | q \rangle &= \langle q' | e^{-\frac{1}{\hbar} H(\tau'-\tau_{N-1})} e^{-\frac{1}{\hbar} H(\tau_{N-1}-\tau_{N-2})} \dots e^{-\frac{1}{\hbar} H(\tau_2-\tau_1)} e^{-\frac{1}{\hbar} H(\tau_1-\tau)} | q \rangle = \\ &= \int \prod_{l=1}^{N-1} dq^{(l)} \langle q' | e^{-\frac{1}{\hbar} H\epsilon} | q^{(N-1)} \rangle \dots \langle q^{(1)} | e^{-\frac{1}{\hbar} H\epsilon} | q \rangle = \\ &= \int \prod_{l=1}^{N-1} \prod_{\alpha=1}^n dq_\alpha^{(l)} \langle q' | e^{-\frac{1}{\hbar} H\epsilon} | q^{(N-1)} \rangle \dots \langle q^{(1)} | e^{-\frac{1}{\hbar} H\epsilon} | q \rangle \end{aligned} \quad (1.4)$$

At this point, we need to make some assumptions on the structure of the Hamiltonian we are working with [2]. In particular, we suppose it to be of the form

$$H = \frac{1}{2} \sum_{\alpha=1}^n \hat{P}_\alpha^2 + V(\hat{Q})$$

where $\{\hat{P}_\alpha\}$ are the momenta canonically conjugate to our $\{\hat{Q}_\alpha\}$ and $V(\hat{Q})$ is a polynomial expression in $\{\hat{Q}_\alpha\}$. We can notice that, for small ϵ , we can use the Baker-Campbell-Hausdorff formula to approximate

$$e^{-\frac{1}{\hbar} H \epsilon} \approx e^{-\frac{\epsilon}{2\hbar} \sum_{\alpha=1}^n \hat{P}_\alpha^2} e^{-\frac{\epsilon}{\hbar} V(\hat{Q})}$$

and obtain

$$\langle q^{(l+1)} | e^{-\frac{1}{\hbar} H \epsilon} | q^{(l)} \rangle \approx \langle q^{(l+1)} | e^{-\frac{\epsilon}{2\hbar} \sum_{\alpha=1}^n \hat{P}_\alpha^2} | q^{(l)} \rangle e^{-\frac{\epsilon}{\hbar} V(q^{(l)})} \quad (1.5)$$

Then, recalling the expression of the scalar product between simultaneous position and momentum eigenstates

$$\langle q | p \rangle = \prod_{\alpha=1}^n \frac{1}{\sqrt{2\pi}} e^{ip_\alpha q_\alpha}$$

and making use of the identity $\mathbb{I} = \int dp^{(l)} |p^{(l)}\rangle \langle p^{(l)}|$, with l corresponding to the different time steps, one has that

$$\begin{aligned} \langle q^{(l+1)} | e^{-\frac{1}{\hbar} H \epsilon} | q^{(l)} \rangle &\approx e^{-\frac{\epsilon}{\hbar} V(q^{(l)})} \int dp^{(l)} \langle q^{(l+1)} | e^{-\frac{\epsilon}{2\hbar} \sum_{\alpha=1}^n \hat{P}_\alpha^2} | p^{(l)} \rangle \langle p^{(l)} | q^{(l)} \rangle = \\ &= e^{-\frac{\epsilon}{\hbar} V(q^{(l)})} \int dp^{(l)} \prod_{\alpha=1}^n \frac{1}{\sqrt{2\pi}} e^{ip_\alpha^{(l)} q_\alpha^{(l+1)}} e^{-\frac{\epsilon p_\alpha^{(l)2}}{2\hbar}} \frac{1}{\sqrt{2\pi}} e^{ip_\alpha^{(l)} q_\alpha^{(l)}} = \\ &= e^{-\frac{\epsilon}{\hbar} V(q^{(l)})} \int \frac{dp^{(l)}}{2\pi} \prod_{\alpha=1}^n e^{-\frac{\epsilon}{\hbar} \left[\frac{p_\alpha^{(l)2}}{2} - ip_\alpha^{(l)} \left(\frac{q_\alpha^{(l+1)} - q_\alpha^{(l)}}{\epsilon} \right) \right]} \\ &= e^{-\frac{\epsilon}{\hbar} V(q^{(l)})} \int \prod_{\beta=1}^n \frac{dp_\beta^{(l)}}{2\pi} \prod_{\alpha=1}^n e^{-\frac{\epsilon}{\hbar} \left[\frac{p_\alpha^{(l)2}}{2} - ip_\alpha^{(l)} \left(\frac{q_\alpha^{(l+1)} - q_\alpha^{(l)}}{\epsilon} \right) \right]} \end{aligned}$$

Substituting these calculations into Eq. (1.4), we find that

$$\langle q' | e^{-\frac{1}{\hbar} H(\tau' - \tau)} | q \rangle \approx \int Dq Dp e^{-\frac{\epsilon}{\hbar} ip_\alpha^{(l)} \left(\frac{q_\alpha^{(l+1)} - q_\alpha^{(l)}}{\epsilon} \right)} e^{-\frac{\epsilon}{\hbar} H(q^{(l)}, p^{(l)})} \quad (1.6)$$

where we wrote

$$Dq = \prod_{\alpha=1}^n \prod_{l=1}^{N-1} dq_\alpha^{(l)} \quad Dp = \prod_{\alpha=1}^n \prod_{l=0}^{N-1} \frac{dp_\alpha^{(l)}}{2\pi} \quad H(q^{(l)}, p^{(l)}) = \sum_{\alpha=1}^n \frac{1}{2} p_\alpha^{(l)2} + \hat{V}(q^{(l)})$$

At this point, we underline that the number of momentum integration exceeds that of the coordinates: in fact, its application refers to the interval between two intermediate

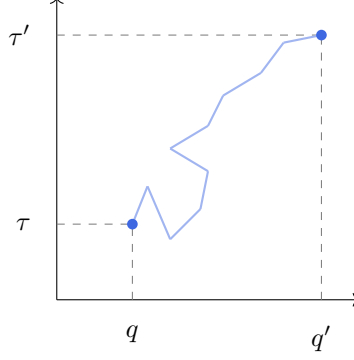


Figure 1.1: Path in q -space connecting two space-time points

points while the coordinate integration excludes $-i\tau$ and $-i\tau'$, which are the initial and the final time respectively. We specify also that the above formula actually holds for every Hamiltonian of the form $H(\hat{Q}, \hat{P}) = T(\hat{P}) + V(\hat{Q})$ in which $T(\hat{P})$ is a polynomial in the canonical momenta. However, for the case where $T(\hat{P})$ is a quadratic form of the type written before, we can also obtain a configuration space Path-Integral representation by carrying out the Gaussian integration over the momenta. Indeed, using the identity

$$\int_{-\infty}^{+\infty} dx e^{(ax^2+bx+c)} = \sqrt{\frac{\pi}{a}} e^{-\left(c-\frac{b^2}{4a}\right)}$$

that is proven in Proposition (B.1.1), we can develop Eq. (1.6) as

$$\begin{aligned} \langle q' | e^{-\frac{1}{\hbar} H(\tau' - \tau)} | q \rangle &\approx \int Dq e^{-\frac{\epsilon}{\hbar} V(q^{(l)})} \int Dp e^{-\frac{\epsilon}{2\hbar} p_\alpha^{(l)2} - \frac{\epsilon}{\hbar} i p_\alpha^{(l)} \left(\frac{q_\alpha^{(l+1)} - q_\alpha^{(l)}}{\epsilon} \right)} = \\ &= \sqrt{\frac{\hbar}{2\pi\epsilon}} \int Dq e^{-\frac{\epsilon}{\hbar} V(q^{(l)})} e^{-\frac{\epsilon}{2\hbar} \left(\frac{q_\alpha^{(l+1)} - q_\alpha^{(l)}}{\epsilon} \right)^2} = \\ &= \sqrt{\frac{\hbar}{2\pi\epsilon}} \int \prod_{l'=1}^{N-1} \prod_{\alpha=1}^n dq_\alpha^{(l')} e^{-\epsilon \sum_{l=0}^{N-1} \hat{L}_E(q^{(l)}, \dot{q}^{(l)})} \end{aligned} \quad (1.7)$$

where we introduced

$$\hat{L}_E(q^{(l)}, \dot{q}^{(l)}) := \sum_{\alpha=1}^n \frac{1}{2} \dot{q}_\alpha^{(l)2} + V(q^{(l)}) \quad \text{where} \quad \dot{q}_\alpha^{(l)} := \left(\frac{q_\alpha^{(l+1)} - q_\alpha^{(l)}}{\epsilon} \right)$$

More precisely, if we consider an arbitrary path in q -space connecting the space-time points (q, τ) and (q', τ') , consisting of a succession of infinitesimal straight line segments for every infinitesimal time interval as the one schematically shown in Fig 1.1, we can provide a precise interpretation of these two quantities. Since $q^{(l)}$ denotes the set of coordinates of the system at time τ_l , it is immediate to notice that $\dot{q}^{(l)}$ represents the euclidean velocity of the system in the time interval $[\tau_l, \tau_{l+1}]$ while moving in a n -dimensional configuration space; thus, it follows that $\hat{L}_E$ is the discretized version of the classical Lagrangian in the euclidean formulation. The action associated with the path is given by

$$S_E[q] = \epsilon \sum_{l=0}^{N-1} \left[\sum_{\alpha=1}^n \frac{1}{2} \dot{q}_\alpha^{(l)2} + \hat{V}(q^{(l)}) \right]$$

By taking the limits $\epsilon \rightarrow 0$ and $N \rightarrow \infty$ in Eq. (1.7), keeping the product $N\epsilon = (\tau' - \tau)$ fixed, we finally recover the Path-Integral representation of Green functions [2, 13, 3] as

$$G(q', t'; q, t) = \langle q' | e^{-\frac{1}{\hbar} H(\tau' - \tau)} | q \rangle \xrightarrow[N \rightarrow \infty]{\epsilon \rightarrow 0} = \int_q^{q'} Dq e^{-S_E[q]} = \int_q^{q'} Dq e^{-\int_{\tau}^{\tau'} d\tau'' \hat{L}_E(q(\tau''), \dot{q}(\tau''))}$$

We can now analyze the physical meaning of this expression [2, 13]. The Green function is represented as a continuous sum of possible paths connecting (q, τ) and (q', τ') , each weighted with $e^{-S_E[q]}$; thus, it is straightforward to notice that the most important contributions to the sum are expected to come from those paths for which $S_E[q]$ takes values close to the minimum, where $\delta S_E[q] = 0$.

1.2 Green function of the free Dirac field

Quantum Field Theory exhibits fundamental differences from non-relativistic Quantum Mechanics. In Quantum Mechanics, we describe a system (for example, a particle) through a quantum state vector $|\psi\rangle$ in a Hilbert space $\mathcal{H}$, and we evaluate the expectation values of physical quantities on such a state by exploiting a set of physical observables $\{\hat{Q}_\alpha\}$ (where the index $\alpha = 1, \dots, n$ accounts for the degrees of freedom of the system) by performing targeted measurements. Moreover, in standard Quantum Mechanics there is a fundamental asymmetry between space and time: the position $\hat{x}$ is a quantum observable (and, therefore, an operator), while time t is an external continuous parameter. In Quantum Field Theory, instead, the position loses its status as an operator and it is put on an equal footing with time, becoming a space-time coordinate. The space-time is defined as a global structure in which every point is uniquely labeled as $x = (\vec{x}, t)$, where t accounts for the temporal dimension and $\vec{x}$ accounts for the spatial ones. In this context, fields are the primary physical entities and particles emerge as localized and quantized excitations of the underlying field, e.g. $\phi(x)$, created or annihilated on the vacuum state $|\Omega\rangle$ within a Fock space $\mathcal{F}$.

In both theories, the information concerning the conditional probability of the time-evolution of a system is enclosed in the Green functions [2, 13]. In Quantum Field Theory, a Green function consists of an expectation value on the ground state of the T-product of a certain number n of Heisenberg field operator, that is

$$G(x_1, \dots, x_n) = \langle \Omega | T [\phi(x_1) \dots \phi(x_n)] | \Omega \rangle$$

where the time-ordered product (T-product) orders the operators from left to right according to descending time. In Quantum Field Theory, however, in order to reconstruct the analytical form of the action, the single Green function is no longer sufficient. In fact, in this case we need an infinite collection [2, 13] of Green functions: the information is no longer encoded in the single eigenstate of the system, but in the action of the fields operators on all the different points of the space-time. In fact, in Quantum Field Theory, every type of physical process (such as particle interactions or particle propagation) is encoded by a different n -point Green function; therefore, knowing all the n -point correlators for every n is equivalent to mastering the entire dynamics of the theory.

In order to derive a Path Integral representation for Green functions in Quantum Field Theory we have to make a fundamental distinction between bosonic Green functions and the fermionic ones, which refers to bosonic or fermionic field operators

respectively. Specifically, the main difference between these correlators relies in the commutation rules that affect bosonic and fermionic fields: the fermionic field operators anticommute with each other, while the bosonic ones commute with each other. This aspect is anticipated to make an important difference in the development of a Path Integral representation of their Green functions.

In order to construct a Path-Integral representation that involves the integration over anticommuting variables, we can introduce the formalism of Grassmann variables [2, 3]. In particular, Grassmann variables are introduced as the counterparts of fermionic fields in functional integration in order to preserve Fermi-Dirac statistics. A Grassmann algebra $\mathfrak{A}$ over $\mathbb{R}$ or $\mathbb{C}$ is an associative algebra constructed from a unit 1 and a set of generators η_i which anticommute with each other as

$$\{\eta_i, \eta_j\} = \eta_i \eta_j + \eta_j \eta_i = 0 \quad \forall i, j$$

As a direct consequence, $\eta_i^2 = 0 \quad \forall i$. A general element of a Grassman algebra $f(\eta)$ is defined as a power series in the η_i 's that has only a finite number of terms (since η_i^2 are always zero $\forall i$) and can be written as

$$f(\eta) = f_0 + \sum_i f_i \eta_i + \sum_{i \neq j} f_{ij} \eta_i \eta_j + \dots$$

To define a functional representation for fermions, we must also introduce a notion of integration over Grassmann variables [2, 3]. Since these variables are purely algebraic constructions, the integral is defined not as a measure-theoretic limit, but through simple algebraic rules. For a single generator η_i , the fundamental rules are

$$\int d\eta_i = 0 \quad \int d\eta_i \eta_i = 1.$$

These two fundamental expressions are the only mathematically consistent rules that allow linear integration over anticommuting variables while maintaining translational invariance. Moreover, these rules reflect the property that the integral of a total derivative must vanish, but they also imply a remarkable identity in Grassmann calculus: integration and differentiation are identical operations. Additionally, it is important to specify that, when dealing with multiple variables, the integration measures are themselves anticommuting. Specifically, we are interested in deriving a Path integral representation of fermionic two-points correlation functions exploiting the formalism of Grassmann variables [2, 3]. In this regard, we define the two-point correlation function in Grassmann algebra as

$$\langle \eta_i \bar{\eta}_j \rangle = \frac{\int D(\bar{\eta} \eta) \eta_i \bar{\eta}_j e^{-\sum_{i,j} \bar{\eta}_i A_{ij} \eta_j}}{\int D(\bar{\eta} \eta) e^{-\sum_{i,j} \bar{\eta}_i A_{ij} \eta_j}} \quad \text{where} \quad D(\bar{\eta} \eta) = \prod_{l=1}^N d\bar{\eta}_l d\eta_l$$

and generalize it as an arbitrary n -point correlation function as

$$\langle \eta_{i_1} \dots \eta_{i_l} \bar{\eta}_{i'_1} \dots \bar{\eta}_{i'_l} \rangle = \frac{\int D(\bar{\eta} \eta) \eta_{i_1} \dots \eta_{i_l} \bar{\eta}_{i'_1} \dots \bar{\eta}_{i'_l} e^{-\sum_{i,j} \bar{\eta}_i A_{ij} \eta_j}}{\int D(\bar{\eta} \eta) e^{-\sum_{i,j} \bar{\eta}_i A_{ij} \eta_j}} \quad (1.8)$$

Algebraically, for a system of N degrees of freedom, the barred variables $\bar{\eta}_1, \dots, \bar{\eta}_N$ are defined as an independent set of N Grassmann generators alongside $\eta_1, \dots, \eta_N$.

Together, they form a complete set of $2N$ independent Grassmann variables. The bar notation is adopted rather than labeling them from 1 to $2N$ because it distinguishes the two sets of generators and highlights their paired structure; for convenience, they can be identified as the Dirac adjoint of the quantum Dirac field. Instead, A_{ij} represents the matrix element of the dynamical operator (or Dirac matrix) that couples the degrees of freedom i and j within the bilinear action.

Keeping this brief aside in mind, let us now introduce the free Dirac field. The free Dirac field $\psi(x)$ is a four-component complex spinor describing relativistic spin-1/2 particles (such as electrons, or quarks) and their corresponding antiparticles. The four complex components combine positive and negative energy solutions with two physical spin projections. The corresponding action in Minkowski is given by

$$S_F[\psi, \bar{\psi}] = \int d^4x \bar{\psi}(x)(i\gamma^\mu \partial_\mu - M)\psi(x) = \sum_{\alpha, \beta} \int d^4x d^4y \bar{\psi}_\alpha(x) K_{\alpha\beta}(x, y) \psi_\beta(y)$$

where

$$K_{\alpha\beta}(x, y) = (i\gamma^\mu \partial_\mu - M)_{\alpha\beta} \delta^{(4)}(x - y)$$

Here, γ^μ (where $\mu = 0, 1, 2, 3$) denotes the 4×4 Dirac matrices, which satisfy the Clifford algebra anticommutation relations $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ with $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ being the Minkowski metric tensor. The term $\gamma^\mu \partial_\mu$ employs Einstein summation convention over spacetime indices, expressing a linear combination of temporal and spatial partial derivatives coupled via the Dirac matrices. Instead, $\psi(x)$ is a 4-component Dirac field of mass M and its components are generally labeled using greek letters as indices. In general, we have that the Path Integral representation of the Green functions involving an equal number of Dirac fields of the type ψ and $\bar{\psi}$ is given by

$$\begin{aligned} \langle \Omega | T[\psi_{\alpha_1}(x_1) \dots \psi_{\alpha_l}(x_l) \bar{\psi}_{\beta_1}(y_1) \dots \bar{\psi}_{\beta_l}(y_l)] | \Omega \rangle &= \\ &= \frac{\int D(\psi \bar{\psi}) \psi_{\alpha_1}(x_1) \dots \psi_{\alpha_l}(x_l) \bar{\psi}_{\beta_1}(y_1) \dots \bar{\psi}_{\beta_l}(y_l) e^{iS_F[\psi, \bar{\psi}]} }{\int D(\psi \bar{\psi}) e^{iS_F[\psi, \bar{\psi}]}} \end{aligned}$$

In this case, we can show that the corresponding two-points correlation function is related to the inverse of the matrix $K_{\alpha\beta}$. In fact, by demonstrating that

$$iK_{\alpha\beta}^{-1}(x, y) = \frac{\int D(\psi \bar{\psi}) \psi_\alpha(x) \bar{\psi}_\beta(y) e^{iS_F[\psi, \bar{\psi}]} }{\int D(\psi \bar{\psi}) e^{iS_F[\psi, \bar{\psi}]}}$$

one has that

$$\langle \Omega | T[\psi_\alpha(x) \bar{\psi}_\beta(y)] | \Omega \rangle = iK_{\alpha\beta}^{-1}(x, y) \quad (1.9)$$

In order to demonstrate Eq. (1.9), it is necessary to know how to calculate two different types of integrals. For the first one, we have to compute the expression given by

$$\begin{aligned} I_N &= \int D(\bar{\eta} \eta) K(\bar{\eta}, \eta) = \int \prod_{l=1}^N d\bar{\eta}_l d\eta_l e^{-\sum_{i,j} \bar{\eta}_i A_{ji} \eta_j} = \\ &= \int \prod_{l=1}^N \prod_{j=1}^N d\bar{\eta}_l d\eta_l e^{-\bar{\eta}_j \sum_{i=1}^N A_{ij} \eta_i} \end{aligned} \quad (1.10)$$

Since $\eta_i^2 = 0$, only the first two terms in the expansion of the exponential in Eq. (1.10) will contribute. Hence, we can rewrite it as

$$\prod_{j=1}^N e^{-\sum_i \bar{\eta}_j A_{ji} \eta_i} = (1 - \bar{\eta}_1 A_{1i_1} \eta_{i_1}) (1 - \bar{\eta}_2 A_{2i_2} \eta_{i_2}) \dots (1 - \bar{\eta}_N A_{Ni_N} \eta_{i_N})$$

Moreover, according to Grassmann integration rules, the integrand must involve the product of all the Grassmann variables; thus, we can rewrite it as

$$K(\bar{\eta}, \eta) = \eta_1 \bar{\eta}_1 \eta_2 \bar{\eta}_2 \dots \eta_N \bar{\eta}_N \sum_{i_1 \dots i_N} \varepsilon_{i_1 \dots i_N} A_{1i_1} \dots A_{Ni_N}$$

where the $\varepsilon_{i_1 \dots i_N}$ is the ε -tensor in N -dimensions. Recalling the formula for the determinant of a matrix A , we finally obtain that

$$K(\bar{\eta}, \eta) = (\det A) \eta_1 \bar{\eta}_1 \eta_2 \bar{\eta}_2 \dots \eta_N \bar{\eta}_N$$

Substituting this expression back in Eq. (1.10), we conclude that

$$I_N = \left[\prod_{i=1}^N \int d\bar{\eta}_i d\eta_i \eta_i \bar{\eta}_i \right] \det A = \det A \quad (1.11)$$

Secondly, we can consider the following expression given by

$$I_L = \int D(\bar{\eta}\eta) \eta_{i_1} \dots \eta_{i_l} \bar{\eta}_{i_1} \dots \bar{\eta}_{i_l} e^{-\sum_{i,j} \bar{\eta}_i A_{ij} \eta_j} \quad (1.12)$$

We can compute (1.12) by defining a generational functional $Z[\bar{\rho}, \rho]$, which depends on external sources. The generating functional represents a compact tool to derive all the correlations and expectation values of the system. It is defined as

$$Z[\bar{\rho}, \rho] = \int D(\bar{\eta}\eta) e^{-\sum_{i,j} \bar{\eta}_i A_{ij} \eta_j + \sum_i (\bar{\rho}_i \eta_i + \bar{\eta}_i \rho_i)}$$

We can compute it explicitly by completing the square in the exponential in Eq. (1.12). In fact, by noticing that

$$-\bar{\eta} A \eta + \bar{\rho} \eta + \bar{\eta} \rho = -(\bar{\eta} - \bar{\rho} A^{-1}) A (\eta - A^{-1} \rho) + \bar{\rho} A^{-1} \rho$$

since

$$\begin{aligned} -\bar{\eta} A \eta + \bar{\rho} \eta + \bar{\eta} \rho &= -\bar{\eta} A \eta + \bar{\rho} \eta + \bar{\eta} \rho - \bar{\rho} A^{-1} \rho + \bar{\rho} A^{-1} \rho = \\ &= -(\bar{\eta} A \eta - \bar{\rho} A^{-1} A \eta - \bar{\eta} A A^{-1} \rho + \bar{\rho} A^{-1} A A^{-1} \rho) + \bar{\rho} A^{-1} \rho = \\ &= -A [\bar{\eta} (\eta - A^{-1} \rho) - \bar{\rho} A^{-1} (\eta - A^{-1} \rho)] + \bar{\rho} A^{-1} \rho = \\ &= -(\bar{\eta} - \bar{\rho} A^{-1}) A (\eta - A^{-1} \rho) + \bar{\rho} A^{-1} \rho \end{aligned}$$

and by doing a change of variables such as $\bar{\eta}' = \bar{\eta} - \bar{\rho} A^{-1}$ and $\eta' = \eta - A^{-1} \rho$, it is straightforward that

$$Z[\bar{\rho}, \rho] = (\det A) e^{\sum_{i,j} \bar{\rho}_i A_{ij}^{-1} \rho_j}$$

If we derive the generating functional with respect to $\rho, \bar{\rho}$, we can finally solve Eq. (1.12). In order to derive $Z[\bar{\rho}, \rho]$, we need to exploit the properties of Grassmann algebra. In fact, since $\bar{\rho}_m^2 = 0$ and $\rho_n^2 = 0 \forall m, n$ we can expand the exponential and stop at the first order in the argument. By doing that, we obtain the following expression

$$\begin{aligned} Z[\bar{\rho}, \rho] &= (\det A) e^{\sum_{m,n} \bar{\rho}_m A_{mn}^{-1} \rho_n} = (\det A) \prod_m e^{\bar{\rho}_m \sum_n A_{mn}^{-1} \rho_n} = \\ &= (\det A) \prod_m \left[1 + \bar{\rho}_m \sum_n A_{mn}^{-1} \rho_n \right] \end{aligned}$$

At this point, we can start by calculate all the derivatives starting from those with respect to ρ_i . Thus, we have that

$$\begin{aligned} \left. \frac{\partial}{\partial \bar{\rho}_{j_1}} \cdots \frac{\partial}{\partial \bar{\rho}_{j_l}} Z[\bar{\rho}, \rho] \frac{\overleftarrow{\partial}}{\partial \rho_{i_1}} \cdots \frac{\overleftarrow{\partial}}{\partial \rho_{i_l}} \right|_{\rho=\bar{\rho}=0} &= \\ &= \frac{\partial}{\partial \bar{\rho}_{j_1}} \cdots \frac{\partial}{\partial \bar{\rho}_{j_l}} \prod_m \left[1 + \bar{\rho}_m \sum_n (A^{-1})_{mn} \rho_n \right] \frac{\overleftarrow{\partial}}{\partial \rho_{i_1}} \cdots \frac{\overleftarrow{\partial}}{\partial \rho_{i_l}} = \\ &= \frac{\partial}{\partial \bar{\rho}_{j_1}} \cdots \frac{\partial}{\partial \bar{\rho}_{j_l}} \sum_m \left[(\bar{\rho}_m A_{mi_1}^{-1}) \prod_{r \neq m} (1 + \bar{\rho}_r \sum_n A_{rn}^{-1} \rho_n) \right] \frac{\overleftarrow{\partial}}{\partial \rho_{i_2}} \cdots \frac{\overleftarrow{\partial}}{\partial \rho_{i_l}} = \\ &= \frac{\partial}{\partial \bar{\rho}_{j_1}} \cdots \frac{\partial}{\partial \bar{\rho}_{j_l}} \sum_m \left[(\bar{\rho}_m A_{mi_1}^{-1}) (\bar{\rho}_m A_{mi_2}^{-1}) \prod_{r \neq m} (1 + \bar{\rho}_r \sum_n A_{rn}^{-1} \rho_n) \right] \frac{\overleftarrow{\partial}}{\partial \rho_{i_3}} \cdots \frac{\overleftarrow{\partial}}{\partial \rho_{i_l}} = \\ &= \cdots = \frac{\partial}{\partial \bar{\rho}_{j_1}} \cdots \frac{\partial}{\partial \bar{\rho}_{j_l}} \sum_m \left[\prod_{i_k=i_1}^{i_l} (\bar{\rho}_m A_{mi_k}^{-1}) \prod_{r \neq m} (1 + \bar{\rho}_r \sum_n A_{rn}^{-1} \rho_n) \right] \end{aligned}$$

When performing the derivatives with respect to $\bar{\rho}_j$, instead, we can simplify the calculations by considering the fact that we will impose $\rho = \bar{\rho} = 0$ and so all the terms which contain ρ or $\bar{\rho}$ will vanish. For this reason, we can safely impose

$$\prod_{r \neq m} (1 + \bar{\rho}_r \sum_n A_{rn}^{-1} \rho_n) \equiv 1$$

and proceed with our calculation considering only the other term. Thus, we have

$$\begin{aligned} \frac{\partial}{\partial \bar{\rho}_{j_1}} \cdots \frac{\partial}{\partial \bar{\rho}_{j_l}} \sum_m \left[\prod_{i_k=i_1}^{i_l} \bar{\rho}_m A_{mi_k}^{-1} \right] &= \frac{\partial}{\partial \bar{\rho}_{j_1}} \cdots \frac{\partial}{\partial \bar{\rho}_{j_{l-1}}} \prod_{i_k=i_1}^{i_l} \left[A_{j_l i_k}^{-1} \sum_{m \neq j_l} \bar{\rho}_m A_{mi_k}^{-1} \right] = \\ &= \frac{\partial}{\partial \bar{\rho}_{j_1}} \cdots \frac{\partial}{\partial \bar{\rho}_{j_{l-2}}} \prod_{i_k=i_1}^{i_l} \left[A_{j_l i_k}^{-1} A_{j_{l-1} i_k}^{-1} \sum_{m \neq j_l, j_{l-1}} \bar{\rho}_m A_{mi_k}^{-1} \right] = \\ &= \cdots = \prod_{i_k=i_1}^{i_l} \prod_{j_h=j_1}^{j_l} A_{j_h i_k}^{-1} = \xi_l \prod_{i_k=i_1}^{i_l} \prod_{j_h=j_1}^{j_l} A_{j_h i_k}^{-1} = \\ &= \xi_l \sum_P (-1)^{\sigma_P} \prod_{j_h=j_1}^{j_l} A_{j_h i_{P_k}}^{-1} \end{aligned}$$

We underline that the term $\xi_l = (-1)^{l(l+1)/2}$ is a pure consequence of the fact that we ordered the indices as $j_1 \dots j_l$ and that $A_{j_h i_k}$ is an antisymmetric matrix. Likewise, the term $(-1)^{\sigma_P}$ is a consequence of the fact that we took into account all the possible permutation of the indices. Finally, we can conclude that

$$\int D(\bar{\eta}\eta) \eta_{i_1} \dots \eta_{i_l} \bar{\eta}_{j_1} \dots \bar{\eta}_{j_l} e^{-\sum_{i,j} \bar{\eta}_i A_{ij} \eta_j} = (\det A) \xi_l \sum_P (-1)^{\sigma_P} (A^{-1})_{i_1 j_{P_1}} \dots (A^{-1})_{i_l j_{P_l}}$$

As a particular case of the previous expression [2, 3], we have

$$\int D(\bar{\eta}\eta) \eta_i \bar{\eta}_j e^{-\sum_{i,j} \bar{\eta}_i A_{ij} \eta_j} = (\det A) A_{i,j}^{-1}$$

Thus, recalling Eq. (1.8) and Eq. (1.11), it finally holds that

$$\langle \eta_i \bar{\eta}_j \rangle = A_{ij}^{-1} \quad (1.13)$$

In conclusion, we summarize the main steps leading to this result. Evaluating the multi-point correlation functions defined in Eq. 1.8 begins with introducing a generating functional $Z[\bar{\rho}, \rho]$ coupled to external Grassmann sources. Completing the square via a translation in Grassmann space decouples the source dependence and factors out the Gaussian partition integral $I_N = \det A$ given by Eq. (1.11). Expanding the source exponential $e^{\bar{\rho} A^{-1} \rho}$ through the properties of Grassmann generators and evaluating functional derivatives with respect to the sources yields the full contraction structure and parity of multi-point correlation functions. Finally, for the two-point correlation function, dividing by $I_N = \det A$ cancels the determinant factor, so that recalling Eq. (1.8) and (1.11) bring us to Eq. (1.13).

1.3 Lattice representation of two-point correlation functions

If we want to give the Path Integral a precise meaning within Quantum Field Theory, we necessarily have to discretize the space-time global structure and define a space-time lattice. In fact, in this context, the multiple integrals are only formally defined when dealing with an infinite number of degrees of freedom [2, 13]. In particular, we want to be able to derive an expression for the two-point correlation function as the one in Eq. (1.13), but formulated on a lattice. Eventually, we will remove the lattice structure to study the continuum limit.

However, we have to pay attention to few aspects. While discretization in the context of the free scalar field does not present any particular criticalities, that of the free Dirac field opens the door to non-trivial difficulties. In particular, the discretization of the Dirac action on a space-time lattice and the behavior of the resulting lattice propagator in momentum space are at the origin of what is commonly referred to as *fermion doubling problem*.

1.3.1 Free scalar field on the lattice

Let us start by considering the classical field equation

$$(\square + M^2)\phi(x) = 0 \quad (1.14)$$

where $\phi(x)$ denotes a scalar field of mass M , $\square = \partial_\mu \partial^\mu$ is the d'Alembert operator, and x stands for the space-time position vector with components x_μ for $\mu = 0, 1, 2, 3$. This equation of motion follows from imposing $\delta S = 0$ according to the least action principle, with S given by

$$S = -\frac{1}{2} \int d^4x \phi(x) (\square + M^2) \phi(x)$$

this equation of motion. Therefore, is the one that minimizes the differential of the action and consequently the one that will correctly describe the dynamic of the system with greatest probability [2, 13]. The Lagrangian density associated to the action can be derived according to the fact that

$$\phi \partial_\mu (\partial^\mu \phi) = \partial_\mu (\phi \partial^\mu \phi) - \partial_\mu \phi \partial^\mu \phi$$

and by noticing that the total derivative in the right-hand side of Eq. (1.14) vanishes when evaluated at the boundaries of space-time since $\phi \rightarrow 0$ there. Thus, we find that

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} M^2 \phi^2$$

In this case, the Path Integral representation of the Green function continued to imaginary times is given by the corresponding correlation function, that is

$$\langle \phi(x) \phi(y) \dots \rangle = \frac{\int D\phi (\phi(x) \phi(y) \dots) e^{-S_E[\phi]}}{\int D\phi e^{-S_E[\phi]}}$$

where x and y are euclidean space-time vectors obtained from the Minkowski ones by replacing $x^0 \rightarrow -ix_4$ and $y^0 \rightarrow -iy_4$. The euclidean action $S_E[\phi]$ can be obtained from the one in Minkowski by imposing that $x^0 \rightarrow -ix_4$ and noticing that

$$\frac{\partial}{\partial x_0} = \frac{\partial}{\partial x_4} \frac{\partial x_0}{\partial x_4} = \frac{\partial}{\partial x_4} \left(\frac{\partial x_0}{\partial x_4} \right)^{-1} = (-i)^{-1} \frac{\partial}{\partial x_4} = i \frac{\partial}{\partial x_4}$$

so that $\square$ denotes from now on the 4-dimensional Laplacian given by

$$\square = \partial_0^2 - \nabla^2 = (i\partial_4)^2 - \nabla^2 = -\partial_4^2 - \nabla^2 = -\sum_{\mu=1}^4 \partial_\mu \partial_\mu$$

We can now introduce a space-time lattice with lattice spacing a . Every point on the lattice is uniquely determined by four integers as $n = (n_1, n_2, n_3, n_4)$, where the last component refers to euclidean time. The transition from the continuum to the lattice can be performed by making the following substitutions [2, 13]. First of all, we can safely write $x_\mu \rightarrow n_\mu a$ and $\phi(x) \rightarrow \phi(na)$. Secondly, we can substitute the integral operator with a sum, paying attention to maintaining the correct units of measurements, as

$$\int d^4x \rightarrow a^4 \sum_n$$

As a consequence, the integration measure naturally transforms as

$$D\phi = \prod_{\vec{x}, x_4} \phi(x) \quad \rightarrow \quad D\phi = \prod_n \phi(na)$$

Lastly, we need to substitute the Laplacian with a discretized version. In order to do that, we need to derive a representation for the second derivative that scales at least as $\mathcal{O}(a^2)$. In particular, by performing an expansion around $x = na$, we have that

$$\begin{aligned} \phi(x + \mu a) &= \phi(x) + a\partial_\mu \phi(x) + \frac{a^2}{2}\partial_\mu^2 \phi(x) + \mathcal{O}(a^3) = \sum_{n=0}^{+\infty} \frac{(+a)^n}{n!} \partial_\mu^n \phi(x) \\ \phi(x - \mu a) &= \phi(x) - a\partial_\mu \phi(x) + \frac{a^2}{2}\partial_\mu^2 \phi(x) + \mathcal{O}(a^3) = \sum_{n=0}^{+\infty} \frac{(-a)^n}{n!} \partial_\mu^n \phi(x) \end{aligned}$$

which implies that

$$\begin{aligned} \phi(x + \mu a) + \phi(x - \mu a) &= 2\phi(x) + a^2\partial_\mu^2 \phi(x) + \mathcal{O}(a^4) \\ \phi(x + \mu a) + \phi(x - \mu a) - 2\phi(x) &= a^2\partial_\mu^2 \phi(x) + \mathcal{O}(a^4) \\ \frac{\phi(x + \mu a) + \phi(x - \mu a) - 2\phi(x)}{a^2} &= \partial_\mu^2 \phi(x) + \mathcal{O}(a^2) \end{aligned}$$

Next, we want to obtain a Path Integral expression involving dimensionless variables [2, 13] only: in order to do that, we can rescale the mass and the fields according to their dimension, that is an inverse length for both. Hence, if we put $\phi_n = a\phi(na)$ and $\hat{M} = aM$, we obtain

$$\langle \phi_n \phi_m \dots \rangle = \frac{\int \prod_l d\phi_l (\phi_n \phi_m \dots) e^{-S_E[\phi]}}{\int \prod_l d\phi_l e^{-S_E[\phi]}} \quad (1.15)$$

where

$$S_E = -\frac{1}{2} \sum_{n, \pm\mu} \phi_n \phi_{n+\mu} + \left(\frac{\hat{M}^2}{2} + 4 \right) \sum_n \phi_n \phi_n$$

In particular, the action can be written in the form

$$S_E = \frac{1}{2} \sum_{n,m} \phi_n K_{nm} \phi_m \quad \text{where} \quad K_{nm} = - \sum_\mu [\delta_{n+\mu,m} + \delta_{n-\mu,m} - 2\delta_{nm}] + \hat{M}^2 \delta_{nm}$$

In order to derive a representation of the two-point correlation function as a function of K_{nm} , we have to perform a few steps [2, 13]. First of all, we can start by defining a generating functional and calculate the integral in the denominator of Eq. (1.15). In fact, if we perform a gaussian integral neglecting the overall constants, we obtain that

$$Z_0[J] = \int \prod_l d\phi_l e^{-S_E[\phi] + \sum_n J_n \phi_n} \propto \frac{1}{\sqrt{\det K}} e^{\frac{1}{2} \sum_{n,m} J_n K_{nm}^{-1} J_m} \quad (1.16)$$

Secondly, we can differentiate this expression with respect to two sources and evaluate the final expression in $J_i = 0 \forall i$. In fact, neglecting the overall constant, we have

$$\begin{aligned} \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} Z_0[J] &= \frac{\partial}{\partial J_i} \left[\frac{1}{2} \left(\sum_m K_{im}^{-1} J_m + \sum_n J_n K_{ni}^{-1} \right) e^{\frac{1}{2} \sum_{n,m} J_n K_{nm}^{-1} J_m} \right] \Big|_{J_i=0} = \\ &= \frac{1}{2} (K_{ij}^{-1} + K_{ji}^{-1}) = K_{ij}^{-1} \end{aligned}$$

Specifically, in the second step we have put the exponential of Eq. (1.16) equal to 1 and differentiated only the linear terms in J ; in the third step, instead, we have recalled that the transpose of the inverse of a matrix is equal to the inverse itself if the matrix is symmetric. Finally, we get that

$$\langle \phi_n \phi_m \rangle = K_{nm}^{-1}$$

Now, we want to explicitly compute K_{nm}^{-1} . In order to do this, we can recall that the convolution of K and K^{-1} on a lattice can be written as

$$\sum_l K_{nl} K_{lm}^{-1} = \delta_{nm}$$

Considering the fact that the Fourier transform of the δ function is equal to 1 and that the Fourier transform of a convolution is equal to the product of the two Fourier transforms, we choose to simplify the calculations by shifting to the momenta representation. In particular, we will refer to the momentum variables as $\hat{k}$ in order to underline that we are working with dimensionless variables. That said, we simply obtain that

$$\widetilde{K} \cdot \widetilde{K}^{-1} = 1 \implies \widetilde{K}^{-1} = \frac{1}{\widetilde{K}}$$

Thus, recalling the representation of δ_{nm} in the momentum space as

$$\delta_{nm} = \int_{-\pi}^{+\pi} \frac{d^4 \hat{k}}{(2\pi)^4} e^{i\hat{k} \cdot (n-m)}$$

we get

$$\begin{aligned} K_{nm} &= - \sum_{\mu} [\delta_{n+\mu, m} + \delta_{n-\mu, m} - 2\delta_{nm}] + M^2 \delta_{nm} = \\ &= - \sum_{\mu} \int_{-\pi}^{+\pi} \frac{d^4 \hat{k}}{(2\pi)^4} \left[e^{i\hat{k} \cdot (n+\mu-m)} + e^{i\hat{k} \cdot (n-\mu-m)} - 2e^{i\hat{k} \cdot (n-m)} \right] + \\ &\quad + \hat{M}^2 \int_{-\pi}^{+\pi} \frac{d^4 \hat{k}}{(2\pi)^4} e^{i\hat{k} \cdot (n-m)} = \\ &= \hat{M}^2 \int_{-\pi}^{+\pi} \frac{d^4 \hat{k}}{(2\pi)^4} e^{i\hat{k} \cdot (n-m)} + 2 \sum_{\mu} \int_{-\pi}^{+\pi} \frac{d^4 \hat{k}}{(2\pi)^4} e^{i\hat{k} \cdot (n-m)} \left[1 - \cos(\hat{k} \cdot \mu) \right] = \\ &= \hat{M}^2 \int_{-\pi}^{+\pi} \frac{d^4 \hat{k}}{(2\pi)^4} e^{i\hat{k} \cdot (n-m)} + 4 \sum_{\mu} \int_{-\pi}^{+\pi} \frac{d^4 \hat{k}}{(2\pi)^4} e^{i\hat{k} \cdot (n-m)} \sin^2 \left(\frac{\hat{k} \cdot \mu}{2} \right) = \end{aligned}$$

$$= \int_{-\pi}^{+\pi} \frac{d^4 \hat{k}}{(2\pi)^4} \tilde{K}(\hat{k}) e^{i\hat{k} \cdot (n-m)}$$

where

$$\tilde{K}(\hat{k}) = \hat{M}^2 + 4 \sum_{\mu=1}^4 \sin^2 \left(\frac{\hat{k} \cdot \mu}{2} \right)$$

and so

$$\widetilde{K^{-1}}(\hat{k}) = \frac{1}{\hat{M}^2 + 4 \sum_{\mu=1}^4 \sin^2 \left(\frac{\hat{k} \cdot \mu}{2} \right)}$$

Finally, we can write the two-point correlation function as

$$K_{nm}^{-1} = \langle \phi_n \phi_m \rangle = \int_{-\pi}^{+\pi} \frac{d^4 \hat{k}}{(2\pi)^4} \frac{e^{i\hat{k} \cdot (n-m)}}{\hat{M}^2 + 4 \sum_{\mu=1}^4 \sin^2 \left(\frac{\hat{k} \cdot \mu}{2} \right)} \quad (1.17)$$

Discretizing spacetime onto a lattice with spacing a renders momentum space inherently periodic with a period of $2\pi/a$. Consequently, the First Brillouin Zone $[-\pi/a, \pi/a]$ acts as the fundamental domain of momentum space. Under periodic boundary conditions, all physical observables become smooth periodic functions in momentum space, ensuring that every distinct physical degree of freedom is uniquely and completely contained within a single Brillouin zone. If we take the continuum limit of Eq. (1.17), we can see that we are able to correctly approximate every value of the Brillouin zone [2, 13]. This confirms the correctness of our derivation so far [2, 13]. In order to demonstrate that, we need to perform a substantial change of variables by sending $\hat{k} \rightarrow ak$ with $k \in [-\frac{\pi}{a}, \frac{\pi}{a}]$, $\hat{M} \rightarrow aM$ and $n, m \rightarrow na, ma$. Thus, one has that

$$\begin{aligned} \langle \phi_n \phi_m \rangle &= \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} a^4 \frac{d^4 k}{(2\pi)^4} \frac{e^{ik(n-m)a^2}}{4 \sum_{\mu} \sin^2 \left(\frac{ak_{\mu}}{2} \right) + M^2 a^2} = \\ &= \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} a^2 \frac{d^4 k}{(2\pi)^4} \frac{e^{ik(n-m)a^2}}{\frac{4}{a^2} \sum_{\mu} \sin^2 \left(\frac{ak_{\mu}}{2} \right) + M^2} \end{aligned}$$

After the correct rescaling of the dimensional correlation function, we have

$$\langle \phi(na) \phi(ma) \rangle = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{k}{(2\pi)^4} \frac{e^{ik(n-m)a^2}}{\frac{4}{a^2} \sum_{\mu} \sin^2 \left(\frac{ak_{\mu}}{2} \right) + M^2}$$

Taking the limit $a \rightarrow 0$, we obtain

$$\lim_{a \rightarrow 0} \left[\frac{e^{ik(n-m)a^2}}{\frac{4}{a^2} \sum_{\mu} \sin^2 \left(\frac{ak_{\mu}}{2} \right) + M^2} \right] \simeq \lim_{a \rightarrow 0} \left[\frac{1}{\sum_{\mu} \frac{4}{a^2} \left(\frac{ak_{\mu}}{2} \right)^2 + M^2} \right] = \frac{1}{k_{\mu}^2 + M^2}$$

Thus, we can conclude that

$$\langle \phi(x) \phi(y) \rangle = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 k}{(2\pi)^4} \frac{1}{k_{\mu}^2 + M^2} \quad (1.18)$$

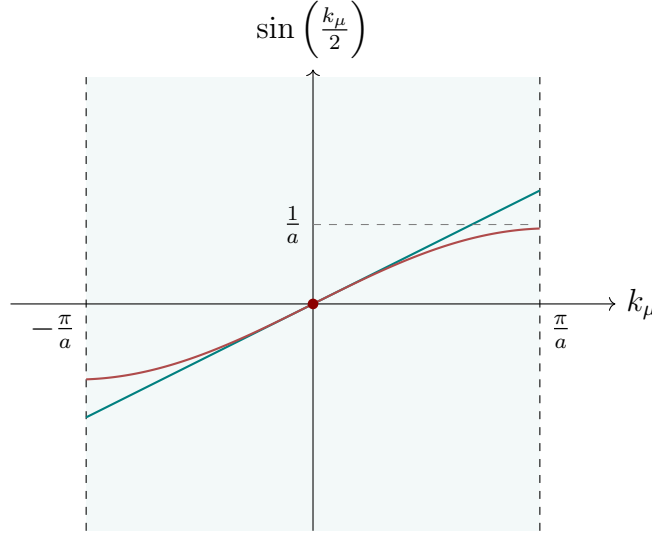


Figure 1.2: Visual representation of the First Brillouin Zone $k_\mu \in [-\pi/a, \pi/a]$ (shaded region) for a lattice scalar field. The lattice momentum function $\sin(k_\mu/2)$ (red curve) spans the restricted interval $[-\pi/2, \pi/2]$, accurately matching the linear continuum relation $k_\mu a/2$ (teal line). Due to the halving of the momentum argument, the sine function vanishes exclusively at the origin $k_\mu = 0$ (red dot) and reaches non-zero extremal values at the boundaries $k_\mu = \pm\pi/a$, guaranteeing a single physical pole and ensuring the absence of doubling phenomena.

It follows that Eq. (1.18) correctly approximates the two-point correlation function in every point of the Brillouin zone, even at the boundaries, as we can see in Fig. 1.2. In fact, the standard discretization of a free scalar field exploits the second-order lattice Laplacian. Therefore, the resulting propagator in momentum space is represented by Eq. (1.17). Because the argument of the sine function is halved, as the momentum ranges over the first Brillouin zone in $[-\frac{\pi}{a}, +\frac{\pi}{a}]$, the argument of the sine function only spans the interval $[-\frac{\pi}{2}, +\frac{\pi}{2}]$. Within this restricted interval, the sine function vanishes exclusively at the origin. The bosonic propagator, therefore, possesses only a single physical pole within the Brillouin zone: there are no high-momentum excitations at the corners that cross zero, meaning that no spurious species (commonly known as *doublers*) are generated. Because there are no spurious poles at the edges of the Brillouin zone, the lattice momentum correctly approximates the physical momentum as the lattice spacing tends to zero for the entire integration region. The inevitability of this phenomenon is formalized by the Nielsen–Ninomiya Theorem [17, 2, 13], which states that a lattice fermionic theory cannot simultaneously satisfy locality, hermiticity, translational invariance, and exact chiral symmetry without producing doublers. These doublers serve a physical purpose on the lattice: they cancel the chiral anomaly, ensuring the lattice theory remains globally anomaly-free as the continuum limit is approached.

1.3.2 Free Dirac field on the lattice

Now, we consider the Dirac equation in Minkowski space as

$$(i\gamma^\mu \partial_\mu - M)\psi(x) = 0$$

This equation of motion for $\psi(x)$, or identically for $\bar{\psi}(x) = \psi^\dagger(x)\gamma^0$, follows from imposing $\delta S = 0$ where the action S is given by

$$S = S[\psi, \bar{\psi}] = \int d^4x \bar{\psi}(x)(i\gamma^\mu \partial_\mu - M)\psi(x)$$

We are interested in deriving an expression for the two-point correlation function associated to the free Dirac field as done in the previous paragraphs, so now we recall the same steps [2, 13]. First of all, we need to move from Minkowski spacetime to the Euclidean one. As a consequence, $x, y, \dots$ will be identified with Euclidean four-vectors and the Dirac matrices will be redefined as $\gamma_4^E = \gamma^0$ and $\gamma_i^E = -i\gamma^i$ in order to satisfy the anticommutation relation also in the Euclidean space-time. The action will change accordingly. Thus, we obtain

$$\langle \psi_\alpha(x) \dots \bar{\psi}_\beta(y) \dots \rangle = \frac{\int D\psi D\bar{\psi} (\psi_\alpha(x) \dots \bar{\psi}_\beta(y) \dots) e^{-S_E[\psi, \bar{\psi}]}}{\int D\psi D\bar{\psi} e^{-S_E[\psi, \bar{\psi}]}}$$

with

$$S_E[\psi, \bar{\psi}] = \int d^4x \bar{\psi}(x)(i\gamma_\mu^E \partial_\mu - M)\psi(x)$$

Secondly, we need to introduce a space-time lattice as before. Therefore, we start by putting $x \rightarrow na$ and $\psi(x), \bar{\psi}(x) \rightarrow \psi(na), \bar{\psi}(na)$ so that

$$D\psi D\bar{\psi} = \prod_\alpha d\psi_\alpha \prod_\beta d\bar{\psi}_\beta \rightarrow \prod_{\alpha, n} d\psi_\alpha(na) \prod_{\beta, m} d\bar{\psi}_\beta(ma)$$

We also need to derive a lattice version of the first derivative that scales at least as $\mathcal{O}(a^2)$. In particular, by performing an expansion around $x = na$, we find that

$$\begin{aligned} \psi_\alpha(x + \mu a) &= \psi_\alpha(x) + a\partial_\mu \psi_\alpha(x) + \frac{a^2}{2}\partial_\mu^2 \psi_\alpha(x) + \mathcal{O}(a^3) = \sum_n \frac{(a)^n}{n!} \partial_\mu^n \psi_\alpha(x) \\ \psi_\alpha(x - \mu a) &= \psi_\alpha(x) - a\partial_\mu \psi_\alpha(x) + \frac{a^2}{2}\partial_\mu^2 \psi_\alpha(x) + \mathcal{O}(a^3) = \sum_n \frac{(-a)^n}{n!} \partial_\mu^n \psi_\alpha(x) \end{aligned}$$

which implies that

$$\begin{aligned} \psi_\alpha(x + \mu a) - \psi_\alpha(x - \mu a) &= 2a \cdot \partial_\mu \psi_\alpha(x) + \mathcal{O}(a^3) \\ \frac{\psi_\alpha(x + \mu a) - \psi_\alpha(x - \mu a)}{2a} &= \partial_\mu \psi_\alpha(x) + \mathcal{O}(a^2) \end{aligned}$$

Next, since we want to work with dimensionless variables only, we need to scale every quantity by a accordingly to their dimensions. In fact, we have to send $M \rightarrow a^{-1}\tilde{M}$, $\psi_\alpha(x), \bar{\psi}_\beta(x) \rightarrow a^{-\frac{3}{2}}\psi_\alpha(n), a^{-\frac{3}{2}}\bar{\psi}_\beta(n)$ and consequently also $\partial_\mu \psi_\alpha(x) \rightarrow a^{-\frac{5}{2}}\partial_\mu \psi_\alpha(n)$. Therefore, the lattice dimensionless version of the action can be written as

$$S_E = \sum_{n, m} \sum_{\alpha, \beta} \bar{\psi}_\alpha(n) K_{\alpha\beta}(n, m) \psi_\beta(m)$$

where

$$K_{\alpha\beta}(n, m) = \sum_{\mu} \frac{1}{2} (\gamma_{\mu})_{\alpha\beta} [\delta_{m, n+\mu} - \delta_{m, n-\mu}] + \hat{M} \delta_{nm} \delta_{\alpha\beta}$$

Thus, the correlation function becomes

$$\langle \psi_{\alpha}(n) \dots \bar{\psi}_{\beta}(m) \dots \rangle = \frac{\int D\psi D\bar{\psi} (\psi_{\alpha}(n) \dots \bar{\psi}_{\beta}(m) \dots) e^{-S_E[\psi, \bar{\psi}]}}{\int D\psi D\bar{\psi} e^{-S_E[\psi, \bar{\psi}]}}$$

Lastly, in order to derive a representation of the two point correlation function in function of $K_{\alpha\beta}(n, m)$, we have to proceed exactly as before [2, 13]. Introducing the corresponding generating functional and performing the gaussian integral, we obtain that

$$\begin{aligned} Z[\eta, \bar{\eta}] &= \int D\psi D\bar{\psi} e^{\sum_{n, \alpha} [\bar{\eta}_{\alpha}(n) \psi_{\alpha}(n) + \bar{\psi}_{\alpha}(n) \eta_{\alpha}(n)]} e^{-S_E[\psi, \bar{\psi}]} = \\ &= (\det K) e^{\sum_{n, m} \sum_{\alpha, \beta} \bar{\eta}_{\alpha}(n) K_{\alpha\beta}(n, m) \eta_{\beta}(m)} \end{aligned}$$

Hence, the two-point correlation function is given by

$$\langle \psi_{\alpha}(n) \dots \bar{\psi}_{\beta}(m) \dots \rangle = K_{\alpha\beta}(n, m)^{-1}$$

Now, we want to explicitly compute $K_{\alpha\beta}^{-1}(n, m)$. In order to do this, we can recall that the convolution of K and K^{-1} on a lattice can be written as

$$\sum_{\lambda, l} K_{\alpha\lambda}(n, l) K_{\lambda\beta}^{-1}(l, m) = \delta_{nm} \delta_{\alpha\beta}$$

Considering the fact that the Fourier transform of the δ function is equal to 1 and that the Fourier transform of a convolution is equal to the product of the two Fourier transforms, we choose to simplify the calculations by shifting to the momenta representation. In particular, we will refer to the momentum variables as $\hat{k}$ in order to underline that we are working with dimensionless variables. That said, as before, we simply obtain that

$$\widetilde{K} \cdot \widetilde{K^{-1}} = 1 \implies \widetilde{K^{-1}} = \frac{1}{\widetilde{K}}$$

Thus, recalling the representation of δ_{nm} in the momentum space as

$$\delta_{nm} = \int_{-\pi}^{+\pi} \frac{d^4 \hat{k}}{(2\pi)^4} e^{i\hat{k} \cdot (n-m)}$$

we get

$$\begin{aligned} K_{\alpha\beta}(n, m) &= \sum_{\mu} \frac{1}{2} (\gamma_{\mu})_{\alpha\beta} [\delta_{m, n+\mu} - \delta_{m, n-\mu}] + \hat{M} \delta_{nm} \delta_{\alpha\beta} = \\ &= \frac{1}{2} \sum_{\mu} (\gamma_{\mu})_{\alpha\beta} \int_{-\pi}^{+\pi} \frac{d^4 \hat{k}}{(2\pi)^4} \left[e^{i\hat{k} \cdot (n+\mu-m)} + e^{i\hat{k} \cdot (n-\mu-m)} \right] + \hat{M} \delta_{\alpha\beta} \int_{-\pi}^{+\pi} \frac{d^4 \hat{k}}{(2\pi)^4} e^{i\hat{k} \cdot (n-m)} = \end{aligned}$$

$$\begin{aligned}
&= \hat{M} \delta_{\alpha\beta} \int_{-\pi}^{+\pi} \frac{d^4 \hat{k}}{(2\pi)^4} e^{i\hat{k} \cdot (n-m)} + i \sum_{\mu} (\gamma_{\mu})_{\alpha\beta} \int_{-\pi}^{+\pi} \frac{d^4 \hat{k}}{(2\pi)^4} e^{i\hat{k} \cdot (n-m)} \sin(\hat{k} \cdot \mu) = \\
&= \int_{-\pi}^{+\pi} \frac{d^4 \hat{k}}{(2\pi)^4} \left[\hat{M} \delta_{\alpha\beta} + i \sum_{\mu} (\gamma_{\mu})_{\alpha\beta} \sin(\hat{k} \cdot \mu) \right] e^{i\hat{k} \cdot (n-m)} = \\
&= \int_{-\pi}^{+\pi} \frac{d^4 \hat{k}}{(2\pi)^4} \left[\frac{\hat{M}^2 + \sum_{\mu} \sin^2(\hat{k} \cdot \mu)}{\hat{M} - i \sum_{\mu} \gamma_{\mu} \sin(\hat{k} \cdot \mu)} \right]_{\alpha\beta} e^{i\hat{k} \cdot (n-m)} = \\
&= \int_{-\pi}^{+\pi} \frac{d^4 \hat{k}}{(2\pi)^4} \tilde{K}(\hat{k}) e^{i\hat{k} \cdot (n-m)}
\end{aligned}$$

where

$$\tilde{K}(\hat{k}) = \left[\frac{\hat{M}^2 + \sum_{\mu} \sin^2(\hat{k} \cdot \mu)}{\hat{M} - i \sum_{\mu} \gamma_{\mu} \sin(\hat{k} \cdot \mu)} \right]_{\alpha\beta}$$

and so

$$\widetilde{K^{-1}}(\hat{k}) = \left[\frac{\hat{M} - i \sum_{\mu} \gamma_{\mu} \sin(\hat{k} \cdot \mu)}{\hat{M}^2 + \sum_{\mu} \sin^2(\hat{k} \cdot \mu)} \right]_{\alpha\beta}$$

Finally, we can write the two-point correlation function as

$$K_{\alpha\beta}^{-1}(n, m) = \langle \psi_n \psi_m \rangle = \int_{-\pi}^{+\pi} \frac{d^4 \hat{k}}{(2\pi)^4} \left[\frac{\hat{M} - i \sum_{\mu} \gamma_{\mu} \sin(\hat{k} \cdot \mu)}{\hat{M}^2 + \sum_{\mu} \sin^2(\hat{k} \cdot \mu)} \right]_{\alpha\beta} e^{i\hat{k} \cdot (n-m)} \quad (1.19)$$

If we take the continuum limit of Eq. (1.19), we can verify if we are able to correctly approximate every value of the Brillouin zone [17, 2, 13]. In order to demonstrate that, we need to perform a change of variables by sending $\hat{k} \rightarrow ak$ with $k \in [-\frac{\pi}{a}, \frac{\pi}{a}]$, $\hat{M} \rightarrow aM$ and $n, m \rightarrow na, ma$ as before. Thus, one has that

$$\begin{aligned}
\langle \psi_n \psi_m \rangle &= \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} a^4 \frac{d^4 k}{(2\pi)^4} \left[\frac{aM - i \sum_{\mu} \gamma_{\mu} \sin(ak_{\mu})}{a^2 M^2 + \sum_{\mu} \sin^2(ak_{\mu})} \right]_{\alpha\beta} e^{ik \cdot (n-m)a^2} = \\
&= \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} a^3 \frac{d^4 k}{(2\pi)^4} \left[\frac{M - \frac{i}{a} \sum_{\mu} \gamma_{\mu} \sin(ak_{\mu})}{M^2 + \sum_{\mu} \frac{1}{a^2} \sin^2(ak_{\mu})} \right]_{\alpha\beta} e^{ik \cdot (n-m)a^2}
\end{aligned}$$

After the correct rescaling of the dimensional correlation function, we have

$$\langle \psi(na) \psi(ma) \rangle = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 k}{(2\pi)^4} \left[\frac{M - \frac{i}{a} \sum_{\mu} \gamma_{\mu} \sin(ak_{\mu})}{M^2 + \sum_{\mu} \frac{1}{a^2} \sin^2(ak_{\mu})} \right]_{\alpha\beta} e^{ik \cdot (n-m)a^2}$$

Taking the limit $a \rightarrow 0$, we obtain

$$\lim_{a \rightarrow 0} \left[\frac{M - \frac{i}{a} \sum_{\mu} \gamma_{\mu} \sin(ak_{\mu})}{M^2 + \sum_{\mu} \frac{1}{a^2} \sin^2(ak_{\mu})} \right]_{\alpha\beta} e^{ik \cdot (n-m)a^2} \simeq \lim_{a \rightarrow 0} \left[\frac{M - \frac{i}{a} \sum_{\mu} \gamma_{\mu} ak_{\mu}}{M^2 + \sum_{\mu} \frac{1}{a^2} (ak_{\mu})^2} \right]_{\alpha\beta} e^{ik \cdot (n-m)a^2} =$$

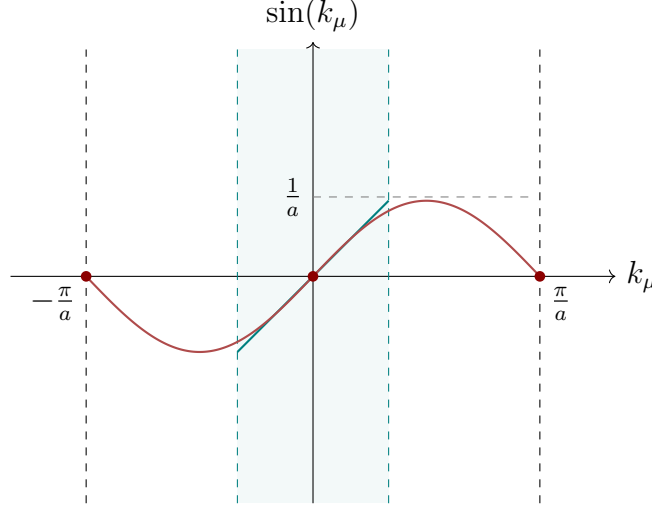


Figure 1.3: Visual representation of the fermion doubling problem in momentum space. The lattice momentum function $\sin(k_\mu a)$ (red curve) is plotted across the First Brillouin Zone $k_\mu \in [-\pi/a, \pi/a]$. Near the origin (highlighted region), the lattice momentum accurately reproduces the physical linear continuum relation k_μ (teal straight line). However, the use of a symmetric lattice derivative introduces spurious zeros of the sine function at the Brillouin zone boundaries $k_\mu = \pm\pi/a$ (red dots), giving rise to unphysical fermionic species (doublers).

$$= \left[\frac{M - i \sum_\mu \gamma_\mu k_\mu}{M^2 + \sum_\mu k_\mu^2} \right]_{\alpha\beta}$$

Thus, we can conclude that

$$\langle \psi(x) \psi(y) \rangle = \int_{-\pi/a}^{\pi/a} \frac{d^4 k}{(2\pi)^4} \left[\frac{M - i \sum_\mu \gamma_\mu k_\mu}{M^2 + \sum_\mu k_\mu^2} \right]_{\alpha\beta} \quad (1.20)$$

We can notice that Eq. (1.20) correctly approximates the two-point correlation function in the central region of the Brillouin zone but not at the boundaries, as depicted in Fig. 1.3. This is a clear symptom of what we refer to as *fermion doubling problem* [17, 2, 13]. More precisely, the fermion doubling problem arises when one discretizes the Dirac action using a symmetric lattice derivative to preserve the hermiticity of the operator. In momentum space, in fact, this choice leads to a propagator as the one in Eq. (1.19). In this case, the sine function has two zeros: one at the origin and another one at the edge of the Brillouin zone. In d space-time dimensions, this creates 2^d degenerate fermion species for every single field introduced: this means that, in four dimensions, we obtain 15 pure lattice artifacts and only one real fermion.

1.4 Abelian gauge field on the lattice

Let us start from the action of the free Dirac field, given by

$$S = S[\psi, \bar{\psi}] = \int d^4 x \bar{\psi}(x) (i\gamma^\mu \partial_\mu - M) \psi(x)$$

Given $G = e^{i\Lambda}$ an element of the abelian $U(1)$ group with Λ independent of x , we can notice that this action is invariant under the transformation given by $\psi(x) \rightarrow G\psi(x)$, $\bar{\psi}(x) \rightarrow \bar{\psi}(x)G^{-1}$. The action, therefore, is invariant under *global* transformations, i.e. transformations that shift the field of the same quantity in every point of the space-time [2, 13]. Since the arbitrary choice of a phase convention point by point in space-time has no measurable physical effect, and demanding local symmetry serves as the fundamental principle to introduce gauge interactions, we are interested in requiring the action to be invariant also under *local* $U(1)$ transformations, with $\psi(x)$, $\bar{\psi}(x)$ rotating by a quantity whose value depends on the considered space-time point [2, 13]. Therefore, given $G(x) = e^{i\Lambda(x)}$ that is an element of the abelian $U(1)$ local group with $\Lambda(x)$ that depends on the point x , we can explicitly verify if each term of the action satisfies the local invariance through the application of the local transformations given by $\psi(x) \rightarrow G(x)\psi(x)$, $\bar{\psi}(x) \rightarrow \bar{\psi}(x)G^{-1}(x)$. First, one can see that the mass term is already locally invariant [2], since

$$\begin{aligned}\bar{\psi}(x)M\psi(x) &= [\psi(x)G^{-1}(x)]M[G(x)\psi(x)] = G(x)[\bar{\psi}(x)M\psi(x)]G^{-1}(x) \\ &= [\bar{\psi}(x)M\psi(x)]G^{-1}(x)G(x) = \bar{\psi}(x)M\psi(x)\end{aligned}$$

The derivative term, instead, is not locally invariant since

$$\begin{aligned}\bar{\psi}(x)i\gamma^\mu\partial_\mu\psi(x) &\neq \bar{\psi}(x)G^{-1}(x)[i\gamma^\mu\partial_\mu]G(x)\psi(x) = \\ &= \bar{\psi}(x)G^{-1}(x)[i\gamma^\mu\partial_\mu(i\Lambda(x))]G(x)\psi(x) = \\ &= \bar{\psi}(x)[i\gamma^\mu\partial_\mu(i\Lambda(x))]G^{-1}(x)G(x)\psi(x) = \bar{\psi}(x)[i\gamma^\mu\partial_\mu(i\Lambda(x))]\psi(x)\end{aligned}$$

Thus, it is necessary to rewrite the action in a way that satisfies the local invariance [2]. In particular, we can modify the action as

$$S[\psi, \bar{\psi}] = \int d^4x \bar{\psi}(x)(i\gamma^\mu D_\mu - M)\psi(x)$$

by defining $D_\mu(x)$ as the operator that transforms under $G(x)$ as $D_\mu \rightarrow G(x)D_\mu G^{-1}(x)$. In particular, we can define $ieA_\mu(x) = D_\mu(x) - \partial_\mu$ as a four-vector field that represents the correction introduced to the previous partial derivative in order to make the action locally invariant [2]; by introducing this correction, we have promoted the derivative to an operator that depends on x . In order to guarantee that $D_\mu(x)$ transforms under $G(x)$ as $D_\mu(x) \rightarrow G(x)D_\mu(x)G^{-1}(x)$ as we desire [2], we have to require that the four-vector field transforms under $G(x)$ as

$$A_\mu \rightarrow G(x)A_\mu(x)G^{-1}(x) - \frac{i}{e}G(x)\partial_\mu G^{-1}(x)$$

We can verify it by computing that

$$D'_\mu = G(x)D_\mu G(x)^{-1} \equiv \partial_\mu + ieA'_\mu(x) \equiv G(x)\partial_\mu G(x)^{-1} + G(x)ieA_\mu G(x)^{-1}$$

which implies that

$$\begin{aligned}ieA'_\mu(x) &= -\partial_\mu + G(x)\partial_\mu G(x)^{-1} + G(x)ieA_\mu G(x)^{-1} = \\ &= -\partial_\mu G(x)G(x)^{-1} + G(x)\partial_\mu G(x)^{-1} + G(x)ieA_\mu G(x)^{-1} =\end{aligned}$$

$$\begin{aligned}
&= -[\partial_\mu G(x)G(x)^{-1} - G(x)\partial_\mu G(x)^{-1}] + G(x)ieA_\mu G(x)^{-1} = \\
&= +G(x)\partial_\mu G(x)^{-1} + G(x)ieA_\mu G(x)^{-1}
\end{aligned}$$

Thus, we can conclude that

$$A'_\mu(x) = G(x)A_\mu(x)G(x)^{-1} = -\frac{i}{e}G(x)\partial_\mu G(x)^{-1} + G(x)A_\mu(x)G(x)^{-1}$$

From now on, we will often use the quantity given by $\delta A_\mu(x) = A'_\mu(x) - A_\mu(x)$. At this point, we have to add to the action a term which allows $A_\mu(x)$ to propagate; the term must obviously be invariant under both global and local transformations [2]. Thus, by introducing the gauge-invariant strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the complete action is given by

$$S = S_F + S_G = \int d^4x \bar{\psi}(x)(i\gamma^\mu D_\mu - M)\psi(x) - \frac{1}{4} \int d^4x F^{\mu\nu} F_{\mu\nu}$$

The Euclidean version of the action can be derived by sending $x_0 \rightarrow -ix_4$ so that $\partial_0 \rightarrow i\partial_4$. Equivalently, we can send $A_0 \rightarrow iA_4$.

1.4.1 Connection between $G(x)$ and $\delta A_\mu(x)$

It is possible to derive an expression for $G(x)$ which depends on the transformation of $A_\mu(x)$ into $A'_\mu(x)$ through the term $\delta A_\mu(x) = A'_\mu(x) - A_\mu(x)$ [2]. The validity of this expression can then be proved by induction. Let us start by writing

$$\begin{aligned}
A'_\mu(x) &= -\frac{i}{e}G(x)\partial_\mu G(x)^{-1} + G(x)A_\mu(x)G(x)^{-1} \\
G(x)^{-1}A'_\mu(x) &= -\frac{i}{e}\partial_\mu G(x)^{-1} + A_\mu G(x)^{-1}
\end{aligned}$$

Thus, recalling the fact that $G(x)$ and $A_\mu(x)$ commute with each other, we have

$$\begin{aligned}
\partial_\mu G(x)^{-1} &= ie \{G(x)^{-1}A'_\mu(x) - A_\mu(x)G(x)^{-1}\} = \\
&= ie \{[G(x)^{-1}, A_\mu(x)] + G^{-1}(x)\delta A_\mu(x)\} = \\
&= ieG^{-1}(x)\delta A_\mu(x)
\end{aligned}$$

Then, we can integrate this expression in $[0, x]$ and obtain that

$$G(x)^{-1} = G^{-1}(0) + ie \int_0^x dx_1 G^{-1}(x_1)\delta A_\mu(x_1)$$

At this point, to evaluate this integral, we can apply a recursive strategy by repeatedly substituting the expression for $G^{-1}(x)$ into itself. Thus, we obtain

$$\begin{aligned}
G(x)^{-1} &= G^{-1}(0) + ie \int_0^x dx_1 \left[G^{-1}(0) + ie \int_0^{x_1} dx_2 G^{-1}(x_2)\delta A_\mu(x_2) \right] \delta A_\mu(x_1) = \\
&= G^{-1}(0) \left[1 + ie \int_0^x dx_1 \delta A_\mu(x_1) \right] + (ie)^2 \int_0^x dx_1 \int_0^{x_1} dx_2 G^{-1}(x_2)\delta A_\mu(x_1)\delta A_\mu(x_2) = \\
&= G^{-1}(0) \left[1 + ie \int_0^x dx_1 \delta A_\mu(x_1) + (ie)^2 \int_0^x dx_1 \int_0^{x_1} dx_2 \delta A_\mu(x_1)\delta A_\mu(x_2) \right] +
\end{aligned}$$

$$\begin{aligned}
& + (ie)^3 \int_0^x dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3 G^{-1}(x_3) \delta A_\mu(x_1) \delta A_\mu(x_2) \delta A_\mu(x_3) = \dots = \\
& = G^{-1}(0) \left[\sum_{n=0}^{+\infty} (ie)^n \int_0^x dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{n-1}} dx_n \delta A_\mu(x_1) \delta A_\mu(x_2) \dots \delta A_\mu(x_n) \right]
\end{aligned}$$

At this point, we want to find a way to rewrite every integral as an integral in $[0, x]$. In order to do that, we can first consider the case of two variables x_1, x_2 for simplicity and then generalize it by induction for x_n . In fact, choosing one specific path in $[0, x]$ and introducing the Path Ordering operator P to keep the variables ordered as $0 \leq x_1 \leq x_2 \leq \dots \leq x_n \leq x$, one can see that

$$\begin{aligned}
P \int_0^x dx_1 \int_0^x dx_2 \delta A_\mu(x_1) \delta A_\mu(x_2) &= \\
&= P \left\{ \int_0^x dx_1 \int_0^{x_1} dx_2 \delta A_\mu(x_1) \delta A_\mu(x_2) + \int_0^x dx_1 \int_{x_1}^x dx_2 \delta A_\mu(x_1) \delta A_\mu(x_2) \right\}
\end{aligned}$$

Since we are working with integrated variables, we can swap x_1 and x_2 in the second term on the right-hand side resulting in

$$\begin{aligned}
P \int_0^x dx_1 \int_0^x dx_2 \delta A_\mu(x_1) \delta A_\mu(x_2) &= \\
&= P \left\{ \int_0^x dx_1 \int_0^{x_1} dx_2 \delta A_\mu(x_1) \delta A_\mu(x_2) + \int_0^x dx_2 \int_{x_2}^x dx_1 \delta A_\mu(x_2) \delta A_\mu(x_1) \right\}
\end{aligned}$$

The right-hand side, at this point, contains two terms. In the first one, the integration domain can be determined by choosing a fixed x_1 and integrating x_2 from 0 to x_1 ; in the second one, instead, the integration domain can be determined by choosing a fixed x_2 and integrating x_1 from x_2 to x . If we represent graphically the integration domains of the two integrals as in Fig. 1.4 and compare them, we observe that they are identical. Therefore, we are allowed to rewrite them as

$$\begin{aligned}
P \int_0^x dx_1 \int_0^x dx_2 \delta A_\mu(x_1) \delta A_\mu(x_2) &= \\
&= P \left\{ \int_0^x dx_1 \int_0^{x_1} dx_2 \delta A_\mu(x_1) \delta A_\mu(x_2) + \int_0^x dx_2 \int_{x_2}^x dx_1 \delta A_\mu(x_2) \delta A_\mu(x_1) \right\} = \\
&= P \left\{ \int_0^x dx_1 \int_0^{x_1} dx_2 \delta A_\mu(x_1) \delta A_\mu(x_2) + \int_0^x dx_1 \int_0^{x_1} dx_2 \delta A_\mu(x_2) \delta A_\mu(x_1) \right\}
\end{aligned}$$

Finally, making the Path Ordering operator P act on both terms, we obtain that

$$\begin{aligned}
P \int_0^x dx_1 \int_0^x dx_2 \delta A_\mu(x_1) \delta A_\mu(x_2) &= \\
&= P \left\{ \int_0^x dx_1 \int_0^{x_1} dx_2 \delta A_\mu(x_1) \delta A_\mu(x_2) + \int_0^x dx_1 \int_0^{x_1} dx_2 \delta A_\mu(x_2) \delta A_\mu(x_1) \right\} = \\
&= \int_0^x dx_1 \int_0^{x_1} dx_2 \delta A_\mu(x_1) \delta A_\mu(x_2) + \int_0^x dx_1 \int_0^{x_1} dx_2 \delta A_\mu(x_1) \delta A_\mu(x_2) = \\
&= 2 \int_0^x dx_1 \int_0^{x_1} dx_2 \delta A_\mu(x_1) \delta A_\mu(x_2)
\end{aligned}$$

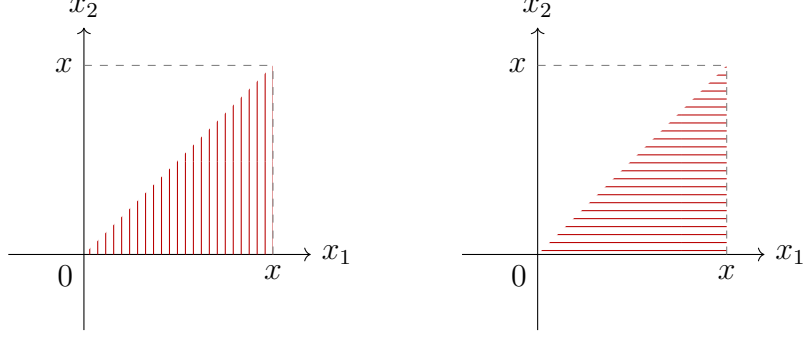


Figure 1.4: A visual representation of the two identical domains

Thus, we conclude that

$$\int_0^x dx_1 \int_0^{x_1} dx_2 \delta A_\mu(x_1) \delta A_\mu(x_2) = \frac{1}{2} P \int_0^x dx_1 \int_0^x dx_2 \delta A_\mu(x_1) \delta A_\mu(x_2)$$

The same procedure can be also performed in the case of three variables x_1, x_2 and x_3 from which we obtain that

$$\begin{aligned} \int_0^x dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3 \delta A_\mu(x_1) \delta A_\mu(x_2) \delta A_\mu(x_3) &= \\ &= \frac{1}{3!} P \int_0^x dx_1 \int_0^x dx_2 \int_0^x dx_3 \delta A_\mu(x_1) \delta A_\mu(x_2) \delta A_\mu(x_3) \end{aligned}$$

Thus, we conclude that

$$\begin{aligned} \int_0^x dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-1}} dx_n \delta A_\mu(x_1) \delta A_\mu(x_2) \cdots \delta A_\mu(x_n) &= \\ &= \frac{1}{n!} P \int_0^x dx_1 \int_0^x dx_2 \cdots \int_0^x dx_n \delta A_\mu(x_1) \delta A_\mu(x_2) \cdots \delta A_\mu(x_n) \end{aligned}$$

The previous formula can be proved by induction. Since we have explicitly calculated it for two variables, we assume that it holds for $n - 1$ integration variables in order to prove it for n integration variables. In fact, since it holds that

$$\begin{aligned} I_{n-1}(x) &= \int_0^x dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-2}} dx_{n-1} \delta A_\mu(x_1) \delta A_\mu(x_2) \cdots \delta A_\mu(x_{n-1}) = \\ &= \frac{1}{(n-1)!} P \int_0^x dx_1 \int_0^x dx_2 \cdots \int_0^x dx_{n-1} \delta A_\mu(x_1) \delta A_\mu(x_2) \cdots \delta A_\mu(x_{n-1}) \end{aligned}$$

we can write that

$$\begin{aligned} \int_0^x dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-1}} dx_n \prod_{i=1}^n \delta A_\mu(x_i) &= \\ &= \int_0^x dx_1 \delta A_\mu(x_1) \left\{ \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-1}} dx_n \prod_{i=2}^n \delta A_\mu(x_i) \right\} = \\ &= \int_0^x dx_1 \delta A_\mu(x_1) I_{n-1}(x_1) = \end{aligned}$$

$$\begin{aligned}
&= \int_0^x dx_1 \delta A_\mu(x_1) \left\{ \frac{1}{(n-1)!} P \int_0^x dx_2 \dots dx_n \prod_{i=2}^n \delta A_\mu(x_i) \right\} = \\
&= \frac{1}{n!} P \int_0^x dx_1 \dots dx_n \prod_{i=1}^n \delta A_\mu(x_i)
\end{aligned}$$

In the last step we exploited the fact that

$$\delta A_\mu(x_1) P [\delta A_\mu(x_2) \dots \delta A_\mu(x_n)] = P [\delta A_\mu(x_1) \dots \delta A_\mu(x_n)]$$

since $x_1 > x_2 > \dots > x_n$, that is since x_1 is the largest variable. Moreover, we have introduced a factor $1/n$ because in the full integration domain any of the n variables could play the role of the largest one, but here we are restricting to a single choice. Thus, putting all these things together, $G^{-1}(x)$ can be finally written as

$$\begin{aligned}
G^{-1}(x) &= G^{-1}(0) \left\{ \sum_{n=0}^{+\infty} \frac{(ie)^n}{n!} \int_0^x dx_1 \dots \int_0^x dx_n \delta A_\mu(x_1) \dots \delta A_\mu(x_n) \right\} = \\
&= G^{-1}(0) \left\{ \sum_{n=0}^{+\infty} \frac{(ie)^n}{n!} \left[\int_0^x dx' \delta A_\mu(x') \right]^n \right\} = G^{-1}(0) \exp \left\{ ie \int_0^x dx' \delta A_\mu(x') \right\} = \\
&= \exp \left\{ ie \int_0^x dx' \delta A_\mu(x') \right\}
\end{aligned}$$

where in the last step we assumed $G^{-1}(0) = 1$. We can prove this formula by induction. Since for $n = 0$ it holds that $G_0^{-1}(x) = G^{-1}(0)$, we can assume that the previous expression holds for n in order to prove it for $n + 1$. Therefore, we can write that

$$\begin{aligned}
G_{n+1}^{-1}(x) &= ie \int_0^{x_{n+1}} dx_{n+2} G_n^{-1}(x_{n+2}) \delta A_\mu(x_{n+2}) = \\
&= ie \int_0^{x_{n+1}} dx_{n+2} \left\{ \sum_{k=0}^n \frac{(ie)^k}{k!} \left[\int_0^x dx' \delta A_\mu(x') \right]^k \right\} \delta A_\mu(x_{n+2}) = \\
&= \sum_{k=0}^n \frac{(ie)^{k+1}}{k!} \left\{ \int_0^{x_{n+1}} dx_{n+2} \delta A_\mu(x_{n+2}) \left[\int_0^x dx' \delta A_\mu(x') \right]^k \right\} = \\
&= \sum_{k=0}^n \frac{(ie)^{k+1}}{(k+1)!} \left\{ \int_0^x dx_{n+2} \delta A_\mu(x_{n+2}) \left[\int_0^x dx' \delta A_\mu(x') \right]^k \right\} = \\
&= \sum_{k=0}^{n+1} \frac{(ie)^k}{k!} \left[\int_0^x dx' \delta A_\mu(x') \right]^k
\end{aligned}$$

Specifically, in the last step we exploited the Path Ordering properties. We can also immediately determine $G(x)$ from $G^{-1}(x)$, that is given by

$$G(x) = \exp \left\{ -ie \int_0^x dx' \delta A_\mu(x') \right\}$$

1.4.2 Schwinger Line-Integral

While discretizing the locally invariant fermionic action, we want to be able to write it in the form

$$S_F = S[\psi, \bar{\psi}] = a^4 \sum_{n,m} \bar{\psi}(na) K_{nm} \psi(ma) = a^4 \sum_{n,m} \bar{\psi}(na) [K_{nm}^M + K_{nm}^D] \psi(ma)$$

in which K_{nm}^M accounts for the mass term and K_{nm}^D accounts for the derivative term [2]. Based on the calculations done so far, it is straightforward to see that $K_{nm}^M = M\delta_{n,m}$. The explicit form of K_{nm}^D , instead, can be derived [2, 5] by considering the way it should transform under the application of $G(x)$. In fact, since $D'_\mu = GD_\mu G^{-1}$, we impose that that K_{nm}^D transforms as

$$\begin{aligned} K_{nm}^{D'} &= G_n K_{nm}^D G_m^{-1} = e^{-ie \int_0^{x_n} dx' \delta A_\mu(x')} K_{nm}^D e^{ie \int_0^{x_m} dx' \delta A_\mu(x')} = \\ &= P e^{ie \int_{x_n}^{x_m} dx' \delta A_\mu(x')} K_{nm}^D \end{aligned}$$

In order to guarantee that, by putting $e = 0$, K_{nm}^D correspond to the discretized version of ∂_μ , it has necessarily to be of the form

$$K_{nm}^D = P e^{ie \int_{x_n}^{x_m} dx' A_\mu(x')} \left[\frac{\delta_{n,m+\hat{\mu}} - \delta_{n,m-\hat{\mu}}}{2} \right]$$

In fact, by substituting it in the previous expression, one has that

$$\begin{aligned} K_{nm}^{D'} &= P e^{ie \int_{x_n}^{x_m} dx' \delta A_\mu(x')} K_{nm}^D = \\ &= P e^{ie \int_{x_n}^{x_m} dx' \delta A_\mu(x')} e^{ie \int_{x_n}^{x_m} dx' \delta A_\mu(x')} \left[\frac{\delta_{n,m+\hat{\mu}} - \delta_{n,m-\hat{\mu}}}{2} \right] = \\ &= P e^{ie \int_{x_n}^{x_m} dx' A'_\mu(x')} \left[\frac{\delta_{n,m+\hat{\mu}} - \delta_{n,m-\hat{\mu}}}{2} \right] = U_{n,m} \left[\frac{\delta_{n,m+\hat{\mu}} - \delta_{n,m-\hat{\mu}}}{2} \right] \end{aligned}$$

where $U_{n,m}$ is the Schwinger Line-Integral [2, 18, 5]. Thus, the discretized adimensional version of the locally invariant fermionic action is given by

$$S_F = \sum_{n,\mu} \frac{i}{2} \left[\bar{\psi}(n) \gamma^\mu U_{n,n+\hat{\mu}} \psi(n+\hat{\mu}) - \bar{\psi}(n) \gamma^\mu U_{n,n-\hat{\mu}} \psi(n-\hat{\mu}) \right] + \hat{M} \sum_n \bar{\psi}(n) \psi(n)$$

1.4.3 Plaquette variables

In order to obtain the discretized dimensionless version of the locally invariant action [2, 18, 5], we need to work also on S_G . In particular, we need to discretize the four-component strength tensor $F_{\mu\nu}$. We can discretize it explicitly as

$$F_{\mu\nu}(n) = \frac{1}{a} \left[A_\nu(n) - A_\nu(n+\hat{\mu}) - (A_\mu(n) - A_\mu(n+\hat{\nu})) \right] + \mathcal{O}(a^{-3})$$

We can notice also that

$$e^{iea^2 F_{\mu\nu}(n)} = 1 + ie a^2 F_{\mu\nu}(n) - \frac{1}{2} e^2 a^4 F_{\mu\nu}^2(n) - \frac{1}{6} ie^3 a^6 F_{\mu\nu}^3(n) + \mathcal{O}(a^8)$$

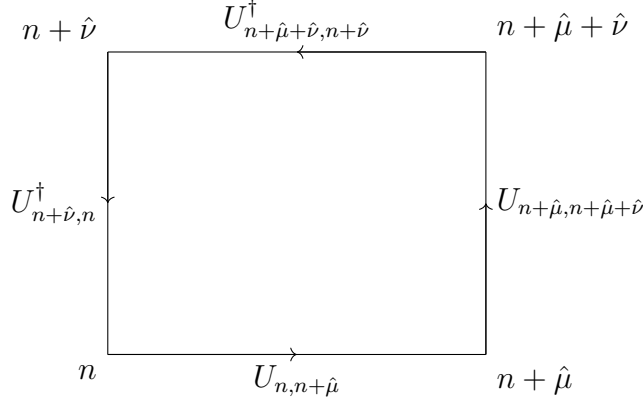


Figure 1.5: Schematic representation of the elementary plaquette variable $U_{\mu\nu}(n)$. The oriented product of four Schwinger line-integrals along the closed contour of width $\hat{\mu}$ and height $\hat{\nu}$ forms the minimal gauge-invariant loop, which reproduces the field strength tensor $F_{\mu\nu}(n)$ in the continuum limit ($a \rightarrow 0$).

$$e^{-iea^2 F_{\mu\nu}(n)} = 1 - iea^2 F_{\mu\nu}(n) - \frac{1}{2}e^2 a^4 F_{\mu\nu}^2(n) + \frac{1}{6}ie^3 a^6 F_{\mu\nu}^3(n) + \mathcal{O}(a^8)$$

so that

$$e^{iea^2 F_{\mu\nu}(n)} + e^{-iea^2 F_{\mu\nu}(n)} = 2 - e^2 a^4 F_{\mu\nu}^2(n) + \mathcal{O}(a^8)$$

$$F_{\mu\nu}^2(n) = \frac{2}{a^4 e^2} \left[1 - \frac{1}{2} (e^{iea^2 F_{\mu\nu}(n)} + e^{-iea^2 F_{\mu\nu}(n)}) \right] + \mathcal{O}(a^2)$$

At this point, we need to work on this expression and rewrite it using the formalism of Schwinger line-integrals [2, 5, 18]. In order to do that, we can start by noticing that, if we choose x_n, x_m as points separated by a singular lattice space, we can assume that A'_μ remains constant along this interval and approximate $U_{n,m}$ as

$$U_{n,m} = e^{ie \int_{x_n}^{x_m} dx' A'_\mu(x')} \simeq e^{iea A'_\mu(n)}$$

Thus, considering the product of Schwinger line-integrals along a rectangle named *plaquette* of width $\hat{\mu}$ and height $\hat{\nu}$, as the one represented in Fig. 1.5, we can write our term as

$$\begin{aligned} e^{iea^2 F_{\mu\nu}(n)} &= e^{iea^2 \frac{1}{a} [A_\mu(n) - A_\mu(n+\hat{\nu}) - (A_\nu(n) - A_\nu(n+\hat{\mu}))]} = \\ &= e^{iea [A_\mu(n) - A_\mu(n+\hat{\nu}) - (A_\nu(n) - A_\nu(n+\hat{\mu}))]} = \\ &= e^{iea [A_\mu(n) + A_\nu(n+\hat{\mu}) - A_\mu(n+\hat{\nu}) - A_\nu(n)]} = \\ &= U_{n,n+\hat{\mu}} U_{n+\hat{\mu},n+\hat{\mu}+\hat{\nu}} U_{n+\hat{\mu}+\hat{\nu},n+\hat{\nu}}^\dagger U_{n+\hat{\nu},n}^\dagger = U_{\mu\nu} \end{aligned}$$

Therefore, one has that

$$F_{\mu\nu}^2(n) = \frac{2}{a^4 e^2} \left[1 - \frac{1}{2} (e^{iea^2 F_{\mu\nu}(n)} + e^{-iea^2 F_{\mu\nu}(n)}) \right] = \frac{2}{a^4 e^2} \left[1 - \frac{1}{2} (U_{\mu\nu}(n) + U_{\mu\nu}^\dagger(n)) \right]$$

Finally, we can write S_G as

$$S_G = \sum_n \sum_{\substack{\mu, \nu \\ \mu < \nu}} \frac{1}{e^2} \left[1 - \frac{1}{2} (U_{\mu\nu}(n) + U_{\mu\nu}^\dagger(n)) \right]$$

1.4.4 Continuum limit

In light of what we have demonstrated in the previous sections, we can write the locally invariant action associated with the free Dirac field [2, 5, 18] as

$$S = S_F + S_G = \sum_{n,\mu} \frac{i}{2} \left[\bar{\psi}(n) \gamma^\mu U_{n,n+\mu} \psi(n+\mu) - \bar{\psi}(n) \gamma^\mu U_{n,n-\mu} \psi(n-\mu) \right] + \\ + \hat{M} \sum_n \bar{\psi}(n) \psi(n) + \sum_n \sum_{\substack{\mu,\nu \\ \mu < \nu}} \frac{1}{e^2} \left[1 - \frac{1}{2} (U_{\mu\nu}(n) + U_{\mu\nu}^\dagger(n)) \right]$$

We can prove the correctness of this expression by taking the continuum limit [2, 5]. Starting from the fermionic side, recalling the rules of the adimensional notation, one has that

$$\lim_{a \rightarrow 0} \left[\hat{M} \sum_n \bar{\psi}(n) \psi(n) \right] = \lim_{a \rightarrow 0} \left[M a \sum_n (a^{3/2})^2 \bar{\psi}(na) \psi(na) \right] = \\ = \lim_{a \rightarrow 0} \left[M a^4 \sum_n \bar{\psi}(na) \psi(na) \right] = \\ = M \int d^4x \bar{\psi}(x) \psi(x)$$

Then, recalling that $U_{n,n+\mu} = 1 + ie a A_\mu + \mathcal{O}(a^2)$, it follows also that

$$\sum_{n,\mu} \frac{i}{2} \left[\bar{\psi}(n) \gamma^\mu U_{n,n+\hat{\mu}} \psi(n+\hat{\mu}) - \bar{\psi}(n) \gamma^\mu U_{n,n-\hat{\mu}} \psi(n-\hat{\mu}) \right] = \\ = a^3 \sum_{n,\mu} \frac{i}{2} \left[\bar{\psi}(na) \gamma^\mu U_{n,n+\hat{\mu}} \psi(na+\mu a) - \bar{\psi}(na) \gamma^\mu U_{n,n-\hat{\mu}} \psi(na-\mu a) \right] = \\ = a^4 \sum_{n,\mu} \frac{i}{2} \left[ie A_\mu (\bar{\psi}(na) \gamma^\mu \psi(na+\mu a) + \bar{\psi}(na+\mu a) \gamma^\mu \psi(na)) + \right. \\ \left. + \frac{1}{a} (\bar{\psi}(na) \gamma^\mu \psi(na+\mu a) - \bar{\psi}(na+\mu a) \gamma^\mu \psi(na)) \right]$$

By exploiting the fact that $x = na$ and so that $\psi(na + \mu a) = \psi(x) + a \partial_\mu \psi(x) + \mathcal{O}(a)^2$, we find also that

$$\sum_{n,\mu} \frac{i}{2} \left[\bar{\psi}(n) \gamma^\mu U_{n,n+\hat{\mu}} \psi(n+\hat{\mu}) - \bar{\psi}(n) \gamma^\mu U_{n,n-\hat{\mu}} \psi(n-\hat{\mu}) \right] = \\ = \int d^4x i \left[ie a A_\mu (\bar{\psi}(x) \gamma^\mu \partial_\mu \psi(x) + \partial_\mu \bar{\psi}(x) \gamma^\mu \psi(x)) + \bar{\psi}(x) \gamma^\mu ie A_\mu \psi(x) + \right. \\ \left. + \frac{1}{2} (\bar{\psi}(x) \gamma^\mu \partial_\mu \psi(x) - \partial_\mu \bar{\psi}(x) \gamma^\mu \psi(x)) \right]$$

Finally, by taking the limit $a \rightarrow 0$, the linear terms in a disappear and we correctly obtain that

$$\lim_{a \rightarrow 0} \left\{ \sum_{n,\mu} \frac{i}{2} \left[\bar{\psi}(n) \gamma^\mu U_{n,n+\hat{\mu}} \psi(n+\hat{\mu}) - \bar{\psi}(n) \gamma^\mu U_{n,n-\hat{\mu}} \psi(n-\hat{\mu}) \right] \right\} =$$

$$\begin{aligned}
&= \int d^4x i \left[\bar{\psi}(x) \gamma^\mu i e A_\mu \psi(x) + \frac{1}{2} (\bar{\psi}(x) \gamma^\mu \partial_\mu \psi(x) - \partial_\mu \bar{\psi}(x) \gamma^\mu \psi(x)) \right] = \\
&= \int d^4x i \left[\bar{\psi}(x) \gamma^\mu i e A_\mu \psi(x) + \bar{\psi}(x) \gamma^\mu \partial_\mu \psi(x) - \frac{1}{2} (\partial_\mu \bar{\psi}(x)) \gamma^\mu \psi(x) \right] = \\
&= \int d^4x \bar{\psi}(x) i \gamma^\mu (\partial_\mu + i e A_\mu) \psi(x)
\end{aligned}$$

Specifically, in the last step we used the fact that the derivative of $\psi(x)$ and $\bar{\psi}(x)$ vanishes when integrated all over the space-time. To recover also the bosonic side of the locally-invariant action, we can exploit the fact that

$$\begin{aligned}
U_{\mu\nu} &= e^{i e a^2 F_{\mu\nu}(n)} = 1 + i e a^2 F_{\mu\nu}(n) - \frac{1}{2} e^2 a^4 F_{\mu\nu}^2(n) + \mathcal{O}(a^6) \\
U_{\mu\nu}^\dagger &= e^{-i e a^2 F_{\mu\nu}(n)} = 1 - i e a^2 F_{\mu\nu}(n) - \frac{1}{2} e^2 a^4 F_{\mu\nu}^2(n) + \mathcal{O}(a^6)
\end{aligned}$$

and so one has that

$$\begin{aligned}
&\sum_n \sum_{\substack{\mu, \nu \\ \mu < \nu}} \frac{1}{e^2} \left[1 - \frac{1}{2} (U_{\mu\nu}(n) + U_{\mu\nu}^\dagger(n)) \right] = \\
&= \sum_n \sum_{\substack{\mu, \nu \\ \mu < \nu}} \frac{1}{e^2} \left[\frac{1}{2} e^2 a^4 F_{\mu\nu}^2 \right] \propto \frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu} + \mathcal{O}(a^2)
\end{aligned}$$

Therefore, we proved that it is possible to correctly recover the original locally invariant action by simply taking the continuum limit.

1.4.5 From the action to the Hamiltonian

Our goal is now to derive the locally invariant Hamiltonian from the corresponding action and then discretize it [2, 19]. Therefore, starting from

$$S = \int d^4x \mathcal{L}(x) = \int d^4x \bar{\psi}(x) (i \gamma^\mu D_\mu - M) \psi(x) - \frac{1}{4} \int d^4x F^{\mu\nu} F_{\mu\nu}$$

we can derive the Hamiltonian by exploiting the fact that

$$H = \int dx \mathcal{H}(x) = \int dx \left\{ \sum_i \pi_i q_i - \mathcal{L}(q_i, \partial_\mu q_i) \right\} \quad \text{where} \quad \pi_i = \frac{\partial \mathcal{L}}{\partial (\partial_\mu q_i)}$$

Specifically, we denote as π_i the conjugate momenta associated to the canonic variables $q_i = A_\mu(x), \psi(x), \bar{\psi}(x)$ with $i = 0, 1, 2, 3$. Assuming the temporal gauge where $A_0(x) = 0 \forall x$, the Lagrangian associated to the locally invariant action can be written as

$$\begin{aligned}
\mathcal{L} &= \bar{\psi}(x) (i \gamma^\mu D_\mu - M) \psi(x) - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \\
&= \bar{\psi}(x) (i \gamma^\mu \partial_\mu - e \gamma^\mu A_\mu - M) \psi(x) + \frac{1}{2} (F_{\mu\nu})_{\mu, \nu \neq 0}^2 + \frac{1}{2} F^{\mu 0} F_{\mu 0} = \\
&= \bar{\psi}(x) (i \gamma^\mu \partial_\mu - e \gamma^\mu A_\mu - M) \psi(x) + \frac{1}{2} (F_{\mu\nu})_{\mu, \nu \neq 0}^2 + \frac{1}{2} (\partial_\mu A_0 - \partial_0 A_\mu)^2 =
\end{aligned}$$

$$\begin{aligned}
&= \bar{\psi}(x)(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - M)\psi(x) + \frac{1}{2}(F_{\mu\nu})_{\mu,\nu \neq 0}^2 + \frac{1}{2}(-\partial_0 A_\mu)^2 = \\
&= \bar{\psi}(x)(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - M)\psi(x) + \frac{1}{2}(F_{\mu\nu})_{\mu,\nu \neq 0}^2 + \frac{1}{2}E_\mu^2
\end{aligned}$$

where we exploited the anti-symmetry of the strength tensor and we defined $\partial_0 A_\mu \equiv E_\mu$. Thus, the conjugate momenta associated to the canonic variables $q_i = A_\mu(x), \psi(x), \bar{\psi}(x)$ are given by

$$\pi_{A_\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = \partial_0 A_\mu \quad \pi_\psi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = \bar{\psi} i \gamma^0 = \psi^\dagger i (\gamma^0)^2 = i \psi^\dagger \quad \pi_{\bar{\psi}} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \bar{\psi})} = 0$$

Therefore, putting all these calculations together, we finally obtain that

$$\begin{aligned}
\mathcal{H} &= \sum_i \pi_i \dot{q}_i - \mathcal{L}(q_i, \partial_\mu q_i) = \\
&= (\partial_0 A_\mu)^2 + \bar{\psi} i \gamma^0 (\partial_0 \psi) - \bar{\psi} (i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - M)\psi - \frac{1}{2}(F_{\mu\nu})_{\mu,\nu \neq 0}^2 - \frac{1}{2}(\partial_0 A_\mu)^2 = \\
&= -\bar{\psi} (i\gamma^\mu D_\mu)_{\mu \neq 0} \psi - \frac{1}{2}(F_{\mu\nu})_{\mu,\nu \neq 0}^2 + \frac{1}{2}(\partial_0 A_\mu)^2
\end{aligned}$$

Finally, integrating the previous expression, we get that

$$H = - \int d^4x \bar{\psi} (i\gamma^\mu D_\mu)_{\mu \neq 0} \psi - \frac{1}{2} \int d^4x (F^{\mu\nu})_{\mu,\nu \neq 0}^2 + \frac{1}{2} \int d^4x (\partial_0 A_\mu)^2 \quad (1.21)$$

In order to discretize Eq. (1.21), we can exploit all the calculations performed so far when working on the locally invariant action. In particular, assuming a lattice with lattice spacing a , we finally have that

$$\begin{aligned}
H &= \sum_{n,\mu \neq 0} \frac{i}{2a} \left[\bar{\psi}(n) \gamma^\mu U_{n,n+\hat{\mu}} \psi(n+\hat{\mu}) - \bar{\psi}(n) \gamma^\mu U_{n,n-\hat{\mu}} \psi(n-\hat{\mu}) \right] + \hat{M} \sum_n \bar{\psi}(n) \psi(n) + \\
&\quad - \sum_n \sum_{\substack{\mu,\nu \\ \mu < \nu \\ \mu,\nu \neq 0}} \frac{1}{e^2} \left[1 - \frac{1}{2}(U_{\mu\nu}(n) + U_{\mu\nu}^\dagger(n)) \right] + \frac{1}{2} \sum_{n,\mu} E_\mu^2(n)
\end{aligned}$$

Since we will work with one spatial dimension and since we will consider only interactions between adjacent sites, the previous Hamiltonian reduces to

$$\begin{aligned}
H &= \sum_n \frac{i}{2} \left[\bar{\psi}(n) \gamma^1 U_{n,n+1} \psi(n+1) - \bar{\psi}(n) \gamma^1 U_{n,n-1} \psi(n-1) \right] + \\
&\quad + \hat{M} \sum_n \bar{\psi}(n) \psi(n) + \frac{1}{2} \sum_n E_1^2(n)
\end{aligned} \quad (1.22)$$

1.5 Kogut-Susskind theoretical framework

1.5.1 Solving the doubling problem

The problem of fermion doubling is an unavoidable obstacle when attempting to discretize Dirac fermions on a lattice due to the theory's topological and symmetry

constraints. However, several analytical approaches have been developed to manage or eliminate these spurious species, with the Kogut–Susskind framework based on *staggered fermions* being one of the most prominent solutions [6, 17, 2].

In one-dimensional fermionic field theories, the doubling problem leads to the presence of two fermion species instead of one. The Kogut-Susskind framework exploits the spin diagonalization approach as the formal mathematical procedure used to solve the doubling problem. In the continuum, a 1D Dirac spinor $\psi(x)$ typically has two components. The discretized hopping term in the Hamiltonian of Eq. (1.22) involves Dirac matrices, which couple these spin components at every site. To remove the matrix structure of γ^1 and decouple the spin components [2, 6], we introduce a local and site-dependent unitary transformation $T(n)$ such that $\psi(n) = T(n)\chi(n)$ and $\bar{\psi}(n) = \bar{\chi}(n)T^\dagger(n)$. In d dimensions, $T(n)$ is a product of gamma matrices raised to the power of the site coordinates, but in one dimension this simplifies to $T(n) = (\gamma^1)^n$. Since $(\gamma^1)^2 = \mathbb{I}$, this transformation effectively rotates the spin basis at every site, repeating every two lattice spacings [6, 17, 2]. Substituting this transformation into the hopping term, we use the fundamental identity $T^\dagger(n)\gamma^1 T(n+1) = \eta_1(n)$ which represents a phase factor known as *staggered phase* that, in one dimension, is simply equal to 1 [6, 17, 2]. Therefore, through this process the Dirac matrix γ^1 is replaced by a numerical factor: this means the different components of the new spinor $\chi(n)$ are now completely decoupled and satisfy identical equations of motion. Since all spin components are now identical and independent, we can discard the redundant ones. By keeping only a single-component spinor at each site, this reduction decreases the number of degrees of freedom per site by a factor of $2^{d/2}$ that corresponds to factor 2 in 1+1 dimension, effectively halving the number of doublers [6, 17, 2].

The staggered formalism spreads the two spinor components across adjacent lattice sites, so that each site hosts only a single-component fermion field $\chi(n)$. Consequently, the lattice unit cell must be doubled, consisting of one even and one odd site to represent the full physical degrees of freedom [6, 20, 17, 2]. To reproduce the physics of the continuum, the two sites are assigned different physical identities based on their parity: sites with even n are interpreted as the location for fermions, while those with odd n consist of anti-fermions' locations. In order to account for these differences, it is mandatory to introduce a factor $(-1)^n$ in the mass term: it is a fundamental physical modeling requirement within the staggered fermion framework, since it ensures that the energy contributions are correctly assigned [6, 20, 17, 2].

Therefore, at this point, the (1+1)D Kogut-Susskind Hamiltonian reduces to

$$H = \sum_n \frac{i}{2} \left[\bar{\chi}(n) U_{n,n+1} \chi(n+1) - \bar{\chi}(n) U_{n,n-1} \chi(n-1) \right] + \hat{M} \sum_n \bar{\chi}(n) \chi(n) + \frac{1}{2} \sum_n E_1^2(n) \quad (1.23)$$

From now on, we will refer to each term of Eq. (1.23) individually as

$$H_U = \sum_n \frac{i}{2} \left[\bar{\chi}(n) U_{n,n+1} \chi(n+1) - \bar{\chi}(n) U_{n,n-1} \chi(n-1) \right] \\ H_M = \hat{M} \sum_n \bar{\chi}(n) \chi(n) \quad (1.24)$$

$$H_E = \frac{1}{2} \sum_n E_1^2(n)$$

1.5.2 Elimination of bosonic degrees of freedom

The theoretical framework of (1+1)-dimensional Quantum Electrodynamics developed in the previous sections is precisely what is known in the literature as the *Schwinger model* [18]. To simulate the Schwinger model (encoded in Eq. (1.23)) on a digital quantum architecture, the target Hamiltonian must be mapped into an equivalent system of interacting two-dimensional spin operators [19, 7, 20, 6]. A significant challenge in this process is the presence of gauge bosons, which are associated with an infinite-dimensional Hilbert space. To overcome this, we can perform an analytical elimination of the gauge fields. In particular, we can notice that, in (1+1) dimensions, the gauge fields are not independent degrees of freedom; rather, they are strictly constrained by the matter distribution via Gauss's Law. At this point, we need to rewrite the term given by

$$H_E = \frac{1}{2} \sum_n E_n^2 \quad (1.25)$$

as a function of matter operators only. We can start our reasoning by considering a 1D closed chain made up by N sites, labeled from 0 to $N - 1$. According to Gauss' Law, by only knowing the value of the electric field in one site and the values of the local charge density in every site of the chain, we are able to calculate the value of the electric field in every point of our closed chain. In the Kogut–Susskind framework, the local charge density ρ_l at site l accounts for the staggered fermion arrangement as

$$\rho_l = e \left[\bar{\chi}_l \chi_l - \frac{1 - (-1)^l}{2} \right] = \begin{cases} e \bar{\chi}_l \chi_l & l \text{ even} \\ e(\bar{\chi}_l \chi_l - 1) & l \text{ odd} \end{cases} \quad (1.26)$$

Physically, this staggered arrangement allows for a consistent identification of particles and antiparticles: an occupied even site represents an electron with charge $\rho_l = e$, whereas an unoccupied odd site (acting as a hole in the Dirac sea) represents a positron with charge $\rho_l = -e$. Within this convention, e is understood as the fundamental negative charge of the electron, making $-e$ the positive charge associated with the positron. Consequently, the numerical term in Eq. (1.26) serves to subtract the background vacuum charge, ensuring that the system is charge-neutral when even sites are empty and odd sites are occupied. Then, the value of the electric field in the site $i + 1$ can be derived as

$$E_{i+1} = E_0 + \sum_{l=1}^i \rho_l = E_0 + e \sum_{l=1}^i \left[\bar{\chi}_l \chi_l - \frac{1 - (-1)^l}{2} \right] \quad (1.27)$$

In particular, we assume that within the link between the sites labeled with 0 and 1 the electric field is worth $E_0 = \text{const.}$ Thus, by substituting Eq. (1.27) into Eq. (1.25), we obtain that

$$H_E = \frac{1}{2} \sum_{n=0}^{N-1} E_n^2 = \frac{1}{2} E_0^2 + \frac{1}{2} \sum_{j=1}^{N-1} \left[E_0 + e \sum_{i=1}^j \left[\bar{\chi}_i \chi_i - \frac{1 - (-1)^i}{2} \right] \right]^2 =$$

$$\begin{aligned}
&= \frac{1}{2}E_0^2 + \frac{1}{2} \sum_{j=1}^{N-1} \left[E_0^2 + 2eE_0 \sum_{i=1}^j \left[\bar{\chi}_i \chi_i - \frac{1 - (-1)^i}{2} \right] + \right. \\
&\quad \left. + \left(e \sum_{i=1}^j \left[\bar{\chi}_i \chi_i - \frac{1 - (-1)^i}{2} \right] \right)^2 \right] = \frac{1}{2}E_0^2 + \frac{1}{2} \sum_{j=1}^{N-1} E_0^2 + \\
&\quad + eE_0 \sum_{j=1}^{N-1} \sum_{i=1}^j \left[\bar{\chi}_i \chi_i - \frac{1 - (-1)^i}{2} \right] + \frac{1}{2} \sum_{j=1}^{N-1} \left(e \sum_{i=1}^j \left[\bar{\chi}_i \chi_i - \frac{1 - (-1)^i}{2} \right] \right)^2 = \\
&= \frac{1}{2}E_0^2 + \frac{1}{2} \sum_{j=1}^{N-1} E_0^2 + eE_0 \sum_{j=1}^{N-1} \sum_{i=1}^j \bar{\chi}_i \chi_i - eE_0 \sum_{j=1}^{N-1} \sum_{i=1}^j \left[\frac{1 - (-1)^i}{2} \right] + \\
&\quad + \frac{1}{2} \sum_{j=1}^{N-1} \left(e \sum_{i=1}^j \left[\bar{\chi}_i \chi_i - \frac{1 - (-1)^i}{2} \right] \right)^2 \simeq \\
&\simeq \frac{1}{2}E_0^2 + \frac{1}{2} \sum_{j=1}^{N-1} E_0^2 + eE_0 \sum_{j=1}^{N-1} \sum_{i=1}^j \bar{\chi}_i \chi_i + \frac{1}{2} \sum_{j=1}^{N-1} \left(e \sum_{i=1}^j \left[\bar{\chi}_i \chi_i - \frac{1 - (-1)^i}{2} \right] \right)^2
\end{aligned}$$

Then, by assuming $E_0 \equiv 0$, the previous expression simplifies as

$$H_E = \frac{1}{2} \sum_{j=1}^{N-1} e^2 \left(\sum_{i=1}^j \left[\bar{\chi}_i \chi_i - \frac{1 - (-1)^i}{2} \right] \right)^2 \quad (1.28)$$

This transformation effectively replaces the explicit bosonic degrees of freedom with long-range interactions between the spins. These interactions represent the Coulomb potential in one dimension, where the coupling strength between particles increases linearly with their relative distance. This procedure allows us to simulate the dynamics of N fermions using only N qubits, ensuring that the system remains within the physically-allowed gauge-invariant subspace by construction [19, 7, 20, 6].

1.6 Summary

In summary, this chapter has systematically established the field-theoretic foundation of the (1+1)-dimensional Schwinger model on the lattice, following a rigorous step-by-step path toward digital quantum simulation.

Beginning with the Path Integral formulation of Green functions and the Grassmann variable formalism required for fermionic functional integration (see Sec. 1.1–1.2), we introduced spacetime discretization as a non-perturbative regularization scheme. Comparing two-point correlation functions in the First Brillouin Zone highlighted the fundamental distinction between scalar fields and Dirac fields, where symmetric lattice derivatives give rise to the fermion doubling problem (see Sec. 1.3).

Secondly, to incorporate local $U(1)$ gauge symmetry on the lattice, we introduced Schwinger line-integrals and elementary plaquette variables, verifying that the continuous action is restored in the continuum limit and deriving the lattice Hamiltonian in the temporal gauge (Sec. 1.4).

Finally, within the Kogut-Susskind framework, we solved the doubling problem in 1+1 dimensions, decoupling spinor components into single-component staggered fields.

Moreover, Gauss's Law allowed us to analytically integrate out the infinite-dimensional electric field degrees of freedom, recasting the gauge field energy into effective long-range Coulomb interactions (Sec. 1.5).

Chapter 2

Digital Quantum Simulation

This chapter introduces the conceptual and algorithmic framework of Quantum Computation, leading into the specifics of Digital Quantum Simulation (DQS). Starting from the foundational principles of quantum information (including qubit registers, universal gate decompositions and quantum measurement) we examine how quantum processors overcome the exponential scaling barriers inherent to classical simulation methods. We detail the paradigm of Digital Quantum Simulation, focusing on the theoretical principles of Adiabatic Quantum State Preparation (AQSP) as a robust protocol for synthesizing non-trivial ground states from simple initial configurations. Finally, we focus on time-dependent unitary evolution discretized via first-order Trotter product formulas.

2.1 Introduction to quantum computing

Quantum computing is defined as the study of information processing tasks that can be accomplished using quantum mechanical systems [21, 7]. This field represents a synthesis of physics, computer science and information theory. It is based on the fundamental principle that the laws of physics governing an information carrier determine its processing capabilities. The need for quantum computing originally emerged from the difficulty classical computers face in simulating quantum systems, whose complexity grows exponentially with the size of the system. A quantum computer can overcome these limitations, offering significant computational advantages [21, 7]. To understand how quantum computers process information, it is helpful to build up the model from its most elementary components. This chapter provides a self-contained introduction to the framework of digital quantum computing, moving systematically from basic state spaces to circuit operations and measurement techniques. First, we review the properties of single and multiple qubits, highlighting the role of superposition and entanglement. Second, we introduce the standard set of quantum gates and show how sequences of unitary operations are assembled into quantum circuits. Finally, we address how classical information is retrieved from a quantum register, focusing on readout protocols like the Hadamard test to compute expectation values of physical operators. This progression establishes the fundamental tools used throughout the subsequent chapters.

2.1.1 Pure states and density operators

Before starting the main analysis, we introduce two basic concepts: pure states and density operators [21]. In quantum mechanics, when the exact state vector of a physical system is known, the system is said to be in a pure state described by a unit vector $|\psi\rangle$ in a Hilbert space $\mathcal{H}$. However, in realistic physical scenarios (such as systems subject to classical noise or thermal fluctuations, or when dealing with subsystems of larger entangled states) the quantum state is not completely known. Instead, the system is described by a statistical ensemble $\{p_i, |\psi_i\rangle\}$, where the system has a classical probability p_i of being in the pure state $|\psi_i\rangle$ with $p_i \geq 0$ and $\sum_i p_i = 1$. The density operator ρ describing such an ensemble is defined as

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

An operator ρ acting on a Hilbert space $\mathcal{H}$ is a valid density operator if and only if it is a positive semidefinite linear operator with unit trace. The operator is Hermitian, satisfying $\rho^\dagger = \rho$, which guarantees that its eigenvalues are real. All fundamental postulates of quantum mechanics can be reformulated directly in terms of the density operator.

2.1.2 The fundamental unit

The *bit* is the fundamental concept of classical computation and classical information. Quantum computation and quantum information are built upon an analogous concept, the *quantum bit*, or *qubit* for short. A qubit is defined as a mathematical object with certain properties that can be realized as an actual physical system. Unlike a classical bit, a qubit can exist not only in the states $|0\rangle$ or $|1\rangle$ but also in a linear superposition of both like

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

with $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$. In other words, we can say that the state of a qubit is a unit vector in a two-dimensional complex vector space [21]. The special states $|0\rangle$ and $|1\rangle$ are known as *computational basis states* and form an orthonormal basis for this vector space. Because of that, we can write the state of a qubit also as

$$|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\varphi} \sin \frac{\theta}{2} |1\rangle$$

where θ and φ define a point on the unit three-dimensional sphere known as *Bloch sphere*, as shown in Fig. 2.1. The Bloch sphere provides an immediate and effective way to visualize the state of a qubit [21]. We underline that any kind of global phase factor $e^{i\gamma}$ can be omitted; in fact, since measurement probabilities depend on the squared magnitude of amplitudes, the phase cancels out $|e^{i\gamma}|^2 = 1$. Correspondingly, we can define systems made of two or more qubits and apply the same rules [21]. For example, a two qubits system has four computational basis states denoted as $|00\rangle, |01\rangle, |10\rangle$ and $|11\rangle$ and can exist in a superposition of these four states such that the state vector describing the two qubits is given by

$$|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$$

with $a_{00}, a_{10}, a_{01}, a_{11} \in \mathbb{C}$ and $|a_{00}|^2 + |a_{10}|^2 + |a_{01}|^2 + |a_{11}|^2 = 1$.

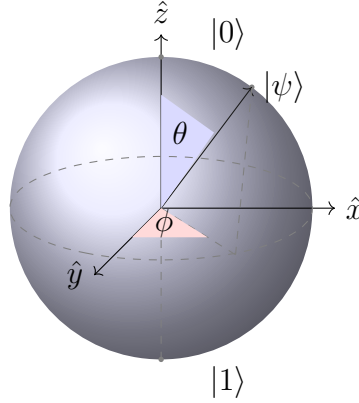


Figure 2.1: Visual geometric representation of a single-qubit pure state $|\psi\rangle$ on the Bloch sphere. The computational basis states $|0\rangle$ and $|1\rangle$ are located at the North and South poles along the $\hat{z}$ -axis, respectively. An arbitrary state vector $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$ is uniquely specified on the unit sphere by the polar angle $\theta \in [0, \pi]$ (measured relative to the $\hat{z}$ -axis) and the azimuthal phase angle $\phi \in [0, 2\pi)$ (in the xy -plane relative to the $\hat{x}$ -axis).

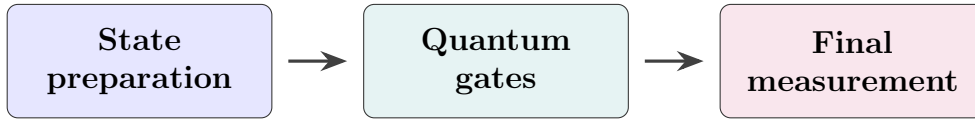


Figure 2.2: Schematic structure of a general quantum circuit workflow, leading to the outcome of a quantum computation.

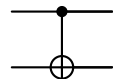
2.1.3 Quantum gates and circuits

Changes occurring to a quantum state can be described using the language of quantum computation. Analogous to the way a classical computer is built from an electrical circuit containing wires and logic gates, a quantum computer is built from a quantum circuit containing wires and elementary quantum gates to carry around and manipulate the quantum information [21, 7, 22]. The structure of a quantum circuit involves different phases: preparation of the system in the desired initial state, applications of quantum gates and final measurement of the result. These steps are summarized in Fig. 2.2. In particular, quantum gates can be divided into two broad categories based on the number of qubits involved: single-qubit gates and multi-qubit gates.

In contrast to the classic case, there are plenty of non-trivial single-qubit quantum gates, each represented by 2×2 unitary matrices. The most relevant ones [21, 7, 22] are represented by the Z -gate, the NOT gate and the Hadamard gate, respectively given by

$$Z \equiv \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad X \equiv \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

The Z -gate leaves $|0\rangle$ unchanged and flips the sign of $|1\rangle$ to give $-|1\rangle$, while the X (NOT) gate flips $|0\rangle$ into $|1\rangle$ and vice versa. The Hadamard gate, instead, turns $|0\rangle$ into $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|1\rangle$ into $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ and so performs a change of basis. In general, we can represent rotations of an arbitrary angle θ about the $\hat{x}$, $\hat{y}$ or



$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Figure 2.3: Circuit representation and matrix form of the CNOT gate in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

$\hat{z}$ -axis in the Bloch sphere by exploiting the exponentiation of the Pauli matrices as

$$\begin{aligned} R_x(\theta) &= e^{-i\hat{\sigma}_x\theta/2} = \cos\frac{\theta}{2}\mathbb{I} - i\sin\frac{\theta}{2}\hat{\sigma}_x \\ R_y(\theta) &= e^{-i\hat{\sigma}_y\theta/2} = \cos\frac{\theta}{2}\mathbb{I} - i\sin\frac{\theta}{2}\hat{\sigma}_y \\ R_z(\theta) &= e^{-i\hat{\sigma}_z\theta/2} = \cos\frac{\theta}{2}\mathbb{I} - i\sin\frac{\theta}{2}\hat{\sigma}_z \end{aligned}$$

Regarding single-qubits gates, it can be demonstrated that every arbitrary 2×2 unitary matrix U can be decomposed into $U = e^{i\alpha}R_z(\beta)R_y(\gamma)R_z(\delta)$.

On the other hand, there exist plenty of multi-qubit gates but the most significant one is definitely represented by the CNOT gate. In fact, any multi-qubit logic gate may be composed from CNOT and single-qubit gates [21, 7, 22]. The action of the CNOT gate may be described as follows: if the control qubit is set to 0 then the target qubit is left untouched, while if it is set to 1 then the control is flipped. The CNOT gate is represented in Fig. 2.3.

2.1.4 Uhlmann Fidelity

To quantify the similarity between quantum states and assess the effects of various quantum operations, it is useful to introduce a measure of distance known as Uhlmann fidelity. For two general density operators ρ and σ , the fidelity [21, 23, 24] is defined as

$$F(\rho, \sigma) \equiv \text{Tr} \left[\sqrt{\rho^{1/2} \sigma \rho^{1/2}} \right]$$

which provides a formal measure of the overlap between the two states. A particularly common scenario involves comparing an arbitrary state σ with a specific pure state $|\psi\rangle$; in this instance, the fidelity simplifies significantly to the square root of the transition probability

$$F(|\psi\rangle, \sigma) = \sqrt{\langle\psi|\sigma|\psi\rangle} \quad (2.1)$$

representing the direct overlap between the target vector and the state ensemble. Moreover, the fidelity has many of the other properties we expect of a good distance measure [21, 23, 24]. For example, the fidelity is symmetric in its inputs and is bounded such that

$$0 \leq F(\rho, \sigma) \leq 1$$

where the upper bound is reached if and only if $\rho = \sigma$. Furthermore, since unitary evolution preserves inner products and trace norms, applying a global unitary operator

to both states leaves the fidelity between them unchanged. Finally, the Uhlmann fidelity exhibits monotonicity since it is non-decreasing under trace-preserving quantum operations, meaning that

$$F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \leq F(\rho, \sigma) \quad \forall \mathcal{E}$$

This result reflects the physical intuition that noise and information loss generally make quantum states progressively less distinguishable.

2.1.5 Hadamard test

In quantum computing, an *ancilla* (or auxiliary qubit) refers to an additional qubit introduced alongside the primary system register to provide extra workspace for quantum operations. Typically initialized in a well-defined state (such as $|0\rangle$), the ancilla interacts with the main register via unitary gates, and it is either measured or disentangled before being discarded. Ancilla qubits are essential for a wide variety of tasks, such as executing non-demolition measurements, as in the case of the Hadamard test. The Hadamard test is a fundamental ancilla-based routine in digital quantum simulation used to estimate the complex expectation value $\langle\psi|U|\psi\rangle$ of a unitary operator U with respect to a quantum state $|\psi\rangle$ [21, 25]. This technique is particularly significant because the direct measurement of the quantum register often collapses the state, whereas the Hadamard test extracts specific information through the measurement on the ancilla qubit. The Hadamard test exploits two circuits to estimate both the real and imaginary part of the expectation value separately.

The real Hadamard test [25, 21] is the quantum circuit in Fig. 2.4 for estimating

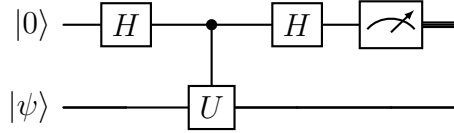


Figure 2.4: Real Hadamard test circuit

$\text{Re}[\langle\psi|U|\psi\rangle]$. To verify this, we find that the circuit transforms the state $|\Psi(0)\rangle = |0\rangle|\psi\rangle$ as

$$\begin{aligned} |\Psi_i\rangle = |0\rangle|\psi\rangle &\xrightarrow{\text{H}_{\text{ancilla}}} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|\psi\rangle \xrightarrow{c-U} \frac{1}{\sqrt{2}}(|0\rangle|\psi\rangle + |1\rangle U|\psi\rangle) \\ &\xrightarrow{\text{H}_{\text{ancilla}}} \frac{1}{\sqrt{2}}(|\psi\rangle + U|\psi\rangle)|0\rangle + \frac{1}{\sqrt{2}}(|\psi\rangle - U|\psi\rangle)|1\rangle = |\Psi_f\rangle \end{aligned}$$

The probabilities of measuring the ancilla to be in state $|0\rangle$ or $|1\rangle$ are therefore given by

$$\begin{aligned} P(0) &= \frac{1}{2} (1 + \text{Re}[\langle\psi|U|\psi\rangle]) \\ P(1) &= \frac{1}{2} (1 - \text{Re}[\langle\psi|U|\psi\rangle]) \end{aligned}$$

so that $P(0) - P(1) = \text{Re}[\langle\psi|U|\psi\rangle]$. For the imaginary Hadamard test [25, 21] the

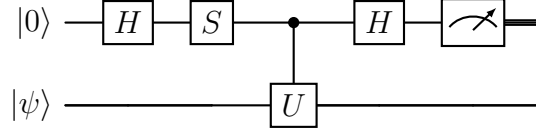


Figure 2.5: Imaginary Hadamard test circuit

procedure is practically identical, except for the fact that before applying the $c - U$ it is required to apply a phase gate S to the ancilla. The phase gate is defined as

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

i.e. it performs a $\hat{z}$ -rotation of $\pi/2$. The imaginary Hadamard test is reported in Fig. 2.5. After the application of S , we obtain that

$$\begin{aligned} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|\psi\rangle &\xrightarrow{S_{\text{ancilla}}} \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)|\psi\rangle \xrightarrow{c-U} \frac{1}{\sqrt{2}}(|0\rangle|\psi\rangle + i|1\rangle U|\psi\rangle) \\ &\xrightarrow{H_{\text{ancilla}}} \frac{1}{\sqrt{2}}(|\psi\rangle + iU|\psi\rangle)|0\rangle + \frac{1}{\sqrt{2}}(|\psi\rangle - iU|\psi\rangle)|1\rangle = |\Psi_f\rangle \end{aligned}$$

The probabilities of measuring the ancilla to be in state $|0\rangle$ or $|1\rangle$ are therefore given by

$$\begin{aligned} P(0) &= \frac{1}{2} (1 + \text{Im} [\langle\psi|U|\psi\rangle]) \\ P(1) &= \frac{1}{2} (1 - \text{Im} [\langle\psi|U|\psi\rangle]) \end{aligned}$$

so that $P(0) - P(1) = \text{Im} [\langle\psi|U|\psi\rangle]$.

2.2 Digital quantum simulation

The concept of digital quantum simulation (DQS) traces its conceptual roots to Richard Feynman's 1982 conjecture, which posited that only a machine operating according to the laws of quantum mechanics could efficiently simulate Nature [7, 8, 9]. Indeed, classical computers face a fundamental bottleneck when describing many-body quantum systems because the size of the system's Hilbert space and the number of complex amplitudes required to describe a state grows exponentially with the number of constituent particles [7, 8, 9]. For instance, representing a system of only 50 qubits requires approximately 10^{15} complex amplitudes, a requirement that already challenges the memory capacity of state-of-the-art supercomputers.

Within this framework, DQS has emerged as a versatile paradigm that utilizes universal programmable quantum computers to calculate the time evolution of various physical models [7, 8, 9]. Unlike analog simulators, which are built to emulate specific target models through direct physical mapping, digital simulators are general-purpose devices that satisfy the DiVincenzo criteria [12, 7]. DiVincenzo's criteria establish the essential theoretical and experimental requirements that a physical architecture must satisfy to realize a universal scalable quantum computer and can be summarized as

follows. First, a universal quantum computer should allow us to work on a scalable physical system with well-characterized qubits, which can be initialized in an arbitrary state. Moreover, it should provide relevant *coherence times*, that is the interval of time before decoherence phenomena degrade the quantum state. The architecture must support also the execution of a universal family of quantum gates and of high-fidelity projective measurements. In DQS, the physical degrees of freedom of interest (such as spins, fermions or gauge fields) are encoded onto a register of qubits and the system's time evolution, governed by the unitary operator $U(t) = e^{-iHt}$, is translated into a discrete sequence of quantum logic gates [7, 8, 9].

A landmark advancement in the field was provided by Seth Lloyd in 1996, who proved that universal quantum computers can efficiently simulate the time evolution of quantum systems by implementing the evolution operator $U(t) = e^{-iHt}$ with time and memory resources that scale polynomially with the system size, provided that the Hamiltonian is composed of a sum of local interaction terms [9, 7]. This efficiency allows researchers to tackle complex problems in quantum chemistry, condensed matter and lattice gauge theories (such as the quantum simulation of the Schwinger model) that are otherwise intractable for classical methods.

To practically implement this simulation pipeline on digital hardware, a structured algorithmic approach is required. This process typically requires a preliminary step, that is the mapping of physical fermionic or bosonic operators into qubit-based Pauli algebra [14]. Then, the application of sophisticated state-preparation protocols, such as Adiabatic Quantum State Preparation (AQSP), can be implemented to reach the target physical state. Finally, because digital hardware operates through discrete gate operations rather than continuous-time evolution, the process necessitates a discretization strategy known as Trotterization [10].

2.2.1 Jordan-Wigner transform

The Jordan–Wigner transformation is the fundamental mathematical tool used to map fermionic systems onto spin systems, represented by qubits. In the context of digital quantum simulation, it acts as the essential bridge that allows intrinsically bosonic quantum hardware operating with qubits and Pauli matrices to represent the antisymmetric statistics and dynamics of fermions [14, 19].

The necessity for the Jordan-Wigner transformation arises from the fundamental difference in how fermionic and spin operators behave. In fact, fermions satisfy the canonical anticommutation relations expressed by $\{\chi_i, \chi_j^\dagger\} = \delta_{ij}$ and $\{\chi_i, \chi_j\} = 0$. This means that exchanging two particles must result in a global sign change in the wavefunction. Instead, qubits behave differently: while spin-operators on the same site anticommute, operators on different sites commute. To emulate fermionic anticommutation in a spin chain, the Jordan-Wigner transform introduces a term known as Pauli string: this product ensures that whenever two fermionic operators are swapped, the cumulative phase picks up a factor of -1, thereby correctly enforcing the global antisymmetry required by the canonical anticommutation relation [14, 19]. Specifically, the spinors $\chi(n)$ and $\chi^\dagger(n)$ are mapped as

$$\chi_n = \prod_{l < n} [i\hat{\sigma}_l^z] \hat{\sigma}_n^- \quad \chi_n^\dagger = \hat{\sigma}_n^+ \prod_{l < n} [-i\hat{\sigma}_l^z]$$

where $\hat{\sigma}_n^{xyz}$ and $\hat{\sigma}_n^\pm$ represent the Pauli matrices and the Ladder operators applied to the n -th qubit, respectively. Thus, without Jordan-Wigner we would obtain

$$\begin{aligned}\{\chi_n^\dagger, \chi_n\}_+ &= \{\hat{\sigma}_n^+, \hat{\sigma}_n^-\}_+ = \mathbb{I} & \{\chi_{n+1}^\dagger, \chi_n\}_+ &= \{\hat{\sigma}_{n+1}^+, \hat{\sigma}_n^-\}_+ = 2\hat{\sigma}_n^+ \hat{\sigma}_{n+1}^- \neq 0 \\ \{\chi_n, \chi_n\}_+ &= \{\hat{\sigma}_n^-, \hat{\sigma}_n^-\}_+ = 0 & \{\chi_{n+1}, \chi_n\}_+ &= \{\hat{\sigma}_{n+1}^-, \hat{\sigma}_n^-\}_+ = 2\hat{\sigma}_n^- \hat{\sigma}_{n+1}^- \neq 0\end{aligned}$$

Applying Jordan-Wigner, instead, we have

$$\begin{aligned}\{\chi_n^\dagger, \chi_n\}_+ &= \{\hat{\sigma}_n^+, \hat{\sigma}_n^-\}_+ = \mathbb{I} & \{\chi_{n+1}^\dagger, \chi_n\}_+ &= \{\hat{\sigma}_{n+1}^+, \hat{\sigma}_n^-\}_+ = 0 \\ \{\chi_n, \chi_n\}_+ &= \{\hat{\sigma}_n^-, \hat{\sigma}_n^-\}_+ = 0 & \{\chi_{n+1}, \chi_n\}_+ &= \{\hat{\sigma}_{n+1}^-, \hat{\sigma}_n^-\}_+ = 0\end{aligned}$$

In fact, for example, exploiting the commutation rules of Pauli operators, one can verify that

$$\begin{aligned}\chi_{n+1}^\dagger \chi_n &= \left[\hat{\sigma}_{n+1}^+ \prod_{l < n+1} [-i\hat{\sigma}_l^z] \right] \cdot \left[\prod_{l < n} [i\hat{\sigma}_l^z] \hat{\sigma}_n^- \right] = \hat{\sigma}_{n+1}^+ \left[\prod_{l < n} (\hat{\sigma}_l^z)^2 \right] [-i\hat{\sigma}_n^z] \hat{\sigma}_n^- = \\ &= \hat{\sigma}_{n+1}^+ [-i\hat{\sigma}_n^z] \hat{\sigma}_n^- = -\hat{\sigma}_{n+1}^+ \hat{\sigma}_n^- [-i\hat{\sigma}_n^z] = i\hat{\sigma}_{n+1}^+ \hat{\sigma}_n^- \\ \chi_n^\dagger \chi_{n+1} &= \left[\hat{\sigma}_n^+ \prod_{l < n} [-i\hat{\sigma}_l^z] \right] \cdot \left[\prod_{l < n+1} [i\hat{\sigma}_l^z] \hat{\sigma}_{n+1}^- \right] = \hat{\sigma}_n^+ \hat{\sigma}_{n+1}^- \left[\prod_{l < n} (\hat{\sigma}_l^z)^2 \right] [i\hat{\sigma}_n^z] = \\ &= \hat{\sigma}_n^+ \hat{\sigma}_{n+1}^- [i\hat{\sigma}_n^z] = -i\hat{\sigma}_n^+ \hat{\sigma}_{n+1}^-\end{aligned}$$

and equivalently do it for

$$\begin{aligned}\chi_n^\dagger \chi_n &= \left[\hat{\sigma}_n^+ \prod_{l < n} [-i\hat{\sigma}_l^z] \right] \cdot \left[\prod_{l < n} [i\hat{\sigma}_l^z] \hat{\sigma}_n^- \right] = \hat{\sigma}_n^+ \left[\prod_{l < n} (\hat{\sigma}_l^z)^2 \right] \hat{\sigma}_n^- = \hat{\sigma}_n^+ \hat{\sigma}_n^- = \frac{1}{2} (\mathbb{I} + \hat{\sigma}_n^z) \\ \chi_n \chi_n^\dagger &= \left[\prod_{l < n} [i\hat{\sigma}_l^z] \hat{\sigma}_n^- \right] \cdot \left[\hat{\sigma}_n^+ \prod_{l < n} [-i\hat{\sigma}_l^z] \right] = \hat{\sigma}_n^- \hat{\sigma}_n^+ \left[\prod_{l < n} (\hat{\sigma}_l^z)^2 \right] = \hat{\sigma}_n^- \hat{\sigma}_n^+ = \frac{1}{2} (\mathbb{I} - \hat{\sigma}_n^z)\end{aligned}$$

Moreover, since we are working on a closed chain, we need to take into account the effects of periodic boundary conditions. In order to satisfy periodic boundary conditions, we need to manually compute the term that connects the sites $N-1$ and 0 since the Jordan-Wigner transformation does not automatically implement this aspect. By putting $n = N-1$ and $n+1 = 0$, we find that the term and its hermitian conjugate are given by

$$\begin{aligned}\chi_0^\dagger \chi_{N-1} &= \hat{\sigma}_0^+ \cdot \left[\prod_{l=0}^{N-2} [i\hat{\sigma}_l^z] \hat{\sigma}_{N-1}^- \right] = i^{N-1} (\hat{\sigma}_0^+ \hat{\sigma}_0^z \hat{\sigma}_{N-1}^-) \left[\prod_{l=1}^{N-2} \hat{\sigma}_l^z \right] = -i^{N-1} \left[\prod_{l=1}^{N-2} \hat{\sigma}_l^z \right] \hat{\sigma}_0^+ \hat{\sigma}_{N-1}^- \\ \chi_{N-1}^\dagger \chi_0 &= \left[\hat{\sigma}_{N-1}^+ \prod_{l=0}^{N-2} [-i\hat{\sigma}_l^z] \right] \cdot \hat{\sigma}_0^- = (-i)^{N-1} \left[\prod_{l=1}^{N-2} \hat{\sigma}_l^z \right] \hat{\sigma}_{N-1}^+ \hat{\sigma}_0^- = \\ &= -(-i)^{N-1} \left[\prod_{l=1}^{N-2} \hat{\sigma}_l^z \right] \hat{\sigma}_{N-1}^+ \hat{\sigma}_0^- = (-1)^N \cdot i^{N-1} \left[\prod_{l=1}^{N-2} \hat{\sigma}_l^z \right] \hat{\sigma}_{N-1}^+ \hat{\sigma}_0^-\end{aligned}$$

By using the fact that N has necessarily to be an even integer in order to correctly close the chain, we finally find that

$$\chi_0^\dagger \chi_{N-1} = -i^{N-1} \left[\prod_{l=1}^{N-2} \hat{\sigma}_l^z \right] \hat{\sigma}_0^+ \hat{\sigma}_{N-1}^- \quad \chi_{N-1}^\dagger \chi_0 = i^{N-1} \left[\prod_{l=1}^{N-2} \hat{\sigma}_l^z \right] \hat{\sigma}_{N-1}^+ \hat{\sigma}_0^-$$

2.2.2 Mapping of Schwinger model through JW transform

At this point, we need to substitute these calculations into our Hamiltonian, mapping every term on the spin-system. Starting with Eq. (1.28), we find that

$$\begin{aligned}
H_E &= \sum_{i=1}^{N-1} \sum_{k=1}^{N-1} \frac{e^2}{2} (N - \max\{i, k\}) \bar{\chi}_i \chi_i \bar{\chi}_k \chi_k + \\
&\quad - \frac{e^2}{2} \sum_{i=0}^{N-1} \sum_{k=1}^{N-1} (N - \max\{i, k\}) (1 - (-1)^k) \bar{\chi}_i \chi_i = \\
&= \sum_{i=1}^{N-1} \sum_{k=0}^{N-1} \frac{e^2}{8} (N - \max\{i, k\}) \hat{\sigma}_i^z \hat{\sigma}_k^z + \frac{e^2}{4} \sum_{i=1}^{N-1} \sum_{k=1}^{N-1} (N - \max\{i, k\}) (-1)^k \hat{\sigma}_i^z
\end{aligned}$$

We can then rewrite this expression in a more convenient way [19] by rearranging the terms as

$$\begin{aligned}
H_E &= \sum_{i=1}^{N-1} \sum_{k=1}^{N-1} \frac{e^2}{8} (N - \max\{i, k\}) \hat{\sigma}_i^z \hat{\sigma}_k^z + \frac{e^2}{4} \sum_{i=1}^{N-1} \sum_{k=1}^{N-1} (N - \max\{i, k\}) (-1)^k \hat{\sigma}_i^z = \\
&= \sum_{i=1}^{N-1} \sum_{k=i+1}^{N-1} \frac{e^2}{4} (N - k) \hat{\sigma}_i^z \hat{\sigma}_k^z + \frac{e^2}{4} \sum_{k=1}^{N-1} (-1)^k \left[\sum_{l=1}^k (N - k) \hat{\sigma}_l^z + \sum_{l=k+1}^{N-1} \sum_{n=l}^{N-1} \hat{\sigma}_l^z \right]
\end{aligned}$$

and conclude that

$$\begin{aligned}
H_E &= \sum_{n=1}^{N-1} \sum_{l=n+1}^{N-1} \frac{e^2}{4} (N - l) \hat{\sigma}_n^z \hat{\sigma}_l^z + \frac{e^2}{4} \sum_{k=1}^{N-1} (-1)^k \left[\sum_{l=1}^k (N - k) \hat{\sigma}_l^z + \sum_{n=k+1}^{N-1} \sum_{l=k+1}^l \hat{\sigma}_l^z \right] = \\
&= \sum_{n=1}^{N-1} \sum_{l=n+1}^{N-1} \frac{e^2}{4} (N - l) \hat{\sigma}_n^z \hat{\sigma}_l^z - \frac{e^2}{4} \sum_{n=1}^{N-1} \frac{1 - (-1)^n}{2} \sum_{l=1}^n \hat{\sigma}_l^z
\end{aligned}$$

The complete derivation can be found in Section B.2. For H_U in Eq. (1.24), instead, we find that

$$\begin{aligned}
H_U &= \sum_{n=0}^{N-1} \frac{i}{2} \left[\chi_n^\dagger \chi_{n+1} U_{n,n+1} - U_{n,n-1} \chi_n^\dagger \chi_{n-1} \right] = \\
&= \sum_{n=0}^{N-2} \frac{i}{2} \left[-U_{n,n+1} i \hat{\sigma}_n^+ \hat{\sigma}_{n+1}^- - U_{n+1,n} i \hat{\sigma}_{n+1}^+ \hat{\sigma}_n^- \right] \\
&\quad + \frac{i^N}{2} \left[\prod_{l=1}^{N-2} \hat{\sigma}_l^z \right] (U_{N-1,0} \hat{\sigma}_{N-1}^+ \hat{\sigma}_0^- + U_{0,N-1} \hat{\sigma}_0^+ \hat{\sigma}_{N-1}^-) = \\
&= \sum_{n=0}^{N-2} \frac{1}{2} \left[U_{n,n+1} \hat{\sigma}_n^+ \hat{\sigma}_{n+1}^- + U_{n+1,n} \hat{\sigma}_{n+1}^+ \hat{\sigma}_n^- \right] + \\
&\quad + \frac{i^N}{2} \left[\prod_{l=1}^{N-2} \hat{\sigma}_l^z \right] (U_{N-1,0} \hat{\sigma}_{N-1}^+ \hat{\sigma}_0^- + U_{0,N-1} \hat{\sigma}_0^+ \hat{\sigma}_{N-1}^-)
\end{aligned}$$

where in the second line we ordered the indices in such a way to guarantee a correct interaction between the sites in our closed chain. In addition, the operators $U_{n,n+1}$ or $U_{n+1,n}$ may be absorbed into the definition of the $\hat{\sigma}^\pm$ operators by sending

$$\hat{\sigma}_n^+ = \prod_{l \leq n} U_{l+1,l} \hat{\sigma}_n^+ \quad \hat{\sigma}_n^- = \prod_{l \leq n} U_{l,l+1} \hat{\sigma}_n^-$$

By doing this, we are multiplying every spin operator $\hat{\sigma}_n^\pm$ by a phase factor whose value depends on the gauge operators belonging to all the sites to the left of n . In fact, we have that

$$\begin{aligned} U_{n,n+1} \hat{\sigma}_n^+ \hat{\sigma}_{n+1}^- &= U_{n,n+1} \left[\prod_{l < n} U_{l,l+1} \right] \hat{\sigma}_n^+ \left[\prod_{l < n+1} U_{l+1,l} \right] \hat{\sigma}_{n+1}^- = \\ &= U_{n,n+1} U_{n+1,n} \hat{\sigma}_n^+ \hat{\sigma}_{n+1}^- = \hat{\sigma}_n^+ \hat{\sigma}_{n+1}^- \end{aligned}$$

and similarly we can find that

$$U_{n+1,n} \hat{\sigma}_{n+1}^+ \hat{\sigma}_n^- = \hat{\sigma}_{n+1}^+ \hat{\sigma}_n^- \quad U_{N-1,0} \hat{\sigma}_{N-1}^+ \hat{\sigma}_0^- = \hat{\sigma}_{N-1}^+ \hat{\sigma}_0^- \quad U_{0,N-1} \hat{\sigma}_0^+ \hat{\sigma}_{N-1}^- = \hat{\sigma}_0^+ \hat{\sigma}_{N-1}^-$$

We specify that, in the last step, we exploited the fact that the Schwinger line-integral is a unitary operator. It is important to underline that this transformation leaves $\hat{\sigma}_n^z$ untouched $\forall n$ since

$$\hat{\sigma}_n^z \equiv \mathbb{I} - 2\hat{\sigma}_n^- \hat{\sigma}_n^+ = \mathbb{I} - 2 \left[\prod_{l \leq n} U_{l-1,l} \right] \hat{\sigma}_n^- \left[\prod_{l \leq n} U_{l,l-1} \right] \hat{\sigma}_n^+ = \mathbb{I} - 2\hat{\sigma}_n^- \hat{\sigma}_n^+$$

Thus, we obtain that

$$H_U^{\text{spin}} = \sum_{n=0}^{N-2} \frac{1}{2} \left[\hat{\sigma}_n^+ \hat{\sigma}_{n+1}^- + \hat{\sigma}_{n+1}^+ \hat{\sigma}_n^- \right] + \frac{i^N}{2} \left[\prod_{l=1}^{N-2} \hat{\sigma}_l^z \right] (\hat{\sigma}_{N-1}^+ \hat{\sigma}_0^- + \hat{\sigma}_0^+ \hat{\sigma}_{N-1}^-) \quad (2.2)$$

We can also rewrite Eq (2.2) by exploiting the fact that

$$\hat{\sigma}^+ = \frac{\hat{\sigma}^x + i\hat{\sigma}^y}{2} \quad \hat{\sigma}^- = \frac{\hat{\sigma}^x - i\hat{\sigma}^y}{2}$$

and by noticing that

$$\begin{aligned} \sum_{n=0}^{N-2} \left[\hat{\sigma}_n^+ \hat{\sigma}_{n+1}^- + \hat{\sigma}_{n+1}^+ \hat{\sigma}_n^- \right] &= \underbrace{\hat{\sigma}_0^+ \hat{\sigma}_1^- + \hat{\sigma}_1^+ \hat{\sigma}_0^-}_{n=0} + \underbrace{\hat{\sigma}_1^+ \hat{\sigma}_2^- + \hat{\sigma}_2^+ \hat{\sigma}_1^-}_{n=1} + \underbrace{\hat{\sigma}_2^+ \hat{\sigma}_3^- + \hat{\sigma}_3^+ \hat{\sigma}_2^-}_{n=2} + \dots \\ &\quad \dots + \underbrace{\hat{\sigma}_{N-2}^+ \hat{\sigma}_{N-1}^- + \hat{\sigma}_{N-1}^+ \hat{\sigma}_{N-2}^-}_{n=N-2} \end{aligned}$$

We can start by expanding the following product imposing $k = j + 1$, thus

$$\hat{\sigma}_j^+ \hat{\sigma}_k^- = \left(\frac{\hat{\sigma}_j^x + i\hat{\sigma}_j^y}{2} \right) \left(\frac{\hat{\sigma}_k^x - i\hat{\sigma}_k^y}{2} \right) = \frac{1}{4} (\hat{\sigma}_j^x \hat{\sigma}_k^x + \hat{\sigma}_j^y \hat{\sigma}_k^y + i\hat{\sigma}_j^y \hat{\sigma}_k^x - i\hat{\sigma}_j^x \hat{\sigma}_k^y)$$

$$\hat{\sigma}_k^+ \hat{\sigma}_j^- = \left(\frac{\hat{\sigma}_k^x + i\hat{\sigma}_k^y}{2} \right) \left(\frac{\hat{\sigma}_j^x - i\hat{\sigma}_j^y}{2} \right) = \frac{1}{4} (\hat{\sigma}_k^x \hat{\sigma}_j^x + \hat{\sigma}_k^y \hat{\sigma}_j^y + i\hat{\sigma}_k^y \hat{\sigma}_j^x - i\hat{\sigma}_k^x \hat{\sigma}_j^y)$$

so that

$$\hat{\sigma}_j^+ \hat{\sigma}_k^- + \hat{\sigma}_k^+ \hat{\sigma}_j^- = \frac{1}{2} (\hat{\sigma}_j^x \hat{\sigma}_k^x + \hat{\sigma}_j^y \hat{\sigma}_k^y)$$

By substituting these calculations in Eq. (2.2), we obtain that

$$\begin{aligned} H_U^{\text{spin}} &= \sum_{n=0}^{N-2} \frac{1}{2} \left[\hat{\sigma}_n^+ \hat{\sigma}_{n+1}^- + \hat{\sigma}_{n+1}^+ \hat{\sigma}_n^- \right] + \frac{i^N}{2} \left[\prod_{l=1}^{N-2} \hat{\sigma}_l^z \right] (\hat{\sigma}_{N-1}^+ \hat{\sigma}_0^- + \hat{\sigma}_0^+ \hat{\sigma}_{N-1}^-) = \\ &= \sum_{n=0}^{N-2} \frac{1}{4} \left[\hat{\sigma}_n^x \hat{\sigma}_{n+1}^x + \hat{\sigma}_{n+1}^y \hat{\sigma}_n^y \right] + \frac{i^N}{4} \left[\prod_{l=1}^{N-2} \hat{\sigma}_l^z \right] (\hat{\sigma}_{N-1}^x \hat{\sigma}_0^x + \hat{\sigma}_0^y \hat{\sigma}_{N-1}^y) \end{aligned}$$

Therefore, the final expression for the spin Hamiltonian mapping the model of Eq. (1.23) is given by

$$H^{\text{spin}} = H_M^{\text{spin}} + H_E^{\text{spin}} + H_U^{\text{spin}} \quad (2.3)$$

with

$$\begin{aligned} H_M^{\text{spin}} &= \frac{\hat{M}}{2} \sum_{n=0}^{N-1} (-1)^n \hat{\sigma}_n^z \\ H_E^{\text{spin}} &= \sum_{n=1}^{N-1} \sum_{l=n+1}^{N-1} \frac{e^2}{4} (N-l) \hat{\sigma}_n^z \hat{\sigma}_l^z + \frac{e^2}{8} \sum_{n=1}^{N-1} (1 - (-1)^n) \sum_{l=1}^n \hat{\sigma}_l^z \\ H_U^{\text{spin}} &= \sum_{n=0}^{N-2} \frac{1}{4} \left[\hat{\sigma}_n^x \hat{\sigma}_{n+1}^x + \hat{\sigma}_{n+1}^y \hat{\sigma}_n^y \right] + \frac{i^N}{4} \left[\prod_{l=1}^{N-2} \hat{\sigma}_l^z \right] (\hat{\sigma}_{N-1}^x \hat{\sigma}_0^x + \hat{\sigma}_0^y \hat{\sigma}_{N-1}^y) \end{aligned}$$

In summary, the total spin Hamiltonian encapsulates the three fundamental physical components governing the lattice gauge theory. The mass term H_M^{spin} accounts for the rest energy of the matter fields distributed across the lattice. The electric field term, H_E^{spin} , describes the energy stored in the gauge field, giving rise to effective long-range interactions between charges. Finally, the kinetic or *hopping* term, H_U^{spin} , drives the dynamics of the system, mediating particle transport and pair creation-annihilation processes between adjacent sites while ensuring boundary conditions are respected.

2.2.3 Adiabatic preparation on digital quantum computers

Adiabatic quantum state preparation (AQSP) on a digital quantum computer represents an initialization subroutine embedded directly within the gate-based quantum circuit model. While ideal Adiabatic quantum computation (AQC) relies on continuous-time physical dynamics governed by the Schrödinger equation, digital AQSP translates this continuous trajectory into a discrete sequence of elementary quantum gates applied to a qubit register [26, 21]. To establish a clear theoretical background, this section first introduces the fundamental concepts of adiabatic quantum computing; thereafter,

attention shifts to the crucial discretization step that enables and characterizes adiabatic state preparation within the digital quantum circuit paradigm.

The core principle of this approach consists of mapping the solution of a computational problem onto the ground state of a target Hamiltonian, thereby transforming a discrete logic task into a physical optimization process [26, 21]. A k -local adiabatic quantum computation is formally defined by two k -local Hamiltonians, denoted as H_0 and H_1 , which act on a system of N particles where each particle possesses $p \geq 2$ possible states. The process begins with the system initialized in the ground state of H_0 , which is specifically designed to be unique, easy to prepare and typically represented as a product state. Subsequently, the initial Hamiltonian is slowly transformed into the final Hamiltonian H_1 through an interpolating schedule defined by $H(s) = (1 - s)H_0 + sH_1$, where $s(t)$ parameterizes the path from the initial to the final state over a total time t_f [26, 21]. The primary objective is to ensure that the final state of the system remains ε -close in ℓ_2 -norm to the ground state of H_1 , which encodes the solution to the computational problem [26, 21].

The physical foundation for this model is the Adiabatic Theorem [27], which serves as the backbone of AQC by providing the rigorous sufficient conditions for the success of the computation. The theorem states that if a system is initially prepared in an eigenstate of a time-dependent Hamiltonian, the time evolution will approximately keep the actual state of the system in the corresponding instantaneous eigenstate, provided that the Hamiltonian varies sufficiently slowly [26, 27]. In particular, the Adiabatic Theorem aims at quantifying the exact nature of this slow variation, providing a relationship that quantifies the required run time t_f in terms of the minimum spectral gap $\Delta_{\min}$ between the ground state and the first excited state, alongside the time-derivative of the Hamiltonian [26, 27]. This theorem exists in many variants [26, 27]; here, we recall the one that we choose to follow along this entire journey.

Theorem 2.2.1 (Adiabatic Theorem). *Let $H_{t_f}(t)$ be the Hamiltonian involved in the Schrödinger equation*

$$\frac{\partial}{\partial t}|\varepsilon_i(t)\rangle = -iH_{t_f}(t)|\varepsilon_i(t)\rangle$$

Suppose that $H_{t_f}(t)$ can be written in a t_f -independent form as $H_{t_f}(st_f) = H(s) = A(s)H_0 + B(s)H_1$, where $s \equiv t/t_f \in [0, 1]$ stands for the dimensionless time and $A(s), B(s)$ are monotonically decreasing and increasing functions, respectively. Therefore, the Schrödinger equation then becomes

$$\frac{1}{t_f} \frac{\partial}{\partial s}|\varepsilon_i(s)\rangle = -iH(s)|\varepsilon_i(s)\rangle \quad (2.4)$$

and guarantees that

$$\frac{1}{t_f} \max_{s \in [0, 1]} \frac{|\langle \varepsilon_i(s) | \partial_s H(s) | \varepsilon_j(s) \rangle|}{|\varepsilon_i(s) - \varepsilon_j(s)|^2} \ll 1 \quad \forall j \neq i \quad (2.5)$$

The conditions provided by Eq. (2.4) and Eq. (2.5) give rise to the criterion that $t_f \gg 1/\Delta_{ij}^2$ with $\Delta_{ij} = \min_{s \in [0, 1]}(\varepsilon_i - \varepsilon_j)$.

In fact, most rigorous variants of the theorem [26, 28] suggest that the total execution time scales inversely with the spectral gap typically as $1/\Delta^3$. Aside from this, the inverse

proportionality with the gap implies that a vanishing energy gap or a non-smooth evolution schedule can cause the system to deviate from the instantaneous eigenstate and leak into other states [26, 27].

A crucial property of AQSP is that, if the evolution is maintained as strictly unitary, the process ensures the preservation of the scalar product throughout the entire path. At the conclusion of the process, the system is described by an orthonormal basis composed of the instantaneous eigenstates of the time-dependent Hamiltonian, which theoretically allows researchers to access every state in the system through linear combinations of these eigenstates [21, 26]. This mathematical structure makes AQSP an extremely robust tool for preparing complex quantum states by evolving from simple initial configurations into the high-fidelity physical state of the target theory [19, 21, 26].

This work mainly focuses on the standard linear interpolation schedule, which represents the most straightforward and common framework for implementing adiabatic quantum computation due to its mathematical simplicity. In this approach, the time-dependent Hamiltonian is defined as $H(s) = (1 - s)H_0 + sH_1$, where the parameter $s \equiv t/t_f$ varies at a constant rate from 0 to 1. This linear path is characterized by a constant rate of change, which simplifies the dynamical analysis as all higher-order derivatives of the Hamiltonian with respect to the interpolation parameter vanish. The computational success of such a linear process is strictly dictated by the minimum spectral gap encountered along the path. While linear interpolation is a fundamental benchmark, it is often sub-optimal for complex algorithmic tasks where the energy gap becomes extremely narrow. A paradigmatic example is found in the adiabatic version of Grover’s search algorithm [26], where a strictly linear schedule fails to maintain the quadratic quantum speedup, resulting in a classical-like scaling of $\mathcal{O}(N)$ because the system runs too fast through the critical region of the minimum gap.

To circumvent these limitations, more sophisticated generic or adaptive strategies are employed. For instance, local adiabatic evolution [26] utilizes a non-linear schedule $s(t)$ where the speed of evolution ds/dt is tuned to be proportional to the instantaneous spectral gap $\Delta^2(s)$, effectively slowing down near the minimum gap to preserve adiabaticity while accelerating elsewhere to minimize the total runtime. Beyond optimizing the schedule, the geometric path itself can be modified to enhance the efficiency of state preparation. Techniques such as the addition of catalyst Hamiltonians H_C , which are intermediate terms that vanish at the boundaries, are designed specifically to keep the gap open and prevent problematic perturbative crossings [26]. More recently, symmetry-guided protocols like Branched-Subspaces Adiabatic Preparation (B-SAP) [29] have been introduced. By ensuring that the energy gap remains open within the relevant symmetry subspace, B-SAP allows for the successful preparation of target states even in regimes where a standard linear interpolation might encounter near-degeneracies or level crossings [29]. These diverse strategies demonstrate that the choice of both the adiabatic path and the temporal schedule are critical degrees of freedom for the practical realization of high-fidelity quantum state preparation.

2.2.4 First-order Trotter formula

At this point, let us consider a quantum system whose dynamics is described by a k -local Hamiltonian H and that is found to be in the state $|\psi(0)\rangle$ at time $t = 0$. That

said, the core concepts of AQSP allow us to determine that our state of input at time $t = 0$ will be evolved in our state of output at time $t = \tau$ through the action of a unitary operator of the form

$$|\psi(\tau)\rangle = U(\tau)|\psi(0)\rangle = e^{-i\tau H}|\psi(0)\rangle$$

The problem of computing the corresponding time evolution operator is equivalent to the task of implementing a well defined unitary matrix. A quantum computer endowed with a universal set of quantum gates is by definition able to perform any unitary transformation, albeit not necessarily in an efficient number of elementary operations. In particular, as mentioned above, it was proven [7, 21] that that universal quantum computers can calculate $U(\tau)$ efficiently (i.e. with polynomial time and memory resources in the size of the target system) when the total Hamiltonian is a sum of local terms. The proof is based on two fundamental facts. First, in the circuit model for UQS we can implement generic transformations approximated to arbitrary precision by successively performing elementary unitary operations (quantum gates), and appending one unitary U_A after another U_B in the circuit results in a total unitary, which is mathematically the product $U_A U_B$ being applied to the state of the qubit register [7, 21]. Second, any unitary operation U acting on N qubits can be implemented by performing $\mathcal{O}(2^{2N})$ elementary operations (we recall that the dimension of the Hilbert space associated to N qubits is $d = 2^N$) [7, 21].

Following this reasoning, we suppose that we can write the Hamiltonian of the system as a sum of L local terms, each one acting on at most k of the N particles of the system. Summarizing, this leads us to

$$|\psi(\tau)\rangle = U(\tau)|\psi(0)\rangle = e^{-i\tau H}|\psi(0)\rangle = e^{-i\tau \sum_{l=1}^L H_l}|\psi(0)\rangle$$

This is justified by the fact that we are working with a k -local Hamiltonian. Therefore, k is the maximum dimension of the subsystem over which the action of each H_l is restricted. Typically, we will have $k \ll 2N$ since local terms only involve few-body interactions. In this case, the single unitary operator $U_l(\tau) = e^{-i\tau H_l}$ can be computed at most with a number $\mathcal{O}(k)$ of operations [7, 21]. The overall product

$$\tilde{U}(\tau) := \prod_{l=1}^L U_l(\tau) := \prod_{l=1}^L e^{-i\tau H_l}$$

can therefore be obtained on a universal quantum computer by juxtaposing the circuit implementations of the single $U_l(\tau)$ unitaries, and it would take at most $\mathcal{O}(Lk^2)$ elementary operations [7, 21]. The final step of the reasoning lies in the following mathematical identity, which is known as the first-order Trotter decomposition

$$e^{-i\tau \sum_{l=1}^L H_l} = \lim_{n \rightarrow \infty} \left(\prod_{l=1}^L e^{-iH_l \tau/n} \right)^n$$

where n is known as the number of *Trotter steps* [10, 7]. Unless all the H_l operators commute, in which case the Trotter identity is exact already for $n = 1$, the product of local unitaries will not be exactly equal to the total target unitary $U(\tau)$. However, as proved in Theorem (B.3.1), $\forall n_s, \tau$ it holds that

$$U(\tau) = e^{-i\tau H} = \left(\prod_{l=1}^L e^{-iH_l \tau/n_s} \right)^{n_s} + \mathcal{O}\left(\frac{\tau^2}{n_s}\right)$$

which means that we can approximate arbitrarily well the desired unitary operator by repeating n_s times the sequence of gates corresponding to the product of local terms for time slices τ/n_s . All in all, we were able to break our original problem into smaller pieces, which can now be implemented efficiently using only a limited set of elementary gates and give the correct answer up to an arbitrarily small digital error. Indeed, for any $\varepsilon > 0$ and τ , there exists a $n_s(\varepsilon)$ such that $U(\tau)$ can be computed within an approximation ε in at most $Ln_s(\varepsilon)k^2$ operations. This is polynomial in N whenever $L = \text{poly}(N)$, as for example in the case of nearest neighbors interactions [10, 7].

Following these instructions, we are able to build the entire quantum circuit to perform AQSP up to the desired level of precision by tuning the parameters τ and n_s .

Chapter 3

Ground State Preparation of the Schwinger Model

This chapter presents the numerical implementation and performance evaluation of the digital adiabatic state preparation protocol for the $(1+1)$ -dimensional Schwinger model. Utilizing the spin- $1/2$ Hamiltonian derived from the Kogut–Susskind staggered lattice formulation (see Eq. (2.3)), we discretize the adiabatic time evolution through first-order Trotter product formulas as proved in Theorem B.3.1. We systematically evaluate the state preparation fidelity against exact classical diagonalization across various system sizes N and lattice spacings a , assessing the impact of Trotter discretization errors. Additionally, we evaluate the spatial two-point correlation functions of the electric field, accurately estimating the autocorrelation at $r = 1$ within a 3% relative error and cleanly isolating the deterministic Trotter discretization error. Furthermore, at the end of the Chapter we explore two complementary pathways for extracting the Schwinger vector boson mass m_S both via spatial correlation function decays and direct first excited state preparation, analyzing the technical trade-offs, finite-size scaling behavior and precision limits in ideal digital quantum environments.

3.1 The starting Hamiltonian for AQSP

In order to perform the adiabatic quantum state preparation of the ground state, we have to decide how we want H_0 to be built. In particular, we recall that H_0 has to be designed in a convenient way, allowing us to easily access the eigenvalues and eigenvectors of the system. We choose to define H_0 as the sum of H_M and H_E that, in a massless framework in which $M = 0$, reduces to only H_E . Therefore, we can actually ignore the mass term and focus only on the electric field Hamiltonian.

Our starting point is not the actual electric field spin-Hamiltonian we are going to implement on a digital quantum computer, but the expression given by

$$H_E = \frac{1}{2} \sum_{n=0}^{N-1} E_n^2$$

in which E_n represents the electric field that lives between sites n and $n + 1$ [19, 20]. According to the Gauss' Law, in which we set the field $E_0 = 0$, in Sec. 1.5.2 we have

derived that

$$E_n - E_{n-1} = e \left[\bar{\chi}_n \psi_n - \frac{1 - (-1)^n}{2} \right] = \begin{cases} e \bar{\chi}_n \chi_n & n \text{ even} \\ e(\bar{\chi}_n \chi_n - 1) & n \text{ odd} \end{cases}$$

We can see that in H_E^i we are performing a sum over squared integers numbers that vary from the previous one by ± 1 in e unit. Therefore, the minimum eigenvalue that the electric field Hamiltonian can have (and that represents the ground state we are searching for) is $H_E^{\text{eig}} = 0$. In particular, we can see that $H_E^{\text{eig}} = 0$ if and only if $E_n^2 = 0 \forall n$, that is if $E_n = 0 \forall n$. By imposing this condition, we obtain that

$$E_n - E_{n-1} = 0 = e \left[\bar{\chi}_n \chi_n - \frac{1 - (-1)^n}{2} \right] \implies \bar{\chi}_n \chi_n = \frac{1 - (-1)^n}{2}$$

As a consequence, the energy of the first excited level must be represented by $H_E^{\text{eig}} = e^2/2$, which means that there exists a value of m such that $E_m = e$ and $E_{n \neq m} = 0 \forall n \neq m$. In order to correctly identify them, we recall that, in the staggered formalism we have adopted, we are working with chains made up of N sites occupied in the even positions by fermions and in the odd ones by anti-fermions [19, 20, 6]. For a more precise analysis, we can define the charge density in the n -th site given by ρ_n as

$$\rho_n = e \left[\bar{\chi}_n \chi_n - \frac{1 - (-1)^n}{2} \right]$$

Recalling the expressions derived for the Jordan-Wigner transformation in Sec. 2.2.1, we can substitute the one for $\bar{\chi}_n \chi_n$ in the previous equation obtaining that

$$\begin{aligned} \rho_n &= e \left[\bar{\chi}_n \chi_n - \frac{1 - (-1)^n}{2} \right] = e \left[\frac{1}{2} (\mathbb{I} + \hat{\sigma}_n^z) - \frac{1 - (-1)^n}{2} \right] = \\ &= e \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \frac{e}{2} \begin{pmatrix} 1 - (-1)^n & 0 \\ 0 & 1 - (-1)^n \end{pmatrix} = \frac{e}{2} \begin{pmatrix} (-1)^n + 1 & 0 \\ 0 & (-1)^n - 1 \end{pmatrix} \end{aligned}$$

Thus, we have that for even or odd n the eigenvalues and the eigenstates of ρ_n are given by

$$\begin{aligned} n \text{ even} &\implies \lambda_+ = +e, \lambda_- = 0 \quad \text{with} \quad v_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle, v_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle \\ n \text{ odd} &\implies \lambda_+ = -e, \lambda_- = 0 \quad \text{with} \quad v_+ = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle, v_- = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle \end{aligned}$$

Therefore, the ground state of our starting Hamiltonian is represented by an empty staggered chain in which the even sites are in the state $|1\rangle$ and the odd ones in the state $|0\rangle$ with no degeneracy. This situation is represented in Fig. 3.1. The first excited level, instead, is given by a staggered chain where the situation is similar to the one presented for the ground state except for the fact that two adjacent sites (one even, one odd) are in the states $|0\rangle$ and $|1\rangle$ respectively. Due to the N different possibility to position these particular sites, the first excited level is a N -fold degenerate level. This situation is pictured in Fig. 3.2.

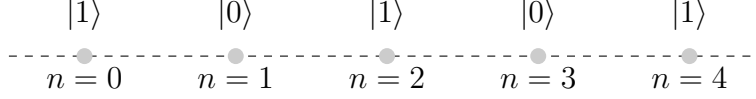


Figure 3.1: A visual representation of the ground state of H_E

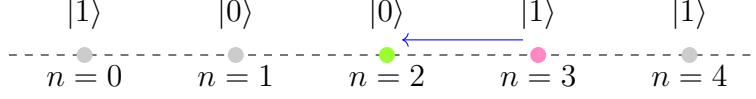


Figure 3.2: A visual representation of the first excited level of H_E

3.2 Trotterization of the Schwinger Hamiltonian

In order to do that, we have to work on the Schwinger Hamiltonian in its dimensional version to underline the dependence on the lattice spacing a that will be significant in the following. The dimensional Schwinger Hamiltonian can be obtained by adjusting the dimensionality of each term independently, thus obtaining

$$\begin{aligned}
H_M^{\text{spin}} &= \frac{M}{2} \sum_{n=0}^{N-1} (-1)^n \hat{\sigma}_n^z \\
H_E^{\text{spin}} &= \sum_{n=1}^{N-1} \sum_{l=n+1}^{N-1} \frac{e^2 a}{4} (N-l) \hat{\sigma}_n^z \hat{\sigma}_l^z + \frac{e^2 a}{8} \sum_{n=1}^{N-1} (1 - (-1)^n) \sum_{l=1}^n \hat{\sigma}_l^z \\
H_U^{\text{spin}} &= \sum_{n=0}^{N-2} \frac{1}{4a} \left[\hat{\sigma}_n^x \hat{\sigma}_{n+1}^x + \hat{\sigma}_{n+1}^y \hat{\sigma}_n^y \right] + \frac{i^N}{4a} \left[\prod_{l=1}^{N-2} \hat{\sigma}_l^z \right] (\hat{\sigma}_{N-1}^x \hat{\sigma}_0^x + \hat{\sigma}_0^y \hat{\sigma}_{N-1}^y)
\end{aligned}$$

The terms may be grouped according to the nature of their Pauli interactions as follows

$$\begin{aligned}
H_Z &= \frac{M}{2} \sum_{n=0}^{N-1} (-1)^n \hat{\sigma}_n^z + \frac{e^2 a}{8} \sum_{n=1}^{N-1} (1 - (-1)^n) \sum_{l=1}^n \hat{\sigma}_l^z \\
H_{ZZ} &= \sum_{n=1}^{N-1} \sum_{l=n+1}^{N-1} \frac{e^2 a}{4} (N-l) \hat{\sigma}_n^z \hat{\sigma}_l^z \\
\Delta \equiv H_U^{\text{spin}} &= \sum_{n=0}^{N-2} \frac{1}{4a} \left[\hat{\sigma}_n^x \hat{\sigma}_{n+1}^x + \hat{\sigma}_{n+1}^y \hat{\sigma}_n^y \right] + \frac{i^N}{4a} \left[\prod_{l=1}^{N-2} \hat{\sigma}_l^z \right] (\hat{\sigma}_{N-1}^x \hat{\sigma}_0^x + \hat{\sigma}_0^y \hat{\sigma}_{N-1}^y)
\end{aligned}$$

We are now ready to perform adiabatic evolution through a Trotterization process [10]. In particular, the first step is to prepare the ground state of the Hamiltonian on quantum computer and then we can act on it with evolution operators. Since the evolution parameter $s \in [0, 1]$, we can imagine to split this interval in a number $1/\delta s = n_s$ of intermediate steps where δs is the amplitude of a single step. Moreover, we assume that the time of the adiabatic evolution is given by τ . Thus, $\forall s$ our goal is to construct the operator given by

$$\text{T}e^{-i\tau \int_0^1 ds H(s)} \simeq \prod_{s=0}^1 e^{-i\tau \delta s H(s)}$$

Then, by specifying the form of our Hamiltonian in this expression, we can write that

$$\prod_{s=0}^1 e^{-i\tau\delta s H(s)} = \prod_{s=0}^1 e^{-i\tau\delta s [H_Z + H_{ZZ} + s\Delta]} \simeq \prod_{s=0}^1 \left\{ e^{-i\tau\delta s H_Z} e^{-i\tau\delta s H_{ZZ}} e^{-i\tau\delta s (s\Delta)} \right\} + \mathcal{O}(\delta s^2)$$

Therefore, we have approximated the evolution operator with the product of three different terms admitting an error that scales up $\propto \delta s^2$. Now, we can work on each of the three single terms in order to derive a clear form ready to be implemented on a quantum computer. Here we present the final results, the full step-by-step derivations can be found in Sec. B.4. For the first term, the final operator decomposition is given by

$$e^{-i\tau\delta s H_Z} = \prod_{n=0}^{N-1} O_Z(\alpha) \prod_{n=1}^{N-1} \prod_{l=1}^n O_Z(\beta)$$

where we assumed $\alpha = \tau\delta s \frac{M}{2}(-1)^n$ and $\beta = \tau\delta s \frac{e^2 a}{8}(1 - (-1)^n)$, and where we defined $O_Z(\alpha) = e^{-i\alpha\hat{\sigma}_n^z}$ and $O_Z(\beta) = e^{-i\beta\hat{\sigma}_l^z}$. The second term, instead, can be written as

$$e^{-i\tau\delta s H_{ZZ}} = \prod_{n=1}^{N-1} \prod_{l=n+1}^{N-1} O_{ZZ}(\theta)$$

where $\theta = \tau\delta s \frac{e^2 a}{4}(N-l)$ and $O_{ZZ}(\theta) = e^{-i\theta\hat{\sigma}_n^z\hat{\sigma}_l^z}$. Finally, for the third one, the operator factorizes into

$$e^{-i\tau\delta s (s\Delta)} = \left[\prod_{n_{\text{even}}=0}^{N-2} O_{XX}(\lambda) \prod_{n_{\text{odd}}=0}^{N-2} O_{XX}(\lambda) \right] \left[\prod_{n_{\text{even}}=0}^{N-2} O_{YY}(\zeta) \prod_{n_{\text{odd}}=0}^{N-2} O_{YY}(\zeta) \right] O_{\text{PBC}}$$

where we assumed $\lambda = \tau\delta s \frac{s}{4a}$ and $\zeta = \tau\delta s \frac{i^N s}{4a}$, and where we defined

$$\begin{aligned} O_{XX}(\lambda) &= e^{-i\lambda\hat{\sigma}_n^x\hat{\sigma}_{n+1}^x} \\ O_{YY}(\lambda) &= e^{-i\lambda\hat{\sigma}_{n+1}^y\hat{\sigma}_n^y} \\ O_{\text{PBC}}(\zeta) &= e^{-i\zeta[\prod_{l=1}^{N-2}\hat{\sigma}_l^z](\hat{\sigma}_{N-1}^x\hat{\sigma}_0^x + \hat{\sigma}_0^y\hat{\sigma}_{N-1}^y)} \end{aligned}$$

Therefore, the complete evolution operator that we have to implement on quantum device is given by

$$\begin{aligned} e^{-i\tau\delta s H(s)} &\simeq \prod_{s=0}^1 \left\{ \left[\prod_{n=0}^{N-1} O_Z(\alpha) \right] \left[\prod_{n=1}^{N-1} \prod_{l=1}^n O_Z(\beta) \right] \left[\prod_{n=1}^{N-1} \prod_{l=n+1}^{N-1} O_{ZZ}(\theta) \right] \right. \\ &\quad \cdot \left[\prod_{n_{\text{even}}=0}^{N-2} O_{XX} \prod_{n_{\text{odd}}=0}^{N-2} O_{XX} \right] \left[\prod_{n_{\text{even}}=0}^{N-2} O_{YY} \prod_{n_{\text{odd}}=0}^{N-2} O_{YY} \right] O_{\text{PBC}} \left. \right\} + \mathcal{O}\left(\frac{\tau^2}{n_s}\right) \end{aligned}$$

As a concrete realization of this Trotter decomposition, Fig. 3.3 illustrates the explicit quantum circuit executing a single first-order Trotter step for a lattice of $N = 4$ sites with $a = 2.84$.

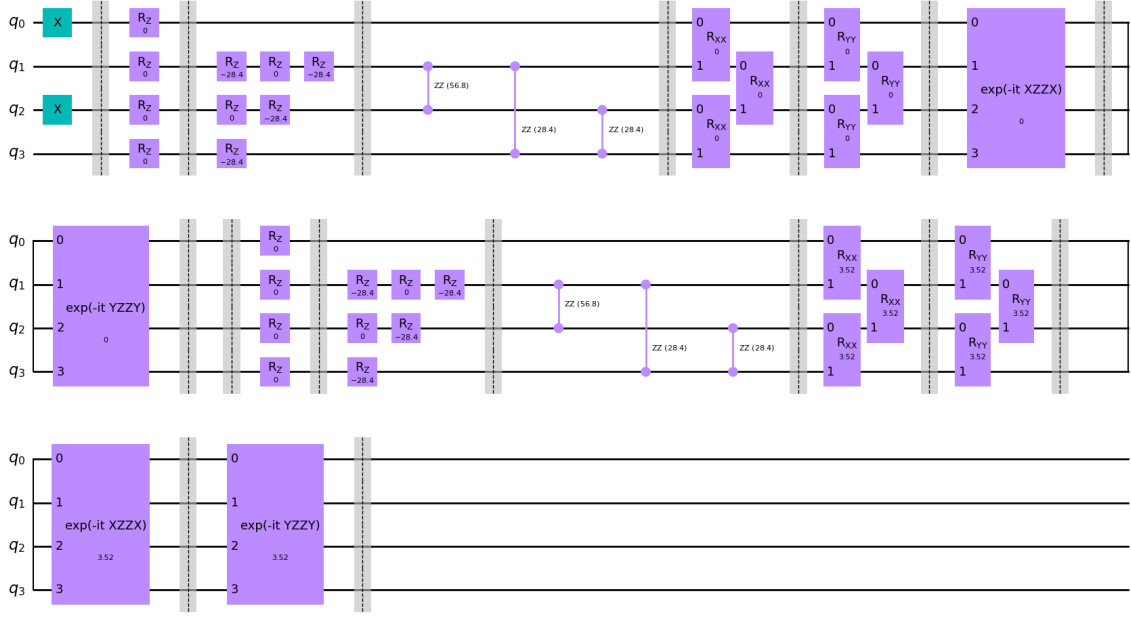


Figure 3.3: Quantum circuit implementation of a single iteration of the first-order Trotter step for the discretized Schwinger model with $N = 4$ lattice sites and $a = 2.84$. The circuit acts on four qubits ($q_0, \dots, q_3$) and decomposes the time-evolution operator into sequence blocks separated by barriers. Single-qubit (X and R_Z) rotations, two-qubit (ZZ , R_{XX} and R_{YY}) gates, as well as four-qubit unitary blocks implementing long-range interactions, systematically encode the terms of the spin Hamiltonian.

3.3 Ensuring gap stability

The fundamental question is: how the values of N and a *should* be chosen to meet the conditions imposed by the Adiabatic Theorem? To address this, we must establish a precise physical bound ensuring that the adiabatic quantum state preparation of a system with N sites and lattice spacing a can be safely performed. In fact, since it is not known a priori for which pairs of parameters (N, a) the adiabatic preparation of a state may be compromised by level crossing phenomena, or by the closing of the spectral gap, a mathematical relationship is required to serve as a guarantee criterion for the success of the adiabatic process [29, 26]. This is particularly critical because bounding the minimum eigenvalue gap for a complex many-body Hamiltonian is a notoriously difficult problem, making it often impossible to predict the system's behavior across the entire parameter space without explicit analytical constraints. Consequently, the formulation of a formal relationship is essential to provide a sufficient condition to ensure that the spectral gap remains open throughout the evolution, thereby guaranteeing the success of the quantum state preparation.

To establish a rigorous safety margin for the adiabatic process, we specifically analyze the preparation of the Schwinger Hamiltonian's ground state. We begin by evaluating the initial spectral gap Δ of our starting Hamiltonian H_0 , defined by the pure electric field term

$$H_0 = H_E = \frac{a}{2} \sum_n E_n^2. \quad (3.1)$$

As previously derived in Sec. 3.1, the gap between the unique vacuum and the first excited state is readily found to be

$$\Delta(0) = \varepsilon_1(0) - \varepsilon_0(0) = \frac{ae^2}{2}. \quad (3.2)$$

During the adiabatic evolution, the instantaneous eigenstates shift from their initial configurations due to the perturbation $s(H_1 - H_0)$ introduced by the hopping terms. The stability of the spectrum under such perturbations can be quantified using Weyl's Inequality [29, 30, 31], which provides a global bound on eigenvalue variations.

Proposition 3.3.1 (Weyl's Inequality). *The spectrum of a Hermitian matrix A remains stable under a perturbation B , such that for each eigenvalue ε_k it holds that*

$$|\varepsilon_k(A + B) - \varepsilon_k(A)| \leq \|B\|_{op}, \quad (3.3)$$

where $\|B\|_{op}$ denotes the operator norm of B .

Applying this to our case, where $A = H_0$ and the full perturbation is $B = H_1 - H_0$, the inequality implies that the maximum energy shift any level can undergo during the path $H(s) = H_0 + s(H_1 - H_0)$ is strictly bounded by $\|H_1 - H_0\|_{op}$. Consequently, a sufficient condition to ensure the spectral gap remains open and avoid any potential level crossings is that the initial gap $\Delta(0)$ must exceed the sum of the maximum possible shifts of the two levels, which is $2\|H_1 - H_0\|_{op}$.

In our model, the difference $H_1^{\text{spin}} - H_0^{\text{spin}}$ is represented by the hopping Hamiltonian H_U^{spin} . Its operator norm can be bounded by applying the triangle inequality to the individual Pauli strings as

$$\begin{aligned} \|H_U^{\text{spin}}\|_{op} &= \left\| \sum_{n=0}^{N-2} \frac{1}{4a} [\hat{\sigma}_n^x \hat{\sigma}_{n+1}^x + \hat{\sigma}_{n+1}^y \hat{\sigma}_n^y] + \frac{i^N}{4a} \left[\prod_{l=1}^{N-2} \hat{\sigma}_l^z \right] (\hat{\sigma}_{N-1}^x \hat{\sigma}_0^x + \hat{\sigma}_0^y \hat{\sigma}_{N-1}^y) \right\|_{op} \leq \\ &\leq \sum_{n=0}^{N-2} \frac{1}{4a} \|\hat{\sigma}_n^x \hat{\sigma}_{n+1}^x + \hat{\sigma}_{n+1}^y \hat{\sigma}_n^y\|_{op} + \frac{1}{4a} \left[\prod_{l=1}^{N-2} \|\hat{\sigma}_l^z\|_{op} \right] \|\hat{\sigma}_{N-1}^x \hat{\sigma}_0^x + \hat{\sigma}_0^y \hat{\sigma}_{N-1}^y\|_{op} \leq \\ &\leq \sum_{n=0}^{N-2} \left(\frac{2}{4a} \right) + \frac{2}{4a} = \frac{N-1}{2a} + \frac{1}{2a} = \frac{N}{2a}. \end{aligned}$$

By combining these results, the adiabatic safety requirement $2\|H_U^{\text{spin}}\|_{op} < \Delta$ yields the physical constraint given by

$$\frac{N}{a} < \frac{ae^2}{2} \implies a > \frac{\sqrt{2N}}{e} \quad (3.4)$$

which, for $e = 1$, simplifies to the condition $a > \sqrt{2N}$. This relationship serves as a mathematical guarantee criterion, ensuring that for a given system size N , the lattice spacing a is large enough to prevent the closing of the spectral gap, thereby safeguarding the fidelity of the state preparation process.

However, it is important to emphasize that a violation of this safety bound does not strictly imply the inevitable occurrence of a level crossing. In fact, the spectral gap may remain sufficiently open even in regions of the parameter space where the condition $a > \sqrt{2N}$ is not met.

3.4 Numerical results

Here we present the numerical results for the Adiabatic Quantum State Preparation (AQSP) of the ground state of the Schwinger model, implemented with Periodic Boundary Conditions (PBC). The primary objective is to evaluate the effectiveness of the adiabatic protocol in steering the system from a simple initial state toward the true vacuum of the target field theory. To rigorously validate the accuracy of our digital simulation, the prepared states are compared against the exact ground state solution obtained through classical diagonalization of the Schwinger Hamiltonian. The performance is quantified using the Uhlmann fidelity, defined as in Eq. (2.1).

A central focus of this analysis is the impact of physical parameter choices on adiabatic safety. As established in our theoretical framework, the stability of the spectral gap during the evolution is critical to avoid level-crossing phenomena. Following a criterion derived from Weyl's Inequality, we examine the behavior of the simulation for pairs of parameters (N, a) that either satisfy the safety bound given in Eq. (3.4) or violate it. By comparing these two regimes, we demonstrate how analytical constraints on lattice spacing and system size serve as a guarantee criterion for the successful preparation of the ground state and the overall reliability of the digital quantum simulation. Indeed, numerical results often show that the final fidelity remains remarkably high even when the bound is violated, suggesting that while the condition ensures absolute safety, it is not a strict requirement for the practical success of the adiabatic preparation.

The results presented confirm the primary characteristics of the Trotter-Suzuki approximation and its impact on the fidelity of digital quantum simulations. In fact, by decomposing the time-evolution operator into a sequence of discrete steps, we introduce a discretization error that depends on the simulation parameters as $\mathcal{O}(\tau^2/n_s)$. In Fig. 3.4 it is shown that the fidelity drops significantly as τ increases if n_s is fixed; instead, in Fig. 3.5 we can notice that the fidelity increases substantially as we make the size of the Trotter step smaller for fixed τ .

As illustrated in Figures 3.4 and 3.5, a pronounced downward trend in fidelity is observed when scaling up the system size. This degradation is a direct consequence of the increased complexity inherent in the quantum circuit structure. In a physical quantum computer, every gate operation is characterized by finite precision and inherent noise; since the number of operations required to simulate the full dynamics of the Schwinger model scales with the number of lattice sites, larger systems necessitate significantly deeper circuits. This increased depth drastically raises the probability of a global failure, as the final state fidelity is inversely related to the total number of operations performed. Furthermore, as the system size grows, individual qubit addressing becomes technically more challenging. It becomes increasingly difficult to target specific qubits without affecting their neighbors (a phenomenon known as *cross-talk*) which leads to the unwanted evolution of the multi-qubit wavefunction.

To mitigate this fidelity degradation as the system size scales up, several strategies can be implemented. First, the discretization error can be significantly reduced by adopting higher-order Trotter-Suzuki product formulas, such as second- or third-order expansions [10, 11]. Alternatively, replacing the conventional linear adiabatic path with a non-linear interpolation schedule optimizes the evolution pace relative to the instantaneous energy gap [26, 21]. Finally, more innovative symmetry-guided techniques, such as Branched-Subspaces Adiabatic Preparation (B-SAP) [29], can be employed in

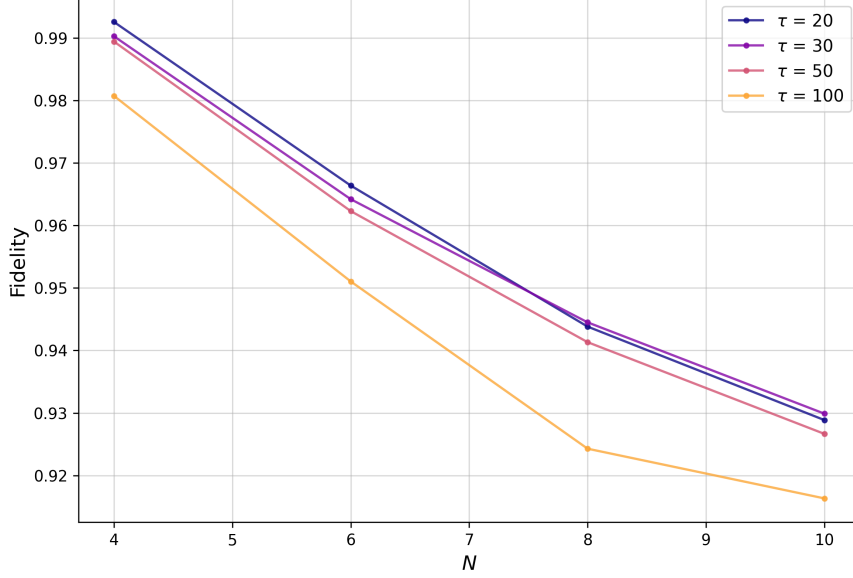


Figure 3.4: This graph shows the fidelity behavior for different system sizes, for $n_s = 2000$ and specific values of τ . One can see that the fidelity drops as τ and N increase, as a consequence of the Suzuki-Trotter approximation and of the increased complexity of the quantum circuit structure.

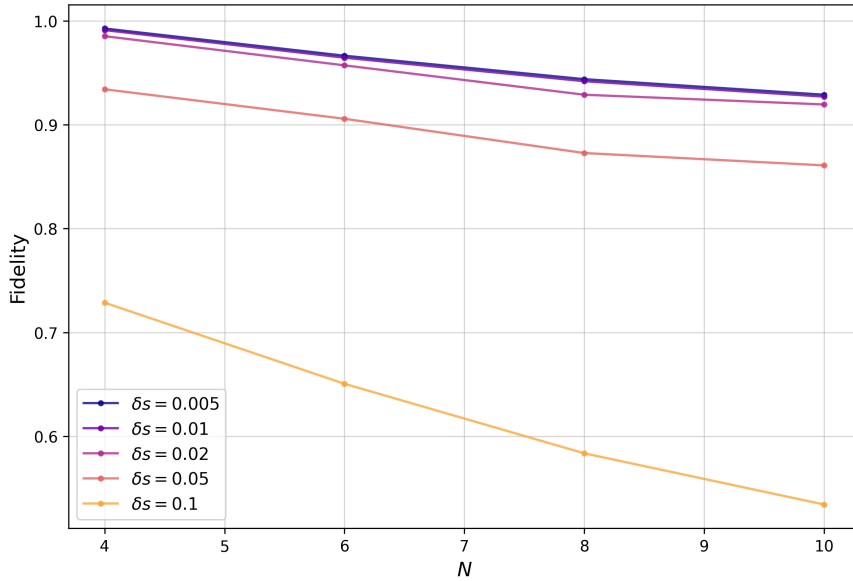


Figure 3.5: This graph shows the fidelity behavior for different system sizes, for $\tau = 20$ and specific values of n_s . One can see that the fidelity drops as N and $\delta s = 1/n_s$ increase, as a consequence of the first-order Trotter approximation and of the increased complexity of the quantum circuit structure.

order to effectively prevent unwanted level crossings and preserve high state fidelity for larger system sizes.

The ideal goal is to achieve high state fidelity across all system sizes N . In principle, this objective can be attained for any arbitrary N by appropriately tuning the total

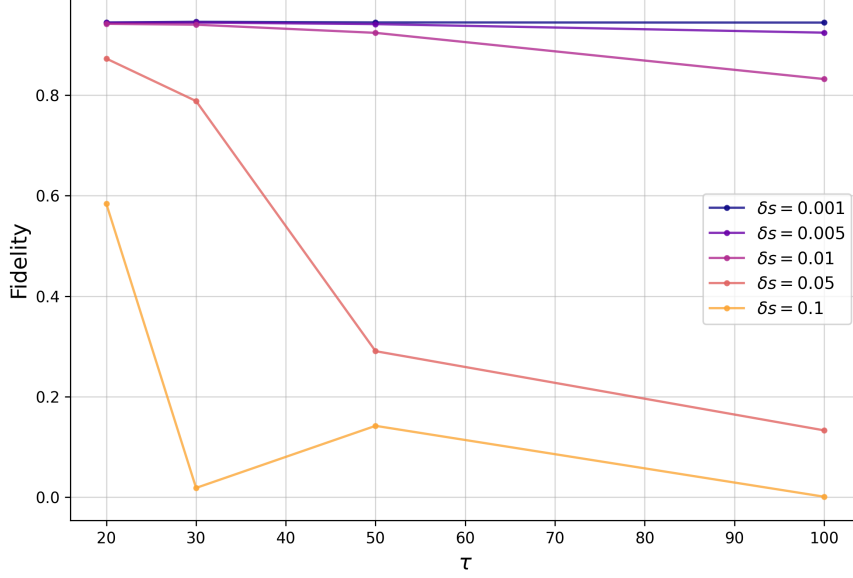


Figure 3.6: This graph shows the fidelity behavior for different values of τ , for $N = 8$ and specific values of n_s . One can see that the fidelity increases as $\delta s = 1/n_s$ decreases, as a consequence of the Suzuki-Trotter approximation. Moreover, as δs decreases, the fidelity's dependence on τ vanishes and the fidelity reach a steady state. This is due to the fact that, when n_s is sufficiently large, the system enters a convergence regime where the digital error is supposed to be negligible compared to the physical scales of the dynamics.

evolution time τ and the number of Trotter steps n_s . However, the values of τ and n_s that can be practically exploited at present are strictly constrained by physical hardware limitations. While τ must be sufficiently large to satisfy the Adiabatic theorem, it is capped from above by finite qubit coherence times. At the same time, increasing n_s to reduce the time step width linearly increases the circuit depth, accelerating the accumulation of gate errors and decoherence. This tight trade-off leaves a residual Trotterization error that inevitably degrades fidelity as N grows, highlighting the need for advanced mitigation strategies. Therefore, projections for the next 5 to 10 years suggest that we will remain limited to simulate systems of approximately 10 or 15 qubits at best. As shown in Figure 3.6, while near-optimal fidelity levels for a register of $N = 8$ qubits are achievable in the near term, the path toward simulating large-scale systems remains strictly dependent on significant advancements in hardware architecture and noise mitigation.

3.5 Electric field correlations

3.5.1 Theoretical approach

As a key physical application of the ground state prepared through digital quantum simulation, we evaluate spatial two-point correlation functions [2, 13, 3]. In a digital quantum simulation workflow, computing a two-point correlation function follows a systematic three-step procedure. First, the selection of a physically relevant target

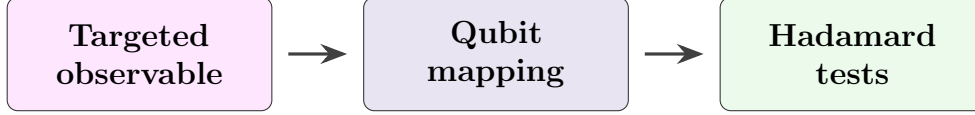


Figure 3.7: Schematic structure of the computation of spatial two-point correlation functions

observable; second, the mapping of the fermionic field operators onto spin-1/2 qubit operators via the Jordan-Wigner transformation; and third, the measurement and evaluation of the connected correlation function on the prepared quantum state. This procedure is summarized in Fig. 3.7.

In this work, we execute this general procedure for the electric field operator in the (1+1)-dimensional Schwinger model formulated on a Kogut-Susskind staggered lattice with PBC across N sites. By integrating out the gauge field degrees of freedom via Gauss's law, the electric field residing on link r (labeled as E_r) is expressed as the cumulative sum of preceding staggered charge density operators as

$$E_r = E_0 + \sum_{n=1}^r \rho_n = \sum_{n=1}^r \rho_n \quad (3.5)$$

since we set $E_0 = 0$ with E_0 being the field residing on the link between sites 0 and 1. Applying the Jordan-Wigner transformation maps the local staggered charge density operator at site n into

$$\rho_n = \frac{1}{2} [\hat{\sigma}_n^z + (-1)^n] \quad (3.6)$$

and therefore, substituting Eq. (3.6) into Eq. (3.5) leads us to

$$E_r = \sum_{n=1}^r \frac{1}{2} [\hat{\sigma}_n^z + (-1)^n] \quad (3.7)$$

The choice of the electric field as our primary target observable is particularly relevant from both a theoretical and a computational perspective. Since E_r represents a cumulative sum of local charge operators, its spatial correlations are intrinsically non-local. In particular, we choose to evaluate the two-point covariance $C(r, 1) = \text{Cov}(E_r, E_l)$ between link r and an arbitrary reference link l (that we choose to set to 1). In particular, as it is proved in Section B.5, this expression expands into a sum of two-point charge-density correlations as

$$C(r, 1) = \sum_{n=1}^r [\langle \rho_n \rho_1 \rangle - \langle \rho_n \rangle \langle \rho_1 \rangle] = \frac{1}{4} \sum_{n=1}^r (\langle \hat{\sigma}_n^z \hat{\sigma}_1^z \rangle - \langle \hat{\sigma}_n^z \rangle \langle \hat{\sigma}_1^z \rangle) \quad (3.8)$$

Consequently, measuring $C(r, 1)$ tests not merely a single isolated pair of qubits, but rather a multi-site string of charge-density operators. This provides a stringent and highly sensitive benchmark for both the fidelity of our digital adiabatic state preparation and the performance of the Hadamard test measurement protocol. In our numerical simulations, we analyze the autocorrelation $C(1, 1)$ and spatial correlations up to separation $r = 3$ relative to the reference site $l = 1$. Under PBC, spatial translation invariance guarantees that selecting $l = 1$ is fully representative and made without

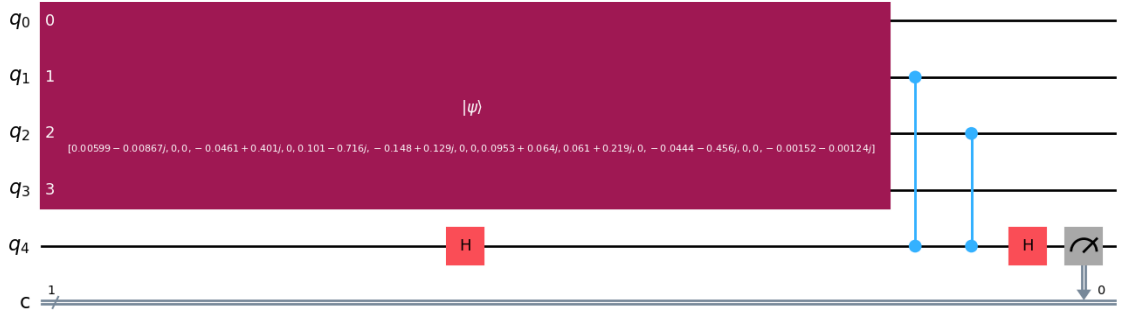


Figure 3.8: Quantum circuit diagram executing an example of a single Hadamard test routine on the prepared Schwinger ground state for $N = 4$ lattice sites and lattice constant $a = 1$. The primary system register (q_0, q_1, q_2, q_3) is initialized in the target ground state vector $|\psi\rangle$. An auxiliary ancilla qubit (q_4) is prepared in $|0\rangle$, transformed into a symmetric superposition state via an initial Hadamard gate H , and used as a control qubit to apply multi-site controlled operations on target qubits (q_1, q_2). A final Hadamard gate on q_4 followed by a measurement in the computational basis.

loss of generality. The restriction to separations $r \leq 3$ is justified by a combination of physical and computational factors. Physically, in the lattice Schwinger model, connected correlation functions exhibit a rapid exponential decay as $C(r, l) \propto e^{-r}$; as a result, for distances beyond $r = 3$, the signal decays toward near-zero values. Computationally, exact classical diagonalization are restricted to moderate system sizes ($N \leq 10$) due to classical memory limitations. Within these finite registers, focusing on $r \leq 3$ captures the dominant physical correlation structure while preventing deep multi-qubit Hadamard test circuits from accumulating circuit depth errors. As a concrete illustration of this measurement scheme, Fig. 3.8 illustrates the explicit quantum circuit executing a single Hadamard test routine for a lattice of $N = 4$ sites with lattice spacing $a = 1$.

3.5.2 Numerical results

The quantitative performance of the digital adiabatic state preparation protocol and the subsequent Hadamard test evaluation of the electric field correlation functions are summarized in Tab. 3.1. A preliminary remarkable aspect lies in the ground state preparation fidelity. Across all investigated system sizes ($N = 4, 6, 8, 10$), the Uhlmann fidelity between the simulated state and the exact classical ground state consistently exceeds 95% (ranging from 95.83% to 98.35%). This high fidelity is achieved even though the chosen pairs of system size N and lattice spacing a do not satisfy the conservative analytical safety bound derived from Weyl's inequality in Eq. (3.4). This result demonstrates that the digital adiabatic preparation protocol is substantially more robust in practice than worst-case theoretical bounds would suggest, enabling high-fidelity state preparation in physically relevant parameter regimes without requiring excessively fine lattice spacings.

As reported in Tab. 3.1, the accuracy of the spatial two-point correlation function is particularly evident in the autocorrelation $C(1, 1)$. Across all system sizes, the ideal

quantum simulation reproduces the exact classical value with an impressive relative error of only 2 to 3 percent (specifically, -2.08% for $N = 4$, -2.34% for $N = 6$, -3.08% for $N = 8$, and -2.25% for $N = 10$). The precision achieved on $C(1, 1)$ is of fundamental importance for two main reasons. Physically, since $E_1 = \rho_1$ under our conventions, $C(1, 1)$ measures the local variance and quantum fluctuations of the staggered charge density in the vacuum state. High accuracy on $C(1, 1)$ proves that the digital adiabatic evolution correctly synthesizes the local vacuum structure and charge distribution. Algorithmically, instead, $C(1, 1)$ serves as a primary sanity check: if the local vacuum fluctuations were not captured accurately at $r = 1$, any subsequent multi-site correlation at larger distances would be certainly unreliable.

Secondly, a detailed comparison between the statistical uncertainty σ and the relative percentage deviation provides crucial insight into the nature of the observed discrepancies. Because the Hadamard test circuits were evaluated on an ideal quantum simulator with a large sample size of 10^6 shots, physical gate errors and decoherence are completely absent and the combined statistical sampling error is extremely small ($\sigma \sim 10^{-4}$). Consequently, the observed absolute differences, though small in magnitude (e.g., a deviation of -0.0036 for $N = 10$), correspond to compatibility scores spanning 8 to 11 standard deviations ($Z \sim 8 - 11 \sigma$). Far from indicating an unphysical instability, this high Z -score confirms that statistical shot noise is effectively negligible, allowing us to cleanly isolate the purely deterministic systematic errors of the digital algorithm. These systematic discrepancies originate from the first-order Suzuki-Trotter discretization step and the finite duration of the adiabatic ramp.

Finally, the spatial behavior of $C(r, 1)$ for separations $r \geq 2$ reflects the interplay between physical correlation decay, Trotter discretization error and finite sampling statistics. In the lattice Schwinger model, correlation functions decay exponentially with distance, causing the exact classical signal to drop rapidly from ~ 0.16 at $r = 1$ down to ~ 0.02 at $r = 2$ and ~ 0.01 at $r = 3$. As the physical signal decays toward zero, the accumulated Trotter error and finite shot-noise resolution yield a small residual numerical offset ($\sim 0.003 - 0.010$). This leads to higher relative percentage deviations at larger separations (e.g., +13.5% for $N = 6$ at $r = 2$, and +48.4% percent for $N = 10$ at $r = 3$). This highlights that while short-range correlations and local vacuum properties can be extracted with high accuracy, measuring long-range correlations remains sensitive to Trotter discretization errors when the physical correlation signal approaches the statistical resolution threshold.

To overcome the systematic limitations highlighted in our analysis (specifically, the first-order Trotter discretization error and the residual numerical offset observed at larger spatial separations), several concrete algorithmic strategies can be implemented in future developments. A primary avenue for systematic improvement is the transition from a first-order Trotter expansion to second- or higher-order Suzuki-Trotter product formulas [10, 11]. By employing symmetric second- or higher-order decompositions, the local discretization error becomes smaller, significantly improving the fidelity of the time-evolved state for a given time-step and effectively mitigating the systematic bias in ground state preparation without requiring an excessively large number of Trotter steps n_s that would otherwise inflate total circuit depth. Additionally, state preparation fidelity could be further enhanced by incorporating symmetry-guided adiabatic protocols such as B-SAP [29]. Summarizing, directly improving the vacuum state preparation fidelity toward near-unity values would refine two-point charge-density correlations and

N	r	$C(1, r)_{\text{class}}$	$C(1, r)_{\text{quant}}$	%	$Z\text{-score}$
$N = 4$	1	0.1722 ± 0.0004	0.1686 ± 0.0002	-2.08%	7.8σ
	2	0.0051 ± 0.0001	0.0133 ± 0.0002	$+162.69\%$	39.3σ
	3	-0.0003 ± 0.0003	0.0034 ± 0.0004	—	7.6σ
$N = 6$	1	0.1649 ± 0.0004	0.1610 ± 0.0002	-2.34%	8.5σ
	2	0.0214 ± 0.0001	0.0243 ± 0.0003	$+13.47\%$	9.4σ
	3	0.0111 ± 0.0002	0.0124 ± 0.0004	$+11.82\%$	2.9σ
$N = 8$	1	0.1622 ± 0.0004	0.1572 ± 0.0002	-3.08%	11.1σ
	2	0.0241 ± 0.0001	0.0277 ± 0.0003	$+14.98\%$	11.8σ
	3	0.0122 ± 0.0002	0.0190 ± 0.0004	$+56.28\%$	15.2σ
$N = 10$	1	0.1615 ± 0.0004	0.1579 ± 0.0002	-2.25%	8.1σ
	2	0.0243 ± 0.0001	0.0285 ± 0.0003	$+17.11\%$	13.7σ
	3	0.0121 ± 0.0002	0.0179 ± 0.0004	$+48.41\%$	13.0σ

Table 3.1: Comparison between exact classical diagonalization values $C(1, r)_{\text{class}}$ and digital quantum simulation outcomes $C(1, r)_{\text{quant}}$ for the evaluation of the electric field two-point correlation function on the Schwinger ground state with lattice spacing $a = 1$ and 10^6 shots per circuit execution. Relative percentage deviations and Z -scores in combined standard deviation units are reported across different system sizes N and spatial distances r .

reduce systematic discretization offsets in Hadamard test measurements, extending the range of reliable spatial correlation functions beyond $r = 3$ and opening a clear path toward precise continuum-limit extrapolations in ideal digital quantum environments.

3.6 The Schwinger vector boson mass

3.6.1 Discussion of the problem

Having successfully demonstrated the digital quantum simulation protocol for preparing the ground state of the lattice Schwinger model, we now outline a compelling physics application enabled by this result which also constitute a prospective continuation of this thesis. In Quantum Field Theory, the ground state plays a fundamental role as the evaluation of vacuum expectation values, physical observables and non-perturbative phenomena strictly hinges upon an accurate representation of the vacuum state $|\Omega\rangle$.

In fact, a particularly prominent application consists in determining the mass of the Schwinger vector boson, m_S , which dynamically emerges through the (1+1)D bosonization mechanism of the underlying fermionic theory [32, 33, 13]. Utilizing our state preparation framework, this mass gap can be extracted via two complementary strategies.

On the one hand, the mass, m_S , can be determined by evaluating the spatial decay of two-point correlation functions in the prepared ground state [32, 33, 13]. In the asymptotic regime, these correlators exhibit an exponential decay governed by the mass of the lowest-lying excitation, proportional to $e^{-m_S r a}$, where a stands for the

lattice spacing constant and r is the index of the sites. By performing a systematic continuum extrapolation taking the limit $N \rightarrow \infty$, or equivalently $1/N \rightarrow 0$, alongside the vanishing lattice spacing limit $a \rightarrow 0$, the physical mass parameter can be precisely determined.

On the other hand, since the Schwinger boson mass represents the fundamental single-particle energy gap above the vacuum, that is $m_S = E_1 - E_0$, one can extend the state preparation protocol to directly target the first excited state $|1\rangle$ [32, 33, 13]. Although preparing $|1\rangle$ introduces additional technical hurdles (such as handling degeneracies and level crossings) it enables a direct spectral measurement of the physical mass via the energy difference $m_S = \langle 1|H|1\rangle - \langle \Omega|H|\Omega\rangle$.

Both approaches highlight how the successful preparation of the ground state of the discretized theory on digital quantum simulators bridges the gap between quantum hardware execution and non-perturbative field theory calculations.

Building upon this theoretical framework, we focused on exploring the first strategy. Although it represents the more technically straightforward approach between the two (as it solely relies on the successful preparation of the ground state), it is by no means free of operational and algorithmic challenges.

First, selecting a suitable observable to compute two-point correlation functions requires careful consideration; the chosen operator must faithfully capture the physical field excitations while properly respecting the imposed periodic boundary conditions to avoid unphysical boundary artifacts [32, 33, 13].

Second, controlling and quantifying the error on the extracted mass parameter presents significant complexities. Operating in an ideal digital quantum simulation regime, the error budget is entirely systematic and algorithmic: on one hand, it is constrained by the maximum spatial lattice size N achievable due to simulator memory and qubit resource limits; on the other hand, it is governed by the finite Trotterization error arising from discrete time-stepping in the state preparation and evolution circuits.

Nevertheless, we remain confident that these systematic hurdles can be effectively addressed through targeted algorithmic refinements and theoretical scaling techniques. For example, to minimize Trotterization errors without incurring into prohibitive circuit depths, higher-order Trotter-Suzuki decompositions or guided state preparation schemes (such as B-SAP) can be employed [11, 29]. Furthermore, the choice of the observable can be optimized by selecting gauge-invariant operators that naturally conform to periodic boundary conditions, while isolating the fundamental mass pole [22, 33]. Altogether, these strategies provide a clear and robust path for accurately extracting the Schwinger boson mass via ideal digital quantum simulation.

Conclusions and perspectives

This thesis has investigated the digital quantum simulation of non-perturbative field-theoretic phenomena, focusing specifically on ground state preparation and spectral characterization of the $(1+1)$ -dimensional Schwinger model. Formulated on a Kogut-Susskind staggered lattice with periodic boundary conditions, the continuous gauge field degrees of freedom were analytically integrated out through the systematic application of Gauss's Law. This procedure transformed the field theory into an effective spin- $1/2$ Hamiltonian featuring long-range, Coulomb-like interactions following the Jordan-Wigner transformation. Within a platform-agnostic digital framework, the Schwinger vacuum state was prepared using Adiabatic Quantum State Preparation discretized via first-order Trotter product formulas. The primary numerical results confirmed the successful preparation of the true Schwinger ground state, validated against exact classical diagonalization through the evaluation of the Uhlmann fidelity. Across various system sizes N and lattice spacings a , state preparation fidelities up to 99% were consistently achieved. Additionally, the evaluation of spatial two-point correlation functions of the electric field demonstrated that the autocorrelation at $r = 1$ was accurately estimated within a 3% relative error, while cleanly isolating the deterministic Trotter discretization error. Furthermore, this work explored two prospective pathways for extracting the mass of the Schwinger vector boson m_S , an observable emerging dynamically through the $(1+1)$ -dimensional bosonization mechanism. Comparing spatial correlation function decays against direct spectral gap measurements via first excited state preparation highlighted critical trade-offs, uncovering operational bottlenecks associated with Trotter errors and finite-size effects in ideal digital quantum environments.

Despite the high fidelities attained, the detailed analysis conducted in this work reveals specific criticalities during adiabatic evolution, which open compelling avenues for algorithmic improvement. A key prospective direction is exploring advanced symmetry-guided adiabatic protocols, such as Branched-Subspaces Adiabatic Preparation [29], to optimize the vacuum state synthesis. By exploiting the underlying global symmetries of the model during adiabatic evolution, such protocols can suppress unwanted transitions into excited sectors and prevent minimum energy gap shrinkages, thereby reducing time overheads and improving state preparation fidelity. Concurrently, evaluating higher-order Suzuki-Trotter product formulas [10, 11], specifically second- and third-order decompositions, represents a crucial step to mitigate discretization errors. Higher-order Trotterization is expected to smooth out the characteristic fidelity drop observed during time evolution, eliminating artificial step-like artifacts and establishing a more stable convergence profile towards the target ground state.

Furthermore, it is worth highlighting that further enhancements in ground state preparation fidelity directly propagate to the precision of spatial correlation functions.

Since measuring two-point electric field covariances relies intrinsically on the structural integrity of the prepared vacuum state, elevating the state preparation fidelity toward near-unity values would significantly improve the quantitative accuracy of spatial correlations across all separations. Moreover, as advanced algorithmic techniques successfully suppress Trotter discretization errors, digital quantum simulation will overcome current circuit-depth limitations and enable the reliable extraction of higher-order multi-point correlation functions. Reaching this regime will open the way to probing complex and non-linear quantum vacuum fluctuations and multi-particle interactions in lattice gauge theories on digital quantum hardware.

Beyond ground state preparation, a structured and comprehensive investigation is required to fully resolve the extraction of the Schwinger vector boson mass. A future dedicated framework must address both the technical and analytical challenges identified in this study. On the technical side, extracting m_S from spatial correlation functions requires managing the statistical noise floor associated with multi-qubit Hadamard test measurements at long ranges, as well as optimizing shot counts and measurement protocols. On the analytical side, a rigorous mass extraction protocol demands systematic finite-size scaling analyses and continuum limit extrapolations ($N \rightarrow \infty, a \rightarrow 0$), ensuring that lattice artifacts and boundary-induced finite-volume shifts are systematically decoupled from the continuum physical observable. Combining direct excited-state preparation with symmetry-guided projections offers a promising, robust strategy for precise energy gap determination.

Looking further ahead, an essential long-term objective is extending this platform-agnostic framework to higher spatial dimensions and more complex field theories. Moving from $(1+1)$ dimensions to $(2+1)$ or $(3+1)$ dimensions introduces substantial theoretical and technical hurdles. While in one spatial dimension Gauss's law allows for the complete analytical elimination of gauge field degrees of freedom, higher dimensions possess intrinsic transverse magnetic fields and propagating gauge degrees of freedom that cannot be fully integrated out. As a consequence, simulating higher-dimensional lattice gauge theories requires explicitly digitizing gauge field variables on the lattice links, imposing strict local Gauss's law constraints directly on the quantum register. Similarly, generalizing the protocol to encompass multiple fermionic flavors or non-Abelian gauge groups, such as $SU(2)$ or $SU(3)$, introduces non-commuting field components and intricate color charge structures. These extensions exponentially increase circuit depth, requiring sophisticated gauge-invariant state preparation methods, and demand advanced Trotterization strategies to manage non-Abelian operator non-commutativity.

An equally important outlook concerns the transition from ideal digital quantum simulators to physical quantum hardware execution. Assessing the performance of these protocols on noisy hardware, as well as on noisy simulator environments incorporating realistic noise models, is essential to quantify resilience against physical decoherence. Testing on architectures such as trapped-ion processors, which naturally support long-range spin interactions, and superconducting circuits, which offer high-speed gate execution, will provide critical insights into gate fidelity thresholds and crosstalk impacts. However, evaluating algorithmic execution on noisy platforms presupposes the availability of a protocol that functions flawlessly under ideal noise-free conditions. Consequently, refining the algorithmic pipeline on ideal simulators remains the mandatory prerequisite prior to real hardware deployment, ensuring that physical hardware noise can be cleanly isolated from algorithmic Trotter or adiabatic errors.

In conclusion, this thesis does not claim to definitively resolve the vast computational complexities inherent to non-perturbative lattice gauge theories. Rather, it establishes a rigorous, benchmarked, platform-agnostic foundation designed to serve as a stepping stone for future research. By demonstrating high-fidelity ground state preparation and identifying the operational boundaries of digital quantum simulation protocols, this work offers a clear roadmap toward managing fundamental large-scale quantum field theories on emerging quantum hardware.

Appendix A

Symmetries of the Schwinger Hamiltonian

A central feature of the target Hamiltonian of Eq. (2.3) is its invariance under a group of symmetry transformations operating across two distinct physical domains: internal spin space and spatial lattice space. By spin-space symmetries, we denote internal transformations acting exclusively on the local Hilbert space of the spin degrees of freedom, leaving the spatial configuration unaltered. Conversely, by lattice-space symmetries, we refer to spatial transformations that act directly on the discrete underlying lattice geometry by transforming or permuting site indices.

A.1 Spin space symmetries

In the spin space, the system is symmetric under generic rotations of each qubit around the $\hat{z}$ -axis of the Bloch sphere represented in Fig. A.1. In other words, the Schwinger Hamiltonian exhibits a global $U(1)$ symmetry, which corresponds to the conservation of the total charge in the fermionic language, or the total magnetization in the spin basis. We can demonstrate it by computing the commutator between the Pauli matrices involved in the Schwinger Hamiltonian and the operator that describes the rotation around the $\hat{z}$ -axis of an angle α of the m -th site, given by

$$\hat{R}_m^z(\alpha) = e^{i\frac{\alpha}{2}\hat{\sigma}_m^z} = \cos\left(\frac{\alpha}{2}\right)\mathbb{I} - i\sin\left(\frac{\alpha}{2}\right)\hat{\sigma}_m^z$$

We can immediately see that this operator commutes with $\hat{\sigma}_n^z$ and $\hat{\sigma}_n^z\hat{\sigma}_l^z$ even when $n = m$ or $n = m = l$ and so they are performed on the same site, because it does not change the $\hat{z}$ -component of the Bloch vector. Thus, since $\hat{\sigma}_n^z$ commute both with $\mathbb{I}$ and itself, it holds that

$$\left[\hat{R}_m^z(\alpha), \hat{\sigma}_n^z\right] = 0 \quad \forall n, m \qquad \left[\hat{R}_m^z(\alpha), \hat{\sigma}_n^z\hat{\sigma}_l^z\right] = 0 \quad \forall n, l, m$$

Instead, the hopping term containing $\hat{\sigma}_n^+\hat{\sigma}_{n+1}^-$ and its Hermitian conjugate requires a more careful analysis. In particular, in this case the symmetry holds only if we perform the rotation in every site (that is if we perform a global rotation of our chain) or if we consider the full term containing also the sum over n . Considering the case of a global

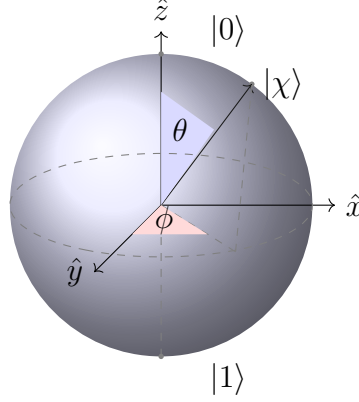


Figure A.1: Visual geometric representation of a single-qubit pure state $|\chi\rangle$ on the Bloch sphere. The computational basis states $|0\rangle$ and $|1\rangle$ are located at the North and South poles along the $\hat{z}$ -axis, respectively. An arbitrary state vector $|\chi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\varphi} \sin \frac{\theta}{2} |1\rangle$ is uniquely specified on the unit sphere by the polar angle $\theta \in [0, \pi]$ (measured relative to the $\hat{z}$ -axis) and the azimuthal phase angle $\phi \in [0, 2\pi)$ (in the xy -plane relative to the $\hat{x}$ -axis).

rotation, we need to work with

$$U_z = \prod_m \hat{R}_m^z(\alpha)$$

Since $\hat{\sigma}_n^+ \hat{\sigma}_{n+1}^-$ and its Hermitian conjugate both commute with $\mathbb{I}$, we only need to perform the commutator with

$$\hat{V}_z = \prod_m \hat{\sigma}_m^z$$

Recalling that $\hat{\sigma}_n^z \hat{\sigma}_n^\pm = \pm \hat{\sigma}_n^\pm$ and that $\hat{\sigma}_n^\pm \hat{\sigma}_n^z = \mp \hat{\sigma}_n^\pm$, we can compute the commutator and show that the symmetry holds. In fact, we obtain that

$$[\hat{V}_z, \hat{\sigma}_n^+ \hat{\sigma}_{n+1}^-] = 0$$

Considering instead the full term containing also the sum over n , when we compute the commutator we immediately see that it vanishes when $m \neq n, n+1$, since we are working on different sites, and that the only terms that survive are those with $m = n$ and $m = n+1$. Thus, we have

$$\sum_n [\hat{\sigma}_n^+ \hat{\sigma}_{n+1}^-, \hat{\sigma}_m^z] = [\hat{\sigma}_n^+, \hat{\sigma}_n^z] \hat{\sigma}_{n+1}^- + \hat{\sigma}_n^+ [\hat{\sigma}_{n+1}^-, \hat{\sigma}_{n+1}^z] = -2\hat{\sigma}_n^+ \hat{\sigma}_{n+1}^- + 2\hat{\sigma}_n^+ \hat{\sigma}_{n+1}^- = 0$$

Summarizing, we have demonstrated that the Schwinger Hamiltonian is symmetric under the application of both a global and a local rotation around the $\hat{z}$ -axis. In particular, this is the only spin symmetry under which the Hamiltonian is invariant in all its terms. Physically, this mathematical property ensures that the simulation remains within the zero-charge subspace throughout the dynamics, provided the initial state is properly prepared in a charge-neutral configuration.

We can also consider the case of a global reflection with respect to the $\hat{x} - \hat{y}$ plane and show that only some terms of the Schwinger Hamiltonian are invariant under this transformation. The global reflection with respect to the $\hat{x} - \hat{y}$ plane can be performed through the action of

$$U_x = \prod_m \hat{\sigma}_m^x$$

Firstly, the commutator with $\hat{\sigma}_n^z$ vanishes when $m \neq n$ since we are working on different sites but when $n = m$ we immediately see that the symmetry does not hold. We can show it by computing

$$[\hat{\sigma}_n^x, \hat{\sigma}_n^z] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 2i\hat{\sigma}_n^y$$

Instead, when we consider the commutator that involves the term $\hat{\sigma}_n^z \hat{\sigma}_l^z$, we can show that the symmetry perfectly holds. In fact, by exploiting the relation $\hat{\sigma}_n^x \hat{\sigma}_n^z = -\hat{\sigma}_n^z \hat{\sigma}_n^x$, it turns out that

$$\begin{aligned} [U_x, \hat{\sigma}_n^z \hat{\sigma}_l^z] &= U_x \cdot \hat{\sigma}_n^z \hat{\sigma}_l^z - \hat{\sigma}_n^z \hat{\sigma}_l^z \cdot U_x = \hat{\sigma}_n^x \hat{\sigma}_n^z \hat{\sigma}_l^x \hat{\sigma}_l^z - \hat{\sigma}_n^z \hat{\sigma}_n^x \hat{\sigma}_l^z \hat{\sigma}_l^x = \\ &= (-\hat{\sigma}_n^z \hat{\sigma}_n^x)(-\hat{\sigma}_l^z \hat{\sigma}_l^x) - \hat{\sigma}_n^z \hat{\sigma}_n^x \hat{\sigma}_l^z \hat{\sigma}_l^x = \hat{\sigma}_n^z \hat{\sigma}_n^x \hat{\sigma}_l^z \hat{\sigma}_l^x - \hat{\sigma}_n^z \hat{\sigma}_n^x \hat{\sigma}_l^z \hat{\sigma}_l^x = 0 \end{aligned}$$

Lastly, when we consider the commutator that involves the term $\hat{\sigma}_n^+ \hat{\sigma}_{n+1}^-$ or its Hermitian conjugate we can show that the symmetry does not hold. To demonstrate it, we can simplify the calculation by considering only the terms with $m = n, n+1$, since when $m \neq n, n+1$ the commutator vanishes. Thus, we have

$$[\hat{\sigma}_n^+ \hat{\sigma}_{n+1}^-, U_x] = \hat{\sigma}_n^+ \hat{\sigma}_{n+1}^- U_x - U_x \hat{\sigma}_n^+ \hat{\sigma}_{n+1}^- = \hat{\sigma}_n^+ \hat{\sigma}_n^x \hat{\sigma}_{n+1}^- \hat{\sigma}_{n+1}^x - \hat{\sigma}_n^x \hat{\sigma}_n^+ \hat{\sigma}_{n+1}^- \hat{\sigma}_{n+1}^x$$

Since $(\hat{\sigma}_n^x)^2 = \mathbb{I}$ and so $(\hat{\sigma}_n^x)^\dagger = \hat{\sigma}_n^x$, we can work on the second term of the commutator by exploiting the conjugation of $\hat{\sigma}_n^\pm$, given by

$$\hat{\sigma}_n^x \hat{\sigma}_n^+ \hat{\sigma}_n^x = \hat{\sigma}_n^- \quad \hat{\sigma}_n^x \hat{\sigma}_n^- \hat{\sigma}_n^x = \hat{\sigma}_n^+$$

and resulting in

$$\hat{\sigma}_n^x \hat{\sigma}_n^+ \hat{\sigma}_{n+1}^- \hat{\sigma}_{n+1}^x = \hat{\sigma}_n^x \hat{\sigma}_n^+ (\hat{\sigma}_n^x)^2 \hat{\sigma}_{n+1}^- \hat{\sigma}_{n+1}^x (\hat{\sigma}_{n+1}^x)^2 = \hat{\sigma}_n^- \hat{\sigma}_n^x \hat{\sigma}_{n+1}^+ \hat{\sigma}_{n+1}^x$$

Putting all these calculations together, we obtain

$$\begin{aligned} [\hat{\sigma}_n^+ \hat{\sigma}_{n+1}^-, U_x] &= \hat{\sigma}_n^+ \hat{\sigma}_n^x \hat{\sigma}_{n+1}^- \hat{\sigma}_{n+1}^x - \hat{\sigma}_n^- \hat{\sigma}_n^x \hat{\sigma}_{n+1}^+ \hat{\sigma}_{n+1}^x = \\ &= (\hat{\sigma}_n^+ - \hat{\sigma}_{n+1}^-) \hat{\sigma}_n^x (\hat{\sigma}_{n+1}^- - \hat{\sigma}_{n+1}^+) \hat{\sigma}_{n+1}^x = (i\hat{\sigma}_n^y) \hat{\sigma}_n^x (-i\hat{\sigma}_{n+1}^y) \hat{\sigma}_{n+1}^x = \\ &= \hat{\sigma}_n^y \hat{\sigma}_n^x \hat{\sigma}_{n+1}^y \hat{\sigma}_{n+1}^x = (-i\hat{\sigma}_n^z) (-i\hat{\sigma}_{n+1}^z) = -\hat{\sigma}_n^z \hat{\sigma}_{n+1}^z \neq 0 \end{aligned}$$

Lastly, when we consider the commutator with the term $H_\pm^{(N-1,0)}$ we can simplify the calculation using the fact that operator acting on different sites always commute. Therefore, we only need to calculate the following two commutators

$$\left[\prod_{l=1}^{N-2} \hat{\sigma}_l^z, \prod_{l=1}^{N-2} \hat{\sigma}_l^x \right] \quad [\hat{\sigma}_{N-1}^+ \hat{\sigma}_0^- - \hat{\sigma}_0^+ \hat{\sigma}_{N-1}^-, \hat{\sigma}_0^x \hat{\sigma}_{N-1}^x]$$

In the first one, we are sure that the symmetry does not hold since $\hat{\sigma}^x$ and $\hat{\sigma}^z$ anticommutes. In the second one, instead, we need to verify that

$$\begin{aligned} & [\hat{\sigma}_{N-1}^+ \hat{\sigma}_0^- + \hat{\sigma}_0^+ \hat{\sigma}_{N-1}^-, \hat{\sigma}_0^x \hat{\sigma}_{N-1}^x] = \\ & = \hat{\sigma}_{N-1}^+ \hat{\sigma}_0^- \hat{\sigma}_0^x \hat{\sigma}_{N-1}^x + \hat{\sigma}_0^+ \hat{\sigma}_{N-1}^- \hat{\sigma}_0^x \hat{\sigma}_{N-1}^x - \hat{\sigma}_0^x \hat{\sigma}_{N-1}^x \hat{\sigma}_{N-1}^+ \hat{\sigma}_0^- - \hat{\sigma}_0^x \hat{\sigma}_{N-1}^x \hat{\sigma}_0^+ \hat{\sigma}_{N-1}^- \end{aligned}$$

and exploiting the conjugation of $\hat{\sigma}_n^\pm$ and the fact that $(\hat{\sigma}_n^x)^2 = \mathbb{I}$, we obtain that

$$\begin{aligned} & \hat{\sigma}_{N-1}^+ \hat{\sigma}_0^- \hat{\sigma}_0^x \hat{\sigma}_{N-1}^x + \hat{\sigma}_0^+ \hat{\sigma}_{N-1}^- \hat{\sigma}_0^x \hat{\sigma}_{N-1}^x - \hat{\sigma}_0^x \hat{\sigma}_{N-1}^x \hat{\sigma}_{N-1}^+ \hat{\sigma}_0^- - \hat{\sigma}_0^x \hat{\sigma}_{N-1}^x \hat{\sigma}_0^+ \hat{\sigma}_{N-1}^- = \\ & = \hat{\sigma}_{N-1}^x \hat{\sigma}_{N-1}^- \hat{\sigma}_0^x \hat{\sigma}_0^+ + \hat{\sigma}_{N-1}^x \hat{\sigma}_{N-1}^+ \hat{\sigma}_0^x \hat{\sigma}_0^- - \hat{\sigma}_{N-1}^- \hat{\sigma}_{N-1}^x \hat{\sigma}_0^+ \hat{\sigma}_0^x - \hat{\sigma}_{N-1}^+ \hat{\sigma}_{N-1}^x \hat{\sigma}_0^- \hat{\sigma}_0^x \end{aligned}$$

Then, exploiting the the following identities

$$\hat{\sigma}_n^x \hat{\sigma}_n^- = \hat{\sigma}_n^+ \hat{\sigma}_n^x = \frac{1}{2} (\mathbb{I}_n + \hat{\sigma}_n^z) \quad \hat{\sigma}_n^x \hat{\sigma}_n^+ = \hat{\sigma}_n^- \hat{\sigma}_n^x = \frac{1}{2} (\mathbb{I}_n - \hat{\sigma}_n^z)$$

we finally compute that

$$[\hat{\sigma}_{N-1}^+ \hat{\sigma}_0^- + \hat{\sigma}_0^+ \hat{\sigma}_{N-1}^-, \hat{\sigma}_0^x \hat{\sigma}_{N-1}^x] = 0$$

Therefore, we can conclude that only certain terms of the Schwinger Hamiltonian are invariant under a global reflection with respect to the $\hat{x} - \hat{y}$ plane.

A.2 Lattice space symmetries

Considering the Kogut-Susskind staggered lattice representation, we have seen that this spatial discretization endows adjacent sites with distinct physical identities based on their index parity: even lattice sites represent the fermionic components of the Dirac field, whereas odd lattice sites correspond to the anti-fermionic components.

We can demonstrate that the closed chain is invariant only under specific spatial reflections across a symmetry axis. To maintain consistency with the staggered fermion framework, this axis must be chosen such that it maps even sites to other even sites and odd sites to other odd sites. This distinction is essential because the staggered discretization assigns distinct physical identities to the lattice sites: even sites represent the electronic component of the field, while odd sites correspond to the positronic component. An axis that exchanges even and odd sites would not constitute a pure spatial symmetry of the encoded Hamiltonian; rather, it would induce a particle-antiparticle swap, which complicates the analysis of the two-component field dynamics. By restricting our study to parity-preserving axes, we ensure that the reflection symmetry strictly acts on the spatial degrees of freedom without shifting the physical interpretation of the matter fields. As an example, the symmetry scheme for a system of size $N=6$ is illustrated in Fig. A.2.

Moreover, we can show that our chain is also invariant under some kinds of rotations. In particular, we can only perform rotations by $2k/N$ fractions of 2π where N is the number of sites in the closed chain and $k \in \mathbb{N}$. As a consequence, only transformations of the form $n \rightarrow (n+2k) \bmod N$ with $k \in \mathbb{N}$ are allowed as shown in Fig. A.3. Instead, the chain is not invariant under transformations of the form $n \rightarrow (n+k) \bmod N$ with $k \in \mathbb{N}$ as shown in Fig. A.4. In fact, by performing this type of transformation we send even sites

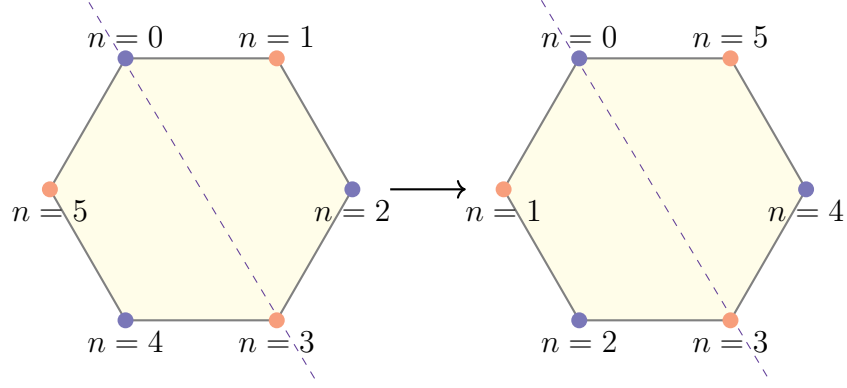


Figure A.2: Allowed reflection of a closed chain with respect to an axis of symmetry

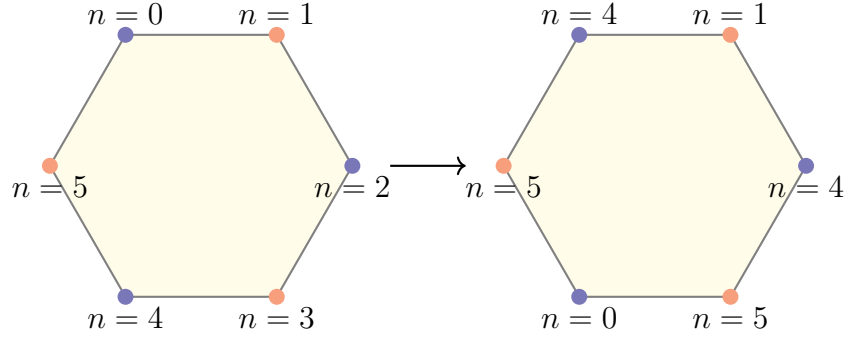


Figure A.3: Rotation of the closed chain under $n \rightarrow (n + 2k) \bmod N$ with $k = 1$

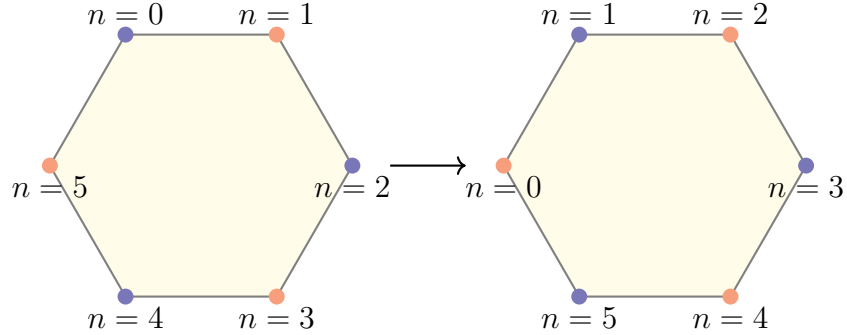


Figure A.4: Rotation of the closed chain under $n \rightarrow (n + k) \bmod N$ with $k = 1$

in odd sites and vice versa: it can cause problems when working with two-component fields where one component is associated with even sites and the other with odd sites, as in our case. However, the local Schwinger Hamiltonian is not invariant under these kinds of symmetries and this is completely due to the fact that the act of mapping a closed chain into a spin system by exploiting the Jordan-Wigner transform breaks them through the introduction of the term $H_{\pm}^{(N-1,0)}$. This term, in fact, represents the key to translate a closed chain into a spin system that has a completely different spatial structure and that, therefore, can not satisfy the lattice symmetries.

Appendix B

Supplementary calculations

B.1 Proof of the Gaussian integral Identity

Proposition B.1.1 (Gaussian integral with quadratic exponent). *Let $a > 0$ and $b, c \in \mathbb{R}$. The Gaussian integral over $\mathbb{R}$ satisfies*

$$\int_{-\infty}^{+\infty} e^{-(ax^2+bx+c)} dx = \sqrt{\frac{\pi}{a}} e^{-\left(c-\frac{b^2}{4a}\right)} \quad (\text{B.1})$$

Proof. In order to demonstrate Eq. (B.1), we begin by completing the square for the quadratic polynomial in the exponent as

$$\begin{aligned} ax^2 + bx + c &= a \left(x^2 + \frac{b}{a}x \right) + c = a \left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} \right) + c = \\ &= a \left(x + \frac{b}{2a} \right)^2 - \frac{b^2}{4a} + c = a \left(x + \frac{b}{2a} \right)^2 + \left(c - \frac{b^2}{4a} \right) \end{aligned}$$

As a consequence, negating this expression yields to

$$-(ax^2 + bx + c) = -a \left(x + \frac{b}{2a} \right)^2 - \left(c - \frac{b^2}{4a} \right) \quad (\text{B.2})$$

Substituting Eq. (B.2) back into the integrand, we notice that we can factor out the constant term independent of x , obtaining that

$$\int_{-\infty}^{+\infty} e^{-(ax^2+bx+c)} dx = \int_{-\infty}^{+\infty} e^{-a\left(x+\frac{b}{2a}\right)^2 - \left(c-\frac{b^2}{4a}\right)} dx = e^{-\left(c-\frac{b^2}{4a}\right)} \int_{-\infty}^{+\infty} e^{-a\left(x+\frac{b}{2a}\right)^2} dx$$

To evaluate the remaining integral, we perform a change of variable as

$$u = \sqrt{a} \left(x + \frac{b}{2a} \right) \implies dx = \frac{du}{\sqrt{a}}$$

As we can see, as $x \rightarrow \pm\infty$, the integration limits for u remain $(-\infty, +\infty)$. Therefore, the integral simplifies to

$$\int_{-\infty}^{+\infty} e^{-a\left(x+\frac{b}{2a}\right)^2} dx = \int_{-\infty}^{+\infty} e^{-u^2} \frac{du}{\sqrt{a}} = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} e^{-u^2} du$$

Recalling the standard Gaussian integral result $\int_{-\infty}^{+\infty} e^{-u^2} du = \sqrt{\pi}$, we obtain

$$\int_{-\infty}^{+\infty} e^{-a(x+\frac{b}{2a})^2} dx = \frac{\sqrt{\pi}}{\sqrt{a}} = \sqrt{\frac{\pi}{a}}$$

Combining this result with the multiplicative constant yields the final identity

$$\int_{-\infty}^{+\infty} e^{-(ax^2+bx+c)} dx = \sqrt{\frac{\pi}{a}} e^{-\left(c-\frac{b^2}{4a}\right)}$$

□

B.2 Complete derivation of H_E^{spin}

$$\begin{aligned} H_E &= \sum_{i=1}^{N-1} \sum_{k=1}^{N-1} \frac{e^2}{2} (N - \max\{i, k\}) \bar{\chi}_i \chi_i \bar{\chi}_k \chi_k + \\ &\quad - \frac{e^2}{2} \sum_{i=0}^{N-1} \sum_{k=1}^{N-1} (N - \max\{i, k\}) (1 - (-1)^k) \bar{\chi}_i \chi_i = \\ &= \sum_{i=1}^{N-1} \sum_{k=1}^{N-1} \frac{e^2}{2} (N - \max\{i, k\}) \frac{1}{4} (\mathbb{I} + \hat{\sigma}_i^z) (\mathbb{I} + \hat{\sigma}_k^z) + \\ &\quad - \frac{e^2}{2} \sum_{i=1}^{N-1} \sum_{k=1}^{N-1} (N - \max\{i, k\}) (1 - (-1)^k) \frac{1}{2} (\mathbb{I} + \hat{\sigma}_i^z) = \\ &\simeq \sum_{i=1}^{N-1} \sum_{k=1}^{N-1} \frac{e^2}{8} (N - \max\{i, k\}) (\hat{\sigma}_i^z + \hat{\sigma}_k^z + \hat{\sigma}_i^z \hat{\sigma}_k^z) + \\ &\quad - \frac{e^2}{4} \sum_{i=1}^{N-1} \sum_{k=0}^{N-1} (N - \max\{i, k\}) (1 - (-1)^k) \hat{\sigma}_i^z = \\ &= \sum_{i=1}^{N-1} \sum_{k=1}^{N-1} \frac{e^2}{4} (N - \max\{i, k\}) \hat{\sigma}_i^z + \sum_{i=1}^{N-1} \sum_{k=1}^{N-1} \frac{e^2}{8} (N - \max\{i, k\}) \hat{\sigma}_i^z \hat{\sigma}_k^z + \\ &\quad - \frac{e^2}{4} \sum_{i=1}^{N-1} \sum_{k=1}^{N-1} (N - \max\{i, k\}) (1 - (-1)^k) \hat{\sigma}_i^z = \\ &= \sum_{i=1}^{N-1} \sum_{k=0}^{N-1} \frac{e^2}{8} (N - \max\{i, k\}) \hat{\sigma}_i^z \hat{\sigma}_k^z + \frac{e^2}{4} \sum_{i=1}^{N-1} \sum_{k=1}^{N-1} (N - \max\{i, k\}) (-1)^k \hat{\sigma}_i^z \\ &= \sum_{i=1}^{N-1} \sum_{k=1}^{N-1} \frac{e^2}{8} (N - \max\{i, k\}) \hat{\sigma}_i^z \hat{\sigma}_k^z + \frac{e^2}{4} \sum_{k=1}^{N-1} (-1)^k \sum_{i=1}^{N-1} (N - \max\{i, k\}) \hat{\sigma}_i^z = \\ &= \sum_{i=1}^{N-1} \sum_{k=1}^i \frac{e^2}{8} (N - i) \hat{\sigma}_i^z \hat{\sigma}_k^z + \sum_{i=1}^{N-1} \sum_{k=i+1}^{N-1} \frac{e^2}{8} (N - k) \hat{\sigma}_i^z \hat{\sigma}_k^z + \\ &\quad + \frac{e^2}{4} \sum_{k=1}^{N-1} (-1)^k \left[\sum_{i=1}^k (N - k) \hat{\sigma}_i^z + \sum_{i=k+1}^{N-1} (N - i) \hat{\sigma}_i^z \right] = \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^{N-1} \sum_{k=i+1}^{N-1} \frac{e^2}{4} (N-k) \hat{\sigma}_i^z \hat{\sigma}_k^z + \frac{e^2}{4} \sum_{k=1}^{N-1} (-1)^k \left[\sum_{l=1}^k (N-k) \hat{\sigma}_l^z + \sum_{l=k+1}^{N-1} \sum_{n=l}^{N-1} \hat{\sigma}_l^z \right] \\
&= \sum_{n=1}^{N-1} \sum_{l=n+1}^{N-1} \frac{e^2}{4} (N-l) \hat{\sigma}_n^z \hat{\sigma}_l^z + \frac{e^2}{4} \sum_{k=1}^{N-1} (-1)^k \left[(N-k) S_k + \sum_{n=k+1}^{N-1} (S_n - S_k) \right] = \\
&= \sum_{n=1}^{N-1} \sum_{l=n+1}^{N-1} \frac{e^2}{4} (N-l) \hat{\sigma}_n^z \hat{\sigma}_l^z + \\
&\quad + \frac{e^2}{4} \sum_{k=1}^{N-1} (-1)^k \left[(N-k) S_k - (N-1-k) S_k + \sum_{n=k+1}^{N-1} S_n \right] = \\
&= \sum_{n=1}^{N-1} \sum_{l=n+1}^{N-1} \frac{e^2}{4} (N-l) \hat{\sigma}_n^z \hat{\sigma}_l^z + \frac{e^2}{4} \sum_{k=1}^{N-1} (-1)^k \left[S_k + \sum_{n=k+1}^{N-1} S_n \right] = \\
&= \sum_{n=1}^{N-1} \sum_{l=n+1}^{N-1} \frac{e^2}{4} (N-l) \hat{\sigma}_n^z \hat{\sigma}_l^z + \frac{e^2}{4} \sum_{k=1}^{N-1} (-1)^k \sum_{n=k}^{N-1} S_n = \\
&= \sum_{n=1}^{N-1} \sum_{l=n+1}^{N-1} \frac{e^2}{4} (N-l) \hat{\sigma}_n^z \hat{\sigma}_l^z + \frac{e^2}{4} \sum_{n=1}^{N-1} \sum_{k=i}^n (-1)^k S_n = \\
&= \sum_{n=1}^{N-1} \sum_{l=n+1}^{N-1} \frac{e^2}{4} (N-l) \hat{\sigma}_n^z \hat{\sigma}_l^z - \frac{e^2}{4} \sum_{n=1}^{N-1} \frac{1 - (-1)^n}{2} S_n = \\
&= \sum_{n=1}^{N-1} \sum_{l=n+1}^{N-1} \frac{e^2}{4} (N-l) \hat{\sigma}_n^z \hat{\sigma}_l^z - \frac{e^2}{4} \sum_{n=1}^{N-1} \frac{1 - (-1)^n}{2} \sum_{l=1}^n \hat{\sigma}_l^z
\end{aligned}$$

B.3 Proof of the first-order Trotter formula

Theorem B.3.1 (First-order Trotter Formula). *Let $H = \sum_{l=1}^L H_l$ be a self-adjoint Hamiltonian on a finite-dimensional Hilbert space, where each H_l is a Hermitian operator. For any total evolution time $\tau > 0$ and any integer number of Trotter steps $n_s \geq 1$, the exact unitary time-evolution operator satisfies the following identity*

$$e^{-i\tau H} = \left(\prod_{l=1}^L e^{-iH_l \tau / n_s} \right)^{n_s} + \mathcal{O}\left(\frac{\tau^2}{n_s}\right) \quad (\text{B.3})$$

Moreover, in the continuum limit $n_s \rightarrow \infty$, the product formula converges to the exact evolution operator.

Proof. In order to prove Eq. (B.3), we start by expanding the exact evolution operator $U(\delta\tau) = e^{-i\delta\tau H}$ for a single time-step in a Taylor series around $\delta\tau = 0$ as

$$\begin{aligned}
U(\delta\tau) &= I - i\delta\tau H - \frac{\delta\tau^2}{2} H^2 + \mathcal{O}(\delta\tau^3) = \\
&= I - i\delta\tau \sum_{l=1}^L H_l - \frac{\delta\tau^2}{2} \left(\sum_{l=1}^L H_l \right)^2 + \mathcal{O}(\delta\tau^3) \quad (\text{B.4})
\end{aligned}$$

By expanding the squared sum, one has that

$$\left(\sum_{l=1}^L H_l\right)^2 = \sum_{l=1}^L H_l^2 + \sum_{1 \leq j < k \leq L} (H_j H_k + H_k H_j)$$

Now, we consider the single-step product of local unitaries $\tilde{U}(\delta t) = \prod_{l=1}^L e^{-i\delta t H_l}$ and we expand it as

$$\tilde{U}(\delta\tau) = \left(I - i\delta\tau H_1 - \frac{\delta\tau^2}{2} H_1^2 + \dots\right) \cdots \left(I - i\delta\tau H_L - \frac{\delta\tau^2}{2} H_L^2 + \dots\right)$$

Multiplying terms order by order up to $\mathcal{O}(\delta\tau^2)$, we find that

$$\tilde{U}(\delta\tau) = I - i\delta\tau \sum_{l=1}^L H_l - \frac{\delta\tau^2}{2} \sum_{l=1}^L H_l^2 - \delta\tau^2 \sum_{1 \leq j < k \leq L} H_j H_k + \mathcal{O}(\delta\tau^3) \quad (\text{B.5})$$

Subtracting the approximate operator $\tilde{U}(\delta\tau)$ in Eq. (B.5) from the exact operator $U(\delta\tau)$ in Eq. (B.4) leads to

$$\begin{aligned} U(\delta\tau) - \tilde{U}(\delta\tau) &= -\frac{\delta\tau^2}{2} \sum_{1 \leq j < k \leq L} (H_j H_k + H_k H_j) + \delta\tau^2 \sum_{1 \leq j < k \leq L} H_j H_k + \mathcal{O}(\delta\tau^3) = \\ &= \frac{\delta\tau^2}{2} \sum_{1 \leq j < k \leq L} (H_j H_k - H_k H_j) + \mathcal{O}(\delta\tau^3) = \\ &= \frac{\delta\tau^2}{2} \sum_{1 \leq j < k \leq L} [H_j, H_k] + \mathcal{O}(\delta\tau^3) \end{aligned}$$

Thus, the local error committed during a single time step is given by

$$U(\delta\tau) - \tilde{U}(\delta\tau) = \mathcal{O}(\delta\tau^2)$$

Let $A = U(\delta\tau)$ and $B = \tilde{U}(\delta\tau)$. Since A and B are unitary operators, $\|A\| = \|B\| = 1$. Therefore, we can exploit the operator telescoping identity given by

$$A^{n_s} - B^{n_s} = \sum_{m=0}^{n_s-1} A^{n_s-1-m} (A - B) B^m$$

and take the operator norm, obtaining

$$\|A^{n_s} - B^{n_s}\| \leq \sum_{m=0}^{n_s-1} \|A^{n_s-1-m}\| \cdot \|A - B\| \cdot \|B^m\| = n_s \cdot \|A - B\|$$

Then, since $\|A - B\| = \mathcal{O}(\delta\tau^2)$, we have that

$$\|A^{n_s} - B^{n_s}\| = n_s \cdot \mathcal{O}(\delta\tau^2)$$

and since $\delta\tau = \tau/n_s$, we conclude that

$$n_s \cdot \mathcal{O}(\delta\tau^2) = n_s \cdot \mathcal{O}\left(\frac{\tau^2}{n_s^2}\right) = \mathcal{O}\left(\frac{\tau^2}{n_s}\right)$$

Reassembling the full expression, we finally find Eq. (B.3).

In order to evaluate Eq. (B.3) in the continuum limit $n_s \rightarrow \infty$, or equivalently $\delta\tau = \tau/n_s \rightarrow 0$), we can exploit the exponential matrix limit identity. We consider the product of local unitaries for a single time step $\delta\tau = \tau/n_s$ and we expand each exponential factor in a Taylor series around $\delta\tau = 0$, obtaining that

$$\begin{aligned}\tilde{U}(\delta\tau) &= (I - i\delta\tau H_1 + \mathcal{O}(\delta\tau^2)) (I - i\delta\tau H_2 + \mathcal{O}(\delta\tau^2)) \cdots (I - i\delta\tau H_L + \mathcal{O}(\delta\tau^2)) = \\ &= I - i\delta\tau \sum_{l=1}^L H_l + \mathcal{O}(\delta\tau^2) = I - i\delta\tau H + \mathcal{O}(\delta\tau^2)\end{aligned}$$

Substituting $\delta\tau = \tau/n_s$ into $\tilde{U}(\delta\tau)$, one has that the expression for a single Trotter step in terms of n_s becomes

$$\tilde{U}\left(\frac{\tau}{n_s}\right) = I - \frac{i\tau H}{n_s} + \mathcal{O}\left(\frac{\tau^2}{n_s^2}\right)$$

Raising this single-step operator to the n_s -th power, we obtain that

$$U_{n_s}(\tau) = \left[\tilde{U}\left(\frac{\tau}{n_s}\right)\right]^{n_s} = \left(I - \frac{i\tau H}{n_s} + \mathcal{O}\left(\frac{\tau^2}{n_s^2}\right)\right)^{n_s}$$

Now, recalling the fundamental operator identity for matrix exponentials, that is

$$\lim_{n \rightarrow \infty} \left(I + \frac{A}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right)^n = e^A$$

and setting $A = -i\tau H$ and $n = n_s$, we can evaluate the continuum limit $n_s \rightarrow \infty$ as

$$\lim_{n_s \rightarrow \infty} U_{n_s}(\tau) = \lim_{n_s \rightarrow \infty} \left(I - \frac{i\tau H}{n_s} + \mathcal{O}\left(\frac{\tau^2}{n_s^2}\right)\right)^{n_s} = e^{-i\tau H}$$

□

B.4 Complete calculations of the Trotterization process

We provide the explicit step-by-step algebraic derivations for the factorization of each term in the Schwinger Hamiltonian. For the first term, we have that

$$\begin{aligned}e^{-i\tau\delta s H_Z} &= e^{-i\tau\delta s \left[\frac{M}{2} \sum_{n=0}^{N-1} (-1)^n \hat{\sigma}_n^z + \frac{e^2 a}{8} \sum_{n=1}^{N-1} (1 - (-1)^n) \sum_{l=1}^n \hat{\sigma}_l^z \right]} = \\ &= \prod_{n=0}^{N-1} e^{-i\tau\delta s \frac{M}{2} (-1)^n \hat{\sigma}_n^z} \prod_{n=1}^{N-1} \prod_{l=1}^n e^{-i\tau\delta s \frac{e^2 a}{8} (1 - (-1)^n) \hat{\sigma}_l^z} = \\ &= \prod_{n=0}^{N-1} e^{-i\alpha \hat{\sigma}_n^z} \prod_{n=1}^{N-1} \prod_{l=1}^n e^{-i\beta \hat{\sigma}_l^z} = \prod_{n=0}^{N-1} O_Z(\alpha) \prod_{n=1}^{N-1} \prod_{l=1}^n O_Z(\beta)\end{aligned}$$

where we assumed $\alpha = \tau\delta s \frac{M}{2}(-1)^n$ and $\beta = \tau\delta s \frac{e^2 a}{8}(1 - (-1)^n)$, and where we called $O_Z(\alpha) = e^{-i\alpha\hat{\sigma}_n^z}$ and $O_Z(\beta) = e^{-i\beta\hat{\sigma}_l^z}$. The second term, instead, can be written as

$$\begin{aligned} e^{-i\tau\delta s H_{ZZ}} &= e^{-i\tau\delta s \sum_{n=1}^{N-1} \sum_{l=n+1}^{N-1} \frac{e^2 a}{4} (N-l) \hat{\sigma}_n^z \hat{\sigma}_l^z} = \prod_{n=1}^{N-1} \prod_{l=n+1}^{N-1} e^{-i\tau\delta s \frac{e^2 a}{4} (N-l) \hat{\sigma}_n^z \hat{\sigma}_l^z} = \\ &= \prod_{n=1}^{N-1} \prod_{l=n+1}^{N-1} e^{-i\theta \hat{\sigma}_n^z \hat{\sigma}_l^z} = \prod_{n=1}^{N-1} \prod_{l=n+1}^{N-1} O_{ZZ}(\theta) \end{aligned}$$

where $\theta = \tau\delta s \frac{e^2 a}{4} (N-l)$ and $O_{ZZ}(\theta) = e^{-i\theta \hat{\sigma}_n^z \hat{\sigma}_l^z}$. Finally, for the third one we have that

$$\begin{aligned} e^{-i\tau\delta s (s\Delta)} &= e^{-i\tau\delta s \cdot s \left[\sum_{n=0}^{N-2} \frac{1}{4a} \left[\hat{\sigma}_n^x \hat{\sigma}_{n+1}^x + \hat{\sigma}_{n+1}^y \hat{\sigma}_n^y \right] + \frac{iN}{4a} \left[\prod_{l=1}^{N-2} \hat{\sigma}_l^z \right] \left(\hat{\sigma}_{N-1}^x \hat{\sigma}_0^x + \hat{\sigma}_0^y \hat{\sigma}_{N-1}^y \right) \right]} = \\ &= e^{-i\tau\delta s \cdot s \sum_{n=0}^{N-2} \frac{1}{4a} \left[\hat{\sigma}_n^x \hat{\sigma}_{n+1}^x + \hat{\sigma}_{n+1}^y \hat{\sigma}_n^y \right]} e^{-i\tau\delta s \cdot s \frac{iN}{4a} \left[\prod_{l=1}^{N-2} \hat{\sigma}_l^z \right] \left(\hat{\sigma}_{N-1}^x \hat{\sigma}_0^x + \hat{\sigma}_0^y \hat{\sigma}_{N-1}^y \right)} = \\ &= \prod_{n=0}^{N-2} e^{-i\tau\delta s \cdot s \frac{1}{4a} \left[\hat{\sigma}_n^x \hat{\sigma}_{n+1}^x + \hat{\sigma}_{n+1}^y \hat{\sigma}_n^y \right]} e^{-i\tau\delta s \cdot s \frac{iN}{4a} \left[\prod_{l=1}^{N-2} \hat{\sigma}_l^z \right] \left(\hat{\sigma}_{N-1}^x \hat{\sigma}_0^x + \hat{\sigma}_0^y \hat{\sigma}_{N-1}^y \right)} = \\ &= \prod_{n=0}^{N-2} e^{-i\lambda \left[\hat{\sigma}_n^x \hat{\sigma}_{n+1}^x + \hat{\sigma}_{n+1}^y \hat{\sigma}_n^y \right]} e^{-i\zeta \left[\prod_{l=1}^{N-2} \hat{\sigma}_l^z \right] \left(\hat{\sigma}_{N-1}^x \hat{\sigma}_0^x + \hat{\sigma}_0^y \hat{\sigma}_{N-1}^y \right)} = \\ &= \prod_{n=0}^{N-2} e^{-i\lambda \hat{\sigma}_n^x \hat{\sigma}_{n+1}^x} e^{-i\lambda \hat{\sigma}_{n+1}^y \hat{\sigma}_n^y} e^{-i\zeta \left[\prod_{l=1}^{N-2} \hat{\sigma}_l^z \right] \left(\hat{\sigma}_{N-1}^x \hat{\sigma}_0^x + \hat{\sigma}_0^y \hat{\sigma}_{N-1}^y \right)} = \\ &= \prod_{n_{\text{even}}=0}^{N-2} \prod_{n_{\text{odd}}=0}^{N-2} e^{-i\lambda \hat{\sigma}_n^x \hat{\sigma}_{n+1}^x} e^{-i\lambda \hat{\sigma}_{n+1}^y \hat{\sigma}_n^y} e^{-i\zeta \left[\prod_{l=1}^{N-2} \hat{\sigma}_l^z \right] \left(\hat{\sigma}_{N-1}^x \hat{\sigma}_0^x + \hat{\sigma}_0^y \hat{\sigma}_{N-1}^y \right)} = \\ &= \left[\prod_{n_{\text{even}}=0}^{N-2} O_{XX} \prod_{n_{\text{odd}}=0}^{N-2} O_{XX} \right] \left[\prod_{n_{\text{even}}=0}^{N-2} O_{YY} \prod_{n_{\text{odd}}=0}^{N-2} O_{YY} \right] O_{\text{PBC}} \end{aligned}$$

where we assumed $\lambda = \tau\delta s \frac{s}{4a}$ and $\zeta = \tau\delta s \frac{iNs}{4a}$, and where we called

$$\begin{aligned} O_{XX} &= e^{-i\lambda \hat{\sigma}_n^x \hat{\sigma}_{n+1}^x} \\ O_{YY} &= e^{-i\lambda \hat{\sigma}_{n+1}^y \hat{\sigma}_n^y} \\ O_{\text{PBC}} &= e^{-i\zeta \left[\prod_{l=1}^{N-2} \hat{\sigma}_l^z \right] \left(\hat{\sigma}_{N-1}^x \hat{\sigma}_0^x + \hat{\sigma}_0^y \hat{\sigma}_{N-1}^y \right)} \end{aligned}$$

Therefore, the complete evolution operator that we have to implement on quantum device is given by

$$\begin{aligned} e^{-i\tau\delta s H(s)} &\simeq \prod_{s=0}^1 \left\{ \left[\prod_{n=0}^{N-1} O_Z(\alpha) \right] \cdot \left[\prod_{n=1}^{N-1} \prod_{l=1}^n O_Z(\beta) \right] \cdot \left[\prod_{n=1}^{N-1} \prod_{l=n+1}^{N-1} O_{ZZ} \right] \cdot \right. \\ &\quad \cdot \left[\prod_{n_{\text{even}}=0}^{N-2} O_{XX} \prod_{n_{\text{odd}}=0}^{N-2} O_{XX} \right] \left[\prod_{n_{\text{even}}=0}^{N-2} O_{YY} \prod_{n_{\text{odd}}=0}^{N-2} O_{YY} \right] O_{\text{PBC}} \left. \right\} + \mathcal{O}\left(\frac{\tau^2}{n_s}\right) \end{aligned}$$

B.5 Derivation of the Electric field Covariance

We report the step-by-step analytical derivation of the correlation function $C(r, l)$ between the electric field E_r at a generic link r and the electric field E_1 at the reference

link $l = 1$. In the Schwinger model on a Kogut-Susskind staggered lattice with PBC, Gauss's law allows us to express the electric field E_r as a cumulative sum of the staggered charge densities of the preceding sites as

$$E_r = \sum_{n=1}^r \rho_n$$

The local staggered charge density operator ρ_n at site n is mapped onto Pauli spin operators σ_z via the Jordan-Wigner transformation as

$$\rho_n = \frac{1}{2} [\hat{\sigma}_n^z + (-1)^n]$$

In particular, since we assumed $E_0 = 0$, for the reference link $r = 1$ the electric field operator coincides with the charge density at site 1

$$E_1 = \sum_{n=1}^1 \rho_n = \rho_1 = \frac{1}{2} [\hat{\sigma}_1^z - 1]$$

The quantum covariance between two observables A and B is defined as

$$\text{Cov}(A, B) \equiv \langle AB \rangle - \langle A \rangle \langle B \rangle \quad (\text{B.6})$$

Substituting $A = E_r = \sum_{n=1}^r \rho_n$ and $B = E_1 = \rho_1$ into Eq. (B.6), we obtain

$$\text{Cov}(E_r, E_1) = \left\langle \left(\sum_{n=1}^r \rho_n \right) \rho_1 \right\rangle - \left\langle \sum_{n=1}^r \rho_n \right\rangle \langle \rho_1 \rangle$$

Exploiting the linearity of the expectation value operator $\langle \cdot \rangle$, we distribute the product in the first term and the summation in the second term as

$$\begin{aligned} \left\langle \left(\sum_{n=1}^r \rho_n \right) \rho_1 \right\rangle &= \left\langle \sum_{n=1}^r (\rho_n \rho_1) \right\rangle = \sum_{n=1}^r \langle \rho_n \rho_1 \rangle \\ \left\langle \sum_{n=1}^r \rho_n \right\rangle \langle \rho_1 \rangle &= \left(\sum_{n=1}^r \langle \rho_n \rangle \right) \langle \rho_1 \rangle = \sum_{n=1}^r \langle \rho_n \rangle \langle \rho_1 \rangle \end{aligned}$$

Grouping the terms within a single summation, the electric field covariance reduces to the sum of the local charge density covariances as

$$\text{Cov}(E_r, E_1) = \sum_{n=1}^r (\langle \rho_n \rho_1 \rangle - \langle \rho_n \rangle \langle \rho_1 \rangle) = \sum_{n=1}^r \text{Cov}(\rho_n, \rho_1). \quad (\text{B.7})$$

We then proceed to express each term of $\text{Cov}(\rho_n, \rho_1)$ individually in terms of $\hat{\sigma}^z$ matrices.

- **Expectation value of the product $\langle \rho_n \rho_1 \rangle$**

$$\begin{aligned} \rho_n \rho_1 &= \frac{1}{2} [\hat{\sigma}_n^z + (-1)^n] \cdot \frac{1}{2} [\hat{\sigma}_1^z - 1] \\ &= \frac{1}{4} [\hat{\sigma}_n^z \hat{\sigma}_1^z - \hat{\sigma}_n^z + (-1)^n \hat{\sigma}_1^z - (-1)^n]. \end{aligned}$$

Taking the expectation value on both sides, one has that

$$\langle \rho_n \rho_1 \rangle = \frac{1}{4} [\langle \hat{\sigma}_n^z \hat{\sigma}_1^z \rangle - \langle \hat{\sigma}_n^z \rangle + (-1)^n \langle \hat{\sigma}_1^z \rangle - (-1)^n] \quad (\text{B.8})$$

- **Product of expectation values** $\langle \rho_n \rangle \langle \rho_1 \rangle$

$$\langle \rho_n \rangle = \frac{1}{2} [\langle \hat{\sigma}_n^z \rangle + (-1)^n] \quad \langle \rho_1 \rangle = \frac{1}{2} [\langle \hat{\sigma}_1^z \rangle - 1]$$

Multiplying the two expressions, one has that

$$\begin{aligned} \langle \rho_n \rangle \langle \rho_1 \rangle &= \frac{1}{4} [\langle \hat{\sigma}_n^z \rangle + (-1)^n] [\langle \hat{\sigma}_1^z \rangle - 1] \\ &= \frac{1}{4} [\langle \hat{\sigma}_n^z \rangle \langle \hat{\sigma}_1^z \rangle - \langle \hat{\sigma}_n^z \rangle + (-1)^n \langle \hat{\sigma}_1^z \rangle - (-1)^n] \end{aligned} \quad (\text{B.9})$$

Subtracting Eq. (B.9) from Eq. (B.8), we observe that some of the linear expectation terms are identical in both factors and cancel out exactly, leading us to

$$\begin{aligned} \text{Cov}(\rho_n, \rho_1) &= \langle \rho_n \rho_1 \rangle - \langle \rho_n \rangle \langle \rho_1 \rangle = \\ &= \frac{1}{4} (\langle \hat{\sigma}_n^z \hat{\sigma}_1^z \rangle - \langle \hat{\sigma}_n^z \rangle \langle \hat{\sigma}_1^z \rangle) = \\ &= \frac{1}{4} \text{Cov}(\hat{\sigma}_n^z, \hat{\sigma}_1^z) \end{aligned}$$

Substituting the obtained expression into Eq. (B.7), we arrive at the final explicit result for the electric field covariance

$$\text{Cov}(E_r, E_1) = \frac{1}{4} \sum_{n=1}^r (\langle \hat{\sigma}_n^z \hat{\sigma}_1^z \rangle - \langle \hat{\sigma}_n^z \rangle \langle \hat{\sigma}_1^z \rangle)$$

The explicit expressions for the separations evaluated in this work ($r = 1, 2, 3$) are reported below.

- **Autocorrelation** ($r = 1$)

$$\text{Cov}(E_1, E_1) = \frac{1}{4} (1 - \langle \hat{\sigma}_1^z \rangle^2)$$

- **Nearest-neighbor correlation** ($r = 2$)

$$\text{Cov}(E_2, E_1) = \frac{1}{4} (1 - \langle \hat{\sigma}_1^z \rangle^2) + \frac{1}{4} (\langle \hat{\sigma}_2^z \hat{\sigma}_1^z \rangle - \langle \hat{\sigma}_2^z \rangle \langle \hat{\sigma}_1^z \rangle)$$

- **Next-to-nearest-neighbor correlation** ($r = 3$)

$$\text{Cov}(E_3, E_1) = \text{Cov}(E_2, E_1) + \frac{1}{4} (\langle \hat{\sigma}_3^z \hat{\sigma}_1^z \rangle - \langle \hat{\sigma}_3^z \rangle \langle \hat{\sigma}_1^z \rangle)$$

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